Chapter 1

My First Chapter

1.1 Limits

1.1.1 Concept of a limit

The value, if any, that function values approach (in the y-direction) as the arguments approach a particular number (in the x-direction). If, from both sides of a particular argument, the function approaches a particular value, that value is the limit. If, from both sides of a particular argument, the function increases (or decreases) without bound, we say the limit is ∞ (or $-\infty$). If the limits from both sides differ, we say the two-sided limit does not exist (DNE).

- Limits can be calculated algebraically or read from a graph.
- Points on graphs marked by open circles indicate a function is not defined by that y-value. Points on graphs marked by closed or filled circles indicate a function is defined by that y-value.
- The two-sided limit takes its value when the values of the two one-sided limits match. Verify that we would conclude that $\lim_{x\to 0} \frac{1}{x^2} = \infty$, but $\lim_{x\to 0} \frac{1}{x} = \text{D.N.E.}$

1.1.2 Computing limits

Limits involving zero} If, when evaluated by substitution, a limit of a rational function is of the form:

- [form "$\frac{0}{\#}$":] the value of the limit is 0
- [form "$\frac{0}{0}$":] the expression in the limit can be simplified by factoring, rationalization
- [form "$\frac{\#}{0}$":] the value of the limit technically does not exist and must be investigated by one-sided limits and/or a test of signs to determine if the value is $\pm\infty$ or DNE

1.1.3 Limits at infinity

- [end behavior:] power functions (those of the form $f(x) = x^n$ for $n > 0$) approach ∞ as $x \to \infty$. Those with even powers approach ∞ as $x \to -\infty$, while those with odd powers approach $-\infty$ as $x \to -\infty$
- [leading terms:] for limits of rational functions, it is sufficient to take the limit of the ratio of leading terms

1.1.4 Continuity

- [limits:] By definition of continuity at a point $\lim_{x\to a} f(x) = f(a)$, meaning we can easily evaluate limits of continuous functions by direct substitution (i.e., function evaluation)
- [discontinuity:] Functions may have jump discontinuities (piecewise functions), holes or removable discontinuities (common factors of zero on top and bottom), infinite discontinuities (division of a number by zero)

1.2 Derivatives (Differentiation)

1.2.1 Rates of change

Given a function $f(x)$ and two points $(x, f(x))$ and $(x+h, f(x+h))$,

- [avg. r.o.c:] $m_{sec} = \dfrac{f(x+h)-f(x)}{h}$ gives the slope of the secant line connecting these points on the graph of the function and describes the average rate of change of the function over the interval
- [inst. r.o.c:] $m_{tan} = f'(x) = \dfrac{df}{dx} = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ gives the slope of the tangent line touching the graph at a point $(x, f(x))$ by the \textit{limit definition of the derivative} and describes the instantaneous rate of change of the function at the point

Notice that $m_{tan} = \lim_{h\to 0} m_{sec}$.

1.2.2 Basic derivatives

- [power rule:] $\dfrac{d}{dx}(x^n) = nx^{n-1}$ (multiply by the old power, then reduce the power by one)
- [polynomials:] Polynomials can be differentiated term by term by repeated application of the power rule to each term.

1.2.3 Products and Quotients

- [product rule:] $\dfrac{d}{dx}(F(x)S(x)) = F(x)S'(x) + S(x)F'(x)$
- [quotient rule:] $\dfrac{d}{dx}\left(\dfrac{T(x)}{B(x)}\right) = \dfrac{B(x)T'(x) - T(x)B'(x)}{(B(x))^2}$

1.2.4 Trigonometric functions (limits and derivative)

The following two special limits are each of the form $\dfrac{0}{0}$ and show up when applying the limit definition of the derivative to trigonometric functions,

- $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$
- $\lim_{x\to 0} \dfrac{1-\cos(x)}{x} = 0$

Since sines and cosines are continuous, we can move the limit inside the sine or cosine, take the limit of the terms inside, and finally evaluate the sine or cosine,

$$\begin{aligned}\lim_{x\to\infty}\sin\left(\frac{3\pi x^5+\text{lower order terms}}{2x^5+\text{lower order terms}}\right) &= \sin\left(\lim_{x\to\infty}\frac{3\pi x^5+\dots}{2x^5+\dots}\right)\\ &= \sin\left(\lim_{x\to\infty}\frac{3\pi x^5}{2x^5}\right)\\ &= \sin\left(\lim_{x\to\infty}\frac{3\pi}{2}\right)\\ &= \dots\end{aligned}$$

To finish the problem take the limit then evaluate the sine.

- Derivatives of sine and cosine can be calculated by the definition.
- Derivatives of the other trigonometric functions can be calculated by the quotient rule.

1.2.5 Chain Rule, implicit differentiation, and related rates

For derivatives of compositions $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x) = \frac{df}{dg}\cdot\frac{dg}{dx}$

Implicit differentiation, an application of the chain rule, is useful when a relationship is defined by an equation too complicated (or inconvenient) to solve for y.

- We differentiate each side of an equation and solve for $\frac{dy}{dx}$.
- One application of this is to find slopes or equations of lines tangent to complicated curves at points on those curves.

Applications of implicit differentiation and the chain rule, related rates problems are used to relate two rates of change by an underlying formula or geometric relationship. This is done with a 5-step process that includes a picture.

1.3 Integrals (Integration)

1.3.1 Antiderivatives and indefinite integrals

We are interested in determining $\frac{d}{dx}\left(\ ?\ \right) = f(x)$.

- [antiderivatives:] We call the function $F(x)$ an antiderivative, and write $F(x) = \int f(x)\,dx$.
- [indefinite integral:] The collection of all possible antiderivatives is called the indefinite integral $\int f(x)\,dx = F(x) + c$.
- [power rule:] For $n \neq 1$, $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ (add one to the power, then divide by the new power).

1.3.2 u-substitution

This classic technique is used to evaluate more complicated integrals. We choose $u = g(x)$ and compute the differential $du = g'(x)dx$. We substitute these into the original integral, and translate from x and dx to u and du. In substituting,

- we sometimes have to multiply or divide both sides of the definition for du by a constant. We \textit{never} solve completely for dx by dividing by expressions involving x.
- Sometimes, for complicated problems, we are required to solve $u = g(x)$ for x as part of the substitution step.

1.3.3 Riemann sums, summation notation, and definite integrals

- [Riemann sums:] The area under a function (above the axis) can be approximated by collections of rectangles, this idea is used to define the definite integral.
- [algebra of sums:] In making the exact calculation, the manipulation of sums is necessary.
  - $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ (\textit{If the original lower index is not $k = 1$, a change of index must be performed!})
  - $\sum_{k=1}^{n} c = nc$ (\textit{Repeated addition gives rise to multiplication.})
  - $\sum_{k=1}^{n} (f(x) \pm g(x)) = \left(\sum_{k=1}^{n} f(x)\right) \pm \left(\sum_{k=1}^{n} g(x)\right)$ (\textit{The sum of a sum or difference is the sum or difference of the sums.})
- [net signed area:] regions below the axis give rise to negative signed areas, the net signed area for a function over a region indicates whether there is more area above or below the axis.

1.3.4 Definite integrals and the Fundamental Theorem of Calculus

For a continuous $f(x)$ and its antiderivative $F(x)$, $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

1.4 Applications of derivatives

1.4.1 Intervals of increasing, decreasing, and concavity

- [critical point:] a location where the derivative is zero or undefined
  - [stationary point:] a location where the derivative is zero, a special type of critical point

    - [undefined derivatives:] may occur, for example, with rational functions where the denominator is zero. To identify, set the denominator equal to zero and attempt to solve.

- [inflection point:] a location where the second derivative is zero and the function changes concavity
- [increase/decrease:] specified on closed intervals, where possible, this is indicated by the sign of the first derivative (derivative is positive, function is increasing; derivative is negative, function is decreasing)
- [concavity:] specified only on open intervals, this is indicated by the sign of the second derivative (second derivative is positive, function is concave up; second derivative is negative, function is concave down)

1.4.2 Relative and absolute extrema

- Relative extrema occur at critical points and can be identified by the first (changes in function behavior) or second (identification of concavity) derivative test
- Absolute extrema can occur at critical points or at endpoints on specified intervals.

1.4.3 Applied optimization

The function to be optimized describes a cost, profit, or amount of materials. Physical or practical constraints help specify the interval for optimization. The regular optimization procedures are followed.