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Population Ecology Models via Predator-Prey Models: Fundamentals and Applications

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I pledge my honor that I have abided by the Stevens Honor System.

Abstract

Differential equations are commonly used to describe natural phenomena such as a pendulum's motion or an object in free fall. Ordinary differential equations (ODEs) are differential equations only containing ordinary derivatives and are commonly used, making them worthwhile to explore. Given some initial values we can model a particular event such as the speed of a ball of certain mass in free fall. These problems where we are given a differential equation and some initial values and are asked to solve are known as initial value problems (IVPs). ODEs can be placed in a system where each equation relies upon each other. A coupled ODE is a special case of this arbitrarily sized system where only two dependent ODEs are taken into consideration. Since they are reliant upon one another, ODEs that are coupled must be solved simultaneously. Doing this directly is sometimes intractable and therefore approximate solutions must be employed. In this project we aim to model predator prey relationships using coupled ODEs and find their approximate solutions using different algorithms.

Introduction

Predator prey models are meant to represent a symbiotic relationship between two parties where one benefits and the other is negatively impacted. The common example widely used in literature is between foxes and hares where the growth of each population is unaffected by any external factors besides their effect on one another. The number of rabbits depends on the population of foxes, and vice-versa. Since the relationship between the two populations is heavily intertwined, it is evident that their populations can be modeled, given certain growth and relationship parameters, using coupled ODEs.

Predator prey models also have other applications outside of biology and population growth. Outside research in behavioural economics utilizes predator prey models and coupled ODEs to model consumer behavior and how price of a product is affected by the supply and how the supply of a product is affected by the price. One such study determined, using predator prey models, that the price plays a less significant role than availability of a product in a product's revenue[1].

Experiment (Accomplished)

Various approximation techniques for solving IVPs for ODEs were implemented in MATLAB to model predator prey relationships. These techniques were modified slightly to solve IVPs on coupled, non-linear, first-order differential equations, whose form was developed independently by both Alfred Lotka and Vito Volterra to model simple cases of predatory-prey scenarios[2]. These differential equations follow the following form:

$$dx/dt = (a-by)x$$
$$dy/dt = (-e+cx)y$$
$$\text{where } a,b,c,e > 0.$$

In these differential equations, x represents the population of the prey species whereas y represents the population of the predator species. Subsequently, dx/dt and dy/dt represent the rate of change in the population of the prey and predator species, respectively. The variables a, b, c, and e all represent different parameters that define the relationship between the two species. In other words, these parameters help define how the population of one species changes given how many of the other species exist in a certain scenario.

The approximation techniques implemented, from lowest expected accuracy to highest expected accuracy, include Euler's method, Modified Euler's method, Runge Kutta method of Order 4, and the Runge-Kutta-Fehlberg method. The built-in MATLAB function called 'ode45' was also used to compare the results of all of these approximation methods.

Euler's method is the most basic approximation technique for these purposes and is seldom used in practice. Using Taylor's Theorem to derive the formula for Euler's method, one can see that the local truncation error is O(h^2), where h is the step size used in the approximation method. This means that the error of the method per step is

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[1] https://link.springer.com/article/10.1007/s10100-019-00656-7#Sec13
[2] See point 1.

proportional to the square of the step size, h. Modified Euler's method has the same local truncation error as Euler's method, but is a Runge-Kutta method of order two that eliminates the need to compute and evaluate derivatives of the functions in ODEs. Runge-Kutta of order four requires four evaluations per step, compared to just one for Euler's and Modified Euler's, and has a local truncation error of O(h^4). Runge-Kutta of order four should provide more accurate results than Euler's method with one-fourth the step size. Runge-Kutta-Fehlberg has a local truncation error of order five and uses varying step sizes within one approximation so that local truncation error is kept within a specified bound[3].

Adjusting the parameters in the Lotka-Volterra differential equations for these predator-prey models, using the code implementing these different approximation methods, provided different and interesting scenarios for the predator and prey species, one of which resulted in seemingly infinite cyclical growth and decline of the two populations and one of which resulted in a flat-lining and extinction of the prey species followed shortly after by a flat-lining and extinction of the predator species.

Observation

In the first type of scenario discovered, from a high-level perspective, the predator consumes members of the prey species, resulting in a decline of the prey population and a growth in the predator population. As the population of the prey declines, the population of the predator population also declines shortly after because of increased competition over the decreasing and limited resource in the prey species. As the population of the predator species declines, the population of the prey slowly increases as there are not as many members of the predator species to decrease the population. As more of the prey species becomes available for consumption by the predator species, the predator population rebounds, and the cycle continues.

This scenario was observed when the absolute value of the growth rates of both species, which correspond to the a and e parameters in the differential equations, are greater than one. A growth rate greater than one is sufficient for a population to stabilize

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[3] Richard L. Burden, J. Douglas Faires: *Numerical Analysis*, 10th Ed, Cengage Learning, 2014.

itself and for a population to meet a non-vanishing equilibrium between reproduction of the species and resource limitation and competition[4]. Another observation from growth rates is that as they become greater, the cycles observed become more frequent on a fixed time scale.

The other scenario observed saw a decline of the prey species as more and more members of the predator species consumed and decreased the prey population. In this scenario, the prey species was unable to rebound and eventually became extinct, inevitably leading to the extinction of the predator species, since they could not consume members of the prey species.

Compared to the previous cyclical scenario, the growth rate of both species is not enough to stabilize either population. This scenario was observed when the absolute value of the growth rates of both species was less than one. A growth rate less than one is not sufficient for a population to stabilize and sustain itself, causing the prey species to always decline, eventually become extinct, and inevitably cause the predator species to decline shortly after[5]. For the prey species, the closer its growth rate is to zero, the faster the prey species becomes extinct. The growth and decline of the predator species and how fast it occurs depends on the prey species, but keeping the prey parameters constant, when the absolute value of the growth rate of the predator species was closer to one, the population of the predator species declined at a faster rate.

Conclusion

Among the approximation methods implemented, Runge-Kutta-Fehlberg performed best, as its local truncation error was best at O(h^5). Phase plots generated[6] for each scenario using Runge-Kutta-Fehlberg also most closely matched the phase plots of the built-in ode45 method in MATLAB.

Different scenarios are encountered when there are changes in the growth rates of both species, and there is a critical point in the growth rates at which the growth of both species could transition to flat-lining and extinction to cyclical growth and decline.

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[4] https://arxiv.org/pdf/nlin/0605029.pdf
[5] See point 4.
[6] https://github.com/seantrinh/predator_prey_model/tree/master/static