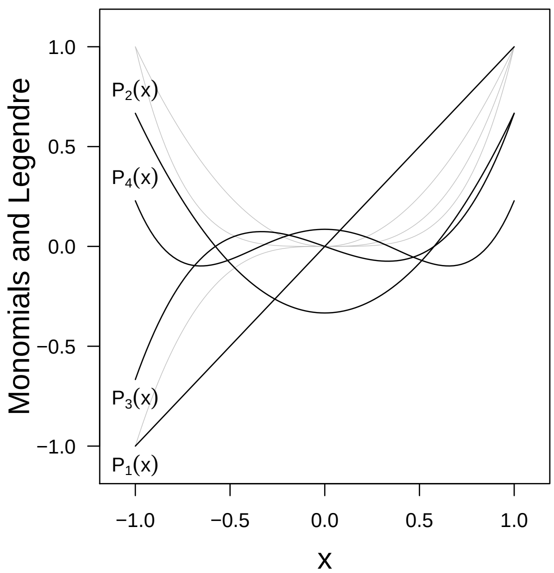
1 Orthogonal polynomials

1.1 Legendre polynomials

The standard monomials for function approximation, shown as the familiar unlabeled gray curves in Figure 1, are $\phi_i(x) = x^i$ for $i = 0, 1, \ldots, n$. These lack the property of orthogonality that is achieved by the Legendre polynomials, also shown Figure 1, but in black and labeled. The first few Legendre polynomials are



$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - \frac{1}{3} \\
P_3(x) &= x^3 - \frac{3}{5}x \\
&\ldots \\
P_n(x) &= (x - B_k)P_{n-1}(x) - C_k P_{x-2}(x)
\end{aligned}
$$

where

$$
B_k = \frac{\int_{-1}^{1} x \cdot 1 \cdot [P_{k-1}(x)]^2\, dx}{\int_{-1}^{1} 1 \cdot [P_{k-1}(x)]^2\, dx}
$$

$$
C_k = \frac{\int_{-1}^{1} x \cdot 1 \cdot [P_{k-1}(x) P_{k-2}(x)]\, dx}{\int_{-1}^{1} 1 \cdot [P_{k-2}(x)]^2\, dx}
$$

Figure 1: In gray the intuitive monomials $\phi_i(x) = x^i$ with the Legendre polynomials $P_i(x)$ in black.

1.2 Chebyshev polynomials

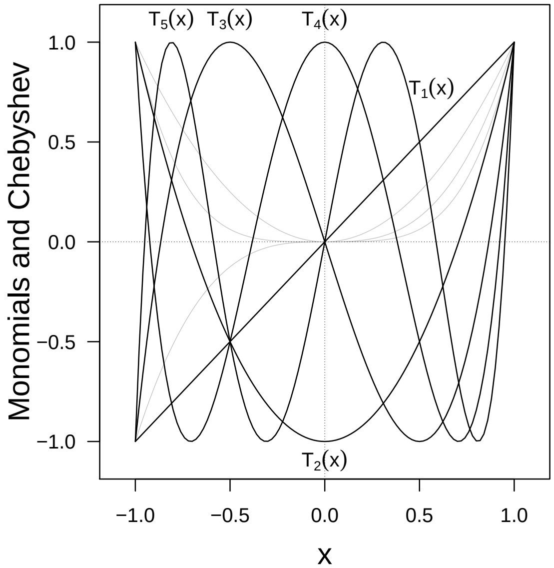
The standard monomials for function approximation, shown as the familiar unlabeled gray curves in Figure 2, are $\phi_i(x) = x^i$ for $i = 0, 1, \ldots, n$. These lack the property of orthogonality that is achieved by the Chebyshev polynomials, also shown Figure 2, but in black and labeled. The first few Chebyshev polynomials are

Since the alternate definition of the Chebyshev polynomials is $T_n(x) = \cos(n \arccos(x))$ for $n \geq 0$, it should seem reasonable that these functions are bounded in magnitude by one, as illustrated in Figure 2. The monic Chebyshev polynomials (not illustrated, but handy for Chebyshev economization of power series), with leading coefficient 1, are defined by

$$
\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x)
$$

For these polynomials the extrema, which occur *at* the same points, are those of $T_n(x)$ but reduced in value by a factor of $\frac{1}{2^{n-1}}$, that is, the extrema are, for $k = 0, 1, \ldots, n$

$$
\tilde{T}'_n(\bar{x}_k) = \frac{(-1)^k}{2^{n-1}}
$$



$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
&\dots \\
T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x)
\end{aligned}
$$

Figure 2: In gray the intuitive monomials $\phi_i(x) = x^i$ with the Chebyshev polynomials $T_i(x)$ in black.

1.2.1 Chebyshev points and Lagrange nodes

By a theorem, we have that the optimal choice of nodes for polynomial approximation by a degree n polynomial is given by the zeros of of the $(n+1)^{\text{st}}$ Chebyshev polynomial $T_{n+1}(x)$. For example, as given in Figure 3, we take the approximation of $f(x) = e^x$ on $[0, 1]$ with three points. We expect to to take, $\tilde{x}_0 = 0$, $\tilde{x}_1 = 0.5$, and $\tilde{x}_2 = 1.0$, but achieve better performance with Chebyshev nodes $\bar{x}_i$. The nodes for interpolation are given by the zeros of the Chebyshev polynomial $T_3(x)$, which in turn are given by $\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ for $k = 1, 2, 3$.

$$
\begin{aligned}
\bar{x}_1 &= \cos\left(\frac{2(1)-1}{2(3)} \cdot \pi\right) = \cos\left(\frac{1 \cdot \pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866 \\
\bar{x}_2 &= \cos\left(\frac{2(2)-1}{2(3)} \cdot \pi\right) = \cos\left(\frac{3 \cdot \pi}{6}\right) = 0 \\
\bar{x}_3 &= \cos\left(\frac{2(3)-1}{2(3)} \cdot \pi\right) = \cos\left(\frac{5 \cdot \pi}{6}\right) = -\frac{\sqrt{3}}{2} \approx -0.866
\end{aligned}
$$

From $\tilde{x}_i \in [a, b]$ we can compute $\bar{x}_i \in [-1, 1]$ by

$$x_i = \frac{2\tilde{x}_i - a - b}{b - a}$$

and in the reverse we can use

$$\tilde{x}_i = \frac{1}{2}\left((b-a)\bar{x}_i + a + b\right)$$

As an example,

i	$\bar{x}_i$	$\tilde{x}_i$
1	0.866	0.933
2	0.0	0.500
3	−0.866	0.067

The näively chosen nodes that include the endpoints (red) are illustrated in the left-hand panel of Figure 3, and as shown give a reasonable approximate. Yet, the approximation is improved by use of the Chebyshev points (blue). Notice, from Figure 4, that the zeros accumulate or cluster near the boundary of the interval. This helps to 'clamp down' the polynomial interpolation near the boundary.

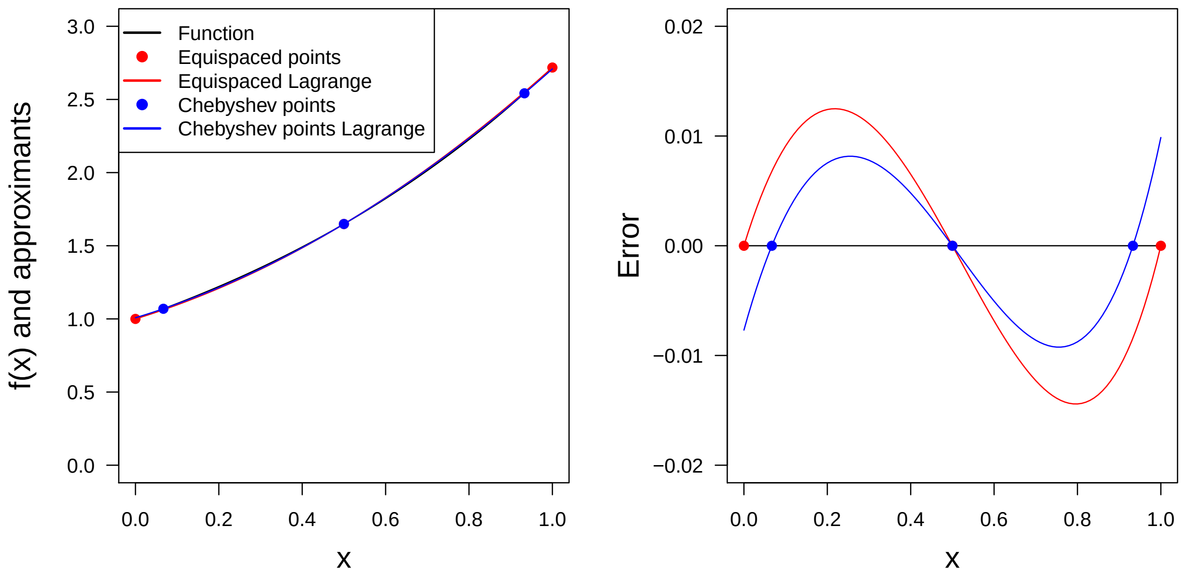


Figure 3: Lagrange interpolation of $f(x) = e^x$ on $[0, 1]$, but with Chebyshev points instead of equispaced points.

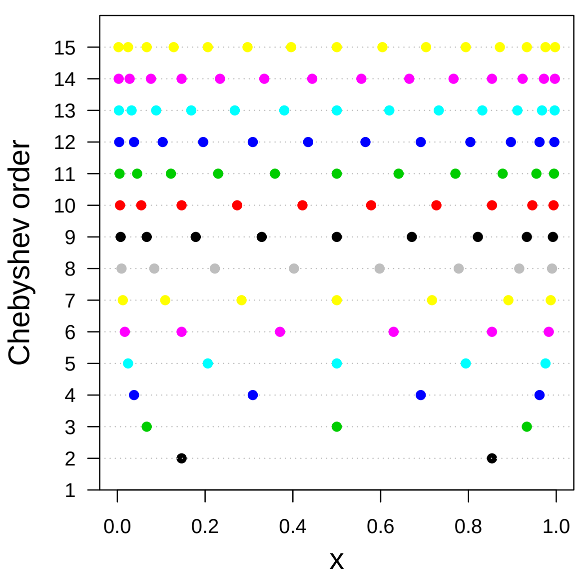


Figure 4: Location of zeros to $T_n(x)$ in the interval $[0, 1]$ for Chebyshev functions of orders from $n = 2, 3, \ldots, 15$.

1.2.2 Chebyshev economization

As we will soon see, the 'best' reduced order polynomial approximation P_{n-1} to a polynomial $P_n(x)$ is given by

$$P_{n-1}(x) = P_n(x) - a_n\tilde{T}_n(x)$$

where $\tilde{T}_n(x)$ is the n^{th} order monic Chebyshev polynomial and a_n is the coefficient of the highest order term in $P_n(x)$. To economize the polynomial approximation to $f(x) = e^x$ given by a 4^{th} order Maclaurin polynomial, we can subtract $\dfrac{1}{24}\tilde{T}_4(x)$ from the original $P_4(x)$ (note that $a_4 = \dfrac{1}{24}$ in the original polynomial representation). This alters coefficients of all powers of x of lower orders that are present in $T_4(x)$ and eliminates the 4^{th} order term entirely. Notice in Figure 5 that the maximum error for the third order function is only slightly worse for $x = 1.0$ than for the fourth order function.

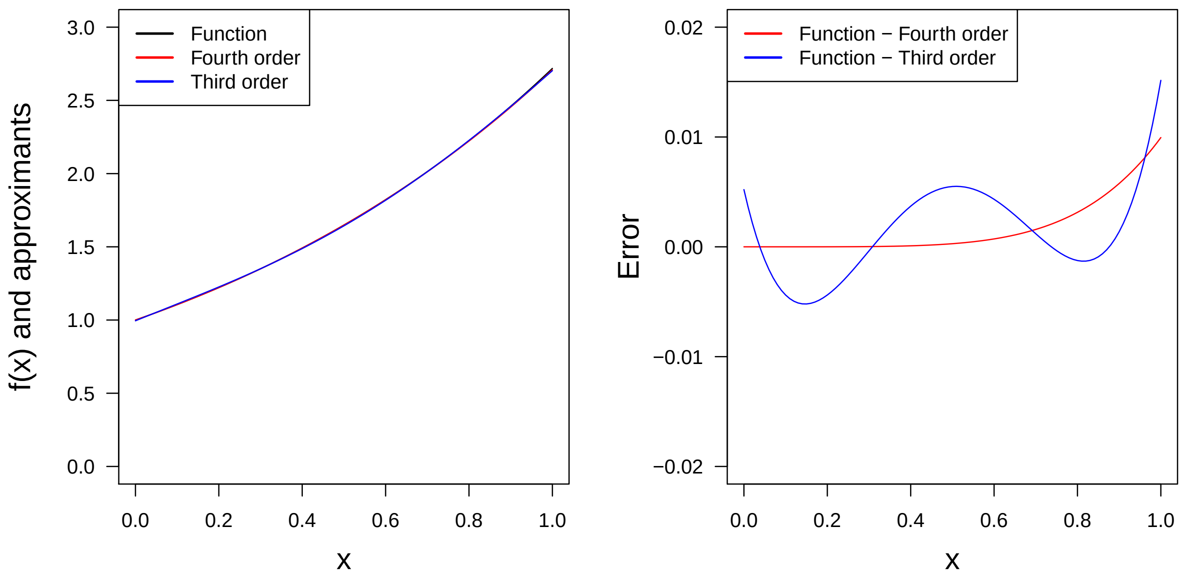


Figure 5: Left: Full- and reduced-order approximations of $f(x) = e^x$ on $[0,1]$ by $P_3(x) = P_4(x) - a_4\tilde{T}_4(x)$. Right: Error in approximations of $f(x) = e^x$ by full- and reduced-order polynomials.