Lagrange: 2 ways

(with a side of Hermite)

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A linear Lagrange polynomial interpolant

Consider linear interpolations to $f(x) = e^x$ using the points $x_0 = 0$ and $x_1 = 1/2$. Here we have

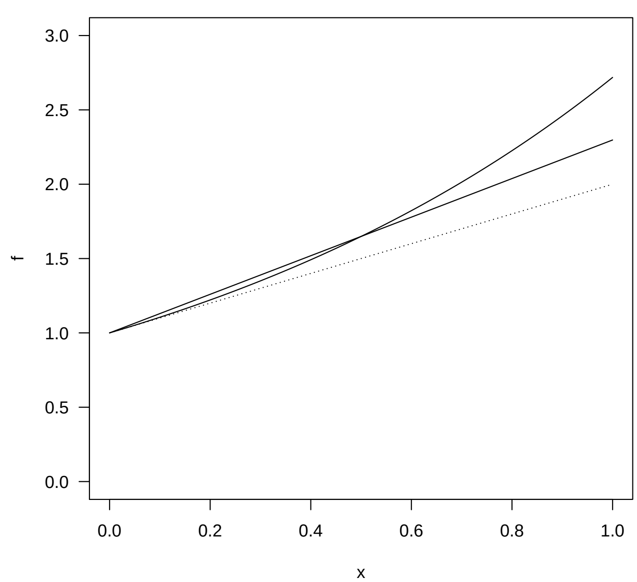
$$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} = \frac{x - 1/2}{0 - 1/2} = \frac{x - 1/2}{-1/2}$$
$$L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)} = \frac{x - 0}{1/2 - 0} = \frac{x - 0}{1/2}$$

The polynomial is given by

$$P(x) = f(0)L_0(x) + f(1/2)L_1(x) = 1\left(\frac{x - 1/2}{-1/2}\right) + \sqrt{e}\left(\frac{x - 0}{1/2}\right)$$

where the function values are $f(0) = e^0 = 1$ and $f(1/2) = e^{1/2} = \sqrt{e}$.

```
xs <- c(0, 1/2, 1)
f <- function(x)exp(x)
L0 <- function(x)(x-xs[[2]])/(xs[[1]]-xs[[2]])
L1 <- function(x)(x-xs[[1]])/(xs[[2]]-xs[[1]])
P <- function(x)f(xs[[1]])*L0(x) + f(xs[[2]])*L1(x)
plot(f, xlim=c(0, 1), ylim=c(0, 3), las=1)
plot(P, xlim=c(0, 1), add=T)
plot(function(x)1 + x, xlim=c(0, 1), add=T, lty=3)
```



Plotted are the function (solid), Lagrange polynomial (solid), and corresponding degree Taylor polyonmial (dashed, and at x_0).

An equivalent polynomial interpolant

We can also consider $\tilde{P}(x) = a_0 + a_1 x$ (this notation a_i where i matches the power of x may be convenient) and specify the unknowns a_0 and a_1. We have

$$\begin{cases} a_0 + a_1 x_0 & = f(x_0) \\ a_0 + a_1 x_1 & = f(x_1) \end{cases}$$

The idea is that for the known points x_0 and x_1 we have known function values of $y_0 = f(x_0)$ and $y_1 = f(x_1)$, assuming that our formula allow us to express y in terms of x. Or,

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \end{pmatrix}$$

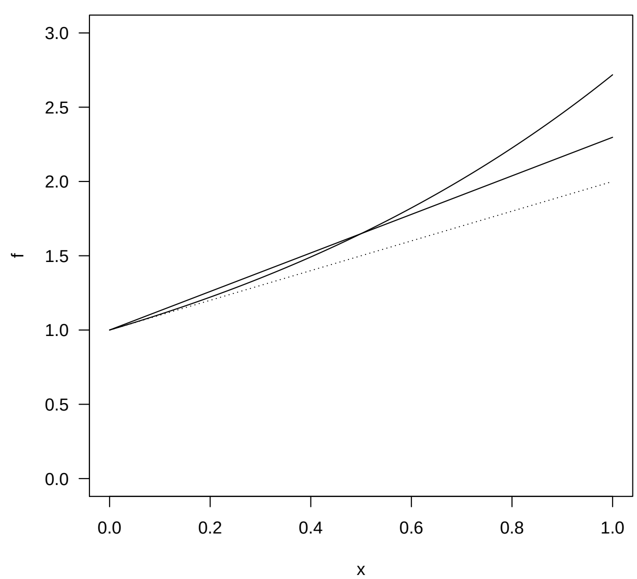
Inverting the matrix, we can solve the system to find

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2(\sqrt{e} - 1) \end{pmatrix}$$

This means $\tilde{P}(x) = 1 + 2(\sqrt{e} - 1)x$, which, following a bit of algebra, is exactly what we found above. Notice that

$$\begin{aligned} P(x) &= f(0)L_0(x) + f(1/2)L_1(x) \\ &= 1\left(\frac{x - 1/2}{-1/2}\right) + \sqrt{e}\left(\frac{x - 0}{1/2}\right) \\ &= -2x + 1 + 2\sqrt{e}x \\ &= 1 + 2(\sqrt{e} - 1)x \\ &= \tilde{P}(x) \end{aligned}$$

```
plot(f, xlim=c(0, 1), ylim=c(0, 3), las=1)
plot(function(x)1 + 2*(sqrt(exp(1))-1)*x, xlim=c(0, 1), add=T)
plot(function(x)1 + x, xlim=c(0, 1), add=T, lty=3)
```



Plotted are the function (solid), Lagrange polynomial (solid), and corresponding degree Taylor polyonmial (dashed, and at x_0).

Hermite

Given the work above and the fact that $f(x) = e^x$ and $f'(x) = e^x$ we can calculate the Hermite polynomial. For this we have,

$$L_0(x) = \frac{x - 1/2}{-1/2} = -2x + 1$$
$$L_0'(x) = \frac{1}{-1/2} = -2$$
$$L_1(x) = \frac{x - 0}{1/2} = 2x$$
$$L_1'(x) = \frac{1}{1/2} = 2$$

Additionally we have, the 4 cubic 'Hermite coefficient polynomials',

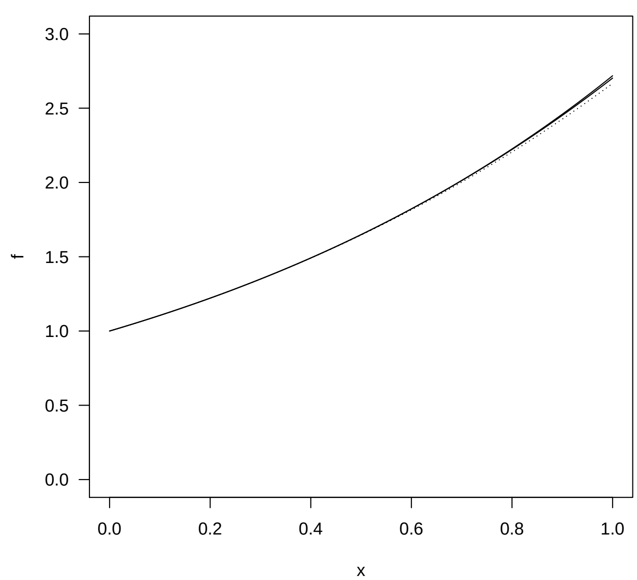
$$H_0(x) = (1 - 2(x - 0)L_0'(0))L_0(x)^2 \qquad = (1 - 2x(-2))L_0(x)^2$$
$$H_1(x) = (1 - 2(x - 1/2)L_1'(1/2))L_1(x)^2 = (1 - 2(x - 1/2)(2))L_1(x)^2$$
$$\hat{H}_0(x) = (x - 0)L_0(x)^2$$
$$\hat{H}_1(x) = (x - 1/2)L_1(x)^2$$

By the definition of the Hermite polynomial we have,

$$H(x) = f(0)H_0(x) + f(1/2)H_1(x) + f'(0)\hat{H}_0(x) + f'(1/2)H_1(x)$$
$$H(x) = 1 + x + 2(5\sqrt{e} - 8)x^2 - 4(3\sqrt{e} - 5)x^3$$

Admittedly putting this together is rather tedious - it takes a few minutes with Matheamtica (which can also help with differentiation) for the desired symbolic portions of the process. Most importantly notice that the linear Lagrange interpolant scales up to a cubic Hermite interpolant.

```
plot(f, xlim=c(0, 1), ylim=c(0, 3), las=1)
plot(function(x)1 + x + 2*(5*exp(1/2)-8)*x^2 - 4*(3*exp(1/2)-5)*x^3, xlim=c(0, 1), add=T)
plot(function(x)1 + x + x^2/2 + x^3/6, xlim=c(0, 1), add=T, lty=3)
```



Plotted are the function (solid), Hermite polynomial (solid), and corresponding degree Taylor polyonmial (dashed, and at x_0). An interesting challenge is to picture what the corresponding matrix representation would be for the Hermite polynomial.