ICSI 210 PRACTICE Midterm Exam SOLUTIONS (110 points)
October 8, 2022 Name: ______________________________

This PRACTICE exam is open book, open note. No electronic devices are allowed. Read all directions carefully and write your answers in the space provided. To receive full credit, you must show all of your work. (But, actually, you don't need to hand in this exam.)

1. (10 points) Construct a truth table for the following proposition: $p \wedge \neg q \implies r$.

Solution:

p	q	r	$p \wedge \neg q$	$p \wedge \neg q \implies r$
F	F	F	F	T
F	F	T	F	T
F	T	F	F	T
F	T	T	F	T
T	F	F	T	F
T	F	T	T	T
T	T	F	F	T
T	T	T	F	T

2. (10 points) Let A, B, C be sets. Show the following set identity. **A Venn diagram or examples are not valid proof techniques.**

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}. \tag{1}$$

Solution:

We start by applying DeMorgan's laws repeatedly:

$$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)} = \overline{A} \cap (\overline{B} \cup \overline{C}). \tag{2}$$

We then apply commutativity of intersection and union repeatedly:

$$\overline{A} \cap (\overline{B} \cup \overline{C}) = \overline{A} \cap (\overline{C} \cup \overline{B}) = (\overline{C} \cup \overline{B}) \cap \overline{A}. \tag{3}$$

This completes the proof.

3. (10 points) Write all elements of the power set of $\{1, 2, 3\}$.

Solution:

The power set of S is the set of all subsets:

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}. \tag{4}$$

4. (10 points) Prove the following claim by contradiction:

Claim 1. *Let $n \in \mathbb{Z}$. If $5n + 2$ is odd, then n is odd.*

Solution: Suppose, for a contradiction, that $5n + 2$ is odd and that n is even. That is, there exist integers r, s such that $5n + 2 = 2r + 1$ and $n = 2s$. Then

$$5n + 2 = 5(2s) + 2 = 2(5s + 1). \tag{5}$$

Since $5s + 1 \in \mathbb{Z}$, this implies that $5n + 2$ is even. But this is a contradiction, since we know that an integer cannot be both even and odd. This completes the proof.

5. (10 points) Let $A_j = \{i \mid i \in \mathbb{Z}, i \geq j\}$ Describe the elements of the following set without reference to set operations:

$$\bigcup_{j=1}^{\infty} A_j. \tag{6}$$

Solution:

This set is equal to $\mathbb{N}$, the set of positive integers.

6. (10 points) Let r, ℓ be even numbers. Prove that $r + \ell$ is even.

Solution: We prove this directly. Since r, ℓ are both even, there exist integers h, k such that $r = 2h$ and $\ell = 2k$. Then

$$r + \ell = 2h + 2k = 2(h + k), \tag{7}$$

and we know that $h + k \in \mathbb{Z}$, since the set of integers is closed under addition. this implies that $r + \ell$ is even, as desired.

7. (10 points) Consider the following relation R on $\mathbb{R}^2$: $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1 \leq x_2$.

Is this a partial order? Prove your answer.

Solution: This is not a partial order, because it fails to satisfy antisymmetry. To show this, it suffices to find a counterexample to antisymmetry. Let $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (1, 3)$. Then

$$(x_1, y_1)R(x_2, y_2), \tag{8}$$

since $1 \leq 1$, and the reverse is also true:

$$(x_2, y_2)R(x_1, y_1), \tag{9}$$

but, clearly, $(1, 2) \neq (1, 3)$.

This completes the solution.

8. (10 points) Prove that

$$(p \wedge q) \implies (p \vee q) \tag{10}$$

is a tautology. That is, prove that the following logical equivalence holds:

$$(p \wedge q) \implies (p \vee q) \equiv T. \tag{11}$$

Do not use a truth table.

Solution: We have

$$\begin{align} (p \wedge q) \implies (p \vee q) &\equiv \neg(p \wedge q) \vee p \vee q \tag{12} \\ &\equiv \neg p \vee \neg q \vee p \vee q \tag{13} \\ &\equiv \neg p \vee p \vee \neg q \vee q \tag{14} \\ &\equiv T \vee \neg q \vee q \tag{15} \\ &\equiv T. \tag{16} \end{align}$$

9. (10 points) Consider a bijection $f : A \to B$. Suppose that $C \subseteq A$. Is it necessarily true that $g : C \to B$ defined by $g(x) = f(x)$ is injective? Is it necessarily true that g is bijective? Justify your answers.

   Solution:

   g is injective: Suppose $x_1, x_2 \in C \subseteq A$ are such that $g(x_1) = g(x_2)$. Then

   $$g(x_1) = g(x_2) \implies f(x_1) = f(x_2) \implies x_1 = x_2, \tag{17}$$

   where the last implication is because f is a bijection. Thus, g is injective.

   g is not necessarily bijective, though, because it's not necessarily surjective. For instance, suppose that $A, B = \{0, 1\}$ and $f(x) = x$ and $C = \{0\}$. Then f is bijective, but g is not surjective, because there is no element of C that maps to $1 \in B$.

10. (10 points) Consider the following recurrence. Determine a solution for it and prove it by induction.

   $$T(n) = 2T(n-1) + 1 \tag{18}$$
   $$T(0) = 0. \tag{19}$$

   To find a solution, we iterate the recurrence. If we iterate $k - 1$ times,

   $$T(n) = 2(2T(n-2)+1)+1 = 2(2(2T(n-3)+1)+1)+1 = ... = 1 + 2 + 2^2 + ...2^k T(n-k). \tag{20}$$

   Setting $k = n$, we have

   $$T(n) = \sum_{j=0}^{n-1} 2^j = \frac{2^n - 1}{2 - 1} = 2^n - 1. \tag{21}$$

   We give an alternative proof by induction:

   Base case: $n = 0$. We check that $T(0) = 2^0 - 1$. We know from the base case of the recurrence that $T(0) = 0$, and, easily enough, $2^0 - 1 = 1 - 1 = 0$.

   Inductive step: Our inductive hypothesis is that $T(k) = 2^k - 1$. We need to show that $T(k+1) = 2^{k+1} - 1$. We have

   $$T(k+1) = 2T(k) + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1, \tag{22}$$

   as desired.

11. (10 points) Prove that the following relation on the integers other than 0 is an equivalence relation:

   $$a \equiv b \iff ab > 0. \tag{23}$$

   Describe all of its equivalence classes.

   Solution:

   We check the properties of equivalence relations.

1. Reflexivity: Let $a \in \mathbb{Z}$, $a \neq 0$. We have $a \cdot a = a^2 > 0$. So $a \equiv a$.
2. Symmetry: Let $a \equiv b$. Then $ab > 0$, and by commutativity of multiplication, this implies that $ba > 0$, which implies that $b \equiv a$.
3. Transitivity: Let $a \equiv b$ and $b \equiv c$. Then $ab, bc > 0$. Now, $ac = ab \cdot bc / b^2$, and $ab > 0, bc > 0$, and $b^2 > 0$, so that $ac > 0$ as well. Thus, $a \equiv c$.

Thus, $\equiv$ is an equivalence relation on $\mathbb{Z} - \{0\}$. It has two equivalence classes:

$$\mathbb{N}, -\mathbb{N}. \tag{24}$$

I.e., all of the positive integers are equivalent to one another, and all of the negative integers are equivalent to one another.