Homework 10 Solutions

Math 198: Math for Machine Learning

Due Date:
Name:
Student ID:

Instructions for Submission

Please include your name and student ID at the top of your homework submission. You may submit handwritten solutions or typed ones (LaTeX preferred). If you at any point write code to help you solve a problem, please include your code at the end of the homework assignment, and mark which code goes with which problem. Homework is due by start of lecture on the due date; it may be submitted in-person at lecture or by emailing a PDF to both facilitators.

1 Probability Practice

1. Let X be a random variable representing the value of a single roll of an n-sided die.
    (a) Define the cumulative distribution function $F(x)$ in terms of n.
    $F(x) = \frac{x}{n}$
    (b) Define the probability distribution function $p(x)$ in terms of n.
    $p(x) = \frac{1}{n}$
    (c) Give the expected value of X in terms of n.
    $\mathbb{E}[X] = \frac{n+1}{2}$
    (d) Give the variance of X in terms of n.
    $\text{Var}(X) = \frac{n^2-1}{12}$
2. Let X be a random variable representing a value uniformly sampled (over the reals) from the range $[a, b]$.
    (a) Define the cumulative distribution function $F(x)$ in terms of a and b.
    $F(x) = \frac{x-a}{b-a}$
    (b) Define the probability distribution function $p(x)$ in terms of a and b.
    $p(x) = \frac{1}{b-a}$
    (c) Give the expected value of X in terms of a and b.
    $\mathbb{E}[X] = \frac{a+b}{2}$
    (d) Give the variance of X in terms of a and b.
    $\text{Var}(X) = \frac{(b-a)^2}{12}$
3. Let Ω be the non-zero integers $\mathbb{Z} \backslash 0$ and $\mathbb{P}(\{\omega\}) = \frac{1}{2^\omega}$. Suppose $X(\omega) = \omega$.
    (a) Define the cumulative distribution function $F(x)$.
    $F(x) = 1 - \frac{1}{2^x}$
    (b) Define the probability distribution function $p(x)$.
    $p(x) = \frac{1}{2^x}$
    (c) Give the expected value of X.
    $\mathbb{E}[X] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{2^i} = 2$

2 Probability Proofs

1. Let A be a generic event. Show that $0 = \mathbb{P}(\varnothing) \leq \mathbb{P}(A) \leq \mathbb{P}(\Omega) = 1$.
   $\varnothing \subseteq A \subseteq \Omega$, so this follows from the fact that if B is an event and $B \subseteq A$, $\mathbb{P}(B) \leq \mathbb{P}(A)$.

2. Prove the union bound, that is, for any countable set of events $\{A_i\} \subseteq \mathcal{F}$, $\mathbb{P}(\bigcup_i A_i) \leq \sum_i \mathbb{P}(A_i)$.
   Define $B_1 = A_1$ and $B_i = A_i \setminus \bigcup_{j<i} A_j$. Note that the B_is are all disjoint, $\bigcup_{j<i} B_i = \bigcup_{j<i} A_i$, and $B_i \subseteq A_i$ for all i. Then

$$\mathbb{P}(\bigcup_i A_i) = \mathbb{P}(\bigcup_i B_i) = \sum_i \mathbb{P}(B_i) \leq \sum_i \mathbb{P}(A_i)$$

3. Show that for any random variable X, $\text{Var}(\alpha X + \beta) = \alpha^2 \text{Var}(X)$.

$$\begin{aligned}
\text{Var}(\alpha X + \beta) &= \mathbb{E}[(\alpha X + \beta)^2] - \mathbb{E}[\alpha X + \beta]^2 \\
&= \mathbb{E}[\alpha^2 X^2 + 2\alpha\beta X + \beta^2] - (\alpha \mathbb{E}[X] + \beta)^2 \\
&= \alpha^2 \mathbb{E}[X^2] + 2\alpha\beta\mathbb{E}[X] + \beta^2 - (\alpha^2 \mathbb{E}[X]^2 + 2\alpha\beta\mathbb{E}[X] + \beta^2) \\
&= \alpha^2 (\mathbb{E}[X^2] - \mathbb{E}[X]^2) \\
&= \alpha^2 \text{Var}(X)
\end{aligned}$$