3.3 Induction

1. Prove the following claim by induction:

Claim 3.1. *For all* $n \in \mathbb{Z} \cap [0, \infty)$, *we have*

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}. \tag{3.2}$$

The domain given for LHS is $[0, \infty)$

Therefore, $\sum_{j=0}^{n} (-\frac{1}{2})^j$, $n \in \mathbb{Z} \cap [0, \infty) \Leftrightarrow \sum_{j=0}^{\infty} (-\frac{1}{2})^j$

Apply Ratio Test : $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-\frac{1}{2})^{j+1}}{(-\frac{1}{2})^j} \right| = \frac{1}{2}$

$\lim_{j \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{j \to \infty} \frac{1}{2} = \frac{1}{2}$, so the series converges.

Find the closed-form of summation : $\sum_{j=0}^{n} (-\frac{1}{2})^j$

$$\sum_{j=0}^{n} = a_0 + \sum_{i=1}^{n}$$

$$= a_0 + \sum_{i=1}^{n} -\frac{1}{2}(-\frac{1}{2})^{j-1} \quad ; \text{ Apply Geometric Formula.}$$

$\frac{a_1}{1-r}$; where $-1 < |r| < 1$; $a_i = -\frac{1}{2}(-\frac{1}{2})^{i-1}$

$$a_1 \frac{1-r^j}{1-r} = -\frac{1}{2} \frac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})} = -\frac{1-(-\frac{1}{2})^n}{3}$$

$$a_0 - \frac{1-(-\frac{1}{2})^n}{3} \Leftrightarrow 1 - \frac{1-(\frac{1}{2})^n}{3} = \boxed{\frac{(-\frac{1}{2})^n + 2}{3}}$$

Proof : $\frac{(-\frac{1}{2})^n + 2}{3} = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$

$$= \frac{2^n \cdot 2 + (-1)^n}{3 \cdot 2^n}$$

$$= \frac{2^n (2 + \frac{(-1)^n}{2^n})}{3 \cdot 2^n}$$

$$= \frac{\cancel{2^n} (2 + (\frac{-1}{2})^n)}{3 \cdot \cancel{2^n}}$$

$$= \boxed{\frac{(-\frac{1}{2})^n + 2}{3}}$$