ECSE 543
Assignment 2

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Contents

Introduction

1 Finite Element Triangles

The equation for the α parameter for a general vertex i of a finite element triangle can be seen in Equation (1), where $i+1$ and $i+2$ implicitly wraps around when exceeding 3.

$$\begin{aligned}\alpha_i(x,y) = \frac{1}{2A}\big[&(x_{i+1}y_{i+2} - x_{i+2}y_{i+1}) \\ &+ (y_{i+1} - y_{i+2})x \\ &+ (x_{i+2} - x_{i+1})y\big]\end{aligned} \tag{1}$$

Using Equation (1), we can solve for the entries of the local S matrix, as shown in Equation (2). This was used in the program to compute every entry for both example triangles.

$$\begin{aligned}S_{ij} &= \int_{\Delta_e} \nabla\alpha_i \cdot \nabla\alpha_j dS \\ &= \frac{1}{4A}\big[(y_{i+1} - y_{i+2})(y_{j+1} - y_{j+2}) \\ &\quad + (x_{i+2} - x_{i+1})(x_{j+2} - x_{j+1})\big]\end{aligned} \tag{2}$$

The local S matrix for the first triangle can be seen in Equation (3).

$$S_1 = \begin{bmatrix} 0.5 & -0.5 & 0.0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix} \tag{3}$$

The local S matrix for the second triangle can be seen in Equation (4).

$$S_2 = \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{bmatrix} \tag{4}$$

The disjoint S matrix is then given by the following:

$$S_{dis} = \begin{bmatrix} 0.5 & -0.5 & 0.0 & 0 & 0 & 0 \\ -0.5 & 1.0 & -0.5 & 0 & 0 & 0 \\ 0.0 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & -0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0.0 \\ 0 & 0 & 0 & -0.5 & 0.0 & 0.5 \end{bmatrix}$$

The connectivity matrix C is given by Equation (5).

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5}$$

The global matrix S is then given by Equation (6).

$$S = C^T S_{dis} C^T \tag{6}$$

Using Equations (5) and (6), we can solve for the global S matrix, giving the value shown in Equation (7), which is computed by the `finite_element_triangles.py` script shown in Listing 3.

$$S = \begin{bmatrix} 1.0 & -0.5 & 0.0 & -0.5 \\ -0.5 & 1.0 & -0.5 & 0.0 \\ 0.0 & -0.5 & 1.0 & -0.5 \\ -0.5 & 0.0 & -0.5 & 1.0 \end{bmatrix} \tag{7}$$

2 Finite Element Coaxial Cable

2.a Mesh

The mesh to be used by the `SIMPLE2D` program is generated by the `finite_element_mesh_generator.py` script shown in Listing 5. This input and output files of the `SIMPLE2D` program are shown in Listings 13 and 14 of Appendix C.

2.b Electrostatic Potential

Based on the results from the `SIMPLE2D` program, the potential at (0.06, 0.04) is 5.5263 V. This corresponds to node 16 in the mesh arrangement we created.

2.c Capacitance

The finite element functional equation for two conjoint finite element triangles forming a square i can be seen in Equation (8).

$$W_i = \frac{1}{2} U_{con_i}^T S U_{con_i} \tag{8}$$

where S is given in Equation (7) and U_{con_i} is the conjoint potential vector for square i, giving the potential at the four corners of the square defining the combination of two finite element triangles. This can be seen in Equation (9).

$$U_{con} = \begin{bmatrix} U_{i_1} \\ U_{i_2} \\ U_{i_3} \\ U_{i_4} \end{bmatrix} \tag{9}$$

To find the total energy function W of the mesh, we must add the contributions from each square and multiply by 4, since our mesh is one quarter of the entire coaxial cable. This yields Equation (10).

$$W = 4\sum_{i}^{N} W_i = 2\sum_{i}^{N} U_{con_i}^T S U_{con_i} \tag{10}$$

where N is the number of finite difference squares in the mesh.

Note that W is not equal to the energy. The relation between the energy per unit length E and W is shown in Equation (11).

$$E = \epsilon_0 W \tag{11}$$

We then know that the energy per unit length E is related to the capacitance per unit length C as shown in Equation (12).

$$E = \frac{1}{2} C V^2 \tag{12}$$

where V is the voltage across the coaxial cable.

Combining Equations (8) and (10) to (12), we obtain an expression for the capacitance per unit length which can be easily calculated, as shown in Equation 13.

$$C = \frac{2E}{V^2} = \frac{4\epsilon_0}{V^2} \sum_{i}^{N} U_{con_i}^T S U_{con_i} \tag{13}$$

The capacitance per unit length is computed as $5.2137 \times 10^{-11}\,\mathrm{F/m}$ by the `finite_element_capacitance.py` script shown in Listing 6 with output shown in Listing 11.

3 Conjugate Gradient Coaxial Cable

3.a Positive Definite Test

To form the A matrix, we must consider all the free nodes in the mesh. The potential at the non-boundary free nodes is given by Equation (14).

$$-4\phi_{i,j} + \phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} = 0 \tag{14}$$

The free nodes along a boundary must satisfy the Neumann boundary condition for symmetry. Since our quarter-mesh is the bottom left corner of the overall mesh, these boundary nodes defining planes of symmetry are along the top and the right. The Neumann boundary condition for the top nodes is given by Equation (15) and that for the right nodes is given by Equation (16).

$$\phi_{i,j+1} - \phi_{i,j-1} = 0 \tag{15}$$

$$\phi_{i+1,j} - \phi_{i-1,j} = 0 \tag{16}$$

Now, the simplified potential for boundary free nodes can be calculated, as seen in Equations (17) and (18).

$$-4\phi_{i,j} + \phi_{i+1,j} + \phi_{i-1,j} + 2\phi_{i,j-1} = 0 \tag{17}$$

$$-4\phi_{i,j} + 2\phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} = 0 \tag{18}$$

The non-free nodes are fixed by the potentials of the conductors, i.e., 15 V and 0 V.

With Equations (14), (17) and (18), we can form the A matrix from every mesh node. This is done in `finite_difference_mesh_generator.py`, as shown in Listing 8. The output A matrix can be seen in Listing 12.

If the matrix A is not positive definite, one can simply multiply both sides of the $Ax = b$ equation by A^T, forming a new equation $A^T Ax = A^T b$. This is equivalent to $A'x = b'$, where $b' = A^T b$ and $A' = A^T A$. Here, A' is now positive definite.

In our case, the matrix A is indeed not positive definite, and multiplying by A^T made it positive definite. The before and after positive definite test can be seen in Listing 12.

3.b Matrix Solution

The matrix equation to be solved can be seen in Equation (19), where A is positive-definite matrix generated previously, ϕ_c is the unknown potential vector and b contains the initial potential values along the boundaries.

$$A\phi_c = b \tag{19}$$

3.c Residual Norm

Consider a vector $\mathbf{v} = \{v_1, \ldots, v_n\}$. The infinity norm $\|\mathbf{v}\|_\infty$ of $\mathbf{v}$ is given by the maximum absolute element of $\mathbf{v}$, as shown in Equation (20).

$$\|\mathbf{v}\|_\infty = \max\{|v_1|, \ldots, |v_n|\} \tag{20}$$

Similarly, the 2-norm $\|\mathbf{v}\|_2$ of $\mathbf{v}$ is given by Equation (21).

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{21}$$

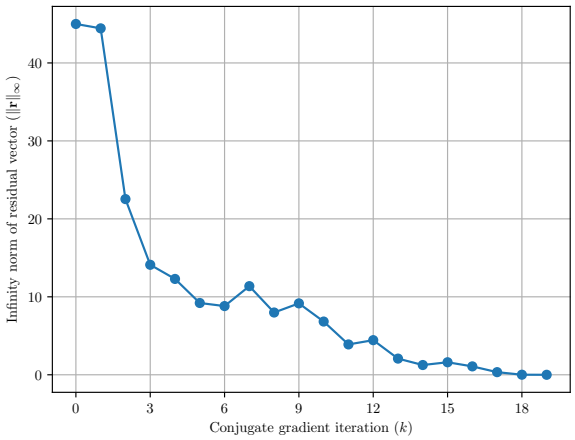

Figure 1: Value of the infinity norm of the residual vector versus iterations of the conjugate gradient algorithm.

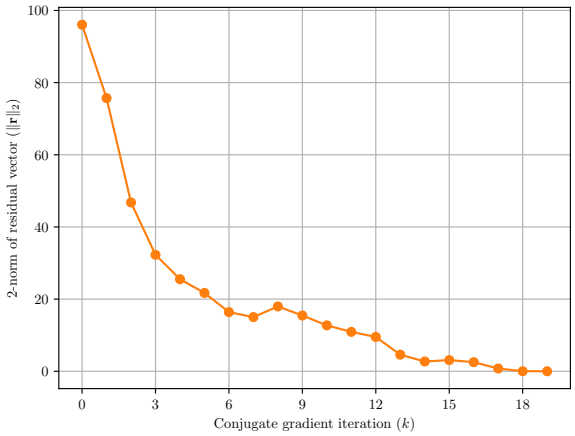

Figure 2: Value of the 2-norm of the residual vector versus iterations of the conjugate gradient algorithm.

3.d Potential Comparison

3.e Capacitance Computation

The capacitance can be calculated in the same way as in Question 2(c), i.e., with Equation (13). The node values must simply be mapped to same mesh used in the finite difference context.

A Code Listings

Listing 1: Custom matrix package (`matrices.py`).

```
from __future__ import division

import copy
import csv
from ast import literal_eval

import math


class Matrix:
    def __init__(self, data):
        self.data = data
        self.num_rows = len(data)
        self.num_cols = len(data[0])

    def __str__(self):
        string = ''
        for row in self.data:
            string += '\n'
            for val in row:
                string += '{:6.2f} '.format(val)
        return string

    def integer_string(self):
        string = ''
        for row in self.data:
            string += '\n'
            for val in row:
                string += '{:3.0f} '.format(val)
        return string

    def precision_string(self):
        string = ''
        for row in self.data:
            string += '\n'
            for val in row:
                string += '{:6.4f} '.format(val)
        return string

    def __add__(self, other):
        if len(self) != len(other) or len(self[0]) != len(other[0]):
            raise ValueError('Incompatible matrix sizes for addition. Matrix A is {}x{}, but matrix B is
            ↪  {}x{}.'
                             .format(len(self), len(self[0]), len(other), len(other[0])))

        return Matrix([[self[row][col] + other[row][col] for col in range(self.num_cols)]
                       for row in range(self.num_rows)])

    def __sub__(self, other):
        if len(self) != len(other) or len(self[0]) != len(other[0]):
            raise ValueError('Incompatible matrix sizes for subtraction. Matrix A is {}x{}, but matrix B
            ↪  is {}x{}.'
                             .format(len(self), len(self[0]), len(other), len(other[0])))

        return Matrix([[self[row][col] - other[row][col] for col in range(self.num_cols)]
                       for row in range(self.num_rows)])

    def __mul__(self, other):
        if type(other) == float or type(other) == int:
            return self.scalar_multiply(other)

        if self.num_cols != other.num_rows:
            raise ValueError('Incompatible matrix sizes for multiplication. Matrix A is {}x{}, but matrix
            ↪  B is {}x{}.'
                             .format(self.num_rows, self.num_cols, other.num_rows, other.num_cols))
```

```
63
64          # Inspired from https://en.wikipedia.org/wiki/Matrix_multiplication
65          product = Matrix.empty(self.num_rows, other.num_cols)
66          for i in range(self.num_rows):
67              for j in range(other.num_cols):
68                  row_sum = 0
69                  for k in range(self.num_cols):
70                      row_sum += self[i][k] * other[k][j]
71                  product[i][j] = row_sum
72          return product
73
74      def scalar_multiply(self, scalar):
75          return Matrix([[self[row][col] * scalar for col in range(self.num_cols)] for row in
            ↪  range(self.num_rows)])
76
77      def __div__(self, other):
78          """
79          Element-wise division.
80          """
81          if self.num_rows != other.num_rows or self.num_cols != other.num_cols:
82              raise ValueError('Incompatible matrix sizes.')
83          return Matrix([[self[row][col] / other[row][col] for col in range(self.num_cols)]
84                         for row in range(self.num_rows)])
85
86      def __neg__(self):
87          return Matrix([[-self[row][col] for col in range(self.num_cols)] for row in range(self.num_rows)])
88
89      def __deepcopy__(self, memo):
90          return Matrix(copy.deepcopy(self.data))
91
92      def __getitem__(self, item):
93          return self.data[item]
94
95      def __len__(self):
96          return len(self.data)
97
98      def item(self):
99          """
100         :return: the single element contained by this matrix, if it is 1x1.
101         """
102         if not (self.num_rows == 1 and self.num_cols == 1):
103             raise ValueError('Matrix is not 1x1')
104         return self.data[0][0]
105
106     def is_positive_definite(self):
107         """
108         :return: True if the matrix if positive-definite, False otherwise.
109         """
110         A = copy.deepcopy(self.data)
111         for j in range(self.num_rows):
112             if A[j][j] <= 0:
113                 return False
114             A[j][j] = math.sqrt(A[j][j])
115             for i in range(j + 1, self.num_rows):
116                 A[i][j] = A[i][j] / A[j][j]
117                 for k in range(j + 1, i + 1):
118                     A[i][k] = A[i][k] - A[i][j] * A[k][j]
119         return True
120
121     def transpose(self):
122         """
123         :return: the transpose of the current matrix
124         """
125         return Matrix([[self.data[row][col] for row in range(self.num_rows)] for col in
            ↪  range(self.num_cols)])
126
127     def mirror_horizontal(self):
128         """
129         :return: the horizontal mirror of the current matrix
130         """
```

```
131             return Matrix([[self.data[self.num_rows - row - 1][col] for col in range(self.num_cols)]
132                            for row in range(self.num_rows)])
133
134     def empty_copy(self):
135         """
136         :return: an empty matrix of the same size as the current matrix.
137         """
138         return Matrix.empty(self.num_rows, self.num_cols)
139
140     def infinity_norm(self):
141         if self.num_cols > 1:
142             raise ValueError('Not a column vector.')
143         return max([abs(x) for x in self.transpose()[0]])
144
145     def two_norm(self):
146         if self.num_cols > 1:
147             raise ValueError('Not a column vector.')
148         return math.sqrt(sum([x ** 2 for x in self.transpose()[0]]))
149
150     def save_to_csv(self, filename):
151         """
152         Saves the current matrix to a CSV file.
153
154         :param filename: the name of the CSV file
155         """
156         with open(filename, "wb") as f:
157             writer = csv.writer(f)
158             for row in self.data:
159                 writer.writerow(row)
160
161     def save_to_latex(self, filename):
162         """
163         Saves the current matrix to a latex-readable matrix.
164
165         :param filename: the name of the CSV file
166         """
167         with open(filename, "wb") as f:
168             for row in range(self.num_rows):
169                 for col in range(self.num_cols):
170                     f.write('{}'.format(self.data[row][col]))
171                     if col < self.num_cols - 1:
172                         f.write('& ')
173                 if row < self.num_rows - 1:
174                     f.write('\\\\\n')
175
176     @staticmethod
177     def multiply(*matrices):
178         """
179         Computes the product of the given matrices.
180
181         :param matrices: the matrix objects
182         :return: the product of the given matrices
183         """
184         n = matrices[0].rows
185         product = Matrix.identity(n)
186         for matrix in matrices:
187             product = product * matrix
188         return product
189
190     @staticmethod
191     def empty(num_rows, num_cols):
192         """
193         Returns an empty matrix (filled with zeroes) with the specified number of columns and rows.
194
195         :param num_rows: number of rows
196         :param num_cols: number of columns
197         :return: the empty matrix
198         """
199         return Matrix([[0 for _ in range(num_cols)] for _ in range(num_rows)])
200
```

```
    @staticmethod
    def identity(n):
        """
        Returns the identity matrix of the given size.

        :param n: the size of the identity matrix (number of rows or columns)
        :return: the identity matrix of size n
        """
        return Matrix.diagonal_single_value(1, n)

    @staticmethod
    def diagonal(values):
        """
        Returns a diagonal matrix with the given values along the main diagonal.

        :param values: the values along the main diagonal
        :return: a diagonal matrix with the given values along the main diagonal
        """
        n = len(values)
        return Matrix([[values[row] if row == col else 0 for col in range(n)] for row in range(n)])

    @staticmethod
    def diagonal_single_value(value, n):
        """
        Returns a diagonal matrix of the given size with the given value along the diagonal.

        :param value: the value of each element on the main diagonal
        :param n: the size of the matrix
        :return: a diagonal matrix of the given size with the given value along the diagonal.
        """
        return Matrix([[value if row == col else 0 for col in range(n)] for row in range(n)])

    @staticmethod
    def column_vector(values):
        """
        Transforms a row vector into a column vector.

        :param values: the values, one for each row of the column vector
        :return: the column vector
        """
        return Matrix([[value] for value in values])

    @staticmethod
    def csv_to_matrix(filename):
        """
        Reads a CSV file to a matrix.

        :param filename: the name of the CSV file
        :return: a matrix containing the values in the CSV file
        """
        with open(filename, 'r') as csv_file:
            reader = csv.reader(csv_file)
            data = []
            for row_number, row in enumerate(reader):
                data.append([literal_eval(val) for val in row])
            return Matrix(data)
```

Listing 2: Question 1 (`q1.py`).

```
from finite_element_triangles import Triangle, find_local_s_matrix, find_global_s_matrix
from matrices import Matrix


def q1():
    print('\n=== Question 1 ===')
    S1 = build_triangle_and_find_local_S(
        [0, 0, 0.02],
        [0.02, 0, 0])
    S1.save_to_latex('report/matrices/S1.txt')
```

```
    print('S1: {}'.format(S1))

    S2 = build_triangle_and_find_local_S(
        [0.02, 0, 0.02],
        [0.02, 0.02, 0])
    S2.save_to_latex('report/matrices/S2.txt')
    print('S2: {}'.format(S2))

    C = Matrix([
        [1, 0, 0, 0],
        [0, 1, 0, 0],
        [0, 0, 1, 0],
        [0, 0, 0, 1],
        [1, 0, 0, 0],
        [0, 0, 1, 0]])
    C.save_to_latex('report/matrices/C.txt')
    print('C: {}'.format(C))

    S = find_global_s_matrix(S1, S2, C)
    S.save_to_latex('report/matrices/S.txt')
    S.save_to_csv('report/csv/S.txt')
    print('S: {}'.format(S))


def build_triangle_and_find_local_S(x, y):
    triangle = Triangle(x, y)
    S = find_local_s_matrix(triangle)
    return S


if __name__ == '__main__':
    q1()
```

Listing 3: Finite element triangles (`finite_element_triangles.py`).

```
from __future__ import division

from matrices import Matrix


class Triangle:
    def __init__(self, x, y):
        self.x = x
        self.y = y
        self.area = (x[1] * y[2] - x[2] * y[1] - x[0] * y[2] + x[2] * y[0] + x[0] * y[1] - x[1] * y[0]) /
        ↪  2


def find_local_s_matrix(triangle):
    x = triangle.x
    y = triangle.y
    S = Matrix.empty(3, 3)

    for i in range(3):
        for j in range(3):
            S[i][j] = ((y[(i + 1) % 3] - y[(i + 2) % 3]) * (y[(j + 1) % 3] - y[(j + 2) % 3])
                       + (x[(i + 1) % 3] - x[(i + 2) % 3]) * (x[(j + 1) % 3] - x[(j + 2) % 3])) / (4 *
                        ↪  triangle.area)

    return S


def find_global_s_matrix(S1, S2, C):
    S_dis = find_disjoint_s_matrix(S1, S2)
    S_dis.save_to_latex('report/matrices/S_dis.txt')
    print('S_dis: {}'.format(S_dis))
    return C.transpose() * S_dis * C
```

```
def find_disjoint_s_matrix(S1, S2):
    n = len(S1)
    S_dis = Matrix.empty(2 * n, 2 * n)
    for row in range(n):
        for col in range(n):
            S_dis[row][col] = S1[row][col]
            S_dis[row + n][col + n] = S2[row][col]
    return S_dis
```

Listing 4: Question 2 (`q2.py`).

```
from finite_element_capacitance import find_capacitance
from matrices import Matrix
from finite_element_mesh_generator import generate_simple_2d_mesh

INNER_CONDUCTOR_POINTS = [28, 29, 30, 34]
OUTER_CONDUCTOR_POINTS = [1, 2, 3, 4, 5, 6, 7, 13, 19, 25, 31]

MESH_SIZE = 6


def q2():
    print('\n=== Question 2 ===')
    q2a()
    q2c()


def q2a():
    generate_simple_2d_mesh(MESH_SIZE, INNER_CONDUCTOR_POINTS, OUTER_CONDUCTOR_POINTS)


def q2c():
    print('\n=== Question 2(c) ===')
    S = Matrix.csv_to_matrix('report/csv/S.txt')
    voltage = 15
    capacitance = find_capacitance(S, voltage, MESH_SIZE)
    print('Capacitance per unit length: {} F/m'.format(capacitance))


if __name__ == '__main__':
    q2()
```

Listing 5: Finite element mesh generator (`finite_element_mesh_generator.py`).

```
def generate_simple_2d_mesh(mesh_size, inner_conductor_points, outer_conductor_points):
    with open('simple2d/mesh.dat', 'w') as f:
        generate_node_positions(f, mesh_size)
        generate_triangle_coordinates(f, mesh_size)
        generate_initial_potentials(f, inner_conductor_points, outer_conductor_points)


def generate_node_positions(f, mesh_size):
    for row in range(mesh_size):
        y = row * 0.02
        for col in range(mesh_size):
            x = col * 0.02
            node = row * mesh_size + (col + 1)
            if node <= 34:  # Inner conductor
                f.write('{} {} {}\n'.format(node, x, y))
    f.write('\n')


def generate_triangle_coordinates(f, mesh_size):
    # Left triangles (left halves of squares)
    for row in range(mesh_size - 1):
        for col in range(mesh_size - 1):
            node = row * mesh_size + (col + 1)
            if node < 28:
```

```
                f.write('{} {} {} 0\n'.format(node, node + 1, node + mesh_size))

    # Right triangles (right halves of squares)
    for row in range(mesh_size - 1):
        for col in range(1, mesh_size):
            node = row * mesh_size + (col + 1)
            if node <= 28:
                f.write('{} {} {} 0\n'.format(node, node + mesh_size - 1, node + mesh_size))

    f.write('\n')


def generate_initial_potentials(f, inner_conductor_points, outer_conductor_points):
    for point in outer_conductor_points:
        f.write('{} {}\n'.format(point, 0))
    for point in inner_conductor_points:
        f.write('{} {}\n'.format(point, 15))
```

Listing 6: Finite element capacitance (`finite_element_capacitance.py`).

```
from matrices import Matrix

E_0 = 8.854187817620E-12


def extract_mesh():
    with open('simple2d/result.dat') as f:
        mesh = {}
        for line_number, line in enumerate(f):
            if line_number >= 2:
                vals = line.split()
                node = int(float(vals[0]))
                voltage = float(vals[3])
                mesh[node] = voltage
    return mesh


def compute_half_energy(S, mesh, mesh_size):
    U_con = Matrix.empty(4, 1)
    half_energy = 0
    for row in range(mesh_size - 1):
        for col in range(mesh_size - 1):
            node = row * mesh_size + (col + 1)  # 1-based
            if node < 28:
                U_con[0][0] = mesh[node + mesh_size]
                U_con[1][0] = mesh[node]
                U_con[2][0] = mesh[node + 1]
                U_con[3][0] = mesh[node + mesh_size + 1]
                half_energy_contribution = U_con.transpose() * S * U_con
                half_energy += half_energy_contribution[0][0]
    return half_energy


def find_capacitance(S, voltage, mesh_size):
    mesh = extract_mesh()
    half_energy = compute_half_energy(S, mesh, mesh_size)
    capacitance = (4 * E_0 * half_energy) / voltage ** 2
    return capacitance
```

Listing 7: Question 3 (`q3.py`).

```
from copy import deepcopy

import matplotlib.pyplot as plt

from matplotlib import rc
from matplotlib.ticker import MaxNLocator

```

```
from choleski import choleski_solve
from conjugate_gradient import conjugate_gradient_solve
from finite_difference_mesh_generator import generate_finite_diff_mesh

MESH_SIZE = 6
NUM_FREE_NODES = 19
rc('font', **{'family': 'serif', 'serif': ['Computer Modern']})
rc('text', usetex=True)

def q3():
    print('\n=== Question 3 ===')
    A, b = q3a()
    choleski_potential, cg_potential, residual_vectors = q3b(A, b)
    q3c(residual_vectors)
    q3d(choleski_potential, cg_potential)


def q3a():
    print('\n=== Question 3(a) ===')
    A, b = generate_finite_diff_mesh(MESH_SIZE, NUM_FREE_NODES)
    print('A: {}'.format(A.integer_string()))
    print('b: {}'.format(b.integer_string()))
    print('A is positive definite: {}'.format(A.is_positive_definite()))
    A_prime = A.transpose() * A
    b_prime = A.transpose() * b
    print("A' is positive definite: {}".format(A_prime.is_positive_definite()))
    return A_prime, b_prime


def q3b(A, b):
    print('\n=== Question 3(b) ===')
    A_copy = deepcopy(A)
    b_copy = deepcopy(b)
    x_choleski = choleski_solve(A_copy, b_copy)
    print('Choleski x: {}'.format(x_choleski))
    residual_vectors = []
    x_cg = conjugate_gradient_solve(A, b, residual_vectors)
    print('Conjugate gradient x: {}'.format(x_cg))
    node_6_4 = 7
    return x_choleski[node_6_4][0], x_cg[node_6_4][0], residual_vectors


def q3c(residual_vectors):
    print('\n=== Question 3(c) ===')
    plot_residual_norms(residual_vectors, infinity_norm=False)
    plot_residual_norms(residual_vectors, infinity_norm=True)


def q3d(choleski_potential, cg_potential):
    print('\n=== Question 3(d) ===')
    print('Choleski potential at (0.06, 0.04): {} V'.format(choleski_potential))
    print('Conjugate gradient potential at (0.06, 0.04): {} V'.format(cg_potential))


def plot_residual_norms(residual_vectors, infinity_norm=False):
    f = plt.figure()
    ax = f.gca()
    ax.xaxis.set_major_locator(MaxNLocator(integer=True))
    x_range = [i for i in range(len(residual_vectors))]
    y_range = [v.infinity_norm() if infinity_norm else v.two_norm() for v in residual_vectors]
    plt.plot(x_range, y_range, 'o-{}'.format('C0' if infinity_norm else 'C1'),
             label=''.format('Infinity norm' if infinity_norm else '2-norm'))
    plt.xlabel('Conjugate gradient iteration ($k$)')
    plt.ylabel('Infinity norm of residual vector $(\\|\\textbf{r}\\|_\\infty)$' if infinity_norm
               else '2-norm of residual vector $(\\|\\textbf{r}\\|_2)$')
    plt.grid(True)
    f.savefig('report/plots/q3c_{}.pdf'.format('infinity' if infinity_norm else '2'), bbox_inches='tight')


if __name__ == '__main__':
```

```
78      q3()
```

Listing 8: Finite difference mesh generator (`finite_difference_mesh_generator.py`).

```
1   from matrices import Matrix
2
3
4   def generate_finite_diff_mesh(mesh_size, num_free_nodes):
5       A = Matrix.empty(num_free_nodes, num_free_nodes)
6       b = Matrix.empty(num_free_nodes, 1)
7       for row in range(mesh_size - 3):
8           for col in range(mesh_size - 1):
9               node = row * (mesh_size - 1) + col
10              A[node][node] = -4
11
12              if row != 0:
13                  A[node][node - mesh_size + 1] = 1
14              if 12 <= node <= 14:
15                  b[node][0] = -15
16              else:
17                  A[node][node + mesh_size - 1] = 1
18
19              # Right Neumann boundary
20              if col == mesh_size - 2:
21                  A[node][node - 1] = 2
22              else:
23                  if col != 0:
24                      A[node][node - 1] = 1
25                  A[node][node + 1] = 1
26
27      # Special nodes
28      A[15][10] = 1
29      A[15][15] = -4
30      A[15][16] = 1
31      A[15][17] = 1
32
33      A[16][11] = 1
34      A[16][15] = 1
35      A[16][16] = -4
36      A[16][18] = 1
37      b[16][0] = -15
38
39      A[17][15] = 2
40      A[17][17] = -4
41      A[17][18] = 1
42
43      A[18][16] = 2
44      A[18][17] = 1
45      A[18][18] = -4
46      b[18][0] = -15
47
48      return A, b
```

Listing 9: Conjugate gradient (`conjugate_gradient.py`).

```
1   from copy import deepcopy
2
3   from matrices import Matrix
4
5
6   def conjugate_gradient_solve(A, b, residual_vectors=None):
7       n = len(A)
8       x = Matrix.empty(n, 1)
9       r = b - A * x
10      p = deepcopy(r)
11      if residual_vectors is not None:
12          residual_vectors.append(r)
13      for _ in range(n):
```

```
        denom = p.transpose() * A * p
        alpha = (p.transpose() * r) / denom
        x = x + p * alpha.item()
        r = b - A * x
        beta = - (p.transpose() * A * r) / denom
        p = r + p * beta.item()
        if residual_vectors is not None:
            residual_vectors.append(r)
    return x
```

B Output Logs

Listing 10: Output of Question 1 program (`q1.txt`).

```
=== Question 1 ===
S1:
  0.50  -0.50   0.00
 -0.50   1.00  -0.50
  0.00  -0.50   0.50
S2:
  1.00  -0.50  -0.50
 -0.50   0.50   0.00
 -0.50   0.00   0.50
C:
  1.00   0.00   0.00   0.00
  0.00   1.00   0.00   0.00
  0.00   0.00   1.00   0.00
  0.00   0.00   0.00   1.00
  1.00   0.00   0.00   0.00
  0.00   0.00   1.00   0.00
S_dis:
  0.50  -0.50   0.00   0.00   0.00   0.00
 -0.50   1.00  -0.50   0.00   0.00   0.00
  0.00  -0.50   0.50   0.00   0.00   0.00
  0.00   0.00   0.00   1.00  -0.50  -0.50
  0.00   0.00   0.00  -0.50   0.50   0.00
  0.00   0.00   0.00  -0.50   0.00   0.50
S:
  1.00  -0.50   0.00  -0.50
 -0.50   1.00  -0.50   0.00
  0.00  -0.50   1.00  -0.50
 -0.50   0.00  -0.50   1.00
```

Listing 11: Output of Question 2 program (`q2.txt`).

```
=== Question 2 ===

=== Question 2(c) ===
Capacitance per unit length: 5.21374340427e-11 F/m
```

Listing 12: Output of Question 3 program (`q3.txt`).

```
=== Question 3 ===

=== Question 3(a) ===
A:
 -4   1   0   0   0   1   0   0   0   0   0   0   0   0   0   0   0   0   0
  1  -4   1   0   0   0   1   0   0   0   0   0   0   0   0   0   0   0   0
  0   1  -4   1   0   0   0   1   0   0   0   0   0   0   0   0   0   0   0
  0   0   1  -4   1   0   0   0   1   0   0   0   0   0   0   0   0   0   0
  0   0   0   2  -4   0   0   0   0   1   0   0   0   0   0   0   0   0   0
  1   0   0   0   0  -4   1   0   0   0   1   0   0   0   0   0   0   0   0
  0   1   0   0   0   1  -4   1   0   0   0   1   0   0   0   0   0   0   0
  0   0   1   0   0   0   1  -4   1   0   0   0   1   0   0   0   0   0   0
```

```
    0   0   0   1   0   0   0   1  -4   1   0   0   0   1   0   0   0   0   0
    0   0   0   0   1   0   0   0   2  -4   0   0   0   0   1   0   0   0   0
    0   0   0   0   0   1   0   0   0   0  -4   1   0   0   0   1   0   0   0
    0   0   0   0   0   0   1   0   0   0   1  -4   1   0   0   0   1   0   0
    0   0   0   0   0   0   0   1   0   0   0   1  -4   1   0   0   0   0   0
    0   0   0   0   0   0   0   0   1   0   0   0   1  -4   1   0   0   0   0
    0   0   0   0   0   0   0   0   0   1   0   0   0   2  -4   0   0   0   0
    0   0   0   0   0   0   0   0   0   0   1   0   0   0   0  -4   1   1   0
    0   0   0   0   0   0   0   0   0   0   0   1   0   0   0   1  -4   0   1
    0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   2   0  -4   1
    0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   2   1  -4
b:
    0
    0
    0
    0
    0
    0
    0
    0
    0
    0
    0
    0
  -15
  -15
  -15
    0
  -15
    0
  -15
A is positive definite: False
A' is positive definite: True

=== Question 3(b) ===
Choleski x:
    0.96
    1.86
    2.61
    3.04
    3.17
    1.97
    3.88
    5.53
    6.37
    6.61
    3.03
    6.18
    9.25
   10.29
   10.55
    3.96
    8.56
    4.25
    9.09
Conjugate gradient x:
    0.96
    1.86
    2.61
    3.04
    3.17
    1.97
    3.88
    5.53
    6.37
    6.61
    3.03
    6.18
    9.25
   10.29
```

```
 10.55
  3.96
  8.56
  4.25
  9.09

=== Question 3(c) ===

=== Question 3(d) ===
Choleski potential at (0.06, 0.04): 5.52634126517 V
Conjugate gradient potential at (0.06, 0.04): 5.52634127414 V
```

C Simple2D Data Files

Listing 13: Input mesh for the `SIMPLE2D` program.

```
1 0.0 0.0
2 0.02 0.0
3 0.04 0.0
4 0.06 0.0
5 0.08 0.0
6 0.1 0.0
7 0.0 0.02
8 0.02 0.02
9 0.04 0.02
10 0.06 0.02
11 0.08 0.02
12 0.1 0.02
13 0.0 0.04
14 0.02 0.04
15 0.04 0.04
16 0.06 0.04
17 0.08 0.04
18 0.1 0.04
19 0.0 0.06
20 0.02 0.06
21 0.04 0.06
22 0.06 0.06
23 0.08 0.06
24 0.1 0.06
25 0.0 0.08
26 0.02 0.08
27 0.04 0.08
28 0.06 0.08
29 0.08 0.08
30 0.1 0.08
31 0.0 0.1
32 0.02 0.1
33 0.04 0.1
34 0.06 0.1

1 2 7 0
2 3 8 0
3 4 9 0
4 5 10 0
5 6 11 0
7 8 13 0
8 9 14 0
9 10 15 0
10 11 16 0
11 12 17 0
13 14 19 0
14 15 20 0
15 16 21 0
16 17 22 0
17 18 23 0
19 20 25 0
```

```
20 21 26 0
21 22 27 0
22 23 28 0
23 24 29 0
25 26 31 0
26 27 32 0
27 28 33 0
2 7 8 0
3 8 9 0
4 9 10 0
5 10 11 0
6 11 12 0
8 13 14 0
9 14 15 0
10 15 16 0
11 16 17 0
12 17 18 0
14 19 20 0
15 20 21 0
16 21 22 0
17 22 23 0
18 23 24 0
20 25 26 0
21 26 27 0
22 27 28 0
23 28 29 0
24 29 30 0
26 31 32 0
27 32 33 0
28 33 34 0

1 0
2 0
3 0
4 0
5 0
6 0
7 0
13 0
19 0
25 0
31 0
28 15
29 15
30 15
34 15
```

Listing 14: Resulting potentials generated by the SIMPLE2D program.

```
ans =

    1.0000         0         0         0
    2.0000    0.0200         0         0
    3.0000    0.0400         0         0
    4.0000    0.0600         0         0
    5.0000    0.0800         0         0
    6.0000    0.1000         0         0
    7.0000         0    0.0200         0
    8.0000    0.0200    0.0200    0.9571
    9.0000    0.0400    0.0200    1.8616
   10.0000    0.0600    0.0200    2.6060
   11.0000    0.0800    0.0200    3.0360
   12.0000    0.1000    0.0200    3.1714
   13.0000         0    0.0400         0
   14.0000    0.0200    0.0400    1.9667
   15.0000    0.0400    0.0400    3.8834
   16.0000    0.0600    0.0400    5.5263
   17.0000    0.0800    0.0400    6.3668
   18.0000    0.1000    0.0400    6.6135
```

19.0000	0	0.0600	0
20.0000	0.0200	0.0600	3.0262
21.0000	0.0400	0.0600	6.1791
22.0000	0.0600	0.0600	9.2492
23.0000	0.0800	0.0600	10.2912
24.0000	0.1000	0.0600	10.5490
25.0000	0	0.0800	0
26.0000	0.0200	0.0800	3.9590
27.0000	0.0400	0.0800	8.5575
28.0000	0.0600	0.0800	15.0000
29.0000	0.0800	0.0800	15.0000
30.0000	0.1000	0.0800	15.0000
31.0000	0	0.1000	0
32.0000	0.0200	0.1000	4.2525
33.0000	0.0400	0.1000	9.0919
34.0000	0.0600	0.1000	15.0000