ICSI 210, discrete structures: Links to useful resources

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Below are some links to potentially useful review resources (though I stress that I have not reviewed these in great detail, and so while I trust that they are very likely correct, I cannot with complete certainty vouch for their efficacy as review aids). I've also included links to interesting lecture notes and books for furthe reading.

1 Interesting books

Here are some books that I really like. Some of them might be available for free online. Google them!

- Concrete Mathematics, by Graham, Knuth, and Patashnik.
- Mathematics for the Analysis of Algorithms, by Greene and Knuth.

Some other books that are a little bit less related to the discrete math course, but still great for sharpening your problem solving abilities, are as follows:

- Programming Challenges, by Skiena & Revilla – Contains very nice programming problems.
- Mathematical Puzzles: A Connoisseur's Collection (AK Peters/CRC Recreational Mathematics Series), by Peter Winkler

Both of these contain problems that range from not so hard to quite challenging.

2 Asymptotic notation

Information about asymptotic notation can be gotten from

- Khan Academy: `https://www.khanacademy.org/computing/computer-science/algorithms/asymptotic-notation/a/asymptotic-notation`
- *Discrete Mathematics and Its Applications, 6th edition*, by Kenneth Rosen, Section 3.2. You may be able to find a pdf version of this book. Updated editions are also published.
- Wikipedia: `https://en.wikipedia.org/wiki/Big_O_notation`

  This contains all of the definitions.
- A very fine (but more mathematical) book: *Concrete Mathematics, 2nd edition*, by Graham, Knuth, and Patashnik. Starting on page 439, there is a chapter devoted to asymptotic notation. You may be able to find a pdf version of this book.

I *strongly* recommend looking at the Khan Academy link.

For completeness, I'll go over the definitions again. I'll start with an example: consider the functions $f(x) = 1000x$ and $g(x) = x^2$. We know that both of these tend to infinity as $x \to \infty$. That is, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x)$. But, intuitively, one of these functions goes to infinity *much faster* than the other – namely, $g(x)$ is much faster than $f(x)$ (here it would be helpful for you to plot these functions in Matlab to see this).

So we want some way of making this notion of "much faster" more explicit. We'll consider what happens with the absolute value of the ratio of $f(x)$ and $g(x)$:

$$\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = \lim_{x\to\infty} \left|\frac{1000x}{x^2}\right| = \lim_{x\to\infty} \left|\frac{1000}{x}\right| = 0. \tag{1}$$

Intuitively, this limit is zero because the numerator function $f(x)$ gets arbitrarily *smaller* than the denominator function $g(x)$ if we take x large enough. We will introduce some special notation that captures this: fix some point y, which may be either some real number or ∞ or $-\infty$. For two functions $f(x)$ and $g(x)$, whenever $\lim_{x\to y} \left|\frac{f(x)}{g(x)}\right| = 0$, we say that $f(x) = o(g(x))$ (pronounced "$f(x)$ is little oh of $g(x)$").

There are a few other cases, besides the case where $f(x) = o(g(x))$:

- $\lim_{x\to y} \left|\frac{f(x)}{g(x)}\right| = C$, where C is some **positive** constant, or the ratio oscillates but remains bounded between two positive constants. In this case, intuitively, $f(x)$ and $g(x)$ have the **same order of growth**. We say that $f(x) = \Theta(g(x))$.

  For example, the function $f(x) = x(|\cos(x)| + 1)$ satisfies $f(x) = \Theta(x)$ as $x \to \infty$, since $\frac{x(|\cos(x)|+1)}{x} = |\cos(x)| + 1$ oscillates but remains bounded between 1 and 2. Additionally, if we multiply by any constant, the same is true. E.g., $200x = \Theta(x)$.

If $\left|\frac{f(x)}{g(x)}\right| \leq C$ whenever x is sufficiently close to y, then we say that $f(x) = O(g(x))$. Note that if $f(x) = O(g(x))$, then it is also true that $f(x) = \Theta(g(x))$.

- $\lim_{x\to y}\left|\frac{f(x)}{g(x)}\right| \to \infty$. This is the mirror case of the $o(\cdot)$ notation, and it indicates that $f(x)$ is asymptotically much larger than $g(x)$. In particular, we say that $f(x) = \omega(g(x))$. Note that in this case we automatically know that $\lim_{x\to y}\left|\frac{g(x)}{f(x)}\right| = 0$, which means that $g(x) = o(f(x))$.

Observe that the choice of the limit point y matters! For instance, consider the functions $f(x) = 200x$ and $g(x) = x^2 + x$. Let us consider $y = 0$ first. We see that

$$\lim_{x\to 0} \frac{200x}{x^2 + x} = \lim_{x\to 0} \frac{200}{x + 1} = 200, \tag{2}$$

which is a constant! Therefore, $f(x) = \Theta(g(x))$ as $x \to 0$. I.e., $f(x)$ is asymptotically the same order of growth as $g(x)$.

Now, consider $y = \infty$:

$$\lim_{x\to \infty} \frac{200x}{x^2 + x} = \lim_{x\to \infty} \frac{200}{x + 1} = 0. \tag{3}$$

Therefore, $f(x) = o(g(x))$ as $x \to \infty$. I.e., $f(x)$ is asymptotically much *smaller* than $g(x)$. Notice that we're dealing with the same functions in both cases!

Let us take another example: what does it mean for $f(x)$ to be $o(1)$ (i.e., here we are taking $g(x) = 1$) as $x \to y$?

As usual, we consider the ratio of $f(x)$ and $g(x)$, which, by definition of $o(\cdot)$, must tend to 0:

$$\lim_{x\to y} \frac{f(x)}{1} = \lim_{x\to y} f(x) = 0. \tag{4}$$

In other words, saying that $f(x) = o(1)$ means exactly the same thing as saying that $\lim_{x\to y} f(x) = 0$.

What does it mean to say that $f(x) = \Theta(1)$? We know from the definition that $|f(x)/1| = |f(x)|$ must either converge to some constant or oscillate but remain bounded and not tend to 0. Intuitively, $f(x) = \Theta(1)$ means that $f(x)$ neither explodes to infinity nor vanishes to 0.

3 Some links to kind of similar courses

- `https://www.cs.purdue.edu/homes/spa/cs182.html`

- https://www.cs.rpi.edu/~magdon/courses/ONLINEfocs.php

4 Some tips for reviewing

There is a lot of material, and it may take time to review all of it. You might find it helpful to take a more directed approach to how you review: during each lecture, or as you do the suggested reading, write down any relevant background material that you don't remember very well and study that. Maintain a list of these topics, and, as time allows, solidify your understanding and write detailed notes. Include in your notes some information about why the topic is important (e.g., in what context you saw it). Make sure to do some practice problems (there should be plenty of problems with solutions from Khan Academy), and include the most enlightening problems in your notes.

Please note that studying math will inevitably, for *everyone*, at some point require sitting and pondering for a while.