# Outline - Quaternions

Sean Turner

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**Abstract**

William Rowan Hamilton first described quaternions in 1843. Quaternions are used to describe transformation in 3-d space and have many applications in aeronautics, robotics, and computer graphics. This manuscript will provide a brief overview of quaternions and spatial geometries, specifically relating to algebra, geometry, and differential calculus. This will be followed by a comparision between quaternions, euler angles, and rotational matrices, and then a discussion of their applications.

# 1 Introduction

Nobody knows how long vectors have been used in mathematics. Some speculate that the parallelogram method for addition of vectors was lost in a work of Aristotle. However quaternions, often defined as the quotient of two vectors, were not described until 1843 by William Rowan Hamilton. Quaternions are now extensively used in aeronautics and computer graphics for their advantages over tradional transformtion methods.

## 1.1 History of Quaternions

On October 17, 1843, William Rowan Hamilton wrote a letter to his friend John t. Graves, Esq. on the subject of "a very curious train of mathematical speculation." The letter details his "theory of quaternions", and follows his mathematical reasoning behind the development of his "quaternions." The day before, while walking across the Royal Canal in Dublin, Hamilton had the idea for the formula of quaternions as shown below in Equation 1.

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

This equations and its implications will be investigated in Section 2. The letter was only a few pages long, but it quite thorough in its' scope, and was eventually published in the *London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* the following year in 1844. Shortly after Hamilton's death in 1865, his son edited and published the longest of Hamilton's books, at over 800 pages. Titled *Elements of Quaternions*, this book was the go-to book on quaternions for several decades. The wake of Hamilton's exploration into quaternions led to research associations like the Quaternion Society who described themselves as an "International Association for Promoting the Study of Quaternions."

### 1.2 Basic Geometric Transformations

The primary topic of this manuscript is the application of quaternions to transformations in 3D space, so some terms will be defined here. There are two methods of transformation: translation and rotation [1].

**Definition 1.2.1** (Translation)**.** A *translation* is a point in space moved from one position to another. Let a point $P \in \mathbb{R}^3$ be denoted as $(x, y, z), x, y, x \in \mathbb{R}$ and the translation by a vector $(\Delta x, \Delta y, \Delta z)$. Then the new position $P'$ is $(x + \Delta x, y + \Delta y, z + \Delta z)$. There is only one translation vector that takes $P$ to $P'$.

A *rotation* can be defined in multiple ways. The following definition is given by Euler, and will be used here.

**Definition 1.2.2** (Euler's Definition of Rotation)**.** Let $O, O' \in \mathbb{R}^3$ be two orientations. Then there exists an axis $l \in \mathbb{R}^3$ and an angle of rotation $\Theta \in [-\pi, \pi]$ such that $O$ yields $O'$ when rotated $\Theta$ about $l$.

It is important to distinguish between *orientation* and *rotation*. Orientation here is the normal vector to an object in 3D space. A rotation comprised of an axis and angle of rotation. Unlike the translation between two points, the rotation between two orientations in 3D space is not unique.

Section 2 will define and discuss quaternions in detail. Section 3 will compare quaternions and alternate methods of expressing rotations, while Section 4 will discuss real world applications. Section 5 will conclude the report.

## 2 Discussion

The quaternion equation was briefly introduced in Equation 1. The following definition is a rigorous definion that follows from that equation.

**Definition 2.0.1** (Quaternion)**.** A *quaternion* is a number of the form

$$a + b\mathbf{i} + c\mathbf{j} + c\mathbf{k}$$

where a, b, c, and d are real numbers, **i**, **j**, **k**, are square roots of -1, and **ijk**= -1.

The following section will describe quaternions in more depth, specifically related to algebra, geometry, and differential calculus.

## 2.1 Algebra & Quaternions

The addition and subtraction of quaternions is the same as 4D vector addition. That is, adding quaternions is simply separately adding the coefficients of **i**, **j**, **k**. For example,

$$(1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + (-2 + 3\mathbf{i} - 1\mathbf{j} + 4\mathbf{k}) = -1 + 5\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$$

Multiplication is a little more involved. Clearly, since **i**, **j**, and **k**are square roots of -1, it is true that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$. However, it is not so clear what, for example, $\mathbf{ij}$ is. We know that $\mathbf{ij} \neq -1$ because $\mathbf{ijk} = -1$, and $\mathbf{k} \neq 1$. To find $\mathbf{ij}$ we must use Equation 1 as shown:

$$\mathbf{ij} = -\mathbf{ij}(-1) = -\mathbf{ijk}^2 = -(\mathbf{ijk})\mathbf{k} = -(-1)\mathbf{k} = \mathbf{k}$$

It is important to note that quaternion multiplication is not commutative, since $\mathbf{ij} = \mathbf{k} \neq \mathbf{ji} = -\mathbf{k}$. A full table of the quaternion relationships between **i**, **j**, and **k** are shown below in Table 1:

Table 1: Quaternion Characteristics

| | **i** | **j** | **k** |
|---|---|---|---|
| **i** | **-1** | **k** | **-j** |
| **j** | **-k** | **-1** | **i** |
| **k** | **-j** | **i** | **-1** |

Quaternion multiplication is not commutative as shown above, but other algebraic properties are satisfied as shown in Theorem 2.1.1

**Theorem 2.1.1.** *Properties of Quaternion Multiplication*

1. Associativity: $\mathbf{q}(\mathbf{rs}) = (\mathbf{qr})\mathbf{s}$
2. Distributivity: $\mathbf{q}(\mathbf{r} + \mathbf{s}) = \mathbf{qr} + \mathbf{qs}$
3. Inverses: $\forall$ quaternions $\mathbf{q} \neq 0$, $\exists$ a quaternion $\mathbf{r}$ s.t. $\mathbf{qr} = 1$
4. Cancellation: If $\mathbf{qr} = \mathbf{qs}$, then $\mathbf{r} = \mathbf{s}$

When quaternions are written in the form $\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, they are said to be in *Cartesian form*, similar to the method of displaying a complex number in the form $a + bi$. Just as we can separate a complex number into real and imaginary parts, so we can split a quaternion $\mathbf{q}$ into a *scalar* part $S\mathbf{q} = a$ and a *vector* part $V\mathbf{q} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. We can also define the *conjugate* of a quaternion as

$$\mathbf{q}^* = S\mathbf{q} - V\mathbf{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k},$$

and the *absolute value* of $\mathbf{q}$ as

$$|\mathbf{q}| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

### 2.2 Geometry & Quaternions

### 2.3 Differential Calculus & Quaternions

## 3 Comparison

### 3.1 Other Non-Euclidean Transformation Methods

### 3.2 Comparisions Between Methods

## 4 Applications

### 4.1 Aeronautics & Quaternions

### 4.2 Computer Graphics & Quaternions

## 5 Conclusion

## References


[1] Erik B. Dam et al, *Quaternions, interpolation, and animation*, 1998, http://web.mit.edu/2.998/www/QuaternionReport1.pdf.

[2] William Hamilton, *Elements of quaternions*, London, Longmans, Green & co., 1866.

[3] Michael Henle, *Modern geometries*, Prentice-Hall, Upper Saddle River, New Jersey, 2001.

[4] Lieut.-Colonel H. W. L. Hime, *Outlines of quaternions*, London, Longmans, Green & co., 1894.

[5] Charles Jasper Joly, *A manual of quaternions*, MacMillan and Co., Limited, New York, NY, 1905.