Q1

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & \end{bmatrix}$$

Q2

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & y \end{pmatrix}$$

rank (A) = number of leading 1's in row reduced echelon form.

Now reduce A

$$\begin{matrix} \text{Row } 2 - \frac{1}{2} \text{ Row } 1 \\ \text{Row } 3 - \frac{4}{2} \text{ Row } 1 \end{matrix} \quad \begin{pmatrix} 2 & 1 & 3 \\ 0 & \frac{17}{2} & -\frac{1}{2} \\ 0 & 4 & y-6 \end{pmatrix}$$

$$\text{Row } 3 - \frac{4}{\left(\frac{17}{2}\right)} \text{ Row } 2 \quad \begin{pmatrix} 2 & 1 & 3 \\ 0 & \frac{17}{2} & -\frac{1}{2} \\ 0 & 0 & y - \frac{98}{17} \end{pmatrix}$$

~~rank~~

$$\text{rank } (A) = 2 \implies y - \frac{98}{17} = 0$$

Since can only have two leading 1's

$$y = \frac{98}{17}$$

$\therefore$ A s.t rank$(A) = 2$ is

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & \frac{98}{17} \end{pmatrix}$$

Q3

$\forall y \neq \frac{98}{17}$ $\exists$ 3 leading 1's

$\therefore$ $y \neq \frac{98}{17} \Rightarrow$ rank$(A) = 3$

choose $y = 5$

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix}$$

rank$(A) = 3$

Q4
a)

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix}$$

[illegible] of A

$$R_2 \to R_2 - \frac{1}{2} R_1 \qquad \begin{pmatrix} 2 & 1 & 3 \\ 0 & 17/2 & -1/2 \\ 0 & 4 & -1 \end{pmatrix}$$

$$R_3 \to R_3 - 2 R_1$$

$$L_{2,1} = \frac{1}{2} \qquad L_{3,1} = 2$$

$$R_3 \to R_3 - \frac{4}{(17/2)} R_2 \qquad \begin{pmatrix} 2 & 1 & 3 \\ 0 & 17/2 & -1/2 \\ 0 & 0 & \frac{-13}{17} \end{pmatrix}$$

$$L_{3,2} = \frac{8}{17}$$

We have

$$U = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 17/2 & -1/2 \\ 0 & 0 & \frac{-13}{17} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & \frac{8}{17} & 1 \end{pmatrix}$$

CamScanner

Q5

a) $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix}$

$$u_1 = a_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 1 \\ 9 \\ 6 \end{pmatrix} \qquad a_3 = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$$

$$u_1 = a_1 \qquad q_1 = \frac{u_1}{\|u_1\|}$$

$$\|u_1\| = \sqrt{21}$$

$$q_1 = \frac{u_1}{\|u_1\|} = \frac{u_1}{\sqrt{21}} = \begin{pmatrix} 0.436 \\ 0.218 \\ 0.873 \end{pmatrix}$$

$$u_2 = a_2 - \frac{a_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

$$\frac{a_2 \cdot u_1}{u_1 \cdot u_1} = \frac{35}{21} = \frac{5}{3}$$

$$u_2 = \begin{pmatrix} 1 \\ 9 \\ 6 \end{pmatrix} - \frac{5}{3}\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} -7/3 \\ 22/3 \\ -2/3 \end{pmatrix}$$

$$q_2 = \frac{u_2}{\|u_2\|} = \frac{u_2}{7.7244} = \begin{pmatrix} -0.302 \\ 0.949 \\ -0.086 \end{pmatrix}$$

$$u_3 = a_3 - \frac{a_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{a_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$u_3 = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} - 1.286 \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + 0.050 \begin{pmatrix} -7/3 \\ 22/3 \\ -2/3 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 0.311 \\ 0.083 \\ -0.176 \end{pmatrix}$$

$$q_3 = \frac{u_3}{\|u_3\|} = \frac{u_3}{0.367} = \begin{pmatrix} 0.847 \\ 0.226 \\ -0.480 \end{pmatrix}$$

$$Q = \begin{pmatrix} q_1 , & q_2 , & q_3 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.436 & -0.302 & 0.847 \\ 0.218 & 0.948 & 0.226 \\ 0.873 & -0.086 & -0.480 \end{pmatrix}$$

$$A = QR$$

$$Q^T A = Q^T Q R$$

$$Q^T A = R$$

$$\begin{pmatrix} 0.436 & 0.218 & 0.873 \\ -0.302 & 0.949 & -0.086 \\ 0.847 & 0.226 & -0.480 \end{pmatrix} \times \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 4.583 & 7.6438 & 5.892 \\ 0 & 7.724 & -0.388 \\ 0 & 0 & 0.367 \end{pmatrix}$$

R

Checked in Python $A = QR$

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 0.436 & -0.302 & 0.847 \\ 0.218 & 0.949 & 0.226 \\ 0.873 & -0.086 & -0.480 \end{pmatrix} \begin{pmatrix} 4.583 & 7.6438 & 5.892 \\ 0 & 7.724 & -0.388 \\ 0 & 0 & 0.367 \end{pmatrix}$$

Q5
b)

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u_1 = a_1 - \|a_1\| e_1$$

$$u_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} - \sqrt{21} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} -2.583 \\ 1 \\ 4 \end{pmatrix}.$$

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{4.865} \begin{pmatrix} -2.583 \\ 1 \\ 4 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} -0.531 \\ 0.206 \\ 0.822 \end{pmatrix} \qquad H_1 = I_3 - 2 v_1 v_1^T.$$

$$H_1 = I_3 - 2 \begin{pmatrix} -0.531 \\ 0.206 \\ 0.822 \end{pmatrix} (-0.531, 0.206, 0.822)$$

$$H_1 = \begin{pmatrix} 0.436 & 0.218 & 0.873 \\ 0.218 & 0.916 & -0.338 \\ 0.873 & -0.338 & -0.352 \end{pmatrix}.$$

$A1 = H1\, A$

$$A1 = \begin{pmatrix} 4.583 & 7.638 & 5.892 \\ 0 & 6.43 & -0.12 \\ 0 & -4.281 & 0.521 \end{pmatrix}$$

$$a_2 = \begin{pmatrix} 6.43 \\ -4.281 \end{pmatrix} \qquad \|a_2\| = 7.724$$

$$u_2 = a_2 - \|a_2\| e_2 \qquad e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 6.43 \\ -4.281 \end{pmatrix} - 7.724 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 14.155 \\ -4.281 \end{pmatrix} \qquad \|u_2\| = 14.787$$

$$v_2 = \frac{u_2}{\|u_2\|} = \frac{1}{14.787} \begin{pmatrix} 14.155 \\ -4.281 \end{pmatrix}$$

$$= \begin{pmatrix} 0.957 \\ -0.289 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & H_{22} & \\ 0 & & \end{pmatrix} \qquad H_2 = I_2 - 2 v_2 v_2^T$$

$$H_{22} = I_2 - 2 \begin{pmatrix} 0.957 \\ -0.289 \end{pmatrix} \begin{pmatrix} 0.957, & -0.289 \end{pmatrix}$$

$$H_{22} = \begin{pmatrix} 0.832 & -0.554 \\ -0.554 & -0.832 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.832 & -0.554 \\ 0 & -0.554 & -0.832 \end{pmatrix}$$

$$A_2 = H_2 A_1$$

$$= \begin{pmatrix} 4.583 & 7.638 & 5.892 \\ 0 & 7.724 & -0.388 \\ 0 & 0 & -0.367 \end{pmatrix}$$

$a_3 = 0.367 \qquad e_3 = 1$

$\|a_3\| = 0.367$

$u_3 = a_3 + \|a_3\| e_3$

$= 0.367 + 0.367(1) = 0.734$

$v_3 = \dfrac{u_3}{\|u_3\|} = \dfrac{0.734}{0.734} = 1$

$$H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & H_{33} \end{pmatrix}$$

$H_{33} = I - 2 v_3 v_3^T$

$= 1 - 2(1)(1) = -1$

$$H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$A_3 = H_3 A_2 = R$

$$R = \begin{pmatrix} 4.583 & 7.638 & 5.892 \\ 0 & 7.724 & -3.884 \\ 0 & 0 & 0.367 \end{pmatrix}$$

$H_3 H_2 H_1 A = R$

$\Rightarrow A_3 = R$

$A = H_1^{-1} H_2^{-1} H_3^{-1} R$

$A = \underbrace{H_1^T H_2^T H_3^T}_{Q} R$ ... since orthogonal

$$Q = \begin{pmatrix} 0.436 & -0.302 & 0.848 \\ 0.218 & 0.949 & 0.226 \\ 0.873 & -0.086 & -0.480 \end{pmatrix}$$

Check

$A = QR$?

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix} = QR \quad ✓$$

in Python

Q6

a) $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix}$

$$A = U \Sigma V^T$$

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T)$$

$$= (V \Sigma^T U^T)(U \Sigma V^T)$$

$$= V \Sigma^T \Sigma V^T$$

$$A^T A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 9 & 6 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 35 & 27 \\ 35 & 118 & 42 \\ 27 & 42 & 35 \end{pmatrix}$$

Want to find eigenvalues of $A^T A$
Have to calculate

$$\det \begin{pmatrix} 21-\lambda & 35 & 27 \\ 35 & 118-\lambda & 42 \\ 27 & 42 & 35-\lambda \end{pmatrix}$$

We get $\lambda_1 = 0.0467$ $\lambda_2 = 24.143$ $\lambda_3 = 149.81$

Now to find eigenvectors

$$(A^TA - \lambda_i I)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

We get

$$V_1 = \begin{pmatrix} -0.808 \\ 0.031 \\ 0.588 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0.496 \\ -0.502 \\ 0.708 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} -0.3107 \\ -0.864 \\ -0.391 \end{pmatrix}$$

these are all unit vectors

$$V = \begin{pmatrix} V_3 & V_2 & V_1 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.3107 & 0.496 & -0.808 \\ -0.864 & -0.502 & 0.031 \\ -0.391 & 0.708 & 0.588 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_3} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_1} \end{pmatrix}$$

$$= \begin{pmatrix} 12.24 & 0 & 0 \\ 0 & 4.914 & 0 \\ 0 & 0 & 0.216 \end{pmatrix}$$

From the SVD derivation we have

$A v_i = \sigma_i u_i$ $\quad U = \text{eigenvectors}(AA^T)$ but this is easier and removes sign error

$$u_i = \frac{1}{\sigma_i} A v_i$$

$$u_1 = \frac{1}{12.24} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix} \begin{pmatrix} -0.317 \\ -0.846 \\ -0.391 \end{pmatrix} = \begin{pmatrix} -0.218 \\ -0.683 \\ -0.687 \end{pmatrix}$$

$$u_2 = \frac{1}{4.914} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix} \begin{pmatrix} 0.496 \\ -0.502 \\ 0.708 \end{pmatrix} = \begin{pmatrix} 0.532 \\ -0.674 \\ 0.512 \end{pmatrix}$$

$$u_3 = \frac{1}{0.216} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 9 & 1 \\ 4 & 6 & 5 \end{pmatrix} \begin{pmatrix} -0.808 \\ 0.031 \\ 0.588 \end{pmatrix} = \begin{pmatrix} 0.818 \\ 0.254 \\ -0.516 \end{pmatrix}$$

$$U = (u_1, u_2, u_3)$$

$$U = \begin{pmatrix} -0.218 & 0.532 & 0.818 \\ -0.693 & -0.674 & 0.254 \\ -0.687 & 0.512 & -0.516 \end{pmatrix}$$

We have

$$A = U \Sigma V^T$$

$$A = \begin{pmatrix} -0.218 & 0.532 & 0.82 \\ -0.693 & -0.674 & 0.254 \\ -0.687 & 0.512 & -0.516 \end{pmatrix} \begin{pmatrix} 12.24 & 0 & 0 \\ 0 & 4.914 & 0 \\ 0 & 0 & 0.216 \end{pmatrix} \begin{pmatrix} -0.317 & -0.864 & -0.391 \\ 0.496 & -0.502 & 0.708 \\ -0.808 & 0.031 & 0.588 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 3 \\ 1 & 8 & 1 \\ 4 & 6 & 5 \end{pmatrix}$$

Check $A = U \Sigma V^T$

Can do opposite way aswell

Find eigenvectors and eigenvalues of AA^T

Then use $A^T u_i = \sigma_i v_i$ eigenvectors give columns of U

$$v_i = \frac{1}{\sigma_i} A^T u_i$$

U = eigenvectors of AA^T

$\Sigma = \sqrt{\text{eigen values}}$ diagonal of AA^T

$$V = \left(\frac{1}{\sigma_1} A^T u_1 , \frac{1}{\sigma_2} A^T u_2 , \frac{1}{\sigma_3} A^T u_3 \right)$$