Discrete case (i.e., 8.1, pg. 486)

Consider the last term

$$\sum_{i=1}^{m} \Big(P_n(x_i)\Big)^2 = \sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j\right)^2$$

in the definition of the error $E = E_2(a_0, a_1, \ldots, a_n)$ for discrete least squares approximation. We have the following (though I do not find this particularly illuminating and instead prefer to differentiate before addressing the sums), where it might initially be helpful to remember that $x_i^0 = 1$ and $x_i^1 = x_i$ (e.g., $a_0 x_i^0 = a_0 \cdot 1 = a_0$ and $a_1 x_i^1 = a_1 x_i$),

$$\begin{aligned}
\sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j\right)^2 &= \sum_{i=1}^{m} (a_0 + a_1 x_i + \cdots + a_n x_i^n)(a_0 + a_1 x_i + \cdots + a_n x_i^n) \\
&= \sum_{i=1}^{m} \Big[a_0(a_0 + \cdots + a_n x_i^n) + \cdots + a_j x_i^j (a_0 + \cdots + a_n x_i^n) + \cdots + a_n x_i^n (a_0 + \cdots + a_n x_i^n)\Big] \\
&= \sum_{i=1}^{m} \sum_{j=0}^{n} a_j x_i^j (a_0 + a_1 x_i + \cdots + a_n x_i^n) \\
&= \sum_{j=0}^{n} a_j \sum_{i=1}^{m} x_i^j (a_0 + a_1 x_i + \cdots + a_n x_i^n) \\
&= \sum_{j=0}^{n} a_j \sum_{i=1}^{m} x_i^j \sum_{k=0}^{n} a_k x_i^k \\
&= \sum_{j=0}^{n} a_j \sum_{i=1}^{m} \sum_{k=0}^{n} a_k x_i^{j+k} \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \Big(\sum_{i=1}^{m} x_i^{j+k}\Big)
\end{aligned}$$

Our goal is the partial derivative of this term with respect to a_j. Notice the following, where key steps are moving differentiation under the sum, applying the product rule, and rearranging the sum,

$$\begin{aligned}
\frac{\partial}{\partial a_j} \left(\sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j\right)^2 \right) &= \frac{\partial}{\partial a_j} \Big(\sum_{i=1}^{m} (a_0 + a_1 x_i + \cdots + a_n x_i^n)(a_0 + a_1 x_i + \cdots + a_n x_i^n) \Big) \\
&= \sum_{i=1}^{m} \frac{\partial}{\partial a_j} \Big((a_0 + a_1 x_i + \cdots + a_n x_i^n)(a_0 + a_1 x_i + \cdots + a_n x_i^n) \Big) \\
&= \sum_{i=1}^{m} \Big(x_i^j (a_0 + a_1 x_i + \cdots + a_n x_i^n) + (a_0 + a_1 x_i + \cdots + a_n x_i^n) x_i^j \Big) \\
&= \sum_{i=1}^{m} 2 \Big(x_i^j (a_0 + a_1 x_i + \cdots + a_n x_i^n) \Big) \\
&= \sum_{i=1}^{m} 2 \left(x_i^j \sum_{k=0}^{n} (a_k x_i^k) \right) \\
&= 2 \sum_{k=0}^{n} a_k \sum_{i=1}^{m} x_i^{j+k}
\end{aligned}$$

Continuous case (i.e., 8.2, pg. 495)

Consider the last term

$$\int_a^b \left(\sum_{k=0}^{n} a_k x^k \right)^2$$

in the definition of the error $E = E_2(a_0, a_1, \ldots, a_n)$ for least squares polynomial approximation. We have,

$$\int_a^b \left(\sum_{k=0}^{n} a_k x^k \right)^2 dx = \int_a^b (a_0 + a_1 x + \cdots + a_n x^n)(a_0 + a_1 x + \cdots + a_n x^n)\, dx$$

Our goal is the partial derivative of this term with respect to a_j. Notice the following, where key steps are moving differentiation under the integral, applying the product rule, and rearranging the sum,

$$\begin{aligned}
\frac{\partial}{\partial a_j} \int_a^b \left(\sum_{k=0}^{n} a_k x^k \right)^2 dx &= \frac{\partial}{\partial a_j} \int_a^b (a_0 + a_1 x + \cdots + a_n x^n)(a_0 + a_1 x + \cdots + a_n x^n)\, dx \\
&= \int_a^b \frac{\partial}{\partial a_j} \Big((a_0 + a_1 x + \cdots + a_n x^n)(a_0 + a_1 x + \cdots + a_n x^n) \Big)\, dx \\
&= \int_a^b \left(x^j (a_0 + a_1 x + \cdots + a_n x^n) + (a_0 + a_1 x + \cdots + a_n x^n) x^j \right)\, dx \\
&= \int_a^b 2x^j (a_0 + a_1 x + \cdots + a_n x^n)\, dx \\
&= 2 \int_a^b x^j \sum_{k=0}^{n} a_k x^k \, dx \\
&= 2 \int_a^b \sum_{k=0}^{n} a_k x^{j+k} \, dx \\
&= 2 \sum_{k=0}^{n} a_k \int_a^b x^{j+k} \, dx
\end{aligned}$$