

ECSE 543

Assignment 2

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Contents

1	Finite Element Triangles	2
2	Finite Element Coaxial Cable	2
2.a	Mesh	2
2.b	Electrostatic Potential	2
2.c	Capacitance	2
3	Conjugate Gradient Coaxial Cable	2
3.a	Positive Definite Test	2
3.b	Matrix Solution	3
3.c	Residual Norm	3
3.d	Potential Comparison	3
3.e	Capacitance Improvement	3
	Appendix A Code Listings	4
	Appendix B Output Logs	4

Introduction

1 Finite Element Triangles

The equation for the α parameter for a general vertex i of a finite element triangle can be seen in Equation (1), where $i+1$ and $i+2$ implicitly wraps around when exceeding 3.

$$\alpha_i(x, y) = \frac{1}{2A} [(x_{i+1}y_{i+2} - x_{i+2}y_{i+1}) + (y_{i+1} - y_{i+2})x + (x_{i+2} - x_{i+1})y] \quad (1)$$

Using Equation (1), we can solve for the entries of the local S matrix, as shown in Equation (2). This was used in the program to compute every entry for both example triangles.

$$\begin{aligned} S_{ij} &= \int_{\Delta_e} \nabla \alpha_i \cdot \nabla \alpha_j dS \\ &= \frac{1}{4A} [(y_{i+1} - y_{i+2})(y_{j+1} - y_{j+2}) + (x_{i+2} - x_{i+1})(x_{j+2} - x_{j+1})] \end{aligned} \quad (2)$$

The local S matrix for the first triangle can be seen in Equation (3).

$$S_1 = \begin{bmatrix} 0.5 & -0.5 & 0.0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix} \quad (3)$$

The local S matrix for the second triangle can be seen in Equation (4).

$$S_2 = \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{bmatrix} \quad (4)$$

The disjoint S matrix is then given by the following:

$$S_{dis} = \begin{bmatrix} 0.5 & -0.5 & 0.0 & 0 & 0 & 0 \\ -0.5 & 1.0 & -0.5 & 0 & 0 & 0 \\ 0.0 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & -0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0.0 \\ 0 & 0 & 0 & -0.5 & 0.0 & 0.5 \end{bmatrix}$$

The connectivity matrix C is given by Section 1.

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The global matrix S is then given by Equation (5).

$$S = C^T S_{dis} C^T \quad (5)$$

Using Sections 1 and 1 and Equation (5), we can solve for the global S matrix, giving the value shown in Equation (6).

$$S = \begin{bmatrix} 1.0 & -0.5 & 0.0 & -0.5 \\ -0.5 & 1.0 & -0.5 & 0.0 \\ 0.0 & -0.5 & 1.0 & -0.5 \\ -0.5 & 0.0 & -0.5 & 1.0 \end{bmatrix} \quad (6)$$

2 Finite Element Coaxial Cable

2.a Mesh

2.b Electrostatic Potential

2.c Capacitance

The finite element functional equation can be seen in Equation (7).

$$W = \frac{1}{2} U_{con}^T S U_{con} \quad (7)$$

The goal of the finite element method is to minimize W to minimize the energy. However, W is not equal to the energy. The relation between the energy per unit length E and W is shown in Equation (8).

$$E = \epsilon_0 W \quad (8)$$

We then know that the energy per unit length E is related to the capacitance per unit length C as shown in Equation (9).

$$E = \frac{1}{2} C V^2 \quad (9)$$

Combining Equations (7) to (9), we obtain an expression for the capacitance per unit length which can be easily calculated, as shown in Equation 10.

$$C = \frac{\epsilon_0 U_{con}^T S U_{con}}{V^2} \quad (10)$$

3 Conjugate Gradient Coaxial Cable

3.a Positive Definite Test

If the matrix A is not positive definite, one can simply multiply both sides of the $Ax = b$ equation

by A^T , forming a new equation $A^T A x = A^T b$. This is equivalent to $A' x = b'$, where $b' = A^T b$ and $A' = A^T A$. Here, A' is now positive definite.

3.b Matrix Solution

3.c Residual Norm

Consider a vector $\mathbf{v} = \{v_1, \dots, v_n\}$. The infinity norm $\|\mathbf{v}\|_\infty$ of \mathbf{v} is given by the maximum absolute element of \mathbf{v} , as shown in Equation (11).

$$\|\mathbf{v}\|_\infty = \max\{|v_1|, \dots, |v_n|\} \quad (11)$$

Similarly, the 2-norm $\|\mathbf{v}\|_2$ of \mathbf{v} is given by Equation (12).

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^n v_i^2} \quad (12)$$

3.d Potential Comparison

3.e Capacitance Improvement

A Code Listings

B Output Logs