Note: Before you begin, please note that there is a typo in the clamped spline example in the book (Example 2, page 148, eighth edition), the correct solution will be given below.

Intro: When computing a cubic spline with $n+1$ grid points, keep in mind that we have n intervals, n spline segments, and an $(n+1)\times(n+1)$ matrix. For example, if the data is known at the grid points x_0, x_1, x_2, x_3, x_4 we have 5 points, 4 intervals, 4 spline segments, and a 5×5 matrix. The spacings between grid points, defined $h_j = x_{j+1} - x_j$, are used below.

General approach: We can verify the conditions of the spline by definition, but ultimately the way to implement this, even for small datasets, is to use matrix algebra. Let's use the vector $\mathbf{r}$ (as in $\mathbf{A}\mathbf{x} = \mathbf{r}$, rather than $\mathbf{A}\mathbf{x} = \mathbf{b}$) for the right-hand side to avoid likely confusion between entries of the right-hand side vector and the coefficients b_j of the splines. The first and last rows of $\mathbf{A}$ and $\mathbf{r}$ depend on the choice of boundary conditions, but the interior rows are illustrated below. On the interior rows of the main diagonal, $\mathbf{A}$ has entries

$$2(h_0+h_1), 2(h_1+h_2), \cdots, 2(h_{n-3}+h_{n-2}), 2(h_{n-2}+h_{n-1})$$

On the interior rows of the sub-diagonal (below main), $\mathbf{A}$ has entries

$$h_0, h_1, \cdots, h_{n-3}, h_{n-2}$$

On the interior rows of the super-diagonal (above main), $\mathbf{A}$ has entries

$$h_1, h_2, \cdots, h_{n-2}, h_{n-1}$$

$$\mathbf{A} = \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & h_{j-1} & 2(h_{j-1}+h_j) & h_j & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} \cdots \\ 3\left(\dfrac{a_2-a_1}{h_1} - \dfrac{a_1-a_0}{h_0}\right) \\ \vdots \\ 3\left(\dfrac{a_n-a_{n-1}}{h_{n-1}} - \dfrac{a_{n-1}-a_{n-2}}{h_{n-2}}\right) \\ \cdots \end{bmatrix}$$

Natural BCs: The interior rows of $\mathbf{A}$ are as described above, but the first row begins with $1, 0, \ldots$ and the last row ends with $\ldots, 0, 1$. To satisfy the boundary condition, the first and last rows of of $\mathbf{r}$ are exactly 0.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Clamped BCs: The interior rows of $\mathbf{A}$ are as described above, but the first row begins with $2h_0, h_0, 0, \ldots$ and the last row ends with $\ldots, 0, h_{n-1}, 2h_{n-1}$. To satisfy the boundary condition, the first and last rows of of $\mathbf{r}$ are exactly $3\left(\frac{a_1 - a_0}{h_0} - f'(a)\right)$ and $3\left(f'(b) - \frac{a_n - a_{n-1}}{h_{n-1}}\right)$, respectively.

$$\mathbf{A} = \begin{bmatrix} 2h_0 & h_0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & h_{n-1} & 2h_{n-1} \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 3\left(\frac{a_1 - a_0}{h_0} - f'(a)\right) \\ \vdots \\ \vdots \\ \vdots \\ 3\left(f'(b) - \frac{a_n - a_{n-1}}{h_{n-1}}\right) \end{bmatrix}$$

Parameterizing the spline: Once the c_j's have been solved, the b_j's and d_j's can be specified *in reverse order* from $j = n-1, n-2, \ldots, 0$ (see Alg. 3.4 (Step 6) or Alg. 3.5 (Step 7)).

$$b_j = \frac{a_{j+1} - a_j}{h_j} - \frac{h_j(c_{j+1} + 2c_j)}{3}$$

$$d_j = \frac{c_{j+1} - c_j}{3h_j}$$

Notice that c_n (from $\mathbf{x}$) is used in the calculation of d_{n-1}, but is never actually used in a spline. Regardless of the boundary condition, the calculated coefficients will appear in the spline of the form

$$S(t) = \begin{cases} \cdots & , \quad \cdots \\ S_j(t) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, & x_j \leq x < x_{j+1} \\ \cdots & , \quad \cdots \end{cases}$$

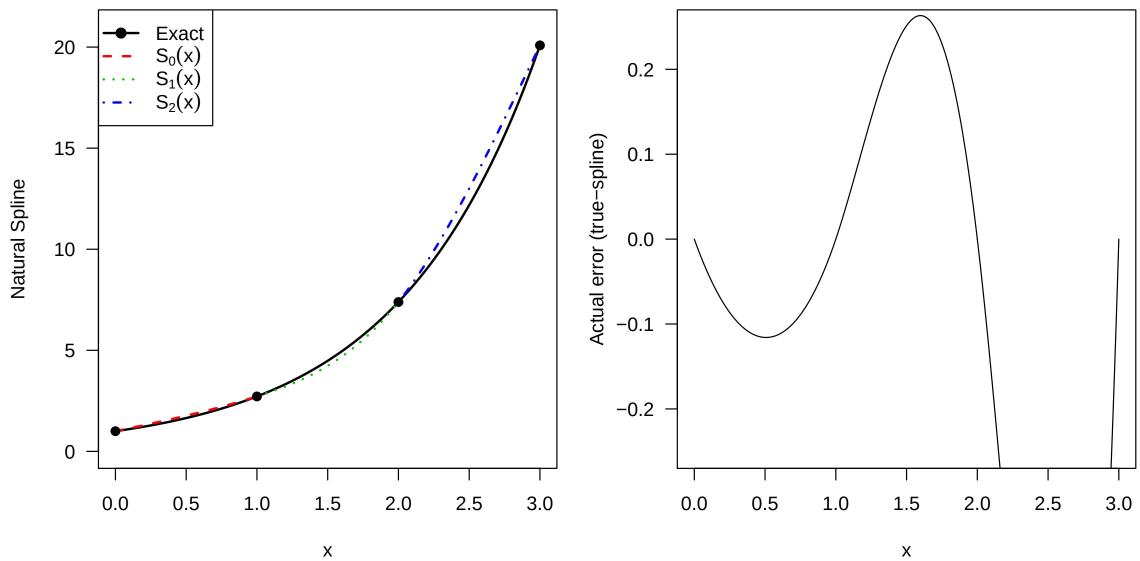
Note that $x_0 = a$ and $x_n = b$.

Keep reading: (next page, please)

Example Consider the data $x_0 = 0, x_1 = 1, x_2, = 2, x_3 = 3$ and $f(x) = e^x$.

A natural spline: See Example 1 on page 143.

j	a_j	b_j	c_j	d_j
0	1.000000	1.465998	0.0000000	0.2522842
1	2.718282	2.222850	0.7568526	1.6910714
2	7.389056	8.809770	5.8300668	-1.9433556



A clamped spline: See Example 2 on page 148.

j	a_j	b_j	c_j	d_j
0	1.000000	1.000000	0.4446825	0.2735993
1	2.718282	2.710163	1.2654805	0.6951308
2	7.389056	7.326516	3.3508729	2.0190916