Homework 8 Solutions

Math 198: Math for Machine Learning

Due Date:
Name:
Student ID:

Instructions for Submission

Please include your name and student ID at the top of your homework submission. You may submit handwritten solutions or typed ones (LaTeX preferred). If you at any point write code to help you solve a problem, please include your code at the end of the homework assignment, and mark which code goes with which problem. Homework is due by start of lecture on the due date; it may be submitted in-person at lecture or by emailing a PDF to both facilitators.

1 Examples of Convex Functions

1. Give an example of a convex function.
   One such example is $f(x) = |x|$. Proofs of this and other examples are omitted.

2. Give an example of a non-convex function.
   $f(x) = x$

3. Give an example of a function which is strictly convex.
   Again, $f(x) = |x|$ is such a function.

4. Give an example of a function which is convex, but not strictly convex.
   $f(x) = 0$

5. Give an example of a function which is strictly convex, but not strongly convex.
   $f(x) = x^4$

6. Give an example of a function which is 2-strongly convex.
   $f(\mathbf{x}) = ||\mathbf{x}||_2^2$

7. Give an example of a function which is convex but has no minima.
   $f(x) = e^x$

2 Feasible Sets

For the following problems, we will consider how the feasible set of solutions $\mathcal{X}$ changes the feasibility of optimization. We will be attempting to optimize the strictly convex function $f(x) = x^2$.

1. Is there a unique global minimum of f if $\mathcal{X} = \mathbb{R}$? If so, what is it? Is this set convex?
   $x = 0$. The feasible set is convex.

2. What if $\mathcal{X} = \{1\}$? Is this set convex?
   $x = 1$. The feasible set is convex.

3. What if $\mathcal{X} = \mathbb{R} \backslash \{0\}$? Is this set convex?
   In this case, there is no unique global minimum. For any point $x \in \mathcal{X}$, there is another point $y \in \mathcal{X}$ such that $f(y) < f(x)$. The feasible set is non-convex.

4. Let $\mathcal{X} = (-\infty, -1] \cup [0, \infty)$. Is this set convex? What are the local minima of f in this set? Are all local minima also global minima?
   This set is not convex. $x = -1$ and $x = 0$ are both local minima, but only $x = 0$ is a global minimum.

5. Let $\mathcal{X} = (-\infty, -1] \cup [1, \infty)$. Is this set convex? What are the local minima of f in this set? Are all local minima also global minima?
   This set is not convex. $x = -1$ and $x = 1$ are both local and global minima.

3 Convexity Proofs

1. Let f and g be convex functions. Show that $h(\mathbf{x}) = \max\{f(\mathbf{x}), g(\mathbf{x})\}$ is convex.
   Noting that $\max\{a + b, c + d\} \le \max\{a, c\} + \max\{b, d\}$,

$$\begin{aligned}
h(t\mathbf{x} + (1-t)\mathbf{y}) &= \max\{f(t\mathbf{x} + (1-t)\mathbf{y}), g(t\mathbf{x} + (1-t)\mathbf{y})\} \\
&\le \max\{tf(\mathbf{x}) + (1-t)f(\mathbf{y}), tg(\mathbf{x}) + (1-t)g(\mathbf{y})\} \\
&\le \max\{tf(\mathbf{x}), tg(\mathbf{x})\} + \max\{(1-t)f(\mathbf{y}), (1-t)g(\mathbf{y})\} \\
&= t\max\{f(\mathbf{x}), g(\mathbf{x})\} + (1-t)\max\{f(\mathbf{y}), g(\mathbf{y})\} \\
&= th(\mathbf{x}) + (1-t)h(\mathbf{y})
\end{aligned}$$

2. Let f be a convex function and g be a strictly convex function. Show that $f + g$ is strictly convex.

$$\begin{aligned}
(f+g)(t\mathbf{x} + (1-t)\mathbf{y}) &= f(t\mathbf{x} + (1-t)\mathbf{y}) + g(t\mathbf{x} + (1-t)\mathbf{y}) \\
&< tf(\mathbf{x}) + (1-t)f(\mathbf{y}) + tg(\mathbf{x}) + (1-t)g(\mathbf{y}) \\
&= t(f(\mathbf{x}) + g(\mathbf{x})) + (1-t)(f(\mathbf{y}) + g(\mathbf{y})) \\
&= t(f+g)(\mathbf{x}) + (1-t)(f+g)(\mathbf{y})
\end{aligned}$$

3. Let f be a convex function and g be an m-strongly convex function. Show that $f + g$ is m-strongly convex.
   Let $h(\mathbf{x}) = g(\mathbf{x}) - \frac{m}{2}||\mathbf{x}||_2^2$. Because g is m-strongly convex, h is convex. Therefore $f + h$ is convex, and so $f + g$ is m-strongly convex.