# ECSE 543
# Assignment 2

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# Introduction

# 1 Finite Element Triangles

The equation for the $\alpha$ parameter for a general vertex $i$ of a finite element triangle can be seen in Equation (1), where $i+1$ and $i+2$ implicitly wraps around when exceeding 3.

$$\begin{aligned}\alpha_i(x,y) = \frac{1}{2A}\big[&(x_{i+1}y_{i+2} - x_{i+2}y_{i+1}) \\ &+ (y_{i+1} - y_{i+2})x \\ &+ (x_{i+2} - x_{i+1})y\big]\end{aligned} \tag{1}$$

Using Equation (1), we can solve for the entries of the local $S$ matrix, as shown in Equation (2). This was used in the program to compute every entry for both example triangles.

$$\begin{aligned}S_{ij} =& \int_{\Delta_e} \nabla\alpha_i \cdot \nabla\alpha_j dS \\ =&\frac{1}{4A}\big[(y_{i+1} - y_{i+2})(y_{j+1} - y_{j+2}) \\ &+ (x_{i+2} - x_{i+1})(x_{j+2} - x_{j+1})\big]\end{aligned} \tag{2}$$

The local $S$ matrix for the first triangle can be seen in Equation (3).

$$S_1 = \begin{bmatrix} 0.5 & -0.5 & 0.0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix} \tag{3}$$

The local $S$ matrix for the second triangle can be seen in Equation (4).

$$S_2 = \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{bmatrix} \tag{4}$$

The disjoint $S$ matrix is then given by the following:

$$S_{dis} = \begin{bmatrix} 0.5 & -0.5 & 0.0 & 0 & 0 & 0 \\ -0.5 & 1.0 & -0.5 & 0 & 0 & 0 \\ 0.0 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & -0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0.0 \\ 0 & 0 & 0 & -0.5 & 0.0 & 0.5 \end{bmatrix}$$

The connectivity matrix $C$ is given by Section 1.

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The global matrix $S$ is then given by Equation (5).

$$S = C^T S_{dis} C^T \tag{5}$$

Using Sections 1 and 1 and Equation (5), we can solve for the global $S$ matrix, giving the value shown in Equation (6).

$$S = \begin{bmatrix} 1.0 & -0.5 & 0.0 & -0.5 \\ -0.5 & 1.0 & -0.5 & 0.0 \\ 0.0 & -0.5 & 1.0 & -0.5 \\ -0.5 & 0.0 & -0.5 & 1.0 \end{bmatrix} \tag{6}$$

# 2 Finite Element Coaxial Cable

## 2.a Mesh

## 2.b Electrostatic Potential

## 2.c Capacitance

The finite element functional equation can be seen in Equation (7).

$$W = \frac{1}{2}U_{con}^T S U_{con} \tag{7}$$

The goal of the finite element method is to minimize $W$ to minimize the energy. However, $W$ is not equal to the energy. The relation between the energy per unit length $E$ and $W$ is shown in Equation (8).

$$E = \epsilon_0 W \tag{8}$$

We then know that the energy per unit length $E$ is related to the capacitance per unit length $C$ as shown in Equation (9).

$$E = \frac{1}{2}CV^2 \tag{9}$$

Combining Equations (7) to (9), we obtain an expression for the capacitance per unit length which can be easily calculated, as shown in Equation 10.

$$C = \frac{\epsilon_0 U_{con}^T S U_{con}}{V^2} \tag{10}$$

# 3 Conjugate Gradient Coaxial Cable

## 3.a Positive Definite Test

If the matrix $A$ is not positive definite, one can simply multiply both sides of the $Ax = b$ equation by $A^T$, forming a new equation $A^T Ax = A^T b$. This is equivalent to $A'x = b'$, where $b' = A^T b$ and $A' = A^T A$. Here, $A'$ is now positive definite.

### 3.b Matrix Solution

### 3.c Residual Norm

Consider a vector $\mathbf{v} = \{v_1, \dots, v_n\}$. The infinity norm $\|\mathbf{v}\|_\infty$ of $\mathbf{v}$ is given by the maximum absolute element of $\mathbf{v}$, as shown in Equation (11).

$$\|\mathbf{v}\|_\infty = \max\{|v_1|, \dots, |v_n|\} \tag{11}$$

Similarly, the 2-norm $\|\mathbf{v}\|_2$ of $\mathbf{v}$ is given by Equation (12).

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{12}$$

### 3.d Potential Comparison

### 3.e Capacitance Improvement

## A Code Listings

## B Output Logs