ECSE 543
Assignment 2

Sean Stappas
260639512

November 13^{th}, 2017

Contents

Introduction

1 Finite Element Triangles

The equation for the α parameter for a general vertex i of a finite element triangle can be seen in Equation (1), where $i+1$ and $i+2$ implicitly wraps around when exceeding 3.

$$\begin{aligned}\alpha_i(x,y) = \frac{1}{2A}\big[&(x_{i+1}y_{i+2} - x_{i+2}y_{i+1}) \\ &+ (y_{i+1} - y_{i+2})x \\ &+ (x_{i+2} - x_{i+1})y\big]\end{aligned} \tag{1}$$

Using Equation (1), we can solve for the entries of the local S matrix, as shown in Equation (2). This was used in the program to compute every entry for both example triangles.

$$\begin{aligned}S_{ij} =& \int_{\Delta_e} \nabla\alpha_i \cdot \nabla\alpha_j dS \\ =& \frac{1}{4A}\big[(y_{i+1} - y_{i+2})(y_{j+1} - y_{j+2}) \\ &+ (x_{i+2} - x_{i+1})(x_{j+2} - x_{j+1})\big]\end{aligned} \tag{2}$$

The local S matrix for the first triangle can be seen in Equation (3).

$$S_1 = \begin{bmatrix} 0.5 & -0.5 & 0.0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix} \tag{3}$$

The local S matrix for the second triangle can be seen in Equation (4).

$$S_2 = \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{bmatrix} \tag{4}$$

The disjoint S matrix is then given by the following:

$$S_{dis} = \begin{bmatrix} 0.5 & -0.5 & 0.0 & 0 & 0 & 0 \\ -0.5 & 1.0 & -0.5 & 0 & 0 & 0 \\ 0.0 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & -0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0.0 \\ 0 & 0 & 0 & -0.5 & 0.0 & 0.5 \end{bmatrix}$$

The connectivity matrix C is given by Equation (5).

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5}$$

The global matrix S is then given by Equation (6).

$$S = C^T S_{dis} C^T \tag{6}$$

Using Equations (5) and (6), we can solve for the global S matrix, giving the value shown in Equation (7), which is computed by the `finite_element_triangles.py` script shown in Listing 3.

$$S = \begin{bmatrix} 1.0 & -0.5 & 0.0 & -0.5 \\ -0.5 & 1.0 & -0.5 & 0.0 \\ 0.0 & -0.5 & 1.0 & -0.5 \\ -0.5 & 0.0 & -0.5 & 1.0 \end{bmatrix} \tag{7}$$

2 Finite Element Coaxial Cable

2.a Mesh

The mesh to be used by the `SIMPLE2D` program is generated by the `finite_element_mesh_generator.py` script shown in Listing 5. This input and output files of the `SIMPLE2D` program are shown in Listings 9 and 10 of Appendix C.

2.b Electrostatic Potential

Based on the results from the `SIMPLE2D` program, the potential at (0.06, 0.04) is 5.5263 V. This corresponds to node 16 in the mesh arrangement we created.

2.c Capacitance

The finite element functional equation for two conjoint finite element triangles forming a square i can be seen in Equation (8).

$$W_i = \frac{1}{2} U_{con_i}^T S U_{con_i} \tag{8}$$

where S is given in Equation (7) and U_{con_i} is the conjoint potential vector for square i, giving the potential at the four corners of the square defining the combination of two finite element triangles. This can be seen in Equation (9).

$$U_{con} = \begin{bmatrix} U_{i_1} \\ U_{i_2} \\ U_{i_3} \\ U_{i_4} \end{bmatrix} \tag{9}$$

To find the total energy function W of the mesh, we must add the contributions from each square and multiply by 4, since our mesh is one quarter of the entire coaxial cable. This yields Equation (10).

$$W = 4 \sum_{i}^{N} W_i = 2 \sum_{i}^{N} U_{con_i}^T S U_{con_i} \tag{10}$$

where N is the number of finite difference squares in the mesh.

Note that W is not equal to the energy. The relation between the energy per unit length E and W is shown in Equation (11).

$$E = \epsilon_0 W \tag{11}$$

We then know that the energy per unit length E is related to the capacitance per unit length C as shown in Equation (12).

$$E = \frac{1}{2} C V^2 \tag{12}$$

where V is the voltage across the coaxial cable.

Combining Equations (8) and (10) to (12), we obtain an expression for the capacitance per unit length which can be easily calculated, as shown in Equation 13.

$$C = \frac{2E}{V^2} = \frac{4\epsilon_0}{V^2} \sum_{i}^{N} U_{con_i}^T S U_{con_i} \tag{13}$$

The capacitance per unit length is computed as 5.2137×10^{-11} F/m by the `finite_element_capacitance.py` script shown in Listing 6 with output shown in Listing 8.

3 Conjugate Gradient Coaxial Cable

3.a Positive Definite Test

If the matrix A is not positive definite, one can simply multiply both sides of the $Ax = b$ equation by A^T, forming a new equation $A^T A x = A^T b$. This is equivalent to $A'x = b'$, where $b' = A^T b$ and $A' = A^T A$. Here, A' is now positive definite.

3.b Matrix Solution

The matrix equation to be solved can be seen in Equation (14), where A is positive-definite matrix generated previously, ϕ_c is the unknown potential vector and b contains the initial potential values along the boundaries.

$$A\phi_c = b \tag{14}$$

3.c Residual Norm

Consider a vector $\mathbf{v} = \{v_1, \ldots, v_n\}$. The infinity norm $\|\mathbf{v}\|_\infty$ of $\mathbf{v}$ is given by the maximum absolute element of $\mathbf{v}$, as shown in Equation (15).

$$\|\mathbf{v}\|_\infty = \max\{|v_1|, \ldots, |v_n|\} \tag{15}$$

Similarly, the 2-norm $\|\mathbf{v}\|_2$ of $\mathbf{v}$ is given by Equation (16).

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{16}$$

3.d Potential Comparison

3.e Capacitance Improvement

A Code Listings

Listing 1: Custom matrix package (`matrices.py`).

```python
from __future__ import division

import copy
import csv
from ast import literal_eval

import math


class Matrix:

    def __init__(self, data):
        self.data = data
        self.rows = len(data)
        self.cols = len(data[0])

    def __str__(self):
        string = ''
        for row in self.data:
            string += '\n'
            for val in row:
                string += '{:6.2f} '.format(val)
        return string

    def __add__(self, other):
        if len(self) != len(other) or len(self[0]) != len(other[0]):
            raise ValueError('Incompatible matrix sizes for addition. Matrix A is {}x{}, but matrix B is {}x{}.'
                             .format(len(self), len(self[0]), len(other), len(other[0])))

        return Matrix([[self[row][col] + other[row][col] for col in range(self.cols)] for row in range(self.rows)])

    def __sub__(self, other):
        if len(self) != len(other) or len(self[0]) != len(other[0]):
            raise ValueError('Incompatible matrix sizes for subtraction. Matrix A is {}x{}, but matrix B is {}x{}.'
                             .format(len(self), len(self[0]), len(other), len(other[0])))

        return Matrix([[self[row][col] - other[row][col] for col in range(self.cols)] for row in range(self.rows)])

    def __mul__(self, other):
        if self.cols != other.rows:
            raise ValueError('Incompatible matrix sizes for multiplication. Matrix A is {}x{}, but matrix B is {}x{}.'
                             .format(self.rows, self.cols, other.rows, other.cols))

        # Inspired from https://en.wikipedia.org/wiki/Matrix_multiplication
        product = Matrix.empty(self.rows, other.cols)
        for i in range(self.rows):
            for j in range(other.cols):
                row_sum = 0
                for k in range(self.cols):
                    row_sum += self[i][k] * other[k][j]
                product[i][j] = row_sum
        return product

    def __deepcopy__(self, memo):
        return Matrix(copy.deepcopy(self.data))

    def __getitem__(self, item):
        return self.data[item]

    def __len__(self):
```

```python
        return len(self.data)

    def is_positive_definite(self):
        """
        :return: True if the matrix if positive-definite, False otherwise.
        """
        A = copy.deepcopy(self.data)
        for j in range(self.rows):
            if A[j][j] <= 0:
                return False
            A[j][j] = math.sqrt(A[j][j])
            for i in range(j + 1, self.rows):
                A[i][j] = A[i][j] / A[j][j]
                for k in range(j + 1, i + 1):
                    A[i][k] = A[i][k] - A[i][j] * A[k][j]
        return True

    def transpose(self):
        """
        :return: the transpose of the current matrix
        """
        return Matrix([[self.data[row][col] for row in range(self.rows)] for col in range(self.cols)])

    def mirror_horizontal(self):
        """
        :return: the horizontal mirror of the current matrix
        """
        return Matrix([[self.data[self.rows - row - 1][col] for col in range(self.cols)] for row in range(self.rows)])

    def empty_copy(self):
        """
        :return: an empty matrix of the same size as the current matrix.
        """
        return Matrix.empty(self.rows, self.cols)

    def save_to_csv(self, filename):
        """
        Saves the current matrix to a CSV file.

        :param filename: the name of the CSV file
        """
        with open(filename, "wb") as f:
            writer = csv.writer(f)
            for row in self.data:
                writer.writerow(row)

    def save_to_latex(self, filename):
        """
        Saves the current matrix to a latex-readable matrix.

        :param filename: the name of the CSV file
        """
        with open(filename, "wb") as f:
            for row in range(self.rows):
                for col in range(self.cols):
                    f.write('{}'.format(self.data[row][col]))
                    if col < self.cols - 1:
                        f.write('& ')
                if row < self.rows - 1:
                    f.write('\\\\\n')

    @staticmethod
    def multiply(*matrices):
        """
        Computes the product of the given matrices.

        :param matrices: the matrix objects
        :return: the product of the given matrices
        """
```

```
        n = matrices[0].rows
        product = Matrix.identity(n)
        for matrix in matrices:
            product = product * matrix
        return product

    @staticmethod
    def empty(num_rows, num_cols):
        """
        Returns an empty matrix (filled with zeroes) with the specified number of columns and rows.

        :param num_rows: number of rows
        :param num_cols: number of columns
        :return: the empty matrix
        """
        return Matrix([[0 for _ in range(num_cols)] for _ in range(num_rows)])

    @staticmethod
    def identity(n):
        """
        Returns the identity matrix of the given size.

        :param n: the size of the identity matrix (number of rows or columns)
        :return: the identity matrix of size n
        """
        return Matrix.diagonal_single_value(1, n)

    @staticmethod
    def diagonal(values):
        """
        Returns a diagonal matrix with the given values along the main diagonal.

        :param values: the values along the main diagonal
        :return: a diagonal matrix with the given values along the main diagonal
        """
        n = len(values)
        return Matrix([[values[row] if row == col else 0 for col in range(n)] for row in range(n)])

    @staticmethod
    def diagonal_single_value(value, n):
        """
        Returns a diagonal matrix of the given size with the given value along the diagonal.

        :param value: the value of each element on the main diagonal
        :param n: the size of the matrix
        :return: a diagonal matrix of the given size with the given value along the diagonal.
        """
        return Matrix([[value if row == col else 0 for col in range(n)] for row in range(n)])

    @staticmethod
    def column_vector(values):
        """
        Transforms a row vector into a column vector.

        :param values: the values, one for each row of the column vector
        :return: the column vector
        """
        return Matrix([[value] for value in values])

    @staticmethod
    def csv_to_matrix(filename):
        """
        Reads a CSV file to a matrix.

        :param filename: the name of the CSV file
        :return: a matrix containing the values in the CSV file
        """
        with open(filename, 'r') as csv_file:
            reader = csv.reader(csv_file)
            data = []
```

```
            for row_number, row in enumerate(reader):
                data.append([literal_eval(val) for val in row])
            return Matrix(data)
```

Listing 2: Question 1 (q1.py).

```
from finite_element_triangles import Triangle, find_local_s_matrix, find_global_s_matrix
from matrices import Matrix


def q1():
    print('\n=== Question 1 ===')
    S1 = build_triangle_and_find_local_S(
        [0, 0, 0.02],
        [0.02, 0, 0])
    S1.save_to_latex('report/matrices/S1.txt')
    print('S1: {}'.format(S1))

    S2 = build_triangle_and_find_local_S(
        [0.02, 0, 0.02],
        [0.02, 0.02, 0])
    S2.save_to_latex('report/matrices/S2.txt')
    print('S2: {}'.format(S2))

    C = Matrix([
        [1, 0, 0, 0],
        [0, 1, 0, 0],
        [0, 0, 1, 0],
        [0, 0, 0, 1],
        [1, 0, 0, 0],
        [0, 0, 1, 0]])
    C.save_to_latex('report/matrices/C.txt')
    print('C: {}'.format(C))

    S = find_global_s_matrix(S1, S2, C)
    S.save_to_latex('report/matrices/S.txt')
    S.save_to_csv('report/csv/S.txt')
    print('S: {}'.format(S))


def build_triangle_and_find_local_S(x, y):
    triangle = Triangle(x, y)
    S = find_local_s_matrix(triangle)
    return S


if __name__ == '__main__':
    q1()
```

Listing 3: Finite element triangles (finite_element_triangles.py).

```
from __future__ import division

from matrices import Matrix


class Triangle:
    def __init__(self, x, y):
        self.x = x
        self.y = y
        self.area = (x[1] * y[2] - x[2] * y[1] - x[0] * y[2] + x[2] * y[0] + x[0] * y[1] - x[1] * y[0]) /
        ↪  2


def find_local_s_matrix(triangle):
    x = triangle.x
    y = triangle.y
    S = Matrix.empty(3, 3)
```

```python

    for i in range(3):
        for j in range(3):
            S[i][j] = ((y[(i + 1) % 3] - y[(i + 2) % 3]) * (y[(j + 1) % 3] - y[(j + 2) % 3])
                       + (x[(i + 1) % 3] - x[(i + 2) % 3]) * (x[(j + 1) % 3] - x[(j + 2) % 3])) / (4 *
                        ↪  triangle.area)

    return S


def find_global_s_matrix(S1, S2, C):
    S_dis = find_disjoint_s_matrix(S1, S2)
    S_dis.save_to_latex('report/matrices/S_dis.txt')
    print('S_dis: {}'.format(S_dis))
    return C.transpose() * S_dis * C


def find_disjoint_s_matrix(S1, S2):
    n = len(S1)
    S_dis = Matrix.empty(2 * n, 2 * n)
    for row in range(n):
        for col in range(n):
            S_dis[row][col] = S1[row][col]
            S_dis[row + n][col + n] = S2[row][col]
    return S_dis
```

Listing 4: Question 2 (`q2.py`).

```python
from finite_element_capacitance import find_capacitance
from matrices import Matrix
from finite_element_mesh_generator import generate_simple_2d_mesh

INNER_CONDUCTOR_POINTS = [28, 29, 30, 34]
OUTER_CONDUCTOR_POINTS = [1, 2, 3, 4, 5, 6, 7, 13, 19, 25, 31]

MESH_SIZE = 6


def q2():
    print('\n=== Question 2 ===')
    q2a()
    q2c()


def q2a():
    generate_simple_2d_mesh(MESH_SIZE, INNER_CONDUCTOR_POINTS, OUTER_CONDUCTOR_POINTS)


def q2c():
    print('\n=== Question 2(c) ===')
    S = Matrix.csv_to_matrix('report/csv/S.txt')
    voltage = 15
    capacitance = find_capacitance(S, voltage, MESH_SIZE)
    print('Capacitance per unit length: {} F/m'.format(capacitance))


if __name__ == '__main__':
    q2()
```

Listing 5: Finite element mesh generator (`finite_element_mesh_generator.py`).

```python
def generate_simple_2d_mesh(mesh_size, inner_conductor_points, outer_conductor_points):
    with open('simple2d/mesh.dat', 'w') as f:
        generate_node_positions(f, mesh_size)
        generate_triangle_coordinates(f, mesh_size)
        generate_initial_potentials(f, inner_conductor_points, outer_conductor_points)


```

```python
def generate_node_positions(f, mesh_size):
    for row in range(mesh_size):
        y = row * 0.02
        for col in range(mesh_size):
            x = col * 0.02
            node = row * mesh_size + (col + 1)
            if node <= 34:  # Inner conductor
                f.write('{} {} {}\n'.format(node, x, y))
    f.write('\n')


def generate_triangle_coordinates(f, mesh_size):
    # Left triangles (left halves of squares)
    for row in range(mesh_size - 1):
        for col in range(mesh_size - 1):
            node = row * mesh_size + (col + 1)
            if node < 28:
                f.write('{} {} {} 0\n'.format(node, node + 1, node + mesh_size))

    # Right triangles (right halves of squares)
    for row in range(mesh_size - 1):
        for col in range(1, mesh_size):
            node = row * mesh_size + (col + 1)
            if node <= 28:
                f.write('{} {} {} 0\n'.format(node, node + mesh_size - 1, node + mesh_size))

    f.write('\n')


def generate_initial_potentials(f, inner_conductor_points, outer_conductor_points):
    for point in outer_conductor_points:
        f.write('{} {}\n'.format(point, 0))
    for point in inner_conductor_points:
        f.write('{} {}\n'.format(point, 15))
```

Listing 6: Finite element capacitance (`finite_element_capacitance.py`).

```python
from matrices import Matrix

E_0 = 8.854187817620E-12


def extract_mesh():
    with open('simple2d/result.dat') as f:
        mesh = {}
        for line_number, line in enumerate(f):
            if line_number >= 2:
                vals = line.split()
                node = int(float(vals[0]))
                voltage = float(vals[3])
                mesh[node] = voltage
    return mesh


def compute_half_energy(S, mesh, mesh_size):
    U_con = Matrix.empty(4, 1)
    half_energy = 0
    for row in range(mesh_size - 1):
        for col in range(mesh_size - 1):
            node = row * mesh_size + (col + 1)  # 1-based
            if node < 28:
                U_con[0][0] = mesh[node + mesh_size]
                U_con[1][0] = mesh[node]
                U_con[2][0] = mesh[node + 1]
                U_con[3][0] = mesh[node + mesh_size + 1]
                half_energy_contribution = U_con.transpose() * S * U_con
                half_energy += half_energy_contribution[0][0]
    return half_energy

```

```python

def find_capacitance(S, voltage, mesh_size):
    mesh = extract_mesh()
    half_energy = compute_half_energy(S, mesh, mesh_size)
    capacitance = (4 * E_0 * half_energy) / voltage ** 2
    return capacitance
```

B Output Logs

Listing 7: Output of Question 1 program (`q1.txt`).

```
=== Question 1 ===
S1:
  0.50  -0.50   0.00
 -0.50   1.00  -0.50
  0.00  -0.50   0.50
S2:
  1.00  -0.50  -0.50
 -0.50   0.50   0.00
 -0.50   0.00   0.50
C:
  1.00   0.00   0.00   0.00
  0.00   1.00   0.00   0.00
  0.00   0.00   1.00   0.00
  0.00   0.00   0.00   1.00
  1.00   0.00   0.00   0.00
  0.00   0.00   1.00   0.00
S_dis:
  0.50  -0.50   0.00   0.00   0.00   0.00
 -0.50   1.00  -0.50   0.00   0.00   0.00
  0.00  -0.50   0.50   0.00   0.00   0.00
  0.00   0.00   0.00   1.00  -0.50  -0.50
  0.00   0.00   0.00  -0.50   0.50   0.00
  0.00   0.00   0.00  -0.50   0.00   0.50
S:
  1.00  -0.50   0.00  -0.50
 -0.50   1.00  -0.50   0.00
  0.00  -0.50   1.00  -0.50
 -0.50   0.00  -0.50   1.00
```

Listing 8: Output of Question 2 program (`q2.txt`).

```
=== Question 2 ===

=== Question 2(c) ===
Capacitance per unit length: 5.21374340427e-11 F/m
```

C Simple2D Data Files

Listing 9: Input mesh for the `SIMPLE2D` program.

```
1 0.0 0.0
2 0.02 0.0
3 0.04 0.0
4 0.06 0.0
5 0.08 0.0
6 0.1 0.0
7 0.0 0.02
8 0.02 0.02
9 0.04 0.02
10 0.06 0.02
11 0.08 0.02
12 0.1 0.02
```

```
13 0.0 0.04
14 0.02 0.04
15 0.04 0.04
16 0.06 0.04
17 0.08 0.04
18 0.1 0.04
19 0.0 0.06
20 0.02 0.06
21 0.04 0.06
22 0.06 0.06
23 0.08 0.06
24 0.1 0.06
25 0.0 0.08
26 0.02 0.08
27 0.04 0.08
28 0.06 0.08
29 0.08 0.08
30 0.1 0.08
31 0.0 0.1
32 0.02 0.1
33 0.04 0.1
34 0.06 0.1

1 2 7 0
2 3 8 0
3 4 9 0
4 5 10 0
5 6 11 0
7 8 13 0
8 9 14 0
9 10 15 0
10 11 16 0
11 12 17 0
13 14 19 0
14 15 20 0
15 16 21 0
16 17 22 0
17 18 23 0
19 20 25 0
20 21 26 0
21 22 27 0
22 23 28 0
23 24 29 0
25 26 31 0
26 27 32 0
27 28 33 0
2 7 8 0
3 8 9 0
4 9 10 0
5 10 11 0
6 11 12 0
8 13 14 0
9 14 15 0
10 15 16 0
11 16 17 0
12 17 18 0
14 19 20 0
15 20 21 0
16 21 22 0
17 22 23 0
18 23 24 0
20 25 26 0
21 26 27 0
22 27 28 0
23 28 29 0
24 29 30 0
26 31 32 0
27 32 33 0
28 33 34 0
```

```
1 0
2 0
3 0
4 0
5 0
6 0
7 0
13 0
19 0
25 0
31 0
28 15
29 15
30 15
34 15
```

Listing 10: Resulting potentials generated by the `SIMPLE2D` *program.*

```
ans =

    1.0000         0         0         0
    2.0000    0.0200         0         0
    3.0000    0.0400         0         0
    4.0000    0.0600         0         0
    5.0000    0.0800         0         0
    6.0000    0.1000         0         0
    7.0000         0    0.0200         0
    8.0000    0.0200    0.0200    0.9571
    9.0000    0.0400    0.0200    1.8616
   10.0000    0.0600    0.0200    2.6060
   11.0000    0.0800    0.0200    3.0360
   12.0000    0.1000    0.0200    3.1714
   13.0000         0    0.0400         0
   14.0000    0.0200    0.0400    1.9667
   15.0000    0.0400    0.0400    3.8834
   16.0000    0.0600    0.0400    5.5263
   17.0000    0.0800    0.0400    6.3668
   18.0000    0.1000    0.0400    6.6135
   19.0000         0    0.0600         0
   20.0000    0.0200    0.0600    3.0262
   21.0000    0.0400    0.0600    6.1791
   22.0000    0.0600    0.0600    9.2492
   23.0000    0.0800    0.0600   10.2912
   24.0000    0.1000    0.0600   10.5490
   25.0000         0    0.0800         0
   26.0000    0.0200    0.0800    3.9590
   27.0000    0.0400    0.0800    8.5575
   28.0000    0.0600    0.0800   15.0000
   29.0000    0.0800    0.0800   15.0000
   30.0000    0.1000    0.0800   15.0000
   31.0000         0    0.1000         0
   32.0000    0.0200    0.1000    4.2525
   33.0000    0.0400    0.1000    9.0919
   34.0000    0.0600    0.1000   15.0000
```