AMAT467 Continuous Probability and Mathematical Statistics.

10/04/2022.

Prof. Kuon. HW Due 10/24.

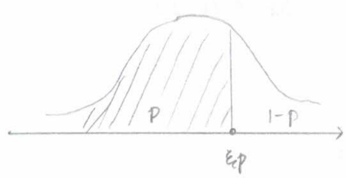
Quantiles

Given $0 < p < 1$, a quantile of order p

(100p) th percentile.

is a value ξ_p such that $F(\xi_p) = p$.

$\xi_{\frac{1}{2}} = \text{median} = q_2$



quartiles $\xi_{\frac{1}{4}}$, $\xi_{\frac{1}{2}}$, $\xi_{\frac{3}{4}}$

$\quad\quad\quad\quad\quad\; \| \quad\;\; \| \quad\;\; \|$

$\quad\quad\quad\quad\quad q_1 \quad q_2 \quad q_3$

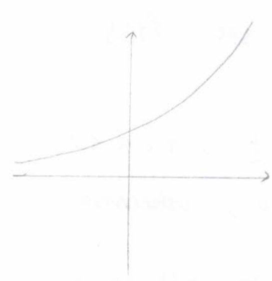
$Zq = q_3 - q_1$ interquartile range of x : used to measure the spread of distribution of x.

ex. X: continuous random variable : w/ pdf $f(x) = \dfrac{e^x}{(1+5e^x)^{1.2}}$

Find the median $F(q_2) = \frac{1}{2}$

$\uparrow$

$$F(x) = \int_{-\infty}^{x} \frac{e^t}{(1+5e^t)^{1.2}}\, dt.$$



$u = 1 + 5e^t$

$du = 5e^t\, dt$.

$$= \frac{1}{5}\int_1^{1+5e^x} \frac{1}{u^{1.2}}\, du$$

$$= \frac{1}{5}\int_1^{1+5e^x} u^{-1.2}\, du.$$

$$= \frac{1}{5}\, \frac{1}{-0.2}\, u^{-0.2}\Big]_1^{1+5e^x}$$

$$= -((1+5e^x)^{-0.2} - 1)$$

$$= 1 - (1+5e^x)^{-0.2}$$

$1 - (1+5e^{q_2})^{-0.2} = \frac{1}{2}$

$\frac{1}{2} = (1+5e^{q_2})^{-0.2}$.

$(\frac{1}{2})^{-5} = (1+5e^{q_2})$

$32 = 1 + 5e^{q_2}$.

$31 = 5e^{q_2}$.

$6.2 = e^{q_2}$

$\ln(6.2) = q_2$

$q_2 = 1.8245$.

differentiate to get the pdf of Y.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[\int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\,dx\right]$$

$$= \frac{d}{dy}\left[\frac{1}{2}\int_{-\sqrt{y}}^{\sqrt{y}} dx\right] = \frac{d}{dy}\left[\frac{1}{2}\cdot \sqrt{y} - (-\sqrt{y})\right]$$

$$= \frac{d}{dy}\left[\frac{1}{2}\cdot 2\sqrt{y}\right]$$

$$= \frac{d}{dy}(\sqrt{y})$$

$$= \frac{d}{dy}(y^{\frac{1}{2}}) = \frac{1}{2}y^{-\frac{1}{2}}$$

$$f_Y(y) = \begin{cases} \frac{1}{2}y^{-\frac{1}{2}}, & 0 < y < 1. \\ 0, & \text{otherwise}. \end{cases}$$

Thm] random variable with $f_X(x)$, S_X.

$Y = g(X)$, $g = 1-1$, differentiable function.

$\Downarrow$

$x = g^{-1}(y)$ exists.

Let $\frac{dx}{dy} = \frac{d}{dy}(g^{-1}(y))$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$y \in S_Y$. absolute value.

$$S_Y = \{ y = g(x) \mid x \in S_X \}$$

$\frac{dx}{dy}$: Jacobian.

$(x, y) \longmapsto (r, \theta)$

$r = \sqrt{x^2+y^2}$. $\tan\theta = \frac{y}{x}$.

$x = r\cos\theta$
$y = r\sin\theta$.

$\iint f(x,y)\,dA$. $\iint (r\cos\theta \cdot r\sin\theta)\, r\,dr\,d\theta$.

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Oct | 5^{th} | 2022.

1.8: Expectation of Random Variable

X: r.v. $\quad E(X) = \sum_x x \cdot p(x)$ discrete

expectation, expected value,

$E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)$ continuous. mean $\mu = E(X)$.

H	T
$\frac{1}{2}$	$\frac{1}{2}$
\$100	-\$100 .
\$200	-\$100

0

$(\$200 \times \frac{1}{2}) + (-\$100 \times \frac{1}{2}) = 100 - 50 = \$50$

ex. $P(k) = 1$

all mass at a constant k.

$E(k) = k \times P(k) = k(1) = k$.

ex.

x	1	2	3	4
$p(x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

$E(X) = \sum_x x \cdot p(x) = (1 \cdot \frac{4}{10}) + (2 \cdot \frac{1}{10}) + (3 \cdot \frac{3}{10}) + (4 \cdot \frac{2}{10})$

$= \frac{4}{10} + \frac{2}{10} + \frac{9}{10} + \frac{8}{10} = \frac{23}{10} = \boxed{2.3}$.

$\frac{a+b}{2} = \frac{1}{2}a + \frac{1}{2}b$.

ex.

$$f(x) = \begin{cases} 4x^3, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$\int_0^1 4x^3 dx = 4 \left. \frac{x^4}{4} \right|_0^1 = 4 \cdot (\frac{1}{4} - 0) = 1$ ✓

$E(X) = \int_0^1 x \cdot 4x^3 dx = 4 \left. \frac{x^5}{5} \right|_0^1 = 4(\frac{1}{5} - 0) = \frac{4}{5}$

$E(X)$ $\quad E(6X+3X^2)$

$= \sum_x (6x+3x^2)\cdot p_X(x)$

$= \sum_x (6x\cdot p_X(x) + 3x^2\cdot p_X(x))$; $\sum(a+b) = \sum a + b$

$= \sum_x 6x\cdot p_X(x) + \sum_x 3x^2\cdot p_X(x)$; $\sum(ka) = k\sum(a)$

$= 6\sum_x x\cdot p_X(x) + 3\sum_x x^2\cdot p_X(x)$

$= 6E(X) + 3E(X^2)$.

Thm. E is a linear operator.; k_1, k_2 are constants.

$E(k_1 g_1(x) + k_2 g_2(x))$

$= k_1 E(g_1(x)) + k_2 E(g_2(x))$

ex. $f(x) = \begin{cases} 2(1-x) & , 0 < x < 1 \\ 0 & , \text{otherwise}. \end{cases}$

$E(6X+3X^2) = 6E(X) + 3E(X^2)$

$= 6\int_0^1 x\cdot 2(1-x)\,dx + 3\int_0^1 x^2\cdot 2(1-x)\,dx$

$= 12\int_0^1 (x-x^2)\,dx + 6\int_0^1 (x^2-x^3)\,dx$

$= \left[12\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1\right] + \left[6\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1\right]$

$= \left[12\cdot\left(\frac{1}{2}-\frac{1}{3}\right)\right] + \left[6\cdot\left(\frac{1}{3}-\frac{1}{4}\right)\right]$

$= (6-4) + \left(2 - \frac{3}{2}\right)$

$= 2 + 2 - \frac{3}{2} = 4 - \frac{3}{2} = \frac{5}{2}$ ✓

ex. $E(6X^2 + X)$.

$p(x) = \begin{cases} \frac{x}{6} & , x = 1, 2, 3 \\ 0 & , \text{otherwise}. \end{cases}$

x	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$E(6X^2 + X) = 6E(X^2) + E(X)$

$= 6\sum_{x\in S_X}(x^2\cdot p_X(x)) + \sum_{x\in S_X} x\cdot p(x)$

$= 6\cdot\left[1^2\cdot\frac{1}{6} + 2^2\cdot\frac{2}{6} + 3^2\cdot\frac{3}{6}\right] + \left(1\cdot\frac{1}{6} + 2\cdot\frac{2}{6} + 3\frac{3}{6}\right)$

$= 6\left[\frac{1}{6} + \frac{8}{6} + \frac{27}{6}\right] + \left(\frac{1}{6} + \frac{4}{6} + \frac{9}{6}\right)$

$= 6\left[\frac{36}{6}\right] + \left(\frac{14}{6}\right) = 6 + \frac{14}{6} = \frac{50}{6}$

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1.9. Some Special Expectations.

mean, expected value, expectation

$\mu = E(X)$.

↗

1^{st} moment at 0.

$E((X-\mu)^m)$

$E((X-\mu)^2)$: variance of X.

$\sigma^2 = Var(X)$.

$\sum_x (x-\mu)^2 \cdot p(x)$. ⟵

$\int_{-\infty}^{+\infty} (x-\mu)^2 \cdot f(x)\, dx$ ⟵

$$E((x-\mu)^2) = E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - E(2\mu X) + \underbrace{E(\mu^2)}_{\text{constant}}$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

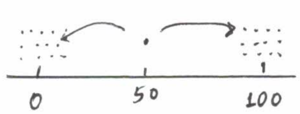
$$= E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$$

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$

↑

$\sigma = \sqrt{\sigma^2}$ = standard deviation of X ⇒ SD_X.

75 ↑ 25+
50
25 ↓ 25−



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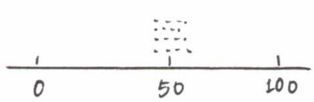

⟵

$\sigma^2 = Var(X) = E((X-\mu))^2$

ex. $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

$$\mu = \int_{-1}^{1} x \cdot \frac{1}{2}(x+1)\, dx = \cancel{\int_{-1}^{1} x \cdot (\frac{1}{2}x + \frac{1}{2})\, dx} = \int_{-1}^{1} \frac{1}{2}x^2 + \frac{1}{2}x\, dx$$

$$\cancel{= \frac{1}{2}\int_{-1}^{1} x^2\, dx + \frac{1}{2}\int_{-1}^{1} x\, dx = \frac{1}{2}\,\frac{1}{3}x^3\Big|_{-1}^{1} + \frac{1}{2}\,\frac{1}{2}x^2\Big|_{-1}^{1}}$$

$$\cancel{= \frac{1}{2}(\frac{1}{3} - \frac{1}{3}) + \frac{1}{2}(1+1)}$$

$$\cancel{= 0 + 1}$$

$$\cancel{= 1}$$

$$\int_{-1}^{1} x \frac{1}{2}(x+1)\, dx = \frac{1}{2}\int_{-1}^{1} x^2 + x\, dx = \frac{1}{2} \cdot [\frac{1}{3}x^3 + \frac{1}{2}x^2]_{-1}^{1}$$

$$= \frac{1}{2}[\frac{1}{3}(1)^3 + \frac{1}{2}(1)^2 - (\frac{1}{3}(-1)^3 + \frac{1}{2}(-1)^2)]$$

$$= \frac{1}{2}[(\frac{1}{3} + \frac{1}{2}) - (-\frac{1}{3} + \frac{1}{2})]$$

$$= \frac{1}{2}[(\frac{2}{6} + \frac{3}{6}) - (-\frac{2}{6} + \frac{3}{6})]$$

$$= \frac{1}{2}[\frac{5}{6} - (\frac{1}{6})]$$

$$= \frac{1}{2}[\frac{4}{6}] = \boxed{\frac{1}{3}}.$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

$$=$$

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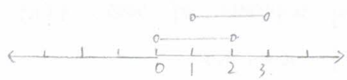
1.2.1.



(b). $C_1 = \{x : 0 < x < 2\}$, $C_2 = \{x : 1 < x < 3\}$.

$C_1 \cup C_2 = \{x : 0 < x < 3\}$.

1.2.2. Find the complement C^c of set C with respect to the space $\mathcal{C}$ if

(c). $C = \{(x,y) : |x| + |y| \le 2\}$, $\mathcal{C} = \{(x,y) ; x^2+y^2 < 2\}$.

1.2.4. Concerning DeMorgan's Laws (1.2.6) and (1.2.7):

(b). Show that the laws are true.

1.2.6. Show that the following sequences of sets, $\{C_K\}$, are nondecreasing, (1.2.16), then find $\lim_{K\to\infty} C_K$.

(b). $C_K = \{(x,y) : 1/K \le x^2+y^2 \le 4 - 1/k\}$, $k = 1,2,3,\ldots$

find C_{K+1}.

$C_{K+1} = \{(x,y) : \frac{1}{K+1} \le x^2 + y^2 \le 4 - \frac{1}{K+1}\}$

From C_K and C_{K+1}, we can say that, C_K is the subset of C_{K+1} that is $C_K \subseteq C_{K+1}$.

Hence, the given set $C_{K+1} = \{(x,y) : \frac{1}{K+1} \le x^2+y^2 \le 4 - \frac{1}{K+1}\}$ is non-decreasing

To Find $\lim\limits_{n\to\infty} C_K$, let us first substitutes values $k = 1,2,3,\ldots$ in the set.

$C_1 = 1 \le x^2+y^2 \le 3$.

$C_2 = \frac{1}{2} \le x^2+y^2 \le \frac{7}{2}$.

$C_3 = \frac{1}{3} \le x^2+y^2 \le \frac{11}{3}$.

In C_1, x varies from 1 to 3, In C_2, x varies from 0.5 to 3.5. From C_1 and C_2, we can say that. $C_1 \subset C_2$. $\lim\limits_{n\to\infty} C_K = C_1 \subset C_2 \subset C_3$.

1.2.9. For every one-dimensional set C for which the integral exists, Let $Q(C) = \int_C f(x)\,dx$. where $f(x) = 6x(1-x)$, $0<x<1$, zero elsewhere; otherwise, let $Q(C)$ be undefined. If $C_1 = \{x: \frac{1}{4}<x<\frac{3}{4}\}$, $C_2 = \{\frac{1}{2}\}$, and $C_3 = \{x: 0<x<10\}$. find $Q(C_1)$, $Q(C_2)$, $Q(C_3)$.

$$Q(C) = \int_C f(x)\,dx.$$

$$f(x) = \begin{cases} 6x(1-x) & 0<x<1. \\ 0 & \text{otherwise.} \end{cases}$$

$C_1 = \{x: \frac{1}{4} < x < \frac{3}{4}\}$.

$$\int_{C_1} f(x)\,dx = \int_{\frac{1}{4}}^{\frac{3}{4}} 6x(1-x)\,dx.$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} 6x - 6x^2\,dx.$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} 6x\,dx - \int_{\frac{1}{4}}^{\frac{3}{4}} 6x^2\,dx$$

$$= \frac{6x^2}{2}\Big|_{\frac{1}{4}}^{\frac{3}{4}} - \frac{6x^3}{3}\Big|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{11}{16}.$$

$C_2 = \{\frac{1}{2}\}$.

For $C_2 = \{x: \frac{1}{2}\}$, plug into the former function.

The probability density function $f(x)$ must have the following properties:

At $x = c$ = $\int_c f(x)dx = 0$ where c is any constant.

Hence

$Q(C_2) = \boxed{0}$. ✓.

$C_3 = \{0<x<10\}$. since C_3 is larger than the field of $f(x)$.

we only use the interval $\{x: 0<x<1\}$.

$$\int_0^1 6x(1-x)\,dx = \int_0^1 6x - 6x^2\,dx$$

$$= \int_0^1 6x\,dx - \int_0^1 6x^2\,dx.$$

$$= \boxed{1}$$ ✓

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10 / 12 / 2022.

Prof. Kwon.

$E(X) = \sum_{x} x \cdot p(x)$ $\quad\quad$ $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$

ex. sometimes the expectation DNE.

$$f(x) = \begin{cases} \frac{1}{x^2}, & 1 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(i). $f(x) \geq 0.$

(ii) $\int_1^{\infty} \frac{1}{x^2}\, dx = 1$?

Δ $\lim_{n\to\infty} \int_1^n \frac{1}{x^2}\, dx = \lim_{n\to\infty} -\frac{1}{x}\Big]_1^n = \lim_{n\to\infty} \left(-\frac{1}{n} + 1\right) = 1.$

$E(X) = \int_1^{\infty} x \cdot \frac{1}{x^2}\, dx = \int_1^{\infty} \frac{1}{x}\, dx = \lim_{n\to\infty} \int_1^n \frac{1}{x}\, dx = \lim_{n\to\infty} \ln x \Big|_1^n = \lim_{n\to\infty} (\ln(n) - \ln(1))$

$= \infty.$

$E(X)$ DNE.

Moment Generating Function (MGF)

on some open interval containing 0 inside.

X: r.v. s.t. $E(e^{tx})$ exists for all $-h < t < h$ for some h, then mgf is defined to be

$M_X(t) = E(e^{tx})$ for $-h < t < h$.

$\cancel{(-1, 0)}$ $\left(-\frac{1}{2}, \frac{1}{2}\right)$ ✓ $\cancel{(1, 5)}$

$$\left.\begin{aligned} M_X(0) = E(e^0) = E(1) &= \int_{-\infty}^{\infty} 1 \cdot f(x)\, dx. && \text{(continuous case)} \\ &= \sum_{x} 1 \cdot p(x) && \text{(discrete case)} \end{aligned}\right\} = 1.$$

$M_X(t) = E(e^{tX})$ $\quad \sum_x e^{tx} \cdot p(x)$

$\int_{-\infty}^{+\infty} e^{tx} \cdot f(x)\, dx.$

Why is Mgf is important ?

① $F_X(z) = F_Y(z)$ for all values z equal in distribution.

$\Downarrow$ $\Uparrow$ yes as well!

$M_X(t) = M_Y(t)$ for t in $(-h, h)$. for some $h > 0$

uniqueness of mgf.

ex.

x	1	2	3	4
$p(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

$$M_X(t) = \sum_{x=1}^{4} e^{tx} \cdot p(x) = e \cdot \frac{1}{10} + e^{2t} \cdot \frac{2}{10} + e^{3t} \cdot \frac{3}{10} + e^{4t} \cdot \frac{4}{10}$$

$$M_Y(t) = \frac{1}{2} + \frac{1}{2} e^{100t}$$

$\frac{1}{2} e^{0 \cdot t}$

y	0	100
$p(y)$	$\frac{1}{2}$	$\frac{1}{2}$

$\Leftarrow$ $M_{Y}(t) = \frac{1}{2} e^{0 \cdot t} + \frac{1}{2} e^{100 \cdot t}$

$= \frac{1}{2} + \frac{1}{2} e^{100t}$.

② generate moments.

$E((X-\mu)^m)$ $\quad m$: positive integer

m^{th} moment about μ:

$m = 2$: $E((X-\mu)^2) = Var(X)$.

$\mu = 0$, $m = 1$: $\mu(X) = \mu = E(X)$

continuous case

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx.$$

$$M_X'(t) = \frac{d}{dt} M_X(t) = \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx.$$

$$= \int_{-\infty}^{\infty} \frac{d}{dt} e^{tx} \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot x \cdot f(x)\, dx.$$

$$= \int_{-\infty}^{\infty} x e^{tx} \cdot f(x)\, dx.$$

discrete case:

$$M_X'(t) = \sum_x x e^{tx} \cdot p(x).$$

Set $t = 0$; $M_X'(0) = \int_{-\infty}^{\infty} x \cdot e^{0 \cdot x} f(x)\, dx$

$= \underbrace{\int_{-\infty}^{\infty} x \cdot f(x)\, dx}_{\text{expectation}}$.

take the 1st derivative of MGF and plug in $t=0$. $\Rightarrow$ expected Value.

ex. There are situations where the MGF DNE.

$$p(x) = \begin{cases} \dfrac{6}{\pi^2 x^2}, & x = 1, 2, 3, \dots \\ 0, & \text{otherwise.} \end{cases}$$

(i) : $p(x) \geq 0$.

(ii) : $\sum_x x \cdot p(x) = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{6}{\pi^2} \cdot \frac{\pi^2}{6} = 1$.

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} \cdot \frac{6}{\pi^2 x^2} = \cancel{\frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{e^{tx}}{x^2}} \quad \sum_{x=1}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}.$$

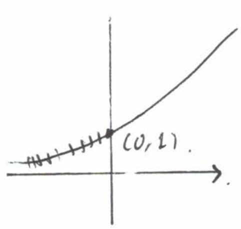
convergence ? (ratio - test).

$$a_{x+1} = \frac{6e^{t(x+1)}}{\pi^2 (x+1)^2}$$

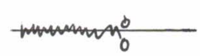
$$a_x = \frac{6e^{tx}}{\pi^2 x^2}.$$

$$\left| \frac{a_{x+1}}{a_x} \right| = \frac{\cancel{6} e^{t(x+1)}}{\cancel{\pi^2} (x+1)^2} \cdot \frac{\pi^2 x^2}{\cancel{6e^{tx}}} = \underbrace{e^t \left(\frac{x}{x+1}\right)^2}_{} \xrightarrow[x \to \infty]{} e^t < 1.$$

$e^{t(x+1)} = \cancel{e^{tx}} \cdot e^t$



$t < 0$.



MGF $\Rightarrow$ DNE. $\because$ the point $t = 0$ is not inside the domain.

HW3: $M_Y(t) = \frac{1}{3} + \frac{1}{2}e^{2t} + \frac{1}{6}e^{4t}$.

Find Variance of Y.

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10/17/2022.

1.10 Important Inequalities

Thm) X: r.v. m: positive integer.

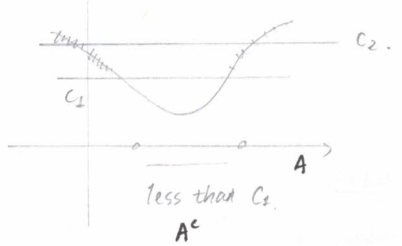
If $E(X^m)$ exists, then for all positive integer $k \le m$, $E(X^k)$ exists also.

Thm) (Markov's Inequality).

△ $u(X)$: nonnegative

If $E(u(X))$ exists, then for every positive constant $c > 0$,

$$P(u(X) \ge c) \le \frac{E(u(X))}{c}$$



$C_2 > C_1$.

$$\frac{E(u(X))}{C_2} \le \frac{E(u(X))}{C_1}.$$

(pf). $A = \{x : u(x) \ge c\}$

$P(X \in A)$ $\qquad f(x)$: pdf for X.

$$E(u(X)) = \int_{-\infty}^{\infty} u(x) f(x)\, dx.$$

$$= \int_A \underbrace{u(x)\, f(x)}_{\ge 0}\, dx + \int_{A^c} \underbrace{u(x)\, f(x)}_{\ge 0}\, dx$$

$$\Downarrow$$

$$E(u(X)) \ge \int_A u(x) f(x)\, dx.$$

$$\ge c \int_A f(x)\, dx.$$

$$E(u(X)) \ge c \int_A f(x)\, dx.$$

$$P(X \in A) = \int_A f(x)\, dx.$$

$$= P(u(X) \ge c)$$

$\Rightarrow$ $E(u(X)) \ge c \cdot P(u(X) \ge c)$

ex. $f(x) = \begin{cases} \frac{1}{2\sqrt{3}} , & \sqrt{3} < x < \sqrt{3} \\ 0 , & \text{otherwise.} \end{cases}$

$\mu = E(X) = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{2\sqrt{3}} x \, dx = 0$

$\sigma^2 = E(X^2) - [E(X)]^2 = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \cdot \frac{1}{2\sqrt{3}} dx - 0 = \frac{1}{2\sqrt{3}} \frac{x^3}{3} \Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{2\sqrt{3}} \left(\frac{3\sqrt{3}(2)}{3} \right) = 1$

$\sigma = \sqrt{\sigma^2} = \sqrt{1} = 1.$

$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$ → chebychev's Inequality.

$P(|X| \geq k) \leq \frac{1}{k^2}$. True for any positive number k.

$k = \frac{3}{2}$. : chebyshev's Inequality gets the upper bond : $\frac{1}{(\frac{3}{2})^2} = \frac{4}{9}$ $= 0.48$

$P(|X| \geq \frac{3}{2}) = 1 - P(-\frac{3}{2} \leq X \leq \frac{3}{2}) = 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} = 1 - \frac{1}{2\sqrt{3}} \left(\frac{3}{2} + \frac{3}{2} \right) = 1 - \frac{\sqrt{3}}{2}$ → precise value. $= 0.15$.

※ Chebyshev's Inequality gives really far off bound.

※ True for any random variable.

· pdf, pmf ⇒ precise proability

ex.

x	-1	0	1
$p(x)$	$\frac{1}{8}$	$\frac{6}{8}$	$\frac{1}{8}$

$E(X) = -1 \cdot \frac{1}{8} + 0 \frac{6}{8} + 1 \frac{1}{8} = 0.$

$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = \cancel{\frac{1}{8}} (1^2 \cdot \frac{1}{8} + 0 + 1^2 \cdot \frac{1}{8}) - 0$

$= \frac{1}{4} \Rightarrow \sigma = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$P(|X - 0| \geq 1)$

μ $\quad k\sigma = 1$

$2 \cdot \frac{1}{2} = 1$ ⇒

$\therefore k = 2.$

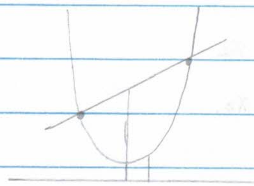
chebshev's Inequality ~~≠~~ upper bound is $\frac{1}{k^2} = \frac{1}{4} = 0.25$.

$P(|X| \geq 1) = P(X=1) + P(X=-1) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4} = 0.25.$

AMAT467 Continuous Probability and Mathematical Statistics.

Oct / 18 / 2022.

Convex function.



Jensen's Inequality

ϕ : convex function

$$\phi(E(X)) \leq E(\phi(X))$$

Proof. Assume that ϕ has a second derivative.

$$\phi(x) = \phi(\mu) + \phi'(\mu)(x-\mu) + \frac{\phi''(\rho)(x-\mu)^2}{2!}$$ for ρ between μ and x.

Taylor Series of ϕ around $\mu = E(X)$.

since ϕ is a convex function $\phi''(\rho) \geq 0$

$\Rightarrow \phi(x) \geq \phi(\mu) + \phi'(\mu)(x-\mu)$

Tabe expectation of both side:

$E(\phi(X)) \geq E(\phi(\mu)) + E(\phi'(\mu)(X-\mu))$

$E(\phi(X)) \geq \phi(\mu) + \phi'(\mu) \underbrace{E(X-\mu)}_{E(X)-\mu}.$

$E(\phi(X)) \geq \phi(E(X)) + 0.$

ex. X r.v. $E(X^2)$

$E(X)$

$(E(X))^2 < E(X^2)$

$Var(X) = E(X^2) - [E(X)]^2 \geq 0$

ϕ: convex

$\phi(E(X)) \leq E(\phi(X)).$

$\phi(t) = t^2.$

① discrete random vector

$\mathcal{D}$ is finite or countable

joint pmf of (X_1, X_2)

$$P_{X_1,X_2}(x_1, x_2) = P(X_1 = x_1, X_2 = x_2)$$

$$0 \le P_{X_1,X_2}(x_1, x_2) \le 1.$$

$$\sum_{x_2}\sum_{x_1} P_{X_1,X_2}(x_1,x_2) = 1.$$

event $B \in \mathcal{A}$, $P((X_1, X_2) \in B) = \sum_B\sum P_{X_1,X_2}(x_1, x_2)$

Video Lecture Part II.

ex. flip a coin three times

X_1. # of heads in the first two flips.

X_2 : # of heads in all three flips.

(X_1, X_2)			
	HHH	$(2,3)$	$\frac{1}{8}$
	HHT	$(2,2)$	$\frac{1}{8}$
	HTH	$(1,2)$	⋮
	THH	$(1,2)$	⋮
	TTH	$(0,1)$	
	THT	$(1,1)$	
	HTT	$(1,1)$	
	TTT.	$(0,0)$	$\frac{1}{8}$

		support of X_2			
		0	1	2	3.
support of X_1.	0	$\frac{1}{8}$	$\frac{1}{8}$.	0	0.
	1	0	$\frac{2}{8}$	$\frac{2}{8}$	0.
	2.	0	0	$\frac{1}{8}$	$\frac{1}{8}$

joint pmf. of (X_1, X_2).

$$P(X_1 \ge 2, X_2 \ge 2) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

$$= P_{X_1,X_2}(2,2) + P_{X_1,X_2}(2,3)$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{2}{8} \Rightarrow \frac{1}{4}$$

support of a discrete r.vector $\{(0,0), (0,1), (1,1), (1,2), (2,2), (2,3)\}$.

ex. $f(x,y) = \begin{cases} 4xy\, e^{-(x^2+y^2)} & x>0,\ y>0 \\ 0, & \text{otherwise} \end{cases}$

$$P\left(x > \tfrac{\sqrt{2}}{2},\ y > \tfrac{\sqrt{2}}{2}\right) = \int_{\frac{\sqrt{2}}{2}}^{\infty} \int_{\frac{\sqrt{2}}{2}}^{\infty} 4xy\, e^{-x^2} e^{-y^2}\, dy\, dx.$$

$$= \int_{\frac{\sqrt{2}}{2}}^{\infty} 4x e^{-x^2} \int_{\frac{\sqrt{2}}{2}}^{\infty} y e^{-y^2}\, dy\, dx.$$

$$\int_{\frac{\sqrt{2}}{2}}^{\infty} y e^{-y^2}\, dy = \int_{\frac{1}{2}}^{\infty} \tfrac{1}{2} e^{-u}\, du. \text{ ; fundamental theorem of Calculus ?}$$

$u = y^2.$ $\quad -\tfrac{1}{2} e^{-u} \Big]_{\frac{1}{2}}^{\infty} = 0 + \tfrac{1}{2} e^{-\frac{1}{2}}$

$du = 2y\,dy.$

$\tfrac{1}{2} du = y\,dy$

$$\int_{\frac{\sqrt{2}}{2}}^{\infty} 4x e^{-x^2} \cdot \tfrac{1}{2} e^{-\frac{1}{2}}\, dx = \int_{\frac{\sqrt{2}}{2}}^{\infty} 2x e^{-x^2} e^{-\frac{1}{2}}\, dx.$$

$$= 2e^{-\frac{1}{2}} \underbrace{\int_{\frac{\sqrt{2}}{2}}^{\infty} x e^{-x^2}\, dx}_{= \frac{1}{2} e^{-\frac{1}{2}}}.$$

$$= 2e^{-\frac{1}{2}} \cdot \tfrac{1}{2} e^{-\frac{1}{2}}$$

$$= (e^{-\frac{1}{2}})^2 = e^{-1} = \boxed{\tfrac{1}{e}}$$

---

support of continuous random vector. (X_1, X_2): points (x_1, x_2) at which $f(x_1, x_2) > 0$.

$S \subseteq D$.

AMAT467 Continuous Probability and Mathematical Statistics.

10/24/2022.

Prof. Kwon

Next Wednesday

Office Hour 12-3

$F(\xi_p) = p$ $\qquad F(x) = \int_{-\infty}^{x} f(t)\,dt$.

$0 < p < 1$

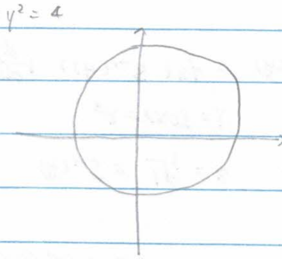
$p = 0.5$

Transformation $\quad X \quad f_X(x) \quad Y = g(X)$.

$p_X(x)$.

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\, dy\, dx$$

$x^2 + y^2 = 4$



$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{2\pi} \int_0^{2} \sqrt{4-r^2}\; r\,dr\,d\theta$$

cdf technique: $\quad X \to f_X(x) \quad Y = g(X)$.

$$\frac{d}{dx}F_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad S_Y = (0,1)$$

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$

$Y = X^2$.

$f_Y(y) = ?$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= P(0 \le X \le \sqrt{y})$$
$$= \int_0^{\sqrt{y}} 2x\,dx = y.$$

g: 1-1 differentiable.

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right|.$$

$Y = g(X) \quad X = g^{-1}(y)$. $\qquad \frac{dx}{dy} = \frac{d}{dy}(g^{-1}(y))$.

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

For this problem: $g^{-1}(y) \Rightarrow \sqrt{y} = X$.

$f_X(g^{-1}(y)) = 2\sqrt{y}$.

$$\frac{dx}{dy} = \frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}} = \quad f_Y(y) = 2\sqrt{y} \cdot \frac{1}{2\sqrt{y}} = \frac{\sqrt{y}}{\sqrt{y}}.$$

$Var(ax+b) = a^2 Var(X)$.

$\delta_{ax+b} = |a| \delta_X$.

Moment Generating Function.

$M_X(t) = E(e^{tX})$.

$\overset{P}{=} \sum e^{tx} \cdot p_X(x)$

$\overset{C}{=} \int_{-\infty}^{+\infty} e^{tx} \cdot f_X(x)\, dx$.

$-h < t < h$. 0 always included. $M_X(0) = 1 =$ sum of all probabilities.

Convergence Test

Why moment generating function: $E(X^m) = M_X^{(m)}(t)\big|_{t=0}$.

$E(X^1) =$ first moment.

$E(X^2) =$ Second Moment.

$V(X) = E(X^2) - (E(X))^2$.

x	0	1	2
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Y	0	3	1000
$p(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$M_X(t) = \frac{1}{2} + \frac{1}{4}e^t + \frac{1}{4}e^{2t}$

$M_Y(t) = \frac{1}{4} + \frac{1}{2}e^{3t} + \frac{1}{4}e^{1000t}$.

$P(|X-\mu| \geq k\delta) \leq \frac{1}{k^2}$ $k > 0$. $k \in \mathbb{Z}^+$

$\mu - 2\delta$. $\mu - \delta$ μ $\mu + \delta$ $\mu + 2\delta$.

convex function. $\Rightarrow$ Jensen's Inequality.

$E(X^2) \geq (E(X))^2$.

AMAT467 Continuous Probability and Mathematical Statistics

10/26/2022

Prof. Kwon

Office Hour Fri 12-3.

chebyshev's Inequality

Annoucement on Wednesday.

Chapter 1 - Marginal Probability.

$P_{X_1, X_2}(x_1, x_2)$; two random variable. $P_{X_1,X_2}(x_1,x_2) = P(X_1 = x_1, X_2 = x_2)$.

$f_{X_1, X_2}(x_1, x_2)$.

$$P((X_1, X_2) \in A) = \iint_A f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2.$$

$$F_{X_1,X_2}(x_1,x_2) = P(X_1 \le x_1 \text{ and } X_2 \le x_2)$$

$$F_{X_1,X_2}(x_1,x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f_{X_1,X_2}(w_1,w_2)\, dw_1\, dw_2.$$

$$\frac{\partial^2 F}{\partial x_1 x_2} = f_{X_1,X_2}(x_1,x_2).$$

Marginal Distributions.

toss a coin 3 times

Ex: X_1 : # of H's on the first two tosses.

X_2 : # of H's on the all three tosses.

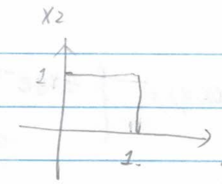
$x_1 \backslash x_2$	0	1	2	3
0	$\frac{1}{8}$	$\frac{1}{8}$	0	0
1	0	$\frac{2}{8}$	$\frac{2}{8}$	0
2	0	0	$\frac{1}{8}$	$\frac{1}{8}$

$P(X_1 = 0, X_2 = 1) \Leftrightarrow P(0,1) = \frac{1}{8}$

$$\begin{aligned} P(X_1 \le 1, X_2 \le 2) &= P(0,0) + P(0,1) + P(1,1) + P(1,2) \\ &= [(\tfrac{1}{8})\cdot 2] + [(\tfrac{2}{8})\cdot 2] \\ &= \tfrac{1}{4} + \tfrac{1}{2} \\ &= 0.25 + 0.5 = \boxed{0.75} \end{aligned}$$

$$\begin{aligned} P(X_1 \le 1) &= P(0,0) + P(0,1) + P(1,1) + P(1,2) \\ &= ((\tfrac{1}{8})\cdot 2) + (\tfrac{2}{8}\cdot 2) \\ &= \tfrac{1}{4} + \tfrac{1}{2} = \boxed{0.75} \end{aligned}$$

ex.



$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1 < 1,\ 0 < x_2 < 1. \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{x_1}(x_1) = \int_0^1 (x_1 + x_2)\, dx_2.$$

$$= x_1 x_2 + \frac{x_2^2}{2} \Big|_0^1$$

$$= \left(x_1(1) + \frac{(1)^2}{2}\right) - 0$$

$$= x_1 + \frac{1}{2}$$

$$f_{x_1}(x_1) = x_1 + \frac{1}{2}, \quad 0 < x < 1.$$

$$\int_0^1 f_{x_1}(x_1)\, dx_1$$

$$= \int_0^1 x_1 + \frac{1}{2}\, dx_1.$$

$$= \frac{1}{2}x_1^2 + \frac{1}{2}x_1 = \left[\frac{1}{2}(1) + \frac{1}{2}(1)\right] - 0 = \boxed{1} \checkmark$$

$$f_{x_2}(x_2) = \int_0^1 (x_1 + x_2)\, dx_1.$$

$$= \frac{1}{2}x_1^2 + x_1 x_2 \Big|_0^1$$

$$= \frac{1}{2}x_1 + x_2$$

$$P(x_1 \le \tfrac{1}{2}) = \int_0^{\frac{1}{2}} \int_0^1 (x_1 + x_2)\, dx_2\, dx_1.$$

x_2 bound is not given.

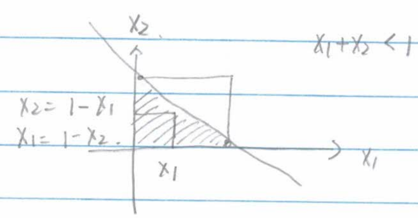
$$\int_0^{\frac{1}{2}} \underbrace{\int_0^1 (x_1 + x_2)\, dx_2}_{f_{x_1}(x_1)}\, dx_1$$

Use Marginal pdf : $\int_0^{\frac{1}{2}} f_{x_1}(x_1)\, dx_1$.

$P(x_1 + x_2 < 1)$

$$= \int_0^1 \int_0^{1-x_1} (x_1 + x_2)\, dx_2\, dx_1.$$

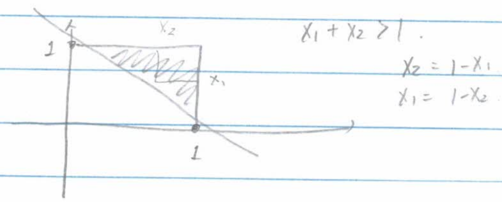
$$\int_0^1 \int_0^{1-x_2} (x_1 + x_2)\, dx_1\, dx_2.$$



$P(x_1 + x_2 > 1)$.

$$= \int_0^1 \int_{1-x_1}^1 (x_1 + x_2)\, dx_2\, dx_1.$$

$$= \int_0^1 \int_{1-x_2}^1 (x_1 + x_2)\, dx_1\, dx_2.$$



AMAT467 - Continuous Probability and Mathematical Statistics.

11 / 07 / 2022.

Prof. Kwon.

※ No Continuous r.v.

Q1. 1, 2, 3, 4, 5

abc = odd.

$$\frac{1}{\binom{5}{3}} = \frac{1}{10}.$$

Q2.

o	e	o	e	o.
↑	↑	↑	↑	↑
$\frac{5}{10}$	$\frac{5}{9}$	$\frac{4}{8}$	$\frac{4}{7}$	$\frac{3}{6}$

Q3. Baye's Theorem.

Q4. X: # of R's.

$\cancel{\binom{5}{3}}$

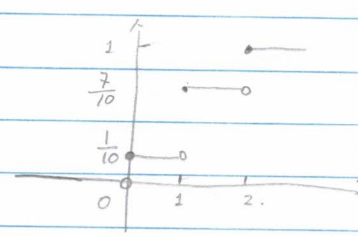
$P(X=0) = \frac{1}{\binom{5}{3}} = \frac{1}{10}$

$P(X=1) = \frac{\binom{2}{1}\binom{3}{2}}{\binom{5}{3}} = \frac{6}{10}.$

$P(X=2) = \frac{3}{10}$

$\frac{1}{10} + \frac{6}{10} + \frac{3}{10} = 1.$

(b). $F_X(x) = P(X \leq x).$



AMAT467 - Continuous Probability and Mathematical Statistics.

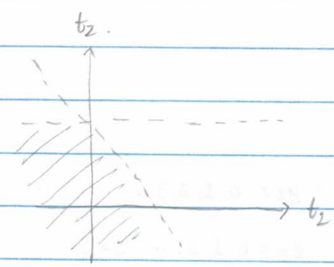
$M(t_1, t_2) = E(e^{t_1X + t_2Y})$

$$f(x,y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$M(t_1,t_2) = \int_0^{\infty}\int_x^{\infty} e^{t_1X + t_2y - y}\, dy\, dx.$$

$$= \int_0^{x}\int_0^{y} e^{t_1X + t_2y - y}\, dx\, dy. = \frac{1}{t_1}\ \frac{t_2 - 1 - t_1 - t_2 + 1}{(t_1 + t_2 - 1)(t_2 - 1)}.$$

$$= \frac{1}{(1 - t_1 - t_2)(1 - t_2)} \qquad t_1 + t_2 < 1. \quad t_2 < 1.$$



Transformation(s):

$P_{X_1, X_2}(x_1, x_2)$ 1-1 transformation.

$\Downarrow$

$P_{Y_1, Y_2}(Y_1, Y_2)$.

$y_1 = u_1(x_1, x_2)$ $\qquad x_1 = w_1(y_1, y_2)$

$y_2 = u_2(x_1, x_2)$ $\qquad x_2 = w_2(y_1, y_2)$.

$$P_{Y_1, Y_2}(Y_1, Y_2) = \begin{cases} P_{X_1, X_2}(w_1(y_1, y_2), w_2(y_1, y_2)), & (y_1, y_2) \in \mathcal{Y} \\ 0, & \text{otherwise}. \end{cases}$$

ex. flu season.

X_1: # of cases for $\quad x_1 = 0, 1, 2, 3.$

X_2: # of cases $\quad x_2 = 0, 1, 2, 3,$

$$P_{X_1, X_2}(x_1, x_2) = \frac{\mu_1^{x_1}\mu_2^{x_2} e^{-\mu_1}}{x_1!\ x_2!} e^{-\mu_2}.$$

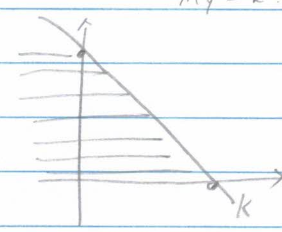
$\mu_1, \mu_2 > 0.$

ex. $$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1, & 0 < x_1,\ x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$Z = X_1 + X_2 \qquad 0 < Z < 2.$

$$F_Z(z) = P(Z \le z) = P(X_1 + X_2 \le z) = \begin{cases} 0, & z \le 0. \\ \dfrac{z^2}{2}, & 0 < z < 1. \\ 1 - \dfrac{(2-z)^2}{2}, & \\ 1, & z \ge 2. \end{cases}$$

$x + y \le k$.



$$f_Z(z) = \begin{cases} z, & 0 < z < 1 \\ 2 - z, & 1 < z < 2 \\ 0, & \text{otherwise} \end{cases}$$

$\frac{d}{dz}\left(\frac{1}{2}z^2\right)$

$X \qquad f_X(x) \qquad y = g(x).$

$Y \qquad \cancel{g(y)}.\ f_Y(y) \qquad x = g^{-1}(y).$

$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right|.$ change ~~unit~~ area.

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \left(\frac{\partial x_1}{\partial y_1}\right)\left(\frac{\partial x_2}{\partial y_2}\right) - \left(\frac{\partial x_1}{\partial y_2}\right)\left(\frac{\partial x_2}{\partial y_1}\right)$$

AMAT467

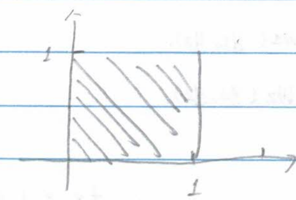
11/09/2022

ex.

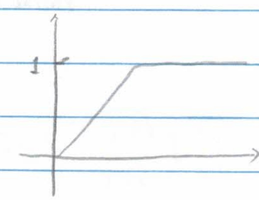
$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1, & 0<x_1,x_2<1 \\ 0, & \text{o.w} \end{cases}$$

$Z = X_1 + X_2$. $0 < Z < 2$

Find pdf of Z



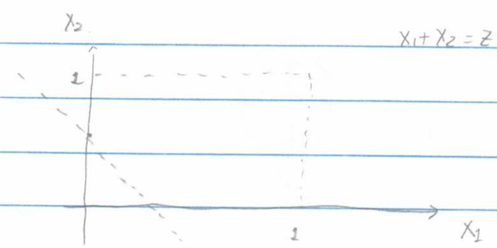
$$F_Z(z) = \begin{cases} 0, & z<0 \\ \quad, & 0<z<2 \\ 1, & z>2 \end{cases}$$



$X_1 + X_2 = Z$

$(Z,0)$ $(0,Z)$

$z = \frac{1}{2}$ $X_1+X_2 = \frac{1}{2}$



$$F_Z(z) : \begin{cases} 0, & z<0 \\ \frac{z^2}{2}, & 0<z\le 1 \\ 1-\frac{(2-z)^2}{2}, & 1\le z<2 \\ 1, & z>2 \end{cases}$$

$\frac{d}{dz}\left(1-\frac{(2-z)^2}{2}\right)$

$= 0 - \frac{d}{dz}\left(\frac{(2-z)^2}{2}\right)$

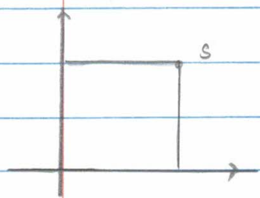
$\boxed{2-z}$ ✓

$$\frac{d}{dz}F_Z(z) \begin{cases} \cancel{\frac{1}{2}z}\ z, & 0<z<1 \\ 2-z, & 1, z<2 \\ 0, & \text{otherwise} \end{cases}$$

ex. $f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1, & 0 < x_1, x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$

$0 < x_1 < 1$

$0 < x_2 < 1.$



$Y_1 = X_1 + X_2$

$Y_2 = X_1 + X_2.$

$X_1 = \dfrac{Y_1 + Y_2}{2}$

$X_2 = \dfrac{Y_1 - Y_2}{2}$

$0 < \frac{1}{2}Y_1 + \frac{1}{2}Y_2 < 1$

$0 < \frac{1}{2}Y_1 - \frac{1}{2}Y_2 < 1$

$\Rightarrow$

$0 < Y_1 + Y_2 < 2$

$0 < Y_1 - Y_2 < 2.$

$f_{Y_1,Y_2}(y_1,y_2) = ?$

$X_1 = Y_1 - X_2$

$Y_2 = Y - X_2$

$|J| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = |-\frac{1}{4} - \frac{1}{4}| = |-\frac{1}{2}| = \frac{1}{2}$

$0 < Y_1 + Y_2 < 2.$

$0 < Y_1 - Y_2 < 2.$

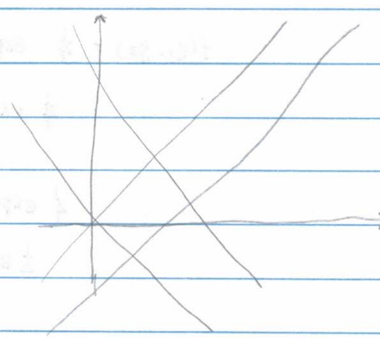
$Y_1 + Y_2 = 0$

$Y_1 + Y_2 = 2$

$Y_1 - Y_2 = 0$

$Y_1 - Y_2 = 2.$

$\Rightarrow$



$\Uparrow$

$\mathcal{Y} = \{(y_1, y_2) : 0 < y_1 + y_2 < 2$

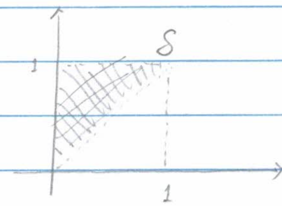
$0 < y_1 - y_2 < 2\}.$

$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 1 \cdot \frac{1}{2}, & (y_1,y_2) \in \mathcal{Y} \\ 0, & \text{otherwise.} \end{cases}$

ex. $f(x_1, x_2) = 10 x_1 x_2^2, \quad 0 < x_1 \leq x_2 < 1.$

$Y_1 = \dfrac{X_1}{X_2}$ $\Rightarrow$ $X_1 = Y_1 Y_2.$

$Y_2 = X_2.$ $\quad X_2 = Y_2$



$$|J| = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = Y_2.$$

$f(y_1, y_2) = 10\, y_1 y_2 (y_2)^2 \, |Y_2|$

$= 10 y_1 \, y_2^4 \qquad (y_1, y_2) \in \mathcal{Y}.$ Find new support.

$\mathcal{Y} = \{ (y_1, y_2) : 0 < y_1, \; y_2 < 1 \}$

AMAT467.

11 / 14 / 2022

2-2.

Wednesday 5-7 CK370.

Office hour Wednesday

MGF Technique Generating moment.

$\mu_1, \mu_2 > 0.$

$$P_{X_1,X_2}(x_1,x_2) = \frac{e^{-\mu_1}\mu_1^{x_1}}{x_1!} \cdot \frac{\mu_2^{x_2} e^{-\mu_2}}{x_2!} \quad , x_2\ x_1, = 0, 1, 2, \ldots$$

Poisson distribution with parameter $\mu > 0$

$$P(x) = \begin{cases} \frac{e^{-\mu}\mu^x}{x!} , & x = 0, 1, 2, \\ 0 , & \text{otherwise} \end{cases}$$

$$M_X(t) = \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\mu}\mu^x}{x!} \quad (\text{Mc} \quad \text{series expansion})$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!} \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{x=0}^{\infty} \frac{x^Y}{x!}$$

$$= e^{-\mu} e^{\mu e^t} = e^{\mu(e^t-1)}$$

$Y = X_1 + X_2.$

$$M_Y(t) = E(e^{t(X_1+X_2)}) = \sum_{x_2=0}^{\infty}\sum_{x_1=0}^{\infty} e^{tx_1} e^{tx_2} \frac{e^{-\mu_1}\mu_1^{x_1}}{x_1!} \frac{\mu_2^{x_2}e^{-\mu_2}}{x_2!}$$

$$e^{-\mu_1-\mu_2} \sum_{x_2=0}^{\infty} \frac{e^{tx_2}\mu_2^{x_2}}{x_2!} \sum_{x_1=0}^{\infty} \frac{e^{tx_1}\mu_1^{x_1}}{x_1!}$$

$$e^{-\mu_1-\mu_2} \sum_{x_2=0}^{\infty} \frac{(\mu_2 e^t)^{x_2}}{x_2!} \sum_{x_1=0}^{\infty} \frac{(\mu_1 e^t)^{x_1}}{x_1!}$$

$$= e^{-\mu_1-\mu_2} \cdot e^{\mu_2 e^t} e^{\mu_1 e^t}$$

$$= e^{-(\mu_1+\mu_2)} e^{e^t(\mu_1+\mu_2)}$$

$$= e^{(\mu_1+\mu_2)(e^t-1)}$$

Y is a poisson distribution.

$P(0 < X_1 < \frac{1}{2}) = \int_0^{\frac{1}{2}} 2(1-X_1)\,dX_1 = 1 - \frac{1}{4} = \frac{3}{4} = \frac{9}{12}$

$f_1(X_1) = \int_{X_2}^{1} 2\,dX_2 = 2(1-X_1)$

$E(X_1 \mid X_2) = \int_0^{X_2} X_1 \, f_{1|2}(X_1 \mid X_2)\,dX_1.$

$= \int_0^{X_2} \frac{X_1}{X_2}\,dX_1 = \frac{X^2}{2}, \quad 0 < X_2 < 1.$

$Var(X_1 \mid X_2) = E(X_1^2 \mid X_2) - (E(X_1 \mid X_2))^2$

$= \frac{X_2^{\,2}}{12}, \quad 0 < X_2 < 1.$

Thm) $E(E(X_2 \mid X_1)) = E(X_2)$ ※ Remard

$Var(E(X_2 \mid X_1)) \leq Var(X_2).$

Recall (2.4) $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$P(A \mid B) = P(A)$

A, B) are independent : $P(A \cap B) = P(A) \cdot P(B)$

$f_{2|1}(X_2 \mid X_1) = \frac{f_{X_1,X_2}(X_1, X_2)}{f_{X_1}(X_1)} \iff f_{X_1,X_2}(X_1, X_2) = f_{2|1}(X_2 \mid X_1) \cdot f_{X_1}(X_1)$

Suppose $f_{2|1}(X_2 \mid X_1)$ doesn't depend on X_1, Then $f_2(X_2) = \int f_{X_1,X_2}(X_1, X_2)\,dX_1.$

$f_2(X_2) = \int f_{2|1}(X_2 \mid X_1) \cdot f_{X_1}(X_1)\,dX_1.$

$= f_{2|1}(X_2 \mid X_1) \int f_{X_1}(X_1)\,dX_1$

$= f_{2|1}(X_2 \mid X_1).$

$f_{X_1,X_2}(X_1, X_2) = f_2(X_2)\, f_{X_1}(X_1)$

X_1 and X_2 are independent ☑

iff

$f(X_1, X_2) = f_1(X_1) \cdot f_2(X_2)$

$P(X_1, X_2) = P_1(X_1) \cdot P_2(X_2).$

AMAT467 Continuously Probability

2.4 Independence

X_1 and X_2 are independent iff $f(x_1, x_2) = f(x_1) f(x_2)$

$p(x_1, x_2) = p(x_1) p(x_2)$

ex. $f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1, x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$

Are X_1 and X_2 independent?

$f_1(x_1) = \int_0^1 (x_1 + x_2)\, dx_2 = x_1 + \frac{1}{2}, \quad 0 < x_1 < 1$

$f_2(x_2) = \int_0^1 (x_1 + x_2)\, dx_1 = x_2 + \frac{1}{2}, \quad 0 < x_2 < 1$

They are dependent since $f(x_1, x_2) \neq f_1(x_1) \cdot f_2(x_2)$

Thm X_1, X_2 independent,

① $\Leftrightarrow F(x_1, x_2) = F_1(x_1) \cdot F_2(x_2) = \int_{-\infty}^{x_1} f_1(t_1)\, dt_1 \cdot \int_{-\infty}^{x_2} f_2(t_2)\, dt_2$

$= P(X_1 \leq x_1, X_2 \leq x_2)$

② $P(a < X_1 \leq b, c < X_2 \leq d)$ $\quad \int_a^b \int_c^d f(x_1, x_2)\, dx_2\, dx_1$

$= P(a < X_1 \leq b) \cdot P(c < X_2 \leq d)$ $\quad \int_a^b f_1(x_1)\, dx_1 \cdot \int_c^d f_2(x_2)\, dx_2$

Thm X_1, X_2 independent,

$\Rightarrow E(u(X_1) v(X_2))$

$= E(u(X_1)) E(v(X_2))$

linear operator special case cor set $u(X_1) = X_1$, $v(X_2) = X_2$.

X_1, X_2 independent.

$\Rightarrow E(X_1 X_2) = E(X_1) E(X_2)$

Thm X_1, X_2 independent.

$\Rightarrow M(t_1, t_2) = M(t_1, 0) \cdot M(0, t_2)$.

$M(t_1, t_2) = E(e^{t_1 X_1 + t_2 X_2})$

Coefficient

Correlation Interpretation

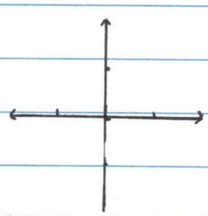
age, test score : The older you get, the better test score.

Thm) If X and Y are independent, then $E(XY) = E(X)E(Y)$. Then $cov(X,Y) = 0$ so that $\rho = 0$.

Contrapositive Statement: $\neg q \to \neg p$.

If $\rho \neq 0$, then X and Y dependent.

ex.



X Y

(1,0), (0,1), (-1,0) (0,-1)

$\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_1 \sigma_2} = 0.$$

$E(XY) = 0$.

$E(X) = \sum x\, p_X(x) = 0$.

$E(Y) = 0$.

X, Y are not correlated.

independence: joint pmf = product of marginal probability.

$p(x,y) = p_X(x) \cdot p_Y(y)$

$P(X=0, Y=0) = 0$.

$P(X=0) = \frac{1}{2}$; $P(Y=0) = \frac{1}{2}$.

$P(X=0, Y=0) \neq P(X=0) \cdot P(Y=0)$.

AMAT467 Continuous Probability and Mathematical Statistics

11/21/2022.

$\rho = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$

Thm) If $E(Y|X)$ is linear in X_1 then $E(Y|X) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1)$

$\overset{\shortparallel}{ax+b}$

$E(Var(Y|X)) = \sigma_2^2(1-\rho^2)$

2.6 Extension to several variables

$Y = u(x_1, \ldots x_n)$

$E(Y) = \Sigma \ldots \Sigma\, u(x_1, \ldots x_n)\, p(x_1, \ldots x_n)$

$= \int \ldots \int u(x_1, \ldots x_n) \cdot p(x_1, \ldots x_n)$

$Var(Y) = E(Y^2) - [E(Y)]^2$

$E(a_1X_1 + a_2X_2 + \cdots a_nX_n)$

$= a_1E(X_1) + a_2E(X_2) + \cdots a_nE(X_n)$

$f(x_1, \ldots x_n)$

$f_1(x_1) = \underbrace{\int \cdots \int}_{(n-1)\text{ integral}} f(x_1, \ldots x_n)\, dx_2, \ldots dx_n.$

$f(x_1, x_2) = f(x_1) \cdot f(x_2)$

$f(x_1, x_2, \ldots x_n) = f_1(x_1) \cdot f_2(x_2) \cdot \ldots \cdot f_n(x_n)$. mutual independence.

$P(a_1 \le x \le b_1)$

$a_2 \le x \le b_2. \quad = \underbrace{\int\int\int\int}_{n} f(x_1 \ldots x_n)\, dx_n$

$\vdots$

$(a_n \le x \le b_n)$

$M(t_1, t_2) = E(e^{t_1X_1 + t_2X_2})$

$M(t_1, t_2 \ldots t_n) = E(e^{t_1X_1 + t_2X_2 + \cdots t_nX_n})$

$-h_i < t_i < h_i.$

$-\min h_i < t_i < \min h_i.$

2.7. Transformation.

Given $x_1, \dots x_n$, $y_1, \dots y_n$. one-to-one transformation.

$f(x_1, \dots x_n)$

$y_1 = u_1(x_1, \dots x_n)$ $p(x_1, \dots x_n) \Rightarrow p(y_1, \dots y_n)$.

$y_2 = u_2(x_1, \dots x_n)$ $x_1 = w_1(y_1 \dots y_n)$.

$\vdots$ $x_2 = w_2(y_1, \dots y_n)$.

$y_n = u_n(x_1, \dots x_n)$. $x_n = w_n(y_1, \dots y_n)$

$f(x_1, \dots x_n) = f(w_1(y_1 \dots y_n) \dots w_n(y_1, \dots y_n)) \, |J|$.

$|J| =$

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 0 \\ 0 & 4 & 5 \end{vmatrix}$$

$$(-1)^{1+1} 0 \begin{vmatrix} -2 & 0 \\ 4 & 5 \end{vmatrix} + (-1)^{1+2}(1) \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} + (-1)^{1+3} \cdot 0 \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{d x_n}{\partial y_n} \end{vmatrix}$$

ex. $f(x_1, x_2, x_3) = \begin{cases} 48 x_1 x_2 x_3 & \overset{S}{\boxed{0 < x_1 < x_2 < x_3 < 1}} \\ 0, & \text{o.w.} \end{cases}$

$Y_1 = \frac{X_1}{X_2}$, $Y_2 = \frac{X_2}{X_3}$, $Y_3 = X_3$.

$y = \{(y_1, y_2, y_3) \mid 0 < y_1 < 1, 0 < y_2 < 1, 0 < y_3 < 1\}$.

$\Uparrow$.

$\frac{x_1}{x_2} \Rightarrow \frac{0 < x_1 < x_2}{x_2} = 0 < Y < 1$.

$X_1 = Y_1 X_2 = Y_1 Y_2 Y_3$

$\begin{cases} X_2 = Y_2 X_3 = Y_2 Y_3 \\ X_3 = Y_3. \end{cases}$

$\Rightarrow$

$\begin{cases} x_1 = y_1 y_2 y_3. \\ x_2 = y_2 y_3. \\ x_3 = Y_3. \end{cases}$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_1} \end{vmatrix}$$

$$= \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ 0 & y_3 & y_2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|J| = \begin{vmatrix} \cancel{y_3} & \cancel{0} & \cancel{y_1} \\ 0 & y_3 & y_2 \\ -y_3 & -y_3 & 1-y_1-y_2 \end{vmatrix} = (-1)^{1+1} y_3 \begin{vmatrix} y_3 & y_2 \\ -y_3 & 1-y_1-y_2 \end{vmatrix} + (-1)^{1+3} (y_1) \begin{vmatrix} 0 & y_3 \\ -y_3 & -y_3 \end{vmatrix}$$

$$= y_3^2$$

$$g(x_1, x_2, x_3) = \begin{cases} e^{-y_3} \cdot y_3^2, & (y_1, y_2, y_3) \in \mathcal{Y} \\ 0, & \text{otherwise} \end{cases}$$

Thank You!

2.8: Linear Combination

$$T = \sum_{i=1}^{n} a_i x_i$$

Suppose $X_i \Rightarrow \mu_i$

σ_i^2.

$E(T) = a_1 E(X_1) + a_2 E(X_2) + \cdots a_n E(X_n)$

$Var(T)$ covariance $cov(T, T) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \, cov(X_i, X_j)$.

$T = \sum_{i=1}^{n} a_i X_i$, $W = \sum_{j=1}^{m} b_j Y_j$.

$$cov(T, W) = E((T - E(T))(W - E(W)))$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \, cov(X_i, Y_j).$$

$$Var(T) = \sum_{i=1}^{n} a_i^2 \, var(a_i) + 2 \sum_{i<j} a_i a_j \, cov(X_i, X_j)$$

$i = j$.

$i \neq j \Rightarrow$ $i<j$

$i>j$.

cor: $X_1, \ldots X_n$: independent.

$\Rightarrow cov(X_i, Y_i) = 0 \Rightarrow \sum_{i=1}^{n} a_i^2 \sigma_i^2$

AMAT467.

11 / 28 / 2022.

Prof. Kwon.

$T = \sum_{i=1}^{n} a_i X_i$

$X_i, \dots X_n$:

iid, mean μ, variance δ^2

from the same population

sample mean

$\bar{X} = \frac{X_1 + \dots + X_n}{n}$

$= \frac{1}{n} X_1 + \dots \frac{1}{n} X_n$

sample variance.

$E(T) = \sum_{i=1}^{n} a_i E(X_i)$

$Var(T) = \Sigma$

$E(\bar{X}) = \frac{1}{n} \mu(n) = \mu$.

$V(\bar{X}) = \sum_n \frac{1}{n^2} \delta^2 = \frac{\delta^2}{n}$

$\bar{X}$ is unbiased estimator for μ.

bias: error term; unbiased estimator: error term is equal to 0.

sample variance:

$E(S^2) = \delta^2$

$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})$

$E\left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2\right) = \frac{1}{n-1} \cdot E\left(\sum_{i=1}^{n} (X_i - \bar{X})^2\right)$

$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} X_i^2 - 2\bar{X} \sum_{i=1}^{n} X_i + \underbrace{\sum_{i=1}^{n} \bar{X}^2}_{\bar{X}^2 \cdot n}\right)$.

$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)$

$-2n\bar{X}^2 + n\bar{X}^2$.

$E(S^2) = \frac{1}{n-1} \left(E\left(\sum_{i=1}^{n} X_i^2\right) - E(n\bar{X}^2)\right)$

$= \frac{1}{n-1} \left(n\delta^2 + n\mu^2 - n\left(\frac{\delta^2}{n} + \mu^2\right)\right)$

$= \frac{1}{n-1} (n\delta^2 + n\mu^2 - \delta^2 - n\mu^2)$

$= \frac{1}{n-1} (n-1)\delta^2 = \delta^2$.

Proof of binomial distribution converge to 1.

$$\sum_{x=0}^{n} p(x)$$

$$\Updownarrow$$

$$\sum_{x=0}^{n} \left[\begin{matrix} n \\ x \end{matrix}\right] p^{x} (1-p)^{n-x} = (p+(1-p))^{n} = 1.$$

$$M(t) = \sum e^{tx} \cdot p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \left[\begin{matrix} n \\ x \end{matrix}\right] p^{x} (1-p)^{n-x}. \qquad \frac{d}{dt}((pe^{t}+1-p)^{n})$$

$$=$$

$$= \sum_{x=0}^{n} \left[\begin{matrix} n \\ x \end{matrix}\right] (pe^{t})^{x} (1-p)^{n-x}. \quad ; \text{ Apply binomial theorem again.}$$

$$= (pe^{t}+1-p)^{n}$$

ex. $M_X(t) = (\frac{2}{3} + \frac{1}{3}e^{t})^{5}$

$$\Downarrow$$

$$\cancel{p=\frac{1}{3}} \quad X \sim bin(5, \frac{1}{3}).$$

$$P(X=2) = \left[\begin{matrix} 5 \\ 2 \end{matrix}\right] (\frac{1}{3})^{2} (\frac{2}{3})^{3}$$

$$= \frac{5!}{2!\,3!} (\frac{1}{9})(\frac{4}{9}).$$

$$= \frac{5 \cdot 4 \cdot 2}{2 \cdot 1} = \cdot \frac{40}{81} \therefore \frac{40}{81}$$

<u>Thm</u> $\underbrace{X_1, \dots X_k}_{Y}$ independent $X_i \sim b(n_i, p)$.

$$\Rightarrow X_1 + \cdots X_k \sim b(n_1+n_2+\cdots n_k, P).$$

(Proof) $M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t)\, M_{X_3}(t) \cdots M_{X_k}(t)$

$$= (pe^{t}+1-p)^{n_1} (pe^{t}+1-p)^{n_2} \cdots (pe^{t}+1-p)^{n_k}.$$

$$= (pe^{t}+1-p)^{n_1+\cdots n_k}.$$

Multinomial Distribution.

n- independent trails.

outcomes are from k categories. $c_1, c_2, \dots c_k$.

$\uparrow \quad \uparrow \qquad \uparrow$

$p_1 \quad p_2 \cdots \quad p_k$.

$p_1 + p_2 + \cdots p_k = 1$.

1 5 1 2 41 21 c_1.

X_i: how many time you get c_i.

$x_1 + x_2 + x_3 + \cdots x_k = n, \Rightarrow x_k = n - (x_1 + \cdots x_{k-1})$

Remember this: joint pmf of $(X_1, \dots X_{k-1})$

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3, \dots X_k = x_k) = \frac{n!}{x_1!\, x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \dots p_k^{x_k}.$$

$$\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_1-x_2}{x_3} \cdots \binom{n-x_1 \cdots x_{n}}{x_n} = \frac{n!}{x_1!(n-x_1)!} \cdot \frac{(n-x_1)!}{}$$

Roll a dice 10 times.

X_1

X_2

X_3

X_4

X_5

X_6.

$$p(X_1 = 5, X_4 = 3, X_6 = 2).$$

$$\frac{10!}{5!\,3!\,2!} \left(\frac{1}{6}\right)^5 \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^2$$

binomial case:

$$\frac{n!}{x_1!\, x_2!} p_1^{x_1} p_2^{x_2}. \qquad \begin{matrix} x_1 + x_2 = n \\ p_1 + p_2 = 1. \end{matrix}$$

$$= \frac{n!}{x_1!\,(n-x_1)!} p_1^{x_1} (1-p_1)^{(n-x_1)}.$$

$$= \binom{n}{x_1} p_1^{x_1} (1-p_1)^{n-x_1}.$$

AMAT467 Continuous Probability

Hypergeometric Distribution. $D \le N$, $n \le N$.

N objects, D defective

$$P(x) = \frac{\binom{N-D}{n-x}\binom{D}{x}}{\binom{N}{n}}, \quad x = 0,1,2,\dots n. \quad \text{w/o replacement}$$

with replacement: binomial situation $X \sim Bin(n, \frac{D}{N})$

$$E(X) = n \cdot \frac{D}{N}$$
$$Var(X) = n \cdot \frac{D}{N} \cdot \frac{N-D}{N}$$

$$E(X) = n \cdot \frac{D}{N}$$
$$Var(X) = n \cdot \frac{D}{N} \cdot \frac{N-D}{N} \boxed{\left(\frac{N-n}{N-1}\right)} \to 1$$

$N >> n$

Poisson Distribution.

3.2 The Poisson Distribution.

$X \sim Poisson(\lambda) \qquad \lambda > 0$

$$P(x) = \begin{cases} \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2,3,\dots \\ 0 \quad \text{otherwise}. \end{cases}$$

$$\sum_{x=0}^{\infty} p(x) = 1.$$

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} + \dots \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

$$e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = e^{\lambda + (-\lambda)} = e^0 = 1$$

3.3. The Γ, χ^2, and β distributions.

$\alpha > 0$: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

$\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

n: positive integer: $\Gamma(n) = (n-1)\Gamma(n-1)$ $\quad\Gamma(n) = (n-1)!$

$\Gamma(3) = (3-1)\Gamma(3-1)$

$= 2\Gamma(2) = 2(2-1)\Gamma(1)$

$= 2 \cdot 1 \cdot 1 = \boxed{2}$

$\Gamma(\alpha, \beta)$ $\quad \alpha > 0, \beta > 0$

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x > 0 \\ 0, & \text{otherwise}. \end{cases}$$

$$\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = 1.$$

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = 1.$$

let $t = \frac{x}{\beta}$.

$\Downarrow$

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

$M(t) = \int_0^\infty e^{tx} \cdot f(x)\, dx$

$\shortparallel$

$(1-\beta t)^{-\alpha}$, $t < \frac{1}{\beta}$.

$E(X) = \alpha\beta$.

$V(X) = \alpha\beta^2$

X: failure time of the device.

$f(x)$, $F(x)$.

hazard function.

$r(x)$ = probability that device fails | device didn't fail at time t.

$$= \frac{f(x)}{1-F(x)} = -\cancel{} - \frac{d}{dx}(\log(1-F(x)))$$

if $r(x)$ = constant, $= \frac{1}{\beta}$, $\beta > 0$.

$$\frac{d}{dx}(\log(1-F(x))) = -\frac{1}{\beta}.$$

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

β distribution

X_1, X_2 : independent, each $\Gamma(\alpha,1)$, $\Gamma(\beta,1)$.

$$h(x_1,x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-(x_1+x_2)} \quad x_1, x_2 > 0.$$

$$Y_1 = X_1 + X_2, \quad Y_2 = \frac{X_1}{X_1+X_2}.$$

$X_1 = Y_1 Y_2 \quad X_2 = Y_1 - Y_1 Y_2. \quad J = -y_1.$

$g(y_1, y_2)$.

$0 < y_1$ $\quad g_2(y_2) = \int_0^\infty g(y_1,y_2)\,dy_1.$

$0 < y_2 < 1$.

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.$$

$$\mu = \frac{\alpha}{\alpha+\beta}.$$

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

The Normal Distribution.

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = 1.$$

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right), \quad -\infty < z < \infty.$$

$Z \Rightarrow E(Z) = 0.$

$V(Z) = 1$.

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2} + tz}\, dz. \quad ; \text{ complete the square.}$$

$$= e^{\frac{t^2}{2}}$$

AMAT467 Continuous Probability and Mathematical Statistics.

Std Normal. $I = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1.$

$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$M_Z(t) = \int_{-\infty}^{+\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$

$I = 1.$ $az^2 + bz + c.$

$= a(z^2 + \frac{b}{a}z + \frac{b^2}{4a^2}) + c - \frac{b^2}{4a}.$

$= a(z + \frac{b}{2a})^2 + c - \frac{b^2}{4a}.$

Normal. $E(Z) = 0$ $X = bZ + a.$

$Var(Z) = 1.$ $z = \frac{x-a}{b}$, $J = \frac{1}{b}.$

$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \quad -\infty < x < \infty.$

$X \sim N(\mu, \sigma^2).$

$\Updownarrow$

$Z = \frac{X-\mu}{\sigma} \sim N(0, 1^2).$

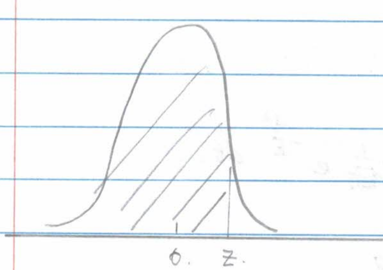


table $\Phi(z) = \int_0^z \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$

$\Phi(-z) = 1 - \Phi(z).$

Thm] $X_1, \dots X_n$ independent random variable. $X_i \sim N(\mu_i, \sigma_i^2)$

$$Y = \sum_{i=1}^{n} a_i X_i.$$

$$\Rightarrow Y \sim N\Big(\underbrace{\sum_{i=1}^{n} a_i \mu_i}_{\text{mean}},\ \underbrace{\sum_{i=1}^{n} a_i^2 \sigma_i^2}_{\text{variance}}\Big).$$

$M_{X_i}(t) = \exp(t\mu_i + \frac{1}{2}t^2\sigma_i^2)$.

(pf). $M_Y(t) = \prod \exp(ta_i\mu_i + \frac{1}{2}t^2 a_i^2 \sigma_i^2)$.

$$\parallel$$

$$e^{ta_1\mu_1 + \frac{1}{2}t^2 a_1^2 \mu_1^2} \; e^{ta_2\mu_2} \cdots$$

$$\parallel$$

$$\exp(t \sum a_i \mu_i + \frac{1}{2}t^2 \sum a_i^2 \sigma_i^2)$$

Cor 1.

$\bar{x} = \frac{1}{n}(X_1 + \dots X_n)$.

$X_1, \dots X_n$: i.i.d $N(\mu, \sigma^2)$

$\Rightarrow \bar{x} \sim N(\mu, \sigma^2/n)$.

$\sum_n \frac{1}{n^2}\sigma^2 = \frac{1}{n^2}\sigma^2 \cdot n \Rightarrow \frac{\sigma^2}{n}$.

Central Limit Theorem $X_1, \dots X_n$: i.i.d. not necessarily normal.

μ, σ^2.

$$\bar{X} \overset{\text{approximately}}{\sim} N(\mu, \frac{\sigma^2}{n}).$$

For n large enough, we have normal distribution.

ex. $\bar{X}$: mean of random sample of size 75 from a distribution w/

pdf $f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

$$P(0.45 < \bar{X} < 0.55) \underset{\text{By CLT,}}{\approx} P\left(\frac{0.45-0.05}{\sqrt{\frac{1}{12(75)}}} < Z < \frac{0.55-0.5}{\sqrt{\frac{1}{12(75)}}} \right)$$

$\mu = \int_0^1 x\,dx = \frac{1}{2}$. ~~[illegible]~~, $\bar{X} \sim N(\frac{1}{2}, \frac{1}{12(75)})$

$\sigma^2 = \int_0^1 x^2 dx - (\frac{1}{2})^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

Monday
3:30 → 5:30 / No bivariate

t distribution and F distribution.

t dist $W \sim N(0,1)$, $V \sim \chi^2(r)$) independent.

form joint pdf. $h(w,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \frac{1}{\Gamma(r/2)\, 2^{r/2}} v^{\frac{r}{2}-1} e^{-\frac{v}{2}}$, $v > 0$.
$-\infty < w < \infty$.

Let $T = \frac{W}{\sqrt{\frac{V}{r}}}$.

$t = \frac{w}{\sqrt{v/r}}$, $u = v$. $w = t\sqrt{\frac{u}{r}}$, $v = u$.

$|J| = \frac{\sqrt{u}}{\sqrt{r}}$ $g(t,u)$.

$g_1(t) = \int g(t,u)\,du = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})} \frac{1}{(1+\frac{t^2}{r})^{\frac{r+1}{2}}}$, $-\infty < t < \infty$.

F distribution.

F dist. $U \sim \chi^2(r_1)$, $V \sim \chi^2(r_2)$.) independent.

$h(u,v) = \frac{1}{\Gamma(\frac{r_1}{2})\,\Gamma(\frac{r_2}{2})\, 2^{\frac{r_1+r_2}{2}}} u^{\frac{r_1}{2}-1} v^{\frac{r_2}{2}-1} e^{-\frac{u+v}{2}}$.

$0 < u, v < \infty$.

$w = \frac{u/r_1}{v/r_2}$, $z = v$. $v = z$, $u = \frac{r_1}{r_2} z w$. $|J| = \frac{r_1}{r_2} z$.

$0 < u, v < \infty$.
$0 < w, z < \infty$.

$g(w,v)$

marginal pdf of w
$g_1(w) = \int_0^\infty g(w,v)\,dv = \frac{\Gamma(\frac{r_1+r_2}{2}) (\frac{r_1}{r_2})^{\frac{r_1}{2}}}{\Gamma(\frac{r_1}{2})\,\Gamma(\frac{r_2}{2})} \frac{w^{\frac{r_1}{2}-1}}{(1+r_1\frac{w}{r_2})^{\frac{r_1+r_2}{2}}}$, $w > 0$.