Rotations in 3D with Quaternions

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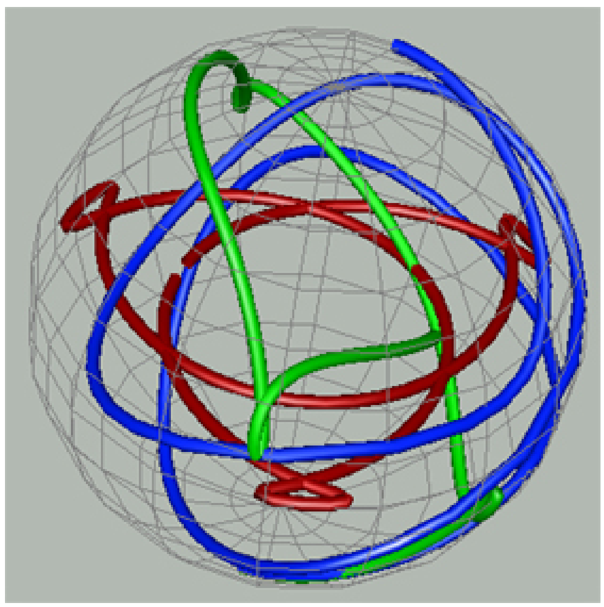

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Abstract

William Rowan Hamilton first described quaternions in 1843. Quaternions are used to describe transformation in 3-d space and have many applications in aeronautics, robotics, and computer graphics. This manuscript will provide a brief overview of quaternions and spatial geometries, specifically relating to algebra, geometry, and differential calculus. This will be followed by a comparision between quaternions, euler angles, and rotational matrices, and then a discussion of their applications.

1 Introduction

Nobody knows how long vectors have been used in mathematics. Some speculate that the parallelogram method for addition of vectors was lost in a work of Aristotle. However quaternions, often defined as the quotient of two vectors, were not described until 1843 by William Rowan Hamilton. Quaternions are now extensively used in aeronautics and computer graphics for their advantages over tradional transformtion methods.

1.1 History of Quaternions

On October 17, 1843, William Rowan Hamilton wrote a letter to his friend John t. Graves, Esq. on the subject of "a very curious train of mathematical speculation." The letter details his "theory of quaternions", and follows his mathematical reasoning behind the development of his "quaternions." The day before, while walking across the Royal Canal in Dublin, Hamilton had the idea for the formula of quaternions as shown below in Equation 1.

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

This equations and its implications will be investigated in Section 2. The letter was only a few pages long, but it quite thorough in its' scope, and was eventually published in the *London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* the following year in 1844. Shortly after Hamilton's death in 1865, his son edited and published the longest of Hamilton's books, at over 800 pages. Titled *Elements of Quaternions*, this book was the go-to book on quaternions for several decades. The wake of Hamilton's exploration into quaternions led to research associations like the Quaternion Society who described themselves as an "International Association for Promoting the Study of Quaternions."

1.2 Basic Geometric Transformations

The primary topic of this manuscript is the application of quaternions to rotations in 3D space, so some terms will be defined here. There are two methods of transformation: translation and rotation [1].

Definition 1.2.1 (Translation)**.** A *translation* is a point in space moved from one position to another. Let a point $P \in \mathbb{R}^3$ be denoted as $(x, y, z), x, y, x \in \mathbb{R}$ and the translation by a vector $(\Delta x, \Delta y, \Delta z)$. Then the new position P' is $(x + \Delta x, y + \Delta y, z + \Delta z)$. There is only one translation vector that takes P to P'.

A *rotation* can be defined in multiple ways. The following definition is given by Euler, and will be used here.

Definition 1.2.2 (Euler's Definition of Rotation)**.** Let $O, O' \in \mathbb{R}^3$ be two orientations. Then there exists an axis $l \in \mathbb{R}^3$ and an angle of rotation $\Theta \in [-\pi, \pi]$ such that O yields O' when rotated Θ about l.

It is important to distinguish between *orientation* and *rotation*. Orientation here is the normal vector to an object in 3D space. A rotation comprised of an axis and angle of rotation. Unlike the translation between two points, the rotation between two orientations in 3D space is not unique.

Section 2 will define and discuss quaternions in detail. Section 3 will compare quaternions and alternate methods of expressing rotations, while Section 4 will discuss real world applications. Section 5 will conclude the report.

2 Discussion

The quaternion equation was briefly introduced in Equation 1. The following definition is a rigorous definion that follows from that equation.

Definition 2.0.1 (Quaternion)**.** A *quaternion* is a number of the form

$$a + b\mathbf{i} + c\mathbf{j} + c\mathbf{k}$$

where a, b, c, and d are real numbers, **i**, **j**, **k**, are square roots of -1, and **ijk**= -1.

The following section will describe quaternions in more depth, specifically related to algebra, geometry, and differential calculus.

2.1 Algebra & Quaternions

The addition and subtraction of quaternions is the same as 4D vector addition. That is, adding quaternions is simply separately adding the coefficients of **i**, **j**, **k**. For example,

$$(1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + (-2 + 3\mathbf{i} - 1\mathbf{j} + 4\mathbf{k}) = -1 + 5\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$$

Multiplication is a little more involved. Clearly, since $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are square roots of -1, it is true that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$. However, it is not so clear what, for example, $\mathbf{ij}$ is. We know that $\mathbf{ij} \neq -1$ because $\mathbf{ijk} = -1$, and $\mathbf{k} \neq 1$. To find $\mathbf{ij}$ we must use Equation 1 as shown:

$$\mathbf{ij} = -\mathbf{ij}(-1) = -\mathbf{ijk}^2 = -(\mathbf{ijk})\mathbf{k} = -(-1)\mathbf{k} = \mathbf{k}$$

It is important to note that quaternion multiplication is not commutative, since $\mathbf{ij} = \mathbf{k} \neq \mathbf{ji} = -\mathbf{k}$. A full table of the quaternion relationships between $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are shown below in Table 1:

Table 1: Quaternion Characteristics

	i	**j**	**k**
i	**-1**	k	-j
j	-k	**-1**	i
k	-j	i	**-1**

Quaternion multiplication is not commutative as shown above, but other algebraic properties are satisfied as shown in Theorem 2.1.1

Theorem 2.1.1. *Properties of Quaternion Multiplication*

1. Associativity: $\mathbf{q}(\mathbf{rs}) = (\mathbf{qr})\mathbf{s}$
2. Distributivity: $\mathbf{q}(\mathbf{r} + \mathbf{s}) = \mathbf{qr} + \mathbf{qs}$
3. Inverses: $\forall$ quaternions $\mathbf{q} \neq 0$, $\exists$ a quaternion $\mathbf{r}$ s.t. $\mathbf{qr} = 1$
4. Cancellation: If $\mathbf{qr} = \mathbf{qs}$, then $\mathbf{r} = \mathbf{s}$

When quaternions are written in the form $\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, they are said to be in *Cartesian form*, similar to the method of displaying a complex number in the form $a + bi$. Just as we can separate a complex number into real and imaginary parts, so we can split a quaternion $\mathbf{q}$ into a *scalar* part $S\mathbf{q} = a$ and a *vector* part $V\mathbf{q} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. We can also define the *conjugate* of a quaternion as

$$\mathbf{q}^* = S\mathbf{q} - V\mathbf{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k},$$

and the *absolute value* of $\mathbf{q}$ as

$$|\mathbf{q}| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Quaternions also appear in abstract algebra as a non-abelian group, which means that $a * b \neq b * a, \forall a, b \in Q$. Figure 1 shows the cycle diagram for the quaternions.

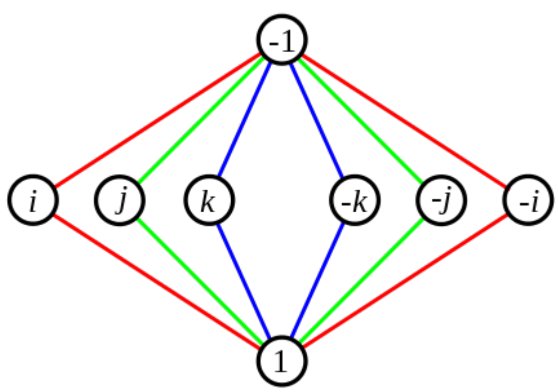

Figure 1: Quaternions Cycle Diagram

Despite being a non-abelian group, every subgroup of the group is normal subgroup. This property is known as being *Hamiltonian*.

2.2 Geometry & Quaternions

As discussed in Section 1.2, the primary methods of transformations that will be dicussed are translation and rotation. When using quaternions for geometry, usually the *Pure Quaternion* is used. A pure quaternion is simply a quaternion in the form of $\mathbf{q} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ because it can be used to represent a point in 3D. We define a *dot product* of two quaternions $\mathbf{q}$and $\mathbf{r}$ by the following equation.

$$-S(\mathbf{qr}) = x_1x_2 + y_1y_2 + z_1z_2 \tag{2}$$

This is analagous to the traditional dot product $\mathbf{q} \cdot \mathbf{r} = |\mathbf{q}||\mathbf{r}|cos(\phi)$, where ϕ is the angle between the vectors $\mathbf{q}$and $\mathbf{r}$in 3D. Similarly, we can define a cross product bewteen $\mathbf{q}$and $\mathbf{r}$as the following:

$$V(\mathbf{qr}) = (y_1z_2 - y_2z_1)\mathbf{i} - (x_1z_2 - x_2z_1)\mathbf{j} - (x_1y_2 - x_2y_1)\mathbf{k} \tag{3}$$

Again, this is the same as the geometric $\mathbf{q} \times \mathbf{r} = |\mathbf{q}||\mathbf{r}|sin(\phi)\mathbf{u}$, where ϕ is the same as before, and $\mathbf{u}$ is a pure unit quaternion and represents a unit vector perpedicular to the plane formed by $\mathbf{q}$ and $\mathbf{r}$. Figure 2 shows the geometric representation of the cross product.

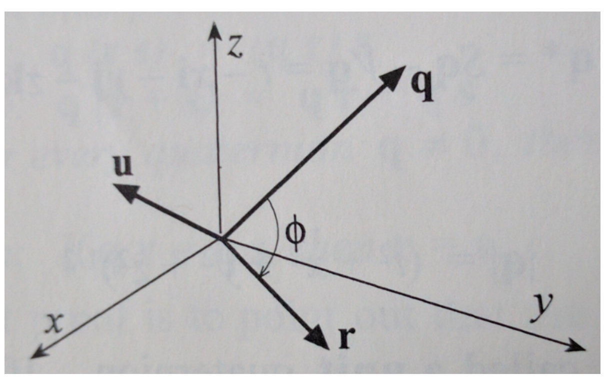

Figure 2: Visualizing $\mathbf{q} \times \mathbf{r}$

By combining the scalar result of the dot product and the vector result of the cross product, we can construct quaternion multiplication:

$$\mathbf{qr} = -(\mathbf{q} \cdot \mathbf{r}) + (\mathbf{q} \times \mathbf{r}) \tag{4}$$

This give us a geometric way of visualizing quaternion multiplication.

There is one more prerequisite before we can do rotations with quaternions. Just like complex numbers can be represented in a polar form $z = |z|(cos(\theta) + isin(\theta))$, we can express quaternions in a similar form of $\mathbf{q} = |\mathbf{q}|(cos(\theta) + \mathbf{u}sin(\theta))$. Here, instead of the imaginary number i, we use the pure unit quaternion $\mathbf{u}$. Therefore, we have the following theorem:

Theorem 2.2.1. Every quaternion can be represented in the form:

$$\mathbf{q} = |\mathbf{q}|(cos(\theta) + \mathbf{u}sin(\theta))$$

With the above theorems and definitions, we are now able to represent rotation in 3D space with quaternions.

Theorem 2.2.2. Let $\mathbf{r}$ be a unit quaternion. Let R be the transformation on the pure quaternion $\mathbf{q}$ defined by

$$\mathrm{R}\mathbf{q} = \mathbf{rqr}^*.$$

Then R is a rotation of the three dimensional space of pure quaternions about an axis passing through the origin. If we take $\mathbf{r}$ in the polar form

$$\mathbf{r} = cos(\theta) + \mathbf{u}sin(\theta),$$

where $\mathbf{u}$ is a unit quaternion, then R$\mathbf{q}$ is the pure quaternion obtained by rotating $\mathbf{q}$ about $\mathbf{u}$ by the angle 2θ. Every rotation about an axis passing through the origin can be expressed this way.

Proof: Let us look at R$\mathbf{q}$ for three cases, $\mathbf{q} = \mathbf{u}$, $\mathbf{q}$ is perpendicular to $\mathbf{u}$, and $\mathbf{q}$ is any pure quaternion.

Case 1: $\mathbf{q} = \mathbf{u}$

Then R$\mathbf{u}$ $= \mathbf{rur}^* = (cos(\theta) + \mathbf{u}sin(\theta))\mathbf{u}(cos(\theta) - \mathbf{u}sin(\theta))$. Expanding this yields

$$\mathrm{R}\mathbf{u} = cos(\theta)^2\mathbf{u} - sin(\theta)^2\mathbf{u}^3 = cos(\theta)^2\mathbf{u} - sin(\theta)^2(-\mathbf{u}) = \mathbf{u}.$$

Thus $\mathbf{u}$ is a fixed point of R. This is expected because R$\mathbf{u}$ is the rotation of $\mathbf{u}$ about the axis $\mathbf{u}$.

Case 2: $\mathbf{q}$ is perpendicular to $\mathbf{u}$.

Then R$\mathbf{u}$ $= \mathbf{rqr}^* = (cos(\theta) + \mathbf{u}sin(\theta))\mathbf{q}(cos(\theta) - \mathbf{u}sin(\theta))$

$= cos(\theta)^2\mathbf{q} + \mathbf{uq}cos(\theta)sin(\theta) - \mathbf{qu}cos(\theta)sin(\theta) - \mathbf{uqu}sin(\theta)^2$. Using Equation 4, we get

$$\mathbf{uq} = -(\mathbf{u} \cdot \mathbf{q}) + (\mathbf{u} \times \mathbf{q}) = (\mathbf{u} \times \mathbf{q}),$$

because $\mathbf{u}$ and $\mathbf{q}$ are perpendicular. Similarly,

$$\mathbf{qu} = (\mathbf{q} \times \mathbf{u}) = -(\mathbf{u} \times \mathbf{q})$$

so

$$\mathbf{uqu} = (\mathbf{u} \cdot (\mathbf{u} \times \mathbf{q})) - (\mathbf{u} \times (\mathbf{u} \times \mathbf{q})) = -(\mathbf{u} \times (\mathbf{u} \times \mathbf{q})).$$

Therefore $\mathbf{uqu} = \mathbf{q}$. When we put this all together we get

$$\text{R}\mathbf{q} = cos(\theta)^2\mathbf{q} + 2cos(\theta)sin(\theta)(\mathbf{u} \times \mathbf{q}) - sin(\theta)^2\mathbf{q} = cos(2\theta)\mathbf{q} + sin(2\theta)(\mathbf{u} \times \mathbf{q}).$$

Therefore, R$\mathbf{q}$ is the rotation of $\mathbf{q}$ through an angle of 2θ about the axis of $\mathbf{u}$.

Case 3: $\mathbf{q}$ is any pure quaternion. First note that R is a linear transformation, that is, $R(\mathbf{q} + \mathbf{r}) = R(\mathbf{q}) + R(\mathbf{r})$ and $R(\alpha\mathbf{q}) = \alpha R(\mathbf{q})$. Secondly notice that the rotation of 3D space is also a linear transformation. To prove that two linear transformations are equal, it is enough to prove that they have the same effect on the vectors from some basis of 3D space. We have shown that the rotation R and the rotation about the axis $\mathbf{u}$ by the angle 2θ are equal when $\mathbf{u} = \mathbf{q}$ and $\mathbf{u}$ is perpendicular to $\mathbf{u}$. However, a basis for 3D space can be found consisting of the vector $\mathbf{u}$ plus two orthogonal vectors. Therefore, R acts like a rotation for all 3D vectors. $\square$

3 Comparison

This section will investigate alternative methods for rotating vectors in 3D, and compare their advantages and disadvantages to quaternions.

3.1 Other Non-Euclidean Transformation Methods

There are two alternate primary methods for expressing rotations of vectors in 3D: Euler angles and rotation matrices. Euler angles, originally developed by Euler as a tool for solving differential equations, have become a widely used method for expressing rotation. Euler angles $(\theta_x, \theta_y, \theta_z)$ are the rotation angles about the x,y, and z axes respectively. There are a few problems with Euler angles that mostly result from ambiguity. The main problem is that there are multiple methods of using Euler angles to achieve the same rotation. Take for example, a rotation about the z-axis by 180°. Clearly, the simplest way to express this is with the Euler angles ($\theta_x = 0°, \theta_y = 0°, \theta_z = 180°$). However, another equally valid solution is ($\theta_x = 180°, \theta_y = 180°, \theta_z = 0°$). It is important to note that the *order* that the Euler angles are applied makes a difference. In the previous case it happened to not make a differnce, but in general, the angles are not commutative. There are various conventions in place to help with the ambiguity. In this paper, we will use the order of x,y, and then z.

The other primary method for expressing rotation in 3D is a rotation matrix. A rotation matrix is usually method for applying Euler angles. Each rotation about the x,y, and z axes are used to generate a 3x3 rotation matrix. These are then multiplied together to get a single rotation matrix. Sometimes a 4x4 homogenous matrix is used because it can hold translation and rotation in the same matrix. We know that matrix multiplication does not commute, which is also true of rotations. The following rotation matrices are used to describe rotation about the given axis:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}, R_y(\theta) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix}, R_z(\theta) = \begin{bmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So a rotation about the z-axis by 180° would be represented by multiplying the three individual rotation matrices together.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that this is the same final rotation matrix as if we had instead rotated the x-axis by 180° and then the y-axis by 180°. This can be verified as shown:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} * \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.2 Comparison Between Methods

Euler angles and rotation matrices have historically been used to represent rotation in 3D, and the mathematics behind them are well known. Euler angles and rotation matrices make rotations about the x,y, or z axes simple. However an arbitrary rotation about an arbitrary axis makes it difficult to derive the Euler angles for. Another disadvantage, as briefly discussed above, is that the order that the rotation matrices are applied in matter. Different conventions lead to different results so care must be taken when coding and using libraries. Another disadvantage is gimbal lock, which will be investiagated in more depth in the Applications section. In addition, given a rotation matrix, it can be difficult to find an inverse. Lastly, the homogenous matrix holds extra information. When coding applications, extra information is a waste of program space.

The biggest advantage to Euler angles and the corresponding rotation matrices, other than for historical reasons, is the ability to encode translation, rotation, scaling, and projection into one matrix. This can be useful

4 Applications

4.1 Aeronautics & Quaternions

4.2 Computer Graphics & Quaternions

5 Conclusion

References

[1] Erik B. Dam et al, *Quaternions, interpolation, and animation*, 1998, http://web.mit.edu/2.998/www/QuaternionReport1.pdf.