Homework 3

Math 198: Math for Machine Learning

Due Date:
Name:
Student ID:

1 Practice with Determinant and Trace

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map that sends $(1, 0) \mapsto (2, 0)$ and $(0, 1) \mapsto (3, 4)$. What is $\det(T)$?
   The matrix representation of T with respect to the standard basis is
   $$\mathbf{T} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$$
   Therefore, $\det(T) = \det(\mathbf{T}) = 2 * 4 - 3 * 0 = 8$.

2. What is $\mathrm{tr}(T)$?
   $\mathrm{tr}(T) = \mathrm{tr}(\mathbf{T}) = 2 + 4 = 6$.

3. Let U be a proper subspace of a vector space V. Let $\mathbf{P}$ be a map onto U. What is $\det(\mathbf{P})$?
   Because U is a proper subspace of V, it has lower dimension than V. Therefore, $\mathbf{P}$ is not invertible, so its determinant is 0.

2 Proofs about Determinant and Trace

Let $\mathbf{A}$ be an arbitrary square matrix.

1. Prove that, if $\mathbf{A}$ is invertible, then $\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$.
   Recall that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$. We have $1 = \det(\mathbf{I}) = \det(\mathbf{AA}^{-1}) = \det(\mathbf{A})\det(\mathbf{A}^{-1})$. Therefore, $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.

2. Conclude that if $\det(\mathbf{A}) = 0$, $\mathbf{A}$ is not invertible.
   Suppose toward a contradiction that $\mathbf{A}$ is invertible and $\det(\mathbf{A}) = 0$. Then $\det(\mathbf{A}^{-1}) = \frac{1}{0}$, so the determinant for $\mathbf{A}^{-1}$ is undefined. But the determinant is defined for all matrices. So, if $\det(\mathbf{A}) = 0$, then $\mathbf{A}$ cannot be invertible.

3. Let $\mathbf{B}$ be an invertible matrix. Prove $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{BAB}^{-1})$.
   Since trace is invariant under cyclic permutations, $\mathrm{tr}(\mathbf{BAB}^{-1}) = \mathrm{tr}(\mathbf{B}^{-1}\mathbf{BA}) = \mathrm{tr}(\mathbf{A})$.

3 Computing Eigenvalues and Eigenvectors

Let
$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -1 \\ \frac{3}{2} & \frac{7}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{5}{2} \end{bmatrix}$$

1. Find $p_{\mathbf{A}}(\lambda)$, the characteristic polynomial of $\mathbf{A}$.

$$\begin{aligned}
p_{\mathbf{A}}(\lambda) &= \det(\mathbf{A} - \lambda\mathbf{I}) \\
&= \det\begin{bmatrix} 4-\lambda & 1 & -1 \\ \frac{3}{2} & \frac{7}{2}-\lambda & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{5}{2}-\lambda \end{bmatrix} \\
&= (4-\lambda)\det\begin{bmatrix} \frac{7}{2}-\lambda & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2}-\lambda \end{bmatrix} - \det\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{5}{2}-\lambda \end{bmatrix} - \det\begin{bmatrix} \frac{3}{2} & \frac{7}{2}-\lambda \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\
&= (4-\lambda)((\frac{7}{2}-\lambda)(\frac{5}{2}-\lambda) - \frac{3}{4}) - (\frac{3}{2}(\frac{5}{2}-\lambda) + \frac{3}{4}) + (\frac{3}{4} + \frac{1}{2}(\frac{7}{2}-\lambda)) \\
&= (4-\lambda)(\lambda^2 - 6\lambda + 8) - (\frac{9}{2} - \frac{3\lambda}{2}) + (\frac{5}{2} - \frac{\lambda}{2}) \\
&= -\lambda^3 + 10\lambda^2 - 31\lambda + 30
\end{aligned}$$

2. Using $p_{\mathbf{A}}(\lambda)$, compute the eigenvalues of $\mathbf{A}$.
Observe that $30 = 2 * 3 * 5$. Therefore, to factor $p_{\mathbf{A}}(\lambda)$, we can start by seeing if any of these values are roots. Using polynomial long division (omitted), we find

$$\frac{-\lambda^3 + 10\lambda^2 - 31\lambda + 30}{\lambda - 2} = -\lambda^2 + 8\lambda - 15 = -(\lambda-3)(\lambda-5)$$

So, we have $p_{\mathbf{A}}(\lambda) = -(\lambda-2)(\lambda-3)(\lambda-5)$. Therefore, the eigenvalues of $\mathbf{A}$ are 2, 3, and 5.

3. Find the eigenvectors of $\mathbf{A}$.
We first find the eigenvector corresponding to $\lambda_1 = 5$:

$$\begin{aligned}
\mathbf{v}_1 &\in \ker(\mathbf{A} - 5\mathbf{I}) \\
&\in \ker\begin{bmatrix} -1 & 1 & -1 \\ \frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}
\end{aligned}$$

Observe that $(\mathbf{A} - 5\mathbf{I})_1 = -(\mathbf{A} - 5\mathbf{I})_2$. Therefore,

$$\begin{bmatrix} -1 & 1 & -1 \\ \frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

so $[1\ 1\ 0]^\top$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_1 = 5$. We turn next to $\lambda_2 = 3$:

$$\begin{aligned}
\mathbf{v}_1 &\in \ker(\mathbf{A} - 3\mathbf{I}) \\
&\in \ker\begin{bmatrix} 1 & 1 & -1 \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

Observe that $(\mathbf{A} - 3\mathbf{I})_1 = -(\mathbf{A} - 3\mathbf{I})_3$. Therefore,

$$\begin{bmatrix} 1 & 1 & -1 \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

so $[1\ 0\ 1]^\top$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_2 = 3$. Finally, for $\lambda_3 = 2$:

$$\begin{aligned}\mathbf{v}_1 &\in \ker(\mathbf{A} - 2\mathbf{I}) \\ &\in \ker \begin{bmatrix} 2 & 1 & -1 \\ \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\end{aligned}$$

Observe that $(\mathbf{A} - 2\mathbf{I})_2 = -(\mathbf{A} - 2\mathbf{I})_3$. Therefore,

$$\begin{bmatrix} 2 & 1 & -1 \\ \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

so $[0\ 1\ 1]^\top$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_3 = 2$.

4 Proofs about Eigenvalues

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix such that the eigenvectors of $\mathbf{A}$ are a basis for $\mathbb{R}^n$. Additionally, let λ_i, $1 \leq i \leq n$ be the eigenvalues of $\mathbf{A}$.

1. Prove that $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$.

   Let $\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ be the eigendecomposition of $\mathbf{A}$. Then $\det(\mathbf{A}) = \det(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}) = \det(\mathbf{Q})\det(\mathbf{\Lambda})\det(\mathbf{Q}^{-1}) = \frac{\det(\mathbf{Q})}{\det(\mathbf{Q})}\det(\mathbf{\Lambda}) = \det(\mathbf{\Lambda})$. But $\mathbf{\Lambda}$ is diagonal, so $\det(\mathbf{\Lambda}) = \prod_{i=1}^{n} \mathbf{\Lambda}_{ii}$. The diagonal elements of $\mathbf{\Lambda}$ are the eigenvalues of $\mathbf{A}$, so therefore $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$.

2. Prove that $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$.

   Recall that trace is invariant under similarity. Since $\mathbf{A} \sim \mathbf{\Lambda}$, $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{\Lambda}) = \sum_{i=1}^{n} \mathbf{\Lambda}_{ii} = \sum_{i=1}^{n} \lambda_i$.