

ECSE 543

Assignment 3

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Introduction

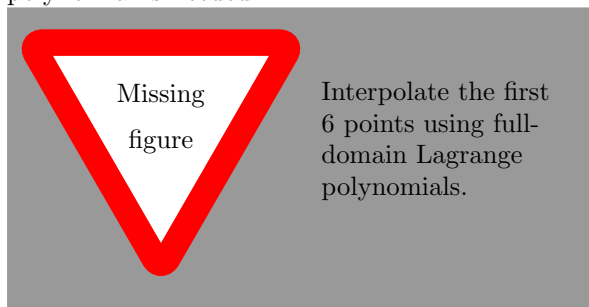
The code for this assignment was created in Python 2.7 and can be seen in Appendix A. To perform the required tasks in this assignment, the `Matrix` class from Assignment 1 was used, with useful methods such as `add`, `multiply`, `transpose`, etc. This package can be seen in the `matrices.py` file shown in Listing 1. The only packages used that are not built-in are those for creating the plots for this report, i.e., `matplotlib` for plotting. The structure of the rest of the code will be discussed as appropriate for each question. Output logs of the program are provided in Appendix B.

1 BH Interpolation

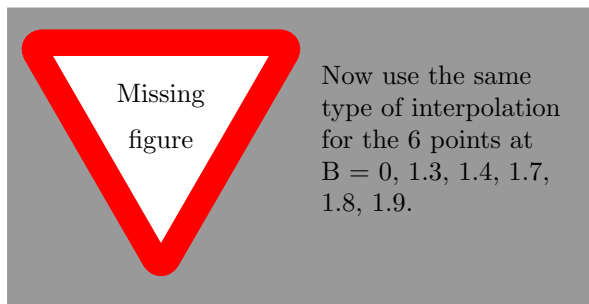
The source code for the Question 1 program can be seen in the `q1.py` file shown in Listing 2.

1.a Lagrange Polynomials

To interpolate 6 points, a 5th-order Lagrange polynomial is needed.



1.b Full-Domain Lagrange Polynomials



The result is not plausible because of the characteristic “wiggles” seen when using full-domain Lagrange polynomials over a wide range.

1.c Cubic Hermite Polynomials

The slopes at each of the 6 points can be approximated by the slope of the straight line passing through the two adjacent points, i.e., the point immediately before and the point after the point of interest. For the boundary points of 0 T and 1.9 T, the slope of the line formed by the point and one adjacent point can be used.

2 Magnetic Circuit

The source code for the Question 2 program can be seen in the `q2.py` file shown in Listing 3.

2.a Flux Equation

The magnetic analog of KVL can be seen in Equation (1).

$$(\mathcal{R}_a + \mathcal{R}_c)\psi = \mathcal{F} \quad (1)$$

where \mathcal{R}_a is the reluctance of the air gap, \mathcal{R}_c is the reluctance of the coil, and \mathcal{F} is the magnetomotive force. Plugging in the relevant variables from the problem, we obtain Equation (2).

$$\left(\frac{L_a}{A\mu_o} + \frac{L_c}{A\mu_c(\psi)} \right) \psi - NI = 0 \quad (2)$$

where $\mu_c(\psi)$ is a function of ψ given by Equation (3).

$$\mu_c(\psi) = \frac{B}{H} = \frac{\psi}{AH} \quad (3)$$

Plugging Equation (3) into Equation (2), we obtain Equation (4).

$$\left(\frac{L_a}{A\mu_o} + \frac{L_c H}{\psi} \right) \psi - NI = 0 \quad (4)$$

Simplifying the terms, we obtain Equation (5).

$$f(\psi) = \frac{L_a \psi}{A\mu_o} + L_c H - NI = 0 \quad (5)$$

Finally, if we plug in the values from the question, we obtain Equation (6), where the coefficients of the terms are calculated in the `q2.py` script shown in Listing 2.

$$f(\psi) = 3.979 \times 10^7 \psi + 0.3H - 8000 = 0 \quad (6)$$

2.b Newton-Raphson

$$B = \frac{\psi}{A} \quad (7)$$

2.c Successive Substitution

3 Diode Circuit

The source code for the Question 3 program can be seen in the `q3.py` file shown in Listing 4.

3.a Voltage Equations

The current-voltage relationship for a diode is given by Equation (8).

$$I = I_s \left(\exp \left[\frac{qv}{kT} \right] - 1 \right) \quad (8)$$

Let the nodal voltage at the anode of the A diode be denoted by v_A and that of the B diode by v_B . Let the current through the circuit be denoted by I . The diode equations for A and B can be seen in Equations (9) and (10).

$$I = I_{sA} \left(\exp \left[\frac{q(v_A - v_B)}{kT} \right] - 1 \right) \quad (9)$$

$$I = I_{sB} \left(\exp \left[\frac{qv_B}{kT} \right] - 1 \right) \quad (10)$$

By KVL, we also have Equation (11), relating V_A and I .

$$I = \frac{E - v_A}{R} \quad (11)$$

Equating Equations (9) and (11), we obtain the nonlinear equation for v_A , shown in Equation (12).

$$\begin{aligned} f_A(v_A, v_B) &= v_A + RI_{sA} \left(\exp \left[\frac{q(v_A - v_B)}{kT} \right] - 1 \right) - E \\ &= 0 \end{aligned} \quad (12)$$

Equating Equations (9) and (10), we obtain the nonlinear equation for v_B , shown in Equation (13).

$$\begin{aligned} f_B(v_A, v_B) &= I_{sA} \left(\exp \left[\frac{q(v_A - v_B)}{kT} \right] - 1 \right) \\ &\quad - I_{sB} \left(\exp \left[\frac{qv_B}{kT} \right] - 1 \right) = 0 \end{aligned} \quad (13)$$

The total system of equations can then be expressed by Equation (14).

$$\mathbf{f}(\mathbf{v}_n) = \begin{bmatrix} f_A(v_A, v_B) \\ f_B(v_A, v_B) \end{bmatrix} = \mathbf{0} \quad (14)$$

3.b Newton-Raphson

To find an expression for the Jacobian matrix \mathbf{F} , we must first find expressions for all the partials of f_A and f_B . These are shown in Equations (15) to (18).

$$\frac{\partial f_A}{\partial v_A} = 1 + RI_{sA} \left(\exp \left[\frac{q(v_A - v_B)}{kT} \right] \frac{q}{kT} \right) \quad (15)$$

$$\frac{\partial f_A}{\partial v_B} = -RI_{sA} \left(\exp \left[\frac{q(v_A - v_B)}{kT} \right] \frac{q}{kT} \right) \quad (16)$$

$$\frac{\partial f_B}{\partial v_A} = I_{sA} \left(\exp \left[\frac{q(v_A - v_B)}{kT} \right] \frac{q}{kT} \right) \quad (17)$$

$$\begin{aligned} \frac{\partial f_B}{\partial v_B} &= -I_{sA} \left(\exp \left[\frac{q(v_A - v_B)}{kT} \right] \frac{q}{kT} \right) \\ &\quad - I_{sB} \left(\exp \left[\frac{qv_B}{kT} \right] \frac{q}{kT} \right) \end{aligned} \quad (18)$$

With these equations, the Jacobian matrix \mathbf{F} is given by Equation (19).

$$\mathbf{F} = \begin{bmatrix} \frac{\partial f_A}{\partial v_A} & \frac{\partial f_A}{\partial v_B} \\ \frac{\partial f_B}{\partial v_A} & \frac{\partial f_B}{\partial v_B} \end{bmatrix} \quad (19)$$

With this information, we can apply the Newton-Raphson update in matrix form, shown in Equation (20).

$$\mathbf{v}_n^{(k+1)} \leftarrow \mathbf{v}_n^{(k)} - (\mathbf{F}^{(k)})^{-1} \mathbf{f}^{(k)} \quad (20)$$

The code performing this update is in the `newton_raphson.py` script and can be seen in Listing 5. The error ϵ in \mathbf{f} is defined as 1×10^{-9} , where the 2-norm of \mathbf{f} is used to compare to the error. The code is executed in the `q3.py` script shown in Listing 4, with output shown in Listing 10. The final solved voltage values are 198.134 mV for v_A and 90.571 mV for v_B .

4 Function Integration

The source code for the Question 4 program can be seen in the `q4.py` file shown in Listing 6.

4.a Cosine Integration

The integral I to be solved by Gauss-Legendre integration is shown in Equation (21).

Table 1: Node voltages and \mathbf{f} values at every iteration of Newton-Raphson.

v_A (mV)	v_B (mV)	$\ \mathbf{f}\ $
0.000	0.000	2.200×10^{-1}
218.254	72.751	9.906×10^{-2}
205.695	81.581	2.837×10^{-2}
200.110	89.250	5.100×10^{-3}
198.211	90.516	1.943×10^{-4}
198.134	90.571	3.088×10^{-7}
198.134	90.571	7.538×10^{-13}

$$I = \int_{x_1}^{x_2} f(x) dx \quad (21)$$

To use Gauss-Legendre integration over N equal segments, the $[x_1, x_2]$ range of x must be mapped to N intervals of size h , with each interval i having a center point x_{0_i} , with x_i ranging from $x_{0_i} - h/2$ to $x_{0_i} + h/2$. Each of these intervals must be mapped to a $[-1, 1]$ range for ζ_i . This mapping between x_i and ζ_i over an interval is given by Equation (22).

$$x_i = x_{0_i} + \frac{h}{2} \zeta_i \quad (22)$$

The integral transformation from x to ζ over an interval i is then given by Equation (23).

$$\begin{aligned} I_i &= \int_{x_{0_i}-h/2}^{x_{0_i}+h/2} f(x_i) dx_i \\ &= \frac{h}{2} \int_{-1}^1 f \left[x_{0_i} + \frac{h}{2} \zeta_i \right] d\zeta_i \end{aligned} \quad (23)$$

The one-point Gauss-Legendre approximation can then be applied for each interval, as shown in Equation (24), where $w_0 = 2$.

$$\begin{aligned} I_i &= \frac{h}{2} \int_{-1}^1 f \left[x_{0_i} + \frac{h}{2} \zeta_i \right] d\zeta_i \\ &= \frac{h}{2} w_0 f(x_{0_i}) \\ &= h f(x_{0_i}) \end{aligned} \quad (24)$$

The equation approximating I is then given by Equation (25).

$$\begin{aligned} I &\approx \sum_{i=0}^{N-1} I_i \\ &= \sum_{i=0}^{N-1} h f(x_{0_i}) \\ &= h \sum_{i=0}^{N-1} f(x_{0_i}) \end{aligned} \quad (25)$$

To summarize, to solve an integral of the form shown in Equation (21) with one-point Gauss-Legendre integration over N intervals, we simply need the width h of each interval and the value $f(x_{0_i})$ of the function f at the midpoint of every interval.

In the context of this question, $x_1 = 0$ and $x_2 = 1$. The width of each interval is $h = 1/N$ and the midpoint of each interval is $x_{0_i} = 1/(2N) + i/N$. This yields the equation shown in Equation (26).

$$\begin{aligned} I &= \int_0^1 f(x) dx \\ &\approx \frac{1}{N} \sum_{i=0}^{N-1} f \left[\frac{1}{N} \left(i + \frac{1}{2} \right) \right] \end{aligned} \quad (26)$$

The code executing Equation (26) for arbitrary $f(x)$ can be seen in the `gauss_legendre.py` script shown in Listing 7.

If we use the fact that $f(x) = \cos x$ as well for this question, we obtain Equation (27).

$$I \approx \frac{1}{N} \sum_{i=0}^{N-1} \cos \left[\frac{1}{N} \left(i + \frac{1}{2} \right) \right] \quad (27)$$

To evaluate the estimation, it can be compared to actual value of the integral, which is given by Equation (28).

$$\int_0^1 \cos x dx = \sin 1 - \sin 0 = \sin 1 \approx 0.841 \quad (28)$$

The equation for the absolute error used is shown in Equation (29), where I_{actual} is the actual value of the integral, and I_{approx} is the approximate value computed by Gauss-Legendre integration.

$$E = |I_{actual} - I_{approx}| \quad (29)$$

The integral of $\cos x$ is computed in the `q4.py` script shown in Listing 6, with output shown in Listing 11. The logarithmic plot of the error versus

N can be seen in Figure 1. The straight-line slope of the plot is indicative of the fact that first-order Gauss-Legendre integration was used.

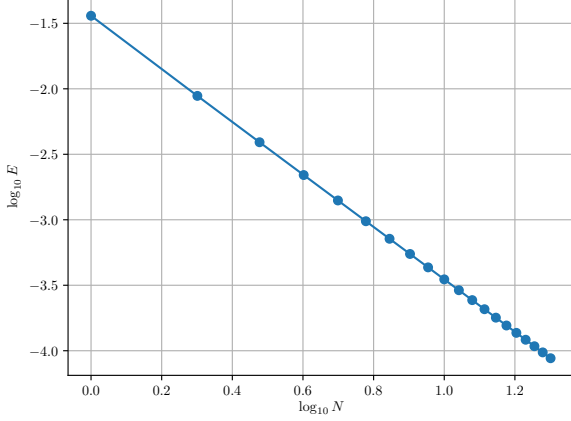


Figure 1: Logarithmic plot of the error E versus number of intervals N for $f(x) = \cos x$.

4.b Log Integration

The integral I to be evaluated is shown in Equation (30).

$$I = \int_0^1 \log_e x dx \quad (30)$$

Using Equation (26) for $f(x) = \log_e x$, we obtain the integral shown in Equation (31).

$$I \approx \frac{1}{N} \sum_{i=0}^{N-1} \log_e \left[\frac{1}{N} \left(i + \frac{1}{2} \right) \right] \quad (31)$$

The actual value of the integral to which the estimation is compared is shown in Equation (32).

$$\int_0^1 \log_e x dx = -1 \quad (32)$$

The integral of $\log_e x$ is computed in the `q4.py` script shown in Listing 6, with output shown in Listing 11. The logarithmic plot of the error versus N can be seen in Figure 2. The straight-line slope of the plot is indicative of the fact that first-order Gauss-Legendre integration was used.

4.c Log Integration Improvement

To have arbitrary interval widths, Equation (25) must be adjusted, as shown in Equation (33), where h_i is the width of interval i .

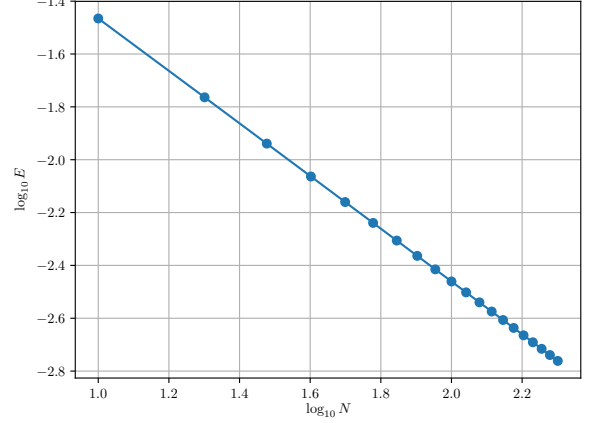


Figure 2: Logarithmic plot of the error E versus number of intervals N for $f(x) = \log_e x$.

$$I \approx \sum_{i=0}^{N-1} h_i f(x_{0_i}) \quad (33)$$

The relative widths used are shown in Equation (34)

$$(h_i)_{i=0}^{N-1} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \quad (34)$$

These relative widths are converted to actual widths and then used to compute the integral in the `gauss_legendre.py` script shown in Listing 7. The code is executed in the `q4.py` script shown in Listing 6, with output shown in Listing 11. How closely the widths approximate the $\log_e x$ curve can be seen in Figure 3. The estimated value of the integral is -0.988377 with absolute error of 0.011622 . This is much more accurate than the equal-segment version in part (b), which obtained a value of -0.965759 and error of 0.034241 .

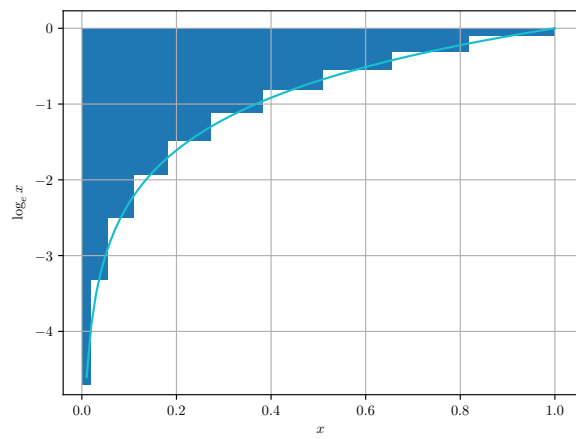


Figure 3: Sizes of intervals used for Question 4(c), with $\log_e x$ shown as reference.

A Code Listings

Listing 1: Custom matrix package (*matrices.py*).

```
1  from __future__ import division
2
3  import copy
4  import csv
5  from ast import literal_eval
6
7  import math
8
9
10 class Matrix:
11     def __init__(self, data):
12         self.data = data
13         self.num_rows = len(data)
14         self.num_cols = len(data[0])
15
16     def __str__(self):
17         string = ''
18         for row in self.data:
19             string += '\n'
20             for val in row:
21                 string += '{:6.3f} '.format(val)
22         return string
23
24     def __add__(self, other):
25         if len(self) != len(other) or len(self[0]) != len(other[0]):
26             raise ValueError('Incompatible matrix sizes for addition. Matrix A is {}x{}, but matrix B is
27                 ↳ {}x{}.'.format(len(self), len(self[0]), len(other), len(other[0])))
28
29         return Matrix([[self[row][col] + other[row][col] for col in range(self.num_cols)]
30                        for row in range(self.num_rows)])
31
32     def __sub__(self, other):
33         if len(self) != len(other) or len(self[0]) != len(other[0]):
34             raise ValueError('Incompatible matrix sizes for subtraction. Matrix A is {}x{}, but matrix B
35                 ↳ is {}x{}.'.format(len(self), len(self[0]), len(other), len(other[0])))
36
37         return Matrix([[self[row][col] - other[row][col] for col in range(self.num_cols)]
38                        for row in range(self.num_rows)])
39
40     def __mul__(self, other):
41         if type(other) == float or type(other) == int:
42             return self.scalar_multiply(other)
43
44         if self.num_cols != other.num_rows:
45             raise ValueError('Incompatible matrix sizes for multiplication. Matrix A is {}x{}, but matrix
46                 ↳ B is {}x{}.'.format(self.num_rows, self.num_cols, other.num_rows, other.num_cols))
47
48         # Inspired from https://en.wikipedia.org/wiki/Matrix_multiplication
49         product = Matrix.empty(self.num_rows, other.num_cols)
50         for i in range(self.num_rows):
51             for j in range(other.num_cols):
52                 row_sum = 0
53                 for k in range(self.num_cols):
54                     row_sum += self[i][k] * other[k][j]
55                 product[i][j] = row_sum
56         return product
57
58     def __div__(self, other):
59         """
60         Element-wise division.
61         """
62         if type(other) == float or type(other) == int:
```



```

63         return self.scalar_divide(other)
64
65     if self.num_rows != other.num_rows or self.num_cols != other.num_cols:
66         raise ValueError('Incompatible matrix sizes.')
67     return Matrix([[self[row][col] / other[row][col] for col in range(self.num_cols)]
68                   for row in range(self.num_rows)])
69
70 def __neg__(self):
71     return Matrix([[-self[row][col] for col in range(self.num_cols)] for row in range(self.num_rows)])
72
73 def __deepcopy__(self, memo):
74     return Matrix(copy.deepcopy(self.data))
75
76 def __getitem__(self, item):
77     return self.data[item]
78
79 def __len__(self):
80     return len(self.data)
81
82 @property
83 def transpose(self):
84     """
85     :return: the transpose of the current matrix
86     """
87     return Matrix([[self.data[row][col] for row in range(self.num_rows)] for col in
88                   ↪ range(self.num_cols)])
89
90 @property
91 def infinity_norm(self):
92     if self.num_cols > 1:
93         raise ValueError('Not a column vector.')
94     return max([abs(x) for x in self.transpose[0]])
95
96 @property
97 def two_norm(self):
98     if self.num_cols > 1:
99         raise ValueError('Not a column vector.')
100     return math.sqrt(sum([x ** 2 for x in self.transpose[0]]))
101
102 @property
103 def values(self):
104     """
105     :return: the values in this matrix, in row-major order.
106     """
107     vals = []
108     for row in self.data:
109         for val in row:
110             vals.append(val)
111     return tuple(vals)
112
113 def scaled_values(self, scale):
114     """
115     :return: the values in this matrix, in row-major order.
116     """
117     vals = []
118     for row in self.data:
119         for val in row:
120             vals.append('{:.3f}'.format(val * scale))
121     return tuple(vals)
122
123 @property
124 def item(self):
125     """
126     :return: the single element contained by this matrix, if it is 1x1.
127     """
128     if not (self.num_rows == 1 and self.num_cols == 1):
129         raise ValueError('Matrix is not 1x1')
130     return self.data[0][0]
131
132 def integer_string(self):

```

```

132         string = ''
133         for row in self.data:
134             string += '\n'
135             for val in row:
136                 string += '{:3.0f} '.format(val)
137         return string
138
139     def scalar_multiply(self, scalar):
140         return Matrix([[self[row][col] * scalar for col in range(self.num_cols)] for row in
141             ↪ range(self.num_rows)])
142
143     def scalar_divide(self, scalar):
144         return Matrix([[self[row][col] / scalar for col in range(self.num_cols)] for row in
145             ↪ range(self.num_rows)])
146
147     def is_positive_definite(self):
148         """
149         :return: True if the matrix is positive-definite, False otherwise.
150         """
151         A = copy.deepcopy(self.data)
152         for j in range(self.num_rows):
153             if A[j][j] <= 0:
154                 return False
155             A[j][j] = math.sqrt(A[j][j])
156             for i in range(j + 1, self.num_rows):
157                 A[i][j] = A[i][j] / A[j][j]
158                 for k in range(j + 1, i + 1):
159                     A[i][k] = A[i][k] - A[i][j] * A[k][j]
160         return True
161
162     def mirror_horizontal(self):
163         """
164         :return: the horizontal mirror of the current matrix
165         """
166         return Matrix([[self.data[self.num_rows - row - 1][col] for col in range(self.num_cols)]
167             ↪ for row in range(self.num_rows)])
168
169     def empty_copy(self):
170         """
171         :return: an empty matrix of the same size as the current matrix.
172         """
173         return Matrix.empty(self.num_rows, self.num_cols)
174
175     def save_to_csv(self, filename):
176         """
177         Saves the current matrix to a CSV file.
178
179         :param filename: the name of the CSV file
180         """
181         with open(filename, "wb") as f:
182             writer = csv.writer(f)
183             for row in self.data:
184                 writer.writerow(row)
185
186     def save_to_latex(self, filename):
187         """
188         Saves the current matrix to a latex-readable matrix.
189
190         :param filename: the name of the CSV file
191         """
192         with open(filename, "wb") as f:
193             for row in range(self.num_rows):
194                 for col in range(self.num_cols):
195                     f.write('{} '.format(self.data[row][col]))
196                     if col < self.num_cols - 1:
197                         f.write('& ')
198                 if row < self.num_rows - 1:
199                     f.write('\n')
200
201     @staticmethod

```

```

200 def multiply(*matrices):
201     """
202     Computes the product of the given matrices.
203
204     :param matrices: the matrix objects
205     :return: the product of the given matrices
206     """
207     n = matrices[0].rows
208     product = Matrix.identity(n)
209     for matrix in matrices:
210         product = product * matrix
211     return product
212
213 @staticmethod
214 def empty(num_rows, num_cols):
215     """
216     Returns an empty matrix (filled with zeroes) with the specified number of columns and rows.
217
218     :param num_rows: number of rows
219     :param num_cols: number of columns
220     :return: the empty matrix
221     """
222     return Matrix([[0 for _ in range(num_cols)] for _ in range(num_rows)])
223
224 @staticmethod
225 def identity(n):
226     """
227     Returns the identity matrix of the given size.
228
229     :param n: the size of the identity matrix (number of rows or columns)
230     :return: the identity matrix of size n
231     """
232     return Matrix.diagonal_single_value(1, n)
233
234 @staticmethod
235 def diagonal(values):
236     """
237     Returns a diagonal matrix with the given values along the main diagonal.
238
239     :param values: the values along the main diagonal
240     :return: a diagonal matrix with the given values along the main diagonal
241     """
242     n = len(values)
243     return Matrix([[values[row] if row == col else 0 for col in range(n)] for row in range(n)])
244
245 @staticmethod
246 def diagonal_single_value(value, n):
247     """
248     Returns a diagonal matrix of the given size with the given value along the diagonal.
249
250     :param value: the value of each element on the main diagonal
251     :param n: the size of the matrix
252     :return: a diagonal matrix of the given size with the given value along the diagonal.
253     """
254     return Matrix([[value if row == col else 0 for col in range(n)] for row in range(n)])
255
256 @staticmethod
257 def column_vector(values):
258     """
259     Transforms a row vector into a column vector.
260
261     :param values: the values, one for each row of the column vector
262     :return: the column vector
263     """
264     return Matrix([[value] for value in values])
265
266 @staticmethod
267 def csv_to_matrix(filename):
268     """
269     Reads a CSV file to a matrix.

```

```

270
271     :param filename: the name of the CSV file
272     :return: a matrix containing the values in the CSV file
273     """
274     with open(filename, 'r') as csv_file:
275         reader = csv.reader(csv_file)
276         data = []
277         for row_number, row in enumerate(reader):
278             data.append([literal_eval(val) for val in row])
279     return Matrix(data)

```

Listing 2: Question 1 (q1.py).

```

1  def q1():
2      print('\n=== Question 1 ===')
3      q1a()
4
5
6  def q1a():
7      pass
8
9
10 if __name__ == '__main__':
11     q1()

```

Listing 3: Question 2 (q2.py).

```

1  import math
2
3  L_a = 5e-3
4  L_c = 0.3
5  A = 1e-4
6  N = 1000
7  I = 8
8  mu_0 = 4e-7 * math.pi
9
10
11 def q2():
12     print('\n=== Question 2 ===')
13     q2b()
14
15
16 def q2b():
17     print('Flux equation: ')
18     coeff_1 = L_a / (A * mu_0)
19     coeff_2 = L_c
20     coeff_3 = N * I
21     eq = 'f(\psi) = \SI{1.3e}{\psi} + {}H - {} = 0'.format(coeff_1, coeff_2, coeff_3)
22     print(eq)
23     with open('report/latex/flux_equation.txt', 'w') as f:
24         f.write(eq)
25
26
27 if __name__ == '__main__':
28     q2()

```

Listing 4: Question 3 (q3.py).

```

1  from __future__ import division
2
3  from data_saver import save_rows_to_latex
4  from newton_raphson import newton_raphson_solve
5
6
7  def q3():
8      print('\n=== Question 3 ===')

```

```

9     v_n, values = newton_raphson_solve()
10    print('Solution: {}'.format(v_n))
11    v_a, v_b = v_n.values
12    print('v_a: {:.3f} mV'.format(v_a * 1000))
13    print('v_b: {:.3f} mV'.format(v_b * 1000))
14    save_rows_to_latex('report/latex/q3.txt', values)
15
16
17    if __name__ == '__main__':
18        q3()

```

Listing 5: Newton-Raphson (newton_raphson.py).

```

1  from __future__ import division
2
3  from math import exp
4
5  from matrices import Matrix
6
7  E = 220e-3
8  R = 500
9  I_SA = 0.6e-6
10 I_SB = 1.2e-6
11 kT_q = 25e-3
12
13 EPSILON = 1e-9
14
15
16 def newton_raphson_solve():
17     values = []
18
19     iteration = 1
20     v_n = Matrix.empty(2, 1)
21     f = Matrix.empty(2, 1)
22     F = Matrix.empty(2, 2)
23     update_f(f, v_n)
24     update_jacobian(F, v_n)
25     values.append(v_n.scaled_values(1000) + ('{:.3e}'.format(f.two_norm), ))
26     while f.two_norm > EPSILON:
27         v_n -= inverse_2x2(F) * f
28         update_f(f, v_n)
29         update_jacobian(F, v_n)
30         iteration += 1
31         values.append(v_n.scaled_values(1000) + ('{:.3e}'.format(f.two_norm), ))
32     return v_n, values
33
34
35 def update_f(f, v_n):
36     v_a, v_b = v_n.values
37     f[0][0] = f_a(v_a, v_b)
38     f[1][0] = f_b(v_a, v_b)
39
40
41 def update_jacobian(F, v_n):
42     v_a, v_b = v_n.values
43     F[0][0] = dfa_dva(v_a, v_b)
44     F[0][1] = dfa_dvb(v_a, v_b)
45     F[1][0] = dfb_dva(v_a, v_b)
46     F[1][1] = dfb_dvb(v_a, v_b)
47
48
49 def f_a(v_a, v_b):
50     return v_a + R * I_SA * exp_f_term(v_a, v_b) - E
51
52
53 def f_b(v_a, v_b):
54     return I_SA * exp_f_term(v_a, v_b) - I_SB * exp_f_term(0, -v_b)
55
56

```

```

57 def dfa_dva(v_a, v_b):
58     return 1 + R * I_SA * exp_df_term(v_a, v_b)
59
60
61 def dfa_dvb(v_a, v_b):
62     return - R * I_SA * exp_df_term(v_a, v_b)
63
64
65 def dfb_dva(v_a, v_b):
66     return I_SA * exp_df_term(v_a, v_b)
67
68
69 def dfb_dvb(v_a, v_b):
70     return - I_SA * exp_df_term(v_a, v_b) - I_SB * exp_df_term(0, -v_b)
71
72
73 def exp_f_term(v_a, v_b):
74     return exp((v_a - v_b) / kT_q) - 1
75
76
77 def exp_df_term(v_a, v_b):
78     return exp((v_a - v_b) / kT_q) / kT_q
79
80
81 def inverse_2x2(A):
82     a = A[0][0]
83     b = A[0][1]
84     c = A[1][0]
85     d = A[1][1]
86     inverse = Matrix([
87         [d, -b],
88         [-c, a]
89     ])
90     return inverse.scalar_divide(a * d - b * c)

```

Listing 6: Question 4 (q4.py).

```

1  from __future__ import division
2
3  from math import cos, log10, sin, log
4
5  import matplotlib.pyplot as plt
6  from matplotlib import rc
7  from matplotlib.patches import Rectangle
8
9  from gauss_legendre import one_point_gauss_legendre, one_point_gauss_legendre_arbitrary_widths, \
10     convert_relative_widths_to_widths
11
12  rc('font', **{'family': 'serif', 'serif': ['Computer Modern']})
13  rc('text', usetex=True)
14
15
16  def q4():
17     print('\n=== Question 4 ===')
18     q4a()
19     q4b()
20     q4c()
21
22
23  def q4a():
24     print('\n=== Question 4(a) ===')
25     n_values = []
26     integrals = []
27     n_max = 20
28     actual_integral = sin(1)
29     print('Actual integral of cos(x): {}'.format(actual_integral))
30     for n in range(1, n_max + 1):
31         integral = one_point_gauss_legendre(n, func=cos)
32         n_values.append(n)

```

```

33     integrals.append(integral)
34     print('Integral of cos(x) with N={}: {}'.format(n, integral))
35     print('Error: {}'.format(abs(actual_integral - integral)))
36     plot_error(n_values, integrals, actual_integral, func=cos, filename='q4a')
37
38
39 def q4b():
40     print('\n=== Question 4(b) ===')
41     n_values = []
42     integrals = []
43     n_max = 200
44     actual_integral = -1
45     print('Actual integral of ln(x): {}'.format(actual_integral))
46     for n in range(10, n_max + 1, 10):
47         integral = one_point_gauss_legendre(n, func=log)
48         n_values.append(n)
49         integrals.append(integral)
50         print('Integral of ln(x) with N={}: {}'.format(n, integral))
51         print('Error: {}'.format(abs(actual_integral - integral)))
52     plot_error(n_values, integrals, actual_integral, func=log, filename='q4b')
53
54
55 def q4c():
56     print('\n=== Question 4(c) ===')
57     actual_integral = -1
58     print('Actual integral of ln(x): {}'.format(actual_integral))
59     relative_widths = [x for x in range(1, 11)]
60     print('Relative widths: {}'.format(relative_widths))
61     widths = convert_relative_widths_to_widths(relative_widths)
62     print('Actual widths: {}'.format(widths))
63     integral = one_point_gauss_legendre_arbitrary_widths(widths, func=log)
64     print('Estimated Integral of ln(x): {}'.format(integral))
65     print('Error: {}'.format(abs(actual_integral - integral)))
66     plot_log_widths(widths)
67
68
69 def plot_error(n_values, integrals, actual_integral, func, filename='q4a'):
70     x_range = [log10(n) for n in n_values]
71     y_range = [log10(abs(actual_integral - integral)) for integral in integrals]
72     f = plt.figure()
73     plt.plot(x_range, y_range, 'o-')
74     plt.xlabel('$\log_{10}\{N\}$')
75     plt.ylabel('$\log_{10}\{E\}$')
76     plt.grid(True)
77     f.savefig('report/plots/{}.pdf'.format(filename), bbox_inches='tight')
78
79
80 def plot_log_widths(widths):
81     x_range = [i / 100 for i in range(1, 101)]
82     y_range = [log(x) for x in x_range]
83     f = plt.figure()
84     plt.plot(x_range, y_range, 'C9')
85     axis = plt.gca()
86     width_sum = 0
87     for w in widths:
88         axis.add_patch(Rectangle((width_sum, 0), w, log(width_sum + w / 2), facecolor='C0'))
89         width_sum += w
90
91     plt.xlabel('$x$')
92     plt.ylabel('$\log_e x$')
93     plt.grid(True)
94     f.savefig('report/plots/q4c2.pdf', bbox_inches='tight')
95
96
97 if __name__ == '__main__':
98     q4()

```

Listing 7: Gauss-Legendre integration (*gauss_legendre.py*).

```

1  from __future__ import division
2
3
4  def one_point_gauss_legendre(n, func):
5      integral = 0
6      for i in range(n):
7          integral += func((i + 0.5) / n)
8      return integral / n
9
10
11 def one_point_gauss_legendre_arbitrary_widths(widths, func):
12     integral = 0
13     width_sum = 0
14     for h in widths:
15         integral += h * func(width_sum + h / 2)
16         width_sum += h
17     return integral
18
19
20 def convert_relative_widths_to_widths(relative_widths):
21     sum_relative_widths = sum(relative_widths)
22     return [r / sum_relative_widths for r in relative_widths]

```

B Output Logs

Listing 8: Output of Question 1 program (q1.txt).

1

Listing 9: Output of Question 2 program (q2.txt).

1

Listing 10: Output of Question 3 program (q3.txt).

```

1  === Question 3 ===
2  Solution:
3      0.198
4      0.091
5  v_a: 198.134 mV
6  v_b: 90.571 mV

```

Listing 11: Output of Question 4 program (q4.txt).

```

1  === Question 4 ===
2
3  === Question 4(a) ===
4  Actual integral of cos(x): 0.841470984808
5  Integral of cos(x) with N=1: 0.87758256189
6  Error: 0.0361115770825
7  Integral of cos(x) with N=2: 0.850300645292
8  Error: 0.00882966048434
9  Integral of cos(x) with N=3: 0.845379345845
10 Error: 0.00390836103756
11 Integral of cos(x) with N=4: 0.843666316703
12 Error: 0.00219533189465
13 Integral of cos(x) with N=5: 0.84287507437
14 Error: 0.00140408956194
15 Integral of cos(x) with N=6: 0.842445699196
16 Error: 0.00097471438853
17 Integral of cos(x) with N=7: 0.842186947503
18 Error: 0.000715962695571
19 Integral of cos(x) with N=8: 0.842019067246

```



```

20 Error: 0.000548082438602
21 Integral of cos(x) with N=9: 0.841903996167
22 Error: 0.000433011359186
23 Integral of cos(x) with N=10: 0.841821700007
24 Error: 0.000350715199399
25 Integral of cos(x) with N=11: 0.841760817405
26 Error: 0.000289832597425
27 Integral of cos(x) with N=12: 0.841714515321
28 Error: 0.000243530512976
29 Integral of cos(x) with N=13: 0.841678483879
30 Error: 0.000207499070943
31 Integral of cos(x) with N=14: 0.841649895569
32 Error: 0.000178910761171
33 Integral of cos(x) with N=15: 0.84162683297
34 Error: 0.000155848162437
35 Integral of cos(x) with N=16: 0.841607958582
36 Error: 0.000136973773665
37 Integral of cos(x) with N=17: 0.841592316399
38 Error: 0.000121331591133
39 Integral of cos(x) with N=18: 0.841579208411
40 Error: 0.000108223603482
41 Integral of cos(x) with N=19: 0.841568115345
42 Error: 9.71305373565e-05
43 Integral of cos(x) with N=20: 0.841558644427
44 Error: 8.76596193864e-05
45
46 === Question 4(b) ===
47 Actual integral of ln(x): -1
48 Integral of ln(x) with N=10: -0.965759065346
49 Error: 0.0342409346539
50 Integral of ln(x) with N=20: -0.982775471974
51 Error: 0.0172245280263
52 Integral of ln(x) with N=30: -0.988493840287
53 Error: 0.0115061597127
54 Integral of ln(x) with N=40: -0.99136170096
55 Error: 0.00863829903958
56 Integral of ln(x) with N=50: -0.993085194472
57 Error: 0.00691480552777
58 Integral of ln(x) with N=60: -0.994235347382
59 Error: 0.00576465261812
60 Integral of ln(x) with N=70: -0.99505745201
61 Error: 0.00494254798958
62 Integral of ln(x) with N=80: -0.995674340479
63 Error: 0.00432565952117
64 Integral of ln(x) with N=90: -0.996154326326
65 Error: 0.0038456736739
66 Integral of ln(x) with N=100: -0.99653843074
67 Error: 0.00346156926044
68 Integral of ln(x) with N=110: -0.996852774507
69 Error: 0.00314722549297
70 Integral of ln(x) with N=120: -0.997114780254
71 Error: 0.00288521974554
72 Integral of ln(x) with N=130: -0.99733651478
73 Error: 0.00266348521974
74 Integral of ln(x) with N=140: -0.997526600199
75 Error: 0.00247339980084
76 Integral of ln(x) with N=150: -0.997691361245
77 Error: 0.00230863875481
78 Integral of ln(x) with N=160: -0.997835542661
79 Error: 0.00216445733879
80 Integral of ln(x) with N=170: -0.997962773572
81 Error: 0.00203722642786
82 Integral of ln(x) with N=180: -0.998075877171
83 Error: 0.00192412282897
84 Integral of ln(x) with N=190: -0.998177082672
85 Error: 0.00182291732836
86 Integral of ln(x) with N=200: -0.998268173714
87 Error: 0.00173182628625
88
89 === Question 4(c) ===

```

```

90 Actual integral of ln(x): -1
91 Relative widths: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
92 Actual widths: [0.01818181818181818, 0.03636363636363636, 0.05454545454545454, 0.07272727272727272,
↪ 0.09090909090909091, 0.10909090909090909, 0.12727272727272726, 0.14545454545454545,
↪ 0.16363636363636364, 0.18181818181818182]
93 Estimated Integral of ln(x): -0.988377436631
94 Error: 0.0116225633689

```