Discrete Structures (Fall 2022) Homework: 02

ICSI 210 – Discrete Structures

Instructor: Abram Magner Date: Fall 2022

Instructions: Please answer the following questions in complete sentences, showing all work (including code and output of programs, when applicable). I prefer that you type your solutions (e.g., using LaTeX, with Overleaf, TeXworks, etc., or Word), but will accept handwritten notes. If the grader cannot read your handwriting, then they cannot award you points.

Due date: Tuesday, 10/11/2022, 11:59 p.m.

2.1 Sets

- Prove that for any sets A and B,

$$A = (A - B) \cup (A \cap B). \tag{2.1}$$

- What is the cardinality of each of the following sets?
  - $\emptyset$
  - $\{\emptyset\}$
  - $\{\emptyset, \{\emptyset\}\}$
  - $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
- If two sets A and B have the same power set, does this imply that A and B are equal? If so, prove it. If not, provide a counterexample.

2.2 Relations

- Consider the following relation on integers: $x \equiv y$ if and only if $xy \geqslant 3$. Determine whether or not $\equiv$ has each of the following properties:
  - Reflexivity
  - Symmetry
  - Antisymmetry
  - Transitivity

  Justify your statements.
- Consider the following relation on integers: $x \equiv y$ if and only if $11 \mid x - y$. Prove that it is an equivalence relation, and give a representative element of each of its equivalence classes.

2.3 Functions

- Let $f : A \to B$ and $g : B \to C$ be two injective functions. Prove that $g \circ f$ is injective.
- Determine which of the following are bijective functions $f : \mathbb{R} \to f(\mathbb{R})$.
  - $f(x) = x^5$.
  - $f(x) = \cos^2(x)$.
  - $f(x) = \frac{x+1}{x-5}, x \neq 5$.

2.4 Summations

- Derive a closed-form formula for the following sums:
  - $\sum_{j=0}^{n}(3^j - 2^j)$.
  - $\sum_{j=1}^{n} a_j - a_{j-1}$, for a sequence a_j of real numbers.

2.5 Proofs by induction

- Use induction to prove that

$$\sum_{i=1}^{n} i2^i = 2^{n+1}(n-1) + 2 \tag{2.2}$$

  for all $i \in \mathbb{N}$.

- Prove that for any sets A and B,

$$A = (A - B) \cup (A \cap B). \tag{2.1}$$

$A = (A-B) \cup (A \cap B)$; $(A-B) = (A \cap B^c)$

$A = (A \cap B^c) \cup (A \cap B)$

$= A \cap (B^c \cup B)$; $B^c \cup B = U$: where U is the universal set

$= A \cap U = A$

number of element.

- What is the cardinality of each of the following sets?
  - $\emptyset$ 0
  - $\{\emptyset\}$ 1
  - $\{\emptyset, \{\emptyset\}\}$ 2
  - $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ 3

- If two sets A and B have the same power set, does this imply that A and B are equal? If so, prove it. If not, provide a counterexample.

$2^A = 2^B$ then $A = B$

1) $A \subseteq B$ 2) $B \subseteq A$.

1). Suppose $x \in A$, then $\{x\} \subseteq A \Rightarrow \{x\} \in 2^A = 2^B$.

$\{x\} \in 2^B \Rightarrow \{x\} \subseteq B \Rightarrow x \in B \Rightarrow A \subseteq B$.

2). By symmetry.

- Consider the following relation on integers: $x \equiv y$ if and only if $xy \geqslant 3$. Determine whether or not $\equiv$ has each of the following properties:
  - Reflexivity
  - Symmetry
  - Antisymmetry
  - Transitivity

  Justify your statements.

Consider $\equiv$ is a relation on the set of integers, $\mathbb{Z}$ defined by $x \equiv y$ if and only if $xy > 3$.

checking $\equiv$ is reflexive or not

clearly $1 \not\equiv 1$ since $(1)(1) > 3$ is false $\boxed{\text{Hence it is not reflexive.}}$

checking relation is symmetric or not :

Let $x, y \in \mathbb{Z}$ such that $x \equiv y$.

$\Rightarrow xy > 3$

$\Rightarrow yx > 3$

implies $y \equiv x$.

$\boxed{\text{Hence relation is symmetric.}}$ By definition, if xRy is symmetric, then xRy is not antisymmetry.

checking relation is transitive or not :

As $(1 \times 4) > 3$ and $(4 \times 1) > 3$ but $(1 \times 1) < 3$.

$\Rightarrow 1 \equiv 4$ and $4 \equiv 1$ but $1 \not\equiv 1$.

$\boxed{\text{Hence the relation is not transitive.}}$

- Consider the following relation on integers: $x \equiv y$ if and only if $11 \mid x - y$. Prove that it is an equivalence relation, and give a representative element of each of its equivalence classes.

Reflexivity: Let $x \in \mathbb{Z}$.

As $11 \mid 0$

$\Rightarrow 11 \mid (x - x)$

$\Rightarrow x \equiv x$.

$\boxed{\text{Hence } \equiv \text{ is reflexive.}}$

Symmetric: Let $x, y \in \mathbb{Z}$ such that $x \equiv y$.

$\Rightarrow 11 \mid (x-y)$

Hence there exists some integer k such that $x-y = 11k$.

$\Rightarrow y-x = 11(-k)$ where $-k$ is an integer.

$\Rightarrow 11 \mid (y-x)$

$\Rightarrow y \equiv x$.

$\boxed{\text{Hence } \equiv \text{ is symmetric.}}$

Proving relation $\equiv$ is Transitive.

Let $x, y, z \in \mathbb{Z}$ such that $x \equiv y$ and $y \equiv z$.

$\Rightarrow 11 \mid (x-y)$ and $11 \mid (y-z)$

There exists integer m and n such that $x-y = 11m$ and $y-x = 11n$.

$x-y+y-z = 11m+11n$

$x-z = 11(m+n)$ where $m+n$ is an integer.

$\Rightarrow 11 \mid (x-z)$.

$\Rightarrow x \equiv z$.

$\boxed{\text{Hence } \equiv \text{ is transitive.}}$

2.3 Functions

- Let $f : A \to B$ and $g : B \to C$ be two injective functions. Prove that $g \circ f$ is injective.
- Determine which of the following are bijective functions $f : \mathbb{R} \to f(\mathbb{R})$.
  - $f(x) = x^5$.
  - $f(x) = \cos^2(x)$.
  - $f(x) = \frac{x+1}{x-5}, x \neq 5$.

$f: A \longrightarrow B$ is injective function means that f is one-to-one function.

$g: B \longrightarrow C$ is injective function means that g is one-to-one function.

Now for $g \circ f$, Let $(x, y) \in A$ such that

$(g \circ f)\, x = (g \circ f)\, y \Rightarrow g(f(x)) = g(f(y))$

$f(x) = f(y)$ because g is one-to-one.

$g(x) = g(y)$ then f is one-to-one.

So we can say that $g \circ f$ is also one to one function.

$\boxed{\text{Hence } g \circ f \text{ is injective function.}}$

$f(x) = x^5$

if $f(x) = x^y$, where $y \in 2k+1$ and $k \in \mathbb{Z}$, then $f(x) = x^y$ is an one-to-one onto function.

$\therefore$ $\boxed{f(x) = x^5 \text{ is bijective}}$

$f(x) = \cos^2(x)$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

① : Proof by example: $\cos^2(0) = 1$; $\cos^2(\pi) = 1$ but $0 \neq \pi$.

Hence $\cos^2(x)$ is not injective (one to one).

② : $\cos^2(x)$ is surjective if for every y in the codomain there is an x in the domain of f such that $f(x) = y$.

Assuming the real numbers are the codomain for $\cos^2(x)$, there is no x s.t. $\cos^2(x) = 2$, Hence, it is not surjective.

$\boxed{\text{Conclusion: } f(x) = \cos^2(x) \text{ is not bijective.}}$

— $f(x) = \frac{x+1}{x-5}$, $x \neq 5$.

Prove Injectivity : $f(x_1) = \frac{x_1+1}{x_1-5}$; $f(x_2) = \frac{x_2+1}{x_2-5}$.

Let $f(x_1) = f(x_2)$ $\frac{x_1+1}{x_1-5} = \frac{x_2+1}{x_2-5}$.

$(x_1+1)(x_2-5) = (x_1-5)(x_2+1)$; cross multiplication.

$\cancel{x_1x_2} - 5x_1 + x_2 - \cancel{5} = \cancel{x_1x_2} + x_1 - 5x_2 - \cancel{5}$

$6x_2 = 6x_1$

$x_2 = x_1$.

Therefore, $f(x)$ is one-to-one.

Prove surjectivity. let $y \in f(\mathbb{R})$. The function f is onto if there exists $x \in R$ such that $f(x) = y$.

$y = \frac{x+1}{x-5}$

$yx - 5y = x + 1$.

$yx - x = 1 + 5y$.

$x(y-1) = 1+5y$

$x = \frac{1+5y}{y-1} \in \mathbb{R}$, the denominator $y-1 \neq 0 \Rightarrow y \neq 1$.

But $1 \in \mathbb{R}$ indeed is defined in our assumption.

Therefore, for $y=1$ there is no such x map with y.

$\boxed{\text{Hence, } f(x) \text{ is not bijective function.}}$

2.4 Summations

- Derive a closed-form formula for the following sums:
  - $\sum_{j=0}^{n}(3^j - 2^j)$.
  - $\sum_{j=1}^{n} a_j - a_{j-1}$, for a sequence a_j of real numbers.

$$\sum_{j=0}^{n} (3^j - 2^j)$$

$$= \sum_{j=0}^{n} 3^j - \sum_{j=0}^{n} 2^j \quad ; \quad \text{by } \sum (a_k + b_k) = \sum a_k + \sum b_k.$$

$$= \frac{3^{n+1}-1}{2} - \left(\frac{2^{n+1}-1}{2-1}\right) ; \quad \text{by } \sum_{k=0}^{k} a^k = \frac{a^{n+1}-1}{a-1}$$

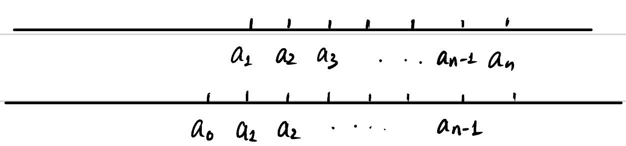
$$= \boxed{\frac{3^{n+1}-1}{2} - 2^{n+1} + 1} \qquad \square$$

$$\sum_{j=1}^{n} (a_j - a_{j-1}) \text{, for a sequence } a_j \text{ of real numbers.}$$

$$= \sum_{j=1}^{n} a_j - \sum_{j=1}^{n} a_{j-1}.$$

$$= (a_1 + \cdots + a_n) - \left(\sum_{j=0}^{n-1} a_j\right) \quad ; \quad \text{by } \sum_{k=m}^{n} a_{k+i} = \sum_{k=m+i}^{n+i} a_k$$

$$= (a_1 + \cdots + a_n) - (a_0 + a_1 + \cdots + a_{n-1})$$



$$= \boxed{a_n - a_0} \qquad \square$$

2.5 Proofs by induction

- Use induction to prove that

$$\sum_{i=1}^{n} i2^i = 2^{n+1}(n-1) + 2 \tag{2.2}$$

for all $i \in \mathbb{N}$.

Base Case: $\sum_{i=1}^{1} i2^i = 1 \cdot 2^1 = 2.$

The following sum of $\sum_{i=1}^{n} i2^i = 2 + 8 + 24 + 64 + \cdots i2^i.$

LHS: $2 + 8 + 24 + 64 + \cdots + \sum i2^i$; RHS $= 2^{n+1}(n-1) + 2.$

Assume $n = k$:

$$2 + 8 + 24 + 64 + \cdots \quad k2^k = 2^{k+1}(k-1) + 2.$$

Inductive Step: Prove S_{k+1} term

$$\underline{2 + 8 + 24 + 64 + \cdots + k2^k} + [(k+1) \cdot 2^{k+1}] = 2^{(k+1)+1}((k-1)+1) + 2$$

$\Downarrow$

As stated above ; $2 + 8 + 24 \cdots + k2^k = 2^{k+1}(k-1) + 2.$

Hence, $2^{k+1}(k-1) + 2 + [(k+1) \cdot 2^{k+1}] = 2^{k+1} \cdot 2(k) + 2.$

Let $x = 2^{k+1}$

$$x(k-1) + 2 + [(k+1) \cdot x] = x \cdot 2k + 2$$

$$xk - x + 2 + xk + x = 2xk + 2$$

$$2xk + 2 = 2xk + 2$$

Since $x = 2^{k+1}$, substitute x back into $2xk + 2$:

$$2(2^{k+1})k + 2 = 2(2^{k+1})k + 2$$

$$\boxed{2^{k+2}k + 2 = 2^{k+2}k + 2} \qquad \square$$

$a|b$ 当 $\exists c : ac = b$. $a, b \in \mathbb{Z}$.

$2|8$ $c \in \mathbb{Z}^+$.

$2c = 8$

$c = 4$.

$4 \in \mathbb{Z}^+$.

$5|13$.

$5c = 13$.

$c = \frac{13}{5} = 2.6 \in \mathbb{Z}^+$.

$5 \nmid 13$. √.

若 $a|b$ and $a|c$ then $a|(b+c)$.

$11c = x - y$.

$c = \frac{x-y}{11} \in \mathbb{Z}^+$.

$11 | 0$.

$11 \cdot c = 0$.

$c = \frac{0}{11}$

$c = 0$.

$11 | (x-y)$.

$11c = x - y$.

$c = \frac{x-y}{11}$