ECSE 543
Assignment 2

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Contents

Introduction

1 Finite Element Triangles

The equation for the α parameter for a general vertex i of a finite element triangle can be seen in Equation (1), where $i+1$ and $i+2$ implicitly wraps around when exceeding 3.

$$\begin{aligned}\alpha_i(x,y) = \frac{1}{2A}\big[&(x_{i+1}y_{i+2} - x_{i+2}y_{i+1}) \\ &+ (y_{i+1} - y_{i+2})x \\ &+ (x_{i+2} - x_{i+1})y\big]\end{aligned} \tag{1}$$

Using Equation (1), we can solve for the entries of the local S matrix, as shown in Equation (2). This was used in the program to compute every entry for both example triangles.

$$\begin{aligned}S_{ij} &= \int_{\Delta_e} \nabla\alpha_i \cdot \nabla\alpha_j dS \\ &= \frac{1}{4A}\big[(y_{i+1} - y_{i+2})(y_{j+1} - y_{j+2}) \\ &\quad + (x_{i+2} - x_{i+1})(x_{j+2} - x_{j+1})\big]\end{aligned} \tag{2}$$

The local S matrix for the first triangle can be seen in Equation (3).

$$S_1 = \begin{bmatrix} 0.5 & -0.5 & 0.0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix} \tag{3}$$

The local S matrix for the second triangle can be seen in Equation (4).

$$S_2 = \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{bmatrix} \tag{4}$$

The disjoint S matrix is then given by the following:

$$S_{dis} = \begin{bmatrix} 0.5 & -0.5 & 0.0 & 0 & 0 & 0 \\ -0.5 & 1.0 & -0.5 & 0 & 0 & 0 \\ 0.0 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & -0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0.0 \\ 0 & 0 & 0 & -0.5 & 0.0 & 0.5 \end{bmatrix}$$

The connectivity matrix C is given by Equation (5).

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5}$$

The global matrix S is then given by Equation (6).

$$S = C^T S_{dis} C^T \tag{6}$$

Using Equations (5) and (6), we can solve for the global S matrix, giving the value shown in Equation (7).

$$S = \begin{bmatrix} 1.0 & -0.5 & 0.0 & -0.5 \\ -0.5 & 1.0 & -0.5 & 0.0 \\ 0.0 & -0.5 & 1.0 & -0.5 \\ -0.5 & 0.0 & -0.5 & 1.0 \end{bmatrix} \tag{7}$$

2 Finite Element Coaxial Cable

2.a Mesh

2.b Electrostatic Potential

Based on the results from the `SIMPLE2D` program, the potential at (0.06, 0.04) is 5.5263 V. This corresponds to node 16 in the mesh arrangement we created.

2.c Capacitance

The finite element functional equation can be seen in Equation (8).

$$W = \frac{1}{2} U_{con}^T S U_{con} \tag{8}$$

The goal of the finite element method is to minimize W to minimize the energy. However, W is not equal to the energy. The relation between the energy per unit length E and W is shown in Equation (9).

$$E = \epsilon_0 W \tag{9}$$

We then know that the energy per unit length E is related to the capacitance per unit length C as shown in Equation (10).

$$E = \frac{1}{2} C V^2 \tag{10}$$

Combining Equations (8) to (10), we obtain an expression for the capacitance per unit length which can be easily calculated, as shown in Equation 11.

$$C = \frac{\epsilon_0 U_{con}^T S U_{con}}{V^2} \tag{11}$$

3 Conjugate Gradient Coaxial Cable

3.a Positive Definite Test

If the matrix A is not positive definite, one can simply multiply both sides of the $Ax = b$ equation by A^T, forming a new equation $A^T Ax = A^T b$. This is equivalent to $A'x = b'$, where $b' = A^T b$ and $A' = A^T A$. Here, A' is now positive definite.

3.b Matrix Solution

The matrix equation to be solved can be seen in Equation (12), where A is positive-definite matrix generated previously, ϕ_c is the unknown potential vector and b contains the initial potential values along the boundaries.

$$A\phi_c = b \tag{12}$$

3.c Residual Norm

Consider a vector $\mathbf{v} = \{v_1, \ldots, v_n\}$. The infinity norm $\|\mathbf{v}\|_\infty$ of $\mathbf{v}$ is given by the maximum absolute element of $\mathbf{v}$, as shown in Equation (13).

$$\|\mathbf{v}\|_\infty = \max\{|v_1|, \ldots, |v_n|\} \tag{13}$$

Similarly, the 2-norm $\|\mathbf{v}\|_2$ of $\mathbf{v}$ is given by Equation (14).

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{14}$$

3.d Potential Comparison

3.e Capacitance Improvement

A Code Listings

B Output Logs