Computing error bounds 8.3.6

$f(x) = xe^x$

$f^{(7)}(x) = 7e^x + xe^x$

$* \ f^{(1)}(x) = e^x + xe^x$

$f^{(2)}(x) = e^x + e^x + xe^x = 2e^x + xe^x$

$f^{(3)}(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$

$$|R_6(x)| = \left| \frac{f^{(7)}(\xi(x))}{7!} x^7 \right|$$

$$= \frac{|f^{(7)}(\xi(x))| \cdot |x^7|}{5040}$$

$$\leq \frac{(7e^1 + 1 \cdot e^1) \cdot 1}{5040}$$

$$|R_6(x)| \leq \frac{8e}{5040} = \frac{e}{630} \approx 0.00431$$

Reducing to 5^{th} order we have:

$$|P_6(x) - P_5(x)| = |a_6 \cdot \tilde{T}_6(x)|$$

$$\leq \frac{1}{120} \cdot \frac{1}{2^{6-1}}$$

$$= \frac{1}{120} \cdot \frac{1}{32} = \frac{1}{3840} \approx 0.000261$$

$$|P_6(x) - P_5(x)| \leq 0.000261$$

The total error bound is $0.00431 + 0.000261 = 0.004571$

Reducing to 4^{th} order we have:

$$|P_5(x) - P_4(x)| = |a_5 \cdot \tilde{T}_5(x)|$$

$$\leq \frac{1}{24} \cdot \frac{1}{2^{5-1}}$$

$$= \frac{1}{24} \cdot \frac{1}{16} = \frac{1}{384} \approx 0.00261$$

The total error bound is $0.004571 + 0.00261 = 0.007181$

Trying once more we have

$$|P_4(x) - P_3(x)| = |a_4 \cdot \widetilde{T}_4(x)|$$

$$\leq \frac{43}{240} \cdot \frac{1}{2^{4-1}} = \frac{43}{1920} \approx 0.0224$$

Reduction to $P_3(x)$ exceeds the error bound.

Reduction to $P_4(x)$ is the best we can do.