ECSE 543
Assignment 2

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Contents

Introduction

1 Finite Element Triangles

The equation for the α parameter for a general vertex i of a finite element triangle can be seen in Equation (1), where $i+1$ and $i+2$ implicitly wraps around when exceeding 3.

$$\begin{aligned}\alpha_i(x,y) = \frac{1}{2A}\big[&(x_{i+1}y_{i+2} - x_{i+2}y_{i+1}) \\ &+ (y_{i+1} - y_{i+2})x \\ &+ (x_{i+2} - x_{i+1})y\big]\end{aligned} \tag{1}$$

Using Equation (1), we can solve for the entries of the local S matrix, as shown in Equation (2). This was used in the program to compute every entry for both example triangles.

$$\begin{aligned}S_{ij} =& \int\limits_{\Delta_e} \nabla\alpha_i \cdot \nabla\alpha_j dS \\ =& \frac{1}{4A}\big[(y_{i+1} - y_{i+2})(y_{j+1} - y_{j+2}) \\ &+ (x_{i+2} - x_{i+1})(x_{j+2} - x_{j+1})\big]\end{aligned} \tag{2}$$

The local S matrix for the first triangle can be seen in Equation (3).

$$S_1 = \begin{bmatrix} 0.5 & -0.5 & 0.0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix} \tag{3}$$

The local S matrix for the second triangle can be seen in Equation (4).

$$S_2 = \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{bmatrix} \tag{4}$$

The disjoint S matrix is then given by the following:

$$S_{dis} = \begin{bmatrix} 0.5 & -0.5 & 0.0 & 0 & 0 & 0 \\ -0.5 & 1.0 & -0.5 & 0 & 0 & 0 \\ 0.0 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & -0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0.0 \\ 0 & 0 & 0 & -0.5 & 0.0 & 0.5 \end{bmatrix}$$

The connectivity matrix C is given by Equation (5).

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5}$$

The global matrix S is then given by Equation (6).

$$S = C^T S_{dis} C^T \tag{6}$$

Using Equations (5) and (6), we can solve for the global S matrix, giving the value shown in Equation (7).

$$S = \begin{bmatrix} 1.0 & -0.5 & 0.0 & -0.5 \\ -0.5 & 1.0 & -0.5 & 0.0 \\ 0.0 & -0.5 & 1.0 & -0.5 \\ -0.5 & 0.0 & -0.5 & 1.0 \end{bmatrix} \tag{7}$$

2 Finite Element Coaxial Cable

2.a Mesh

2.b Electrostatic Potential

2.c Capacitance

The finite element functional equation can be seen in Equation (8).

$$W = \frac{1}{2} U_{con}^T S U_{con} \tag{8}$$

The goal of the finite element method is to minimize W to minimize the energy. However, W is not equal to the energy. The relation between the energy per unit length E and W is shown in Equation (9).

$$E = \epsilon_0 W \tag{9}$$

We then know that the energy per unit length E is related to the capacitance per unit length C as shown in Equation (10).

$$E = \frac{1}{2} C V^2 \tag{10}$$

Combining Equations (8) to (10), we obtain an expression for the capacitance per unit length which can be easily calculated, as shown in Equation 11.

$$C = \frac{\epsilon_0 U_{con}^T S U_{con}}{V^2} \tag{11}$$

3 Conjugate Gradient Coaxial Cable

3.a Positive Definite Test

If the matrix A is not positive definite, one can simply multiply both sides of the $Ax = b$ equation by A^T, forming a new equation $A^T Ax = A^T b$. This is equivalent to $A'x = b'$, where $b' = A^T b$ and $A' = A^T A$. Here, A' is now positive definite.

3.b Matrix Solution

3.c Residual Norm

Consider a vector $\mathbf{v} = \{v_1, \ldots, v_n\}$. The infinity norm $\|\mathbf{v}\|_\infty$ of $\mathbf{v}$ is given by the maximum absolute element of $\mathbf{v}$, as shown in Equation (12).

$$\|\mathbf{v}\|_\infty = \max\{|v_1|, \ldots, |v_n|\} \tag{12}$$

Similarly, the 2-norm $\|\mathbf{v}\|_2$ of $\mathbf{v}$ is given by Equation (13).

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{13}$$

3.d Potential Comparison

3.e Capacitance Improvement

A Code Listings

B Output Logs