Proof of Taylor's Theorem

Recall that we had observed that the Mean Value Theorem was a special case of Taylor's Theorem. As it turns out, the proof of Taylor's Theorem parallels that of the Mean Value Theorem. Both make use of Rolle's Theorem: If g is continuous on the interval $[a, b]$, differentiable on (a, b) and $g(a) = g(b)$, then there is a number $c \in (a, b)$ for which $g'(c) = 0$. As with the proof of the Mean Value Theorem, for a **fixed** $x \in (c - r, c + r)$, we define the function

$$\begin{aligned} g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{1}{2!}f''(t)(x-t)^2 - \frac{1}{3!}f'''(t)(x-t)^3 \\ - \cdots - \frac{1}{n!}f^{(n)}(t)(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-c)^{n+1}}, \end{aligned}$$

where $R_n(x)$ is the remainder term, $R_n(x) = f(x) - P_n(x)$. If we take $t = x$, notice that

$$g(x) = f(x) - f(x) - 0 - 0 - \cdots - 0 = 0$$

and if we take $t = c$, we get

$$\begin{aligned} g(c) &= f(x) - f(c) - f'(c)(x-c) - \frac{1}{2!}f''(c)(x-c)^2 - \frac{1}{3!}f'''(c)(x-c)^3 \\ &\quad - \cdots - \frac{1}{n!}f^{(n)}(c)(x-c)^n - R_n(x)\frac{(x-c)^{n+1}}{(x-c)^{n+1}} \\ &= f(x) - P_n(x) - R_n(x) = R_n(x) - R_n(x) = 0. \end{aligned}$$

By Rolle's Theorem, there must be some number z between x and c for which $g'(z) = 0$. Differentiating our expression for $g(t)$ (with respect to t!), we get (beware of all the product rules!)

$$\begin{aligned} g'(t) &= 0 - f'(t) - f'(t)(-1) - f''(t)(x-t) - \frac{1}{2}f''(t)(2)(x-t)(-1) \\ &\quad - \frac{1}{2}f'''(t)(x-t)^2 - \cdots - \frac{1}{n!}f^{(n)}(t)(n)(x-t)^{n-1}(-1) \\ &\quad - \frac{1}{n!}f^{(n+1)}(t)(x-t)^n - R_n(x)\frac{(n+1)(x-t)^n(-1)}{(x-c)^{n+1}} \\ &= -\frac{1}{n!}f^{(n+1)}(t)(x-t)^n + R_n(x)\frac{(n+1)(x-t)^n}{(x-c)^{n+1}}, \end{aligned}$$

after most of the terms cancel. So, taking $t = z$, we have that

$$0 = g'(z) = -\frac{1}{n!}f^{(n+1)}(z)(x-z)^n + R_n(x)\frac{(n+1)(x-z)^n}{(x-c)^{n+1}}.$$

Solving this for the remainder term, $R_n(x)$, we get

$$R_n(x)\frac{(n+1)(x-z)^n}{(x-c)^{n+1}} = \frac{1}{n!}f^{(n+1)}(z)(x-z)^n$$

and finally,

$$\begin{aligned} R_n(x) &= \frac{1}{n!}f^{(n+1)}(z)(x-z)^n\frac{(x-c)^{n+1}}{(n+1)(x-z)^n} \\ &= \frac{f^{(n+1)}(z)}{(n+1)\,n!}(x-c)^{n+1} \\ &= \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}, \end{aligned}$$

as we had claimed.