ICSI 210 – Discrete Structures

Instructor: Abram Magner Date: Fall 2022

Instructions: Please answer the following questions in complete sentences, showing all work (including code and output of programs, when applicable). I prefer that you type your solutions (e.g., using LaTeX, with Overleaf, TeXworks, etc., or Word), but will accept handwritten notes. If the grader cannot read your handwriting, then they cannot award you points. **If you do not show your work and just give an answer, then you will get at most 1 point.**

Due date: Tuesday, 11/8/2022, 11:59 p.m.

3.1 Sums and products

1. Find a closed-form formula for

$$S_n = \sum_{j=0}^{n} j2^j. \tag{3.1}$$

Hint: Remember how we figured out a closed-form formula for $\sum_{j=0}^{n} r^j$, the geometric series. We exploited the self-similarity of the sum. In particular, you should try to find **two different formulas** for S_{n+1} in terms of S_n. Then you can set these formulas equal to each other and solve for S_n.

Additional note: If you want to know where this sum would come up, here's a story. It requires a little bit of probability, which we haven't learned yet. Sorry about that.

Rooted binary trees are ubiquitous as data structures in computer science. A binary tree is a data structure on "nodes" (just elements of some fixed set V), where each node has 0, 1, or 2 *children* (which are different nodes). In a data structures context, nodes would be associated with data items. A complete binary tree is a connected binary tree such that every node has either 0 or 2 nodes. The *height* of a binary tree is the length of the longest path from any node upward to the root. Suppose we choose a node uniformly at random (call the resulting randomly chosen node X). A measure of how much computation time it takes to retrieve the data item associated with this node is the depth of X. The *expected value* of the depth of X is a decent measure of the average time to retrieve a data item from the tree. It turns out that this expected value has a formula that can be written in terms of the sum (3.1).

3.2 Recurrence relations

1. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

2. What are the initial conditions?

3. Give a closed-form solution for the recurrence.

3.3 Induction

1. Prove the following claim by induction:

   Claim 3.1. *For all* $n \in \mathbb{Z} \cap [0, \infty)$*, we have*

   $$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}. \tag{3.2}$$

2. Give a detailed explanation of what is logically wrong with the following inductive "proof":

   Claim 3.2. *For every positive integer* n*, if* x *and* y *are positive integers with* $\max(x, y) = n$*, then* $x = y$*.*

   Proof. We prove this by induction.

   Base case: $n = 1$. if $\max(x, y) = 1$ and x and y are positive integers, then we must have $x = 1$ and $y = 1$.

   Inductive step: Assume that the claim is true for $n = k$. We need to verify for $n = k + 1$. That is, assume that whenever $\max(x, y) = k$ where x, y are positive integers, we have $x = y$. Now suppose $\max(x, y) = k + 1$ and x, y are positive integers. Then

   $$\max(x - 1, y - 1) = k, \tag{3.3}$$

   so by the inductive hypothesis, $x - 1 = y - 1$, and so $x = y$, which completes the proof. ■

3.4 Counting, permutations, combinations

1. How many bit strings of length 12 contain exactly 3 ones?

2. How many bit strings of length 12 contain the same number of 0s and 1s?

3. How many ways are there to arrange the letters a, b, c, d in such a way that a is not followed immediately by b?

4. How many positive integers less than or equal to 100 are divisible by 2 or 3? Recall that x is divisible by y (or, equivalently, $y \mid x$) if there exists $h \in \mathbb{Z}$ such that $x = hy$.

   Hint: Use inclusion-exclusion.

5. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

6. A professor writes 40 discrete math true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

3.1 Sums and products

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Additional note: If you want to know where this sum would come up, here's a story. It requires a little bit of probability, which we haven't learned yet. Sorry about that.

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Expand the series : $S_n = \sum_{j=0}^{n} j 2^j$

$S_n = 0\cdot 2^0 + 1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + (n-1)\cdot 2^{n-1} + n\cdot 2^n$

Multiply S_n by 2 :

$2\cdot S_n = 1\cdot 2^2 + 2\cdot 2^3 + 3\cdot 2^4 + \cdots + (n+1)2^n + n\cdot 2^{n+1}$

$S_n - 2S_n = 1\cdot 2^1 + (2-1)2^2 + (3-2)2^3 + \cdots (n-n+1)2^n - n2^{n+1}$

$-S_n = 2 + 2^2 + 2^3 + \cdots + 2^n - n2^{n+1} \quad \ldots (1).$

$2+2^2+2^3+\cdots 2^n = \frac{2(2^n-1)}{2-1}$

$\frac{2(2^n-1)}{2-1} - n2^{n+1}$

$2(2^n-1) - n\cdot 2^{n+1}$

$-S_n = 2^{n+1} - 2 - n\cdot 2^{n+1}$

$-1\cdot -S_n = 2 + n\cdot 2^{n+1} - 2^{n+1}$

$S_n = 2 + (n+1)\cdot 2^{n+1}$

1. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

   2. What are the initial conditions?

   3. Give a closed-form solution for the recurrence.

Let a_n be the number of ways to climb n stairs if the person climb the stairs can take one stair or two stairs at a time.

Suppose first time he climbs the single star. the remaining # of stairs would be $(n-1)$ in a_{n-1} ways.

Suppose first time he climbs two stairs, $n \geq 2$. then the remaining # of stairs would be $(n-2)$ and it would take a_{n-2} number of ways.

The total # of ways to climb n stairs is $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$.

Since $\boxed{a_n = a_{n-1} + a_{n-2}, \quad n \geq 2.}$

$\boxed{a_2 = a_1 + a_0}$

The initial condition would be $n \geq 2$.

The closed form formula is $a_n = a_{n-1} + a_{n-2}$ with $n \geq 2$.

3.3 Induction

1. Prove the following claim by induction:

Claim 3.1. *For all $n \in \mathbb{Z} \cap [0, \infty)$, we have*

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}. \tag{3.2}$$

The domain given for LHS is $[0, \infty)$

Therefore, $\sum_{j=0}^{n} (-\frac{1}{2})^j$, $n \in \mathbb{Z} \cap [0, \infty) \Leftrightarrow \sum_{j=0}^{\infty} (-\frac{1}{2})^j$

Apply Ratio Test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-\frac{1}{2})^{j+1}}{(-\frac{1}{2})^j} \right| = \frac{1}{2}$

$\lim_{j \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{j \to \infty} \frac{1}{2} = \frac{1}{2}$, so the series converges.

Find the closed-form of summation: $\sum_{j=0}^{n} (-\frac{1}{2})^j$

$$\sum_{j=0}^{n} = a_0 + \sum_{i=1}^{n}$$

$$= a_0 + \sum_{i=1}^{n} -\frac{1}{2}(-\frac{1}{2})^{j-1}$$; Apply Geometric series formula

$\frac{a_1}{1-r}$; where $-1 < |r| < 1$; $a_i = -\frac{1}{2}(-\frac{1}{2})^{i-1}$

$$a_1 \frac{1-r^j}{1-r} = -\frac{1}{2} \frac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})} = -\frac{1-(-\frac{1}{2})^n}{3}$$

$$a_0 - \frac{1-(-\frac{1}{2})^n}{3} \Leftrightarrow 1 - \frac{1-(\frac{1}{2})^n}{3} = \boxed{\frac{(-\frac{1}{2})^n + 2}{3}}$$

Proof: $\frac{(-\frac{1}{2})^n + 2}{3} = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$

$$= \frac{2^n \cdot 2 + (-1)^n}{3 \cdot 2^n}$$

$$= \frac{2^n (2 + \frac{(-1)^n}{2^n})}{3 \cdot 2^n}$$

$$= \frac{\cancel{2^n} (2 + (\frac{-1}{2})^n)}{3 \cdot \cancel{2^n}}$$

$$= \boxed{\frac{(-\frac{1}{2})^n + 2}{3}}$$

※ This is the first approach to proof, proof by induction at following page.

3.3 Induction

1. Prove the following claim by induction:

Claim 3.1. *For all $n \in \mathbb{Z} \cap [0, \infty)$, we have*

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}. \tag{3.2}$$

Assume that the Left Hand side is.

$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + - \cdots (-\frac{1}{2})^j$, where $(-\frac{1}{2})^j$ is an alternating series

would be depend on the power of the series, if $j = 2k$, where $k \in \mathbb{Z}$, then it would

plus $(-\frac{1}{2})^j$ term, hence, this term would be even since we know that a negative

fraction to the even power would be positive. i.g. $(-\frac{1}{2})^2 = \frac{1}{4}$; $(-\frac{1}{2})^4 = \frac{1}{16}$.

Vice versa, $(-\frac{1}{2})^j$, when $j = 2k+1$ and $k \in \mathbb{Z}$ is negative. i.g $(-\frac{1}{2})^3 = -\frac{1}{8}$.

① Base Case: Assume $n = 1$.

LHS: $\sum_{j=0}^{n=1} (-\frac{1}{2})^j = (-\frac{1}{2})^0 + (-\frac{1}{2})^1 = 1 - \frac{1}{2} = \frac{1}{2}$

RHS: $\frac{2^{1+1} + (-1)^1}{3 \cdot 2^1} = \frac{2^2 + (-1)}{6} = \frac{3}{6} = \frac{1}{2}$

Base Case is correct, next we assume $n = k$.

②. Assume $n = k$.

LHS: $\sum_{j=0}^{n=k} (-\frac{1}{2})^j = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots (-\frac{1}{2})^k$

RHS: $\frac{2^{k+1} + (-1)^k}{3 \cdot 2^k}$ ... ①.

③ S_{k+1} term:

LHS: $\sum_{j=0}^{n=k+1} (-\frac{1}{2})^j = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots (-\frac{1}{2})^k + (-\frac{1}{2})^{k+1}$

Apply ①: $\frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + (-\frac{1}{2})^{k+1}$

RHS: $\frac{2^{k+1+1} + (-1)^{k+1}}{3 \cdot 2^{k+1}} = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$

④ Proof LHS ≡ RHS: $\frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + (-\frac{1}{2})^{k+1} = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$

Proof LHS:

$\longrightarrow$

④ Proof LHS ≡ RHS: $$\frac{2^{k+1}+(-1)^k}{3\cdot 2^k} + \left(-\frac{1}{2}\right)^{k+1} = \frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}}$$

Proof LHS:

$$\frac{2(2^{k+1}+(-1)^k)}{3\cdot 2^k\cdot 2} + \frac{3(-1)^{k+1}}{2^{k+1}\cdot 3} = \frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}}$$

$$\frac{2^{k+2}+2(-1)^k+3(-1)^{k+1}}{3\cdot 2^{k+1}} = \frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}}$$

$$\frac{2^{k+2}+2(-1)^k+3(-1)^k(-1)}{3\cdot 2^{k+1}} = \text{''}$$

$$\frac{[2^{k+2}+2(-1)^k+3(-1)^k(-1)](-1)}{3\cdot 2^{k+1}(-1)} = \text{''}$$

$$\frac{-2^{k+2}+(-1)^k}{3\cdot 2^{k+1}(-1)} = \text{''}$$

$$\frac{(-2^{k+2}+(-1)^k)(-1)}{3\cdot 2^{k+1}(-1)(-1)} = \text{''}$$

$$\frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}} = \text{RHS} \qquad \square$$

Claim 3.2. *For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.*

Proof. We prove this by induction.

Base case: $n = 1$. if $\max(x, y) = 1$ and x and y are positive integers, then we must have $x = 1$ and $y = 1$.

Inductive step: Assume that the claim is true for $n = k$. We need to verify for $n = k + 1$. That is, assume that whenever $\max(x, y) = k$ where x, y are positive integers, we have $x = y$. Now suppose $\max(x, y) = k + 1$ and x, y are positive integers. Then

$$\boxed{\max(x - 1, y - 1) = k,} \tag{3.3}$$

so by the inductive hypothesis, $x - 1 = y - 1$, and so $x = y$, which completes the proof. ■

Let $x = 2$, $y = 3$ then $\max(2,3) = 3 = k+1$

but $\max(x-1, y-1) = \max(1, 2) = k$ does not imply that $x = y$ as $1 \neq 2$.

Hence, from the statement: Base case: $n = 1$. if $\max(x, y) = 1$ and x and y are positive integers,

then we must have $x = 1$ and $y = 1$, this statement is logically flawed.

The inductive step $\max(x-1, y-1) = k$ is wrong as 1 cannot be subtracted from the brackets

and also the steps does not guarantee that $x = y$.

Therefore, it can be said that the step $\max(x-1, y-1) = k$ is where the mistake made,

and we could not use this statement at all.

3.4 Counting, permutations, combinations

1. How many bit strings of length 12 contain exactly 3 ones?
2. How many bit strings of length 12 contain the same number of 0s and 1s?
3. How many ways are there to arrange the letters a, b, c, d in such a way that a is not followed immediately by b?
4. How many positive integers less than or equal to 100 are divisible by 2 or 3? Recall that x is divisible by y (or, equivalently, $y \mid x$) if there exists $h \in \mathbb{Z}$ such that $x = hy$.

   ≤ 100

   Hint: Use inclusion-exclusion.
5. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?
6. A professor writes 40 discrete math true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

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1. Combination, order does not matter: $\frac{12!}{3!(12-3)!} = \frac{12!}{3!\ 9!} = \frac{2\cancel{12}\cdot 11\cdot 10\cdot \cancel{9!}}{\cancel{3\cdot 2}\cdot 1\cdot \cancel{9!}} = \boxed{220}$

2. Combination. order does not matter. If the length of bit strength is 12, then it would have 6 0's and 6 1's.

   so. # of 0 = 6. $\frac{12!}{6!(12-6)!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot \cancel{6!}}{6!\ \cancel{6!}} = \boxed{924}$

3. □□□□

   ab□□

   □ab□

   □□ab.

   There are $4! = 24$ ways to arrange these letter.

   If we are looking for "a is not immediately followed by b"

   Then, we could regard "ab" as a combination, Hence, the following conditions are not satisfied: abXX, XabX, XXab.

   Each of these combination produce 2 unique letter strings, so there are 6 in total.

   $\boxed{24 - 6 = 18.}$

4. $A_2 = \{(2,1), (2,2), \dots (2,50)\} \le 100. \Rightarrow \#(A_2) = 50$

   $A_3 = \{(3,1), (3,2), \dots (3,33)\} \le 100. \Rightarrow \#(A_3) = 33$

   $A_6 = \{(6,1), (6,2), \dots (6,16)\} \le 100 \Rightarrow \#(A_6) = 16.$

   $|A_2 \cup A_3| = |A_2| + |A_3| - |A_6|$

   $= 50 + 33 - 16 = 67$; By Inclusion-Exclusion Theorem.

5. $\{a, b, c, d, e, f, g\}$ end with a ?

7 letter string that is given, so there are $7!$ ways to arrange these letter.

1 of these letter has to be fixed, so it implies that there are 7 ways to fix a in the end.

However, we have to consider the permutation of 6 letter before it reaches the 7^{th} one.

Therefore, the total ways of combining $\{a, b, \ldots g\}$ end with a is $\frac{7!}{7} = \frac{7 \cdot 6!}{7}$

$= \boxed{6! \text{ ways.}}$

6. Given that the total number of questions is 40; 17 of them are true.

Hence, $40 - 17 = 23$, 23 of these question are false. If the question can be positioned in any order, then this is of combination of $n = 40$, $k = 17$,

$$\frac{40!}{17!\,(40-17)!} = \frac{40!}{17!\,23!} = \boxed{8.87323788 \times 10^{10}}$$