Chapter 8: Applications of Newton's Second Law

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Chapter 8

Applications of Newton's Second Law

Those who are in love with practice without knowledge are like the sailor who gets into a ship without rudder or compass and who never can be certain whether he is going. Practice must always be founded on sound theory. . . [1]

Leonardo da Vinci

8.1 Force Laws

There are forces that don't change appreciably from one instant to another, which we refer to as constant in time, and forces that don't change appreciably from one point to another, which we refer to as constant in space. The gravitational force on an object near the surface of the earth is an example of a force that is constant in space.

There are forces that depend on the configuration of a system. When a mass is attached to one end of a spring, the spring force acting on the object increases in strength whether the spring is extended or compressed.

There are forces that spread out in space such that their influence becomes less with distance. Common examples are the gravitational and electrical forces. The gravitational force between two objects falls off as the inverse square of the distance separating the objects provided the objects are of a small dimension compared to the distance between them. More complicated arrangements of attracting and repelling interactions give rise to forces that fall off with other powers of r: constant, $1/r$, $1/r^2$, $1/r^3, \ldots,$.

A force may remain constant in magnitude but change direction; for example the gravitational force acting on a planet undergoing circular motion about a star is directed towards the center of the circle. This type of attractive central force is called a *centripetal force*.

[1] *Notebooks of Leonardo da Vinci Complete*, tr. Jean Paul Richter, 1888, Vol.1.

A *force law* describes the relationship between the force and some measurable property of the objects involved. We shall see that some interactions are describable by force laws and other interactions cannot be so simply described.

8.1.1 Hooke's Law

In order to stretch or compress a spring from its equilibrium length, a force must be exerted on the spring. Consider an object of mass m that is lying on a horizontal surface. Attach one end of a spring to the object and fix the other end of the spring to a wall. Let l_0 denote the equilibrium length of the spring (neither stretched nor compressed). Assume that the contact surface is smooth and hence frictionless in order to consider only the effect of the spring force. If the object is pulled to stretch the spring or pushed to compress the spring, then by Newton's Third Law the force of the spring on the object is equal and opposite to the force that the object exerts on the spring. We shall refer to the force of the spring on the object as the *spring force* and experimentally determine a relationship between that force and the amount of stretching or compression of the spring.

Choose a coordinate system with the origin located at the point of contact of the spring and the object when the spring-object system is in the equilibrium configuration. Choose the unit vector $\hat{\mathbf{i}}$ to point in the direction the object moves when the spring is being stretched. Choose the coordinate function $x(t)$ to denote the position of the object with respect to the origin at time t Figure 8.1.

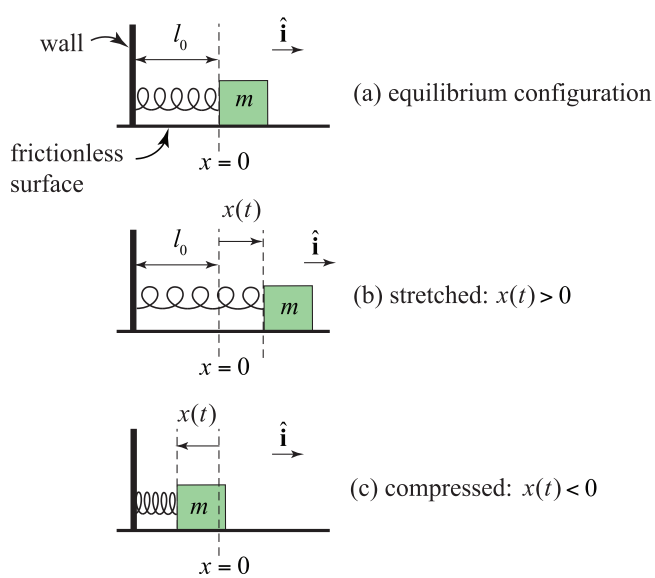

Figure 8.1: Hooke's Law: (a) Equilibrium position; (b) stretched; (c) compressed.

Initially stretch the spring until the object is at position $x(t = 0) = x_0$. Then release the object and measure the acceleration of the object the instant the object is released. The magnitude of the spring force acting on the object is $|\vec{\mathbf{F}}| = m|\vec{\mathbf{a}}|$. Now repeat the experiment for a range of stretches (or compressions). Experiments show that for each spring, there is a range of maximum values $x_{max} > 0$ for stretching and minimum values $x_{min} < 0$ for compressing such that the magnitude of the measured force is proportional to the stretched or compressed length and is given by the formula

$$|\vec{\mathbf{F}}| = k|\vec{\mathbf{x}}|. \tag{8.1}$$

where the *spring constant* k has units $[\mathrm{N} \cdot \mathrm{m}^{-1}]$. The free-body force diagram is shown in Figure 8.2. The constant k is equal to the negative of the slope of the graph of the

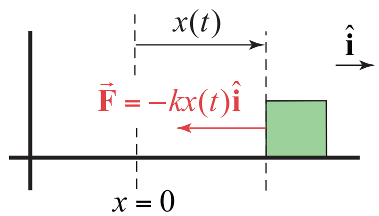

Figure 8.2: Hooke's Law: force on extended spring.

force vs. the compression or stretch (Figure 8.3). The direction of the acceleration is

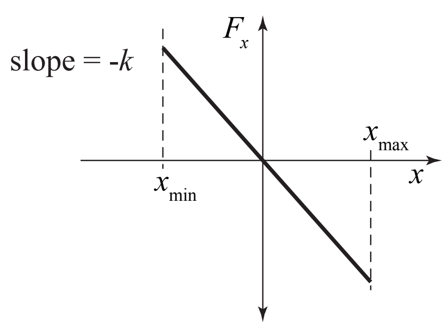

Figure 8.3: Plot of the x-component of the spring force F_x vs. x.

always towards the equilibrium position whether the spring is stretched or compressed. This type of force is called a *restoring force*. Let F_x denote the x-component of the spring force. Then

$$F_x = -kx. \tag{8.2}$$

Now perform similar experiments on other springs. For a range of stretched lengths, each spring exhibits the same proportionality between force and stretched length, although the spring constant may differ for each spring.

It would be extremely impractical to experimentally determine whether this proportionality holds for all springs, and because a modest sampling of springs has confirmed the relation, we shall *infer* that all *ideal springs* will produce a restoring force, which is linearly proportional to the stretched (or compressed) length. This experimental relation regarding force and stretched (or compressed) lengths for a finite set of springs has now been *inductively* generalized into the above mathematical model for ideal springs, a force law known as a *Hooke's Law*.

This inductive step, referred to as *Newtonian induction*, is the critical step that makes physics a predictive science. Suppose a spring, attached to an object of mass m, is stretched by an amount Δx. Use the force law to predict the magnitude of the force between the spring and the object, $|\vec{\mathbf{F}}| = k|\vec{\mathbf{x}}|$, without having to experimentally measure the acceleration. Now use Newton's Second Law to predict the magnitude of the acceleration of the object

$$\left|\vec{\mathbf{a}}\right| = \frac{\left|\vec{\mathbf{F}}\right|}{m} = \frac{k\,|\Delta x|}{m}. \tag{8.3}$$

Carry out the experiment, and measure the acceleration within some error bounds. If the magnitude of the predicted acceleration disagrees with the measured result, then the model for the force law needs modification. The ability to adjust, correct or even reject models based on new experimental results enables a description of forces between objects to cover larger and larger experimental domains.

Many real springs have been wound such that a force of magnitude F_0 must be applied before the spring begins to stretch and a force of magnitude F_1 must be applied before the spring begins to compress. The values F_0 and F_1 are referred to as the *pre-tensions* of the spring. Under these circumstances, Hooke's law must be modified to account for these pretensions,

$$\begin{aligned} F_x &= -F_0 - kx, \quad x > 0, \\ F_x &= +F_1 - kx, \quad x < 0. \end{aligned} \tag{8.4}$$

Note the value of the pre-tensions F_0 and F_1 may differ for compressing or stretching a spring and are determined experimentally.

8.2 Fundamental Laws of Nature

Force laws are mathematical models of physical processes. They arise from observation and experimentation, and they have limited ranges of applicability. Does the linear force law for the spring hold for all springs? Each spring will most likely have a different range of linear behavior. So the model for stretching springs still lacks a universal character. As such, there should be some hesitation to generalize this observation to all springs unless some property of the spring, universal to all springs, is responsible for the force law.

Perhaps springs are made up of very small components, which when pulled apart tend to contract back together. This would suggest that there is some type of force that contracts spring molecules when they are pulled apart. What holds molecules together? Can we find some fundamental property of the interaction between atoms that will suffice to explain the macroscopic force law? This search for *fundamental forces* is a central task of physics.

In the case of springs, this could lead into an investigation of the composition and structural properties of the atoms that compose the steel in the spring. We would investigate the geometric properties of the lattice of atoms and determine whether there is some fundamental property of the atoms that create this lattice. Then we ask how stable is this lattice under deformations. This may lead to an investigation into the electron configurations associated with each atom and how they overlap to form bonds between atoms. These particles carry charges, which obey Coulomb's Law, but also the Laws of Quantum Mechanics. So in order to arrive at a satisfactory explanation of the elastic restoring properties of the spring, we need models that describe the fundamental physics that underline Hooke's Law.

8.2.1 Universal Law of Gravitation

At points significantly far away from the surface of Earth, the gravitational force is no longer constant with respect to the distance to the center of Earth. *Newton's Universal Law of Gravitation* describes the gravitational force between two objects with masses, m_1 and m_2. This force points along the line connecting the objects, is attractive, and its magnitude is proportional to the inverse square of the distance, $r_{1,2}$, between the two point-like objects (Figure 8.4a). The force on object 2 due to the gravitational interaction between the two objects is given by

$$\vec{\mathbf{F}}^{G}_{1,2} = -G\,\frac{m_1\,m_2}{{r_{1,2}}^2}\,\hat{\mathbf{r}}_{1,2}. \tag{8.5}$$

where $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$ is a vector directed from object 1 to object 2, $r_{1,2} = |\vec{\mathbf{r}}_{1,2}|$, and $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2}/|\vec{\mathbf{r}}_{1,2}|$ is a unit vector directed from object 1 to object 2 (Figure 8.4b). The constant of proportionality in SI units is $G = (6.67408 \pm 0.00031) \times 10^{-11}\,\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}$.[2]

8.2.2 Principle of Equivalence

The Principle of Equivalence states that the mass that appears in the Universal Law of Gravity is identical to the inertial mass that is determined with respect to the standard kilogram. From this point on, the equivalence of inertial and gravitational mass will be assumed and the mass will be denoted by the symbol m.

8.2.3 Gravitational force near the surface of the earth

Near the surface of Earth, the gravitational interaction between an object and Earth is mutually attractive and has a magnitude of

[2]Chao Xue and others, *Precision measurement of the Newtonian gravitational constant*, National Science Review, Volume 7, Issue 12, December 2020, Pages 1803–1817, https://doi.org/10.1093/nsr/nwaa165.

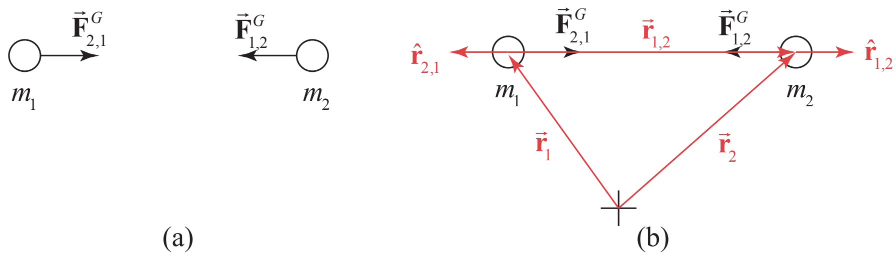

Figure 8.4: (a) Gravitational force between two point-like objects; (b) Coordinate system for the two-body problem.

$$\left|\vec{\mathbf{F}}^G_{earth,object}\right| = m\,g. \tag{8.6}$$

where g is a positive constant. The International Committee on Weights and Measures has adopted as a standard value for the acceleration of an object freely falling in a vacuum $g = 9.80665\ \mathrm{m\cdot s^{-2}}$. The actual value of g varies as a function of elevation and latitude. If ϕ is the latitude and h the elevation in meters then the acceleration of gravity in SI units is

$$g = (9.80616 - 0.025928\ \cos(2\phi) + 0.000069\cos^2(2\phi) - 3.086\times10^{-4}h)\ \mathrm{m{\cdot}s^{-2}}. \tag{8.7}$$

This is known as *Helmert's equation*. The strength of the gravitational force on the standard kilogram at $42°$ latitude is $9.80345|\mathrm{N\cdot kg^{-1}}$, and the acceleration due to gravity at sea level is therefore $9.80345|\mathrm{m\cdot s^{-2}}$for all objects. At the equator $g = 9.78\ \mathrm{m\cdot s^{-2}}$ and at the poles $g = 9.83\ \mathrm{m\cdot s^{-2}}$. This difference is primarily due to the earth's rotation, which introduces an apparent (fictitious) repulsive force that affects the determination of g as given in Equation 8.6 and also flattens the spherical shape of Earth. The distance from the center of Earth is larger at the equator than it is at the poles by about $26.5\ \mathrm{km}$. (In a later chapter on Non-inertial Reference Frames, we shall discuss these effects in more detail). Both the magnitude and the direction of the gravitational force also show variations that depend on local features to an extent that's useful in searching for salt domes, investigating the water table, navigating submerged submarines, and as well as many other practical uses. Such variations in g can be measured with a sensitive spring balance. Local variations have been much studied over the past two decades in attempts to discover a proposed "fifth force" which would fall off faster than the gravitational force that falls off as the inverse square of the distance between the objects.

8.2.4 Electric charge and Coulomb's Law

Matter has properties other than mass. Matter can also carry one of two types of observed *electric charge*, positive and negative. Like charges repel, and opposite charges attract each other. The unit of charge in the SI system of units is called the *coulomb*.

The *elementary charge*, which is the charge of an electron or proton, in SI units is exactly defined as

$$e = 1.602176634 \times 10^{-19}\ \mathrm{C}. \tag{8.8}$$

It has been shown experimentally that charge carried by ordinary objects is quantized in integral multiples of the elementary charge. The electron carries one unit of negative charge, $q_e = -e$ and the proton carries one unit of positive charge $q_p = +e$. In an isolated system, the charge stays constant; in a closed system, an amount of unbalanced charge can neither be created nor destroyed. Charge can only be transferred from one object to another. Consider two point-like objects with opposite charges q_1 and q_2,

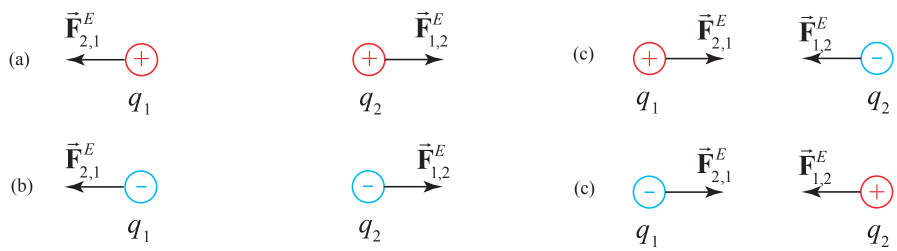

Figure 8.5: Coulombic force. (a) and (b): repulsive interaction between like charged objects; (c) and (d): attractive interaction between oppositely charged objects.

separated by a distance $r_{1,2}$ in vacuum. By experimental observation, the two objects repel each other if they are both positively or negatively charged (Figures 8.5 (a) and (b). They attract each other if they are oppositely charged ((Figures 8.5 (c) and (d)).

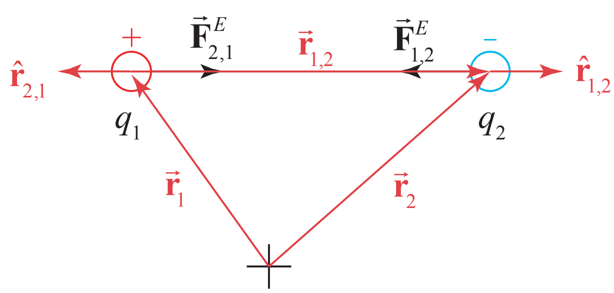

Figure 8.6: Coordinate system for attractive interaction between oppositely charged objects.

The force exerted on object 2 due to the interaction between objects 1 and 2 is given by Coulomb's Law,

$$\vec{\mathbf{F}}^E_{1,2} = \frac{1}{4\pi\epsilon_0}\frac{q_1\, q_2}{r_{1,2}{}^2}\,\hat{r}_{1,2}. \tag{8.9}$$

where $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2}/|\vec{\mathbf{r}}_{1,2}|$ is a unit vector directed from object 1 to object 2, where the *vacuum permeability* in SI units the *vacuum permeability* is $\epsilon_0 = 8.8541878128(13) \times 10^{-12}\ \mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{C}^{-2}$, as illustrated in the Figure 8.6, where $q_1 > 0$ and $q_2 < 0$. The quantity $ke \equiv 1/(4\pi\epsilon_0) \approx 9.0 \times 10^9 \mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{C}^{-2}$. This law was derived empirically by Charles Augustin de Coulomb in the late 18th century.

8.2.5 Example: Coulomb's Law and the Universal Law of Gravitation

Show that both Coulomb's Law and the Universal Law of Gravitation satisfy Newton's Third Law.

Answer

To see this, interchange 1 and 2 in the Universal Law of Gravitation to find the force on object 1 due to the interaction between the objects. The only quantity to change sign is the unit vector $\hat{\mathbf{r}}_{2,1} = -\hat{\mathbf{r}}_{1,2}$. Then

$$\vec{\mathbf{F}}^G_{2,1} = -G\,\frac{m_2\,m_1}{r_{2,1}{}^2}\,\hat{\mathbf{r}}_{2,1} = G\,\frac{m_1\,m_2}{r_{1,2}{}^2}\,\hat{\mathbf{r}}_{1,2} = -\vec{\mathbf{F}}^G_{1,2}.$$

Coulomb's Law also satisfies Newton's Third Law since the only quantity to change sign is the unit vector, just as in the case of the Universal Law of Gravitation.

8.3 Constraint Forces

Knowledge of all the external and internal forces acting on each of the objects in a system and applying Newton's Second Law to each of the objects determine a set of equations of motion. These equations of motion are not necessarily independent due to the fact that the motion of the objects may be limited by equations of constraint. In addition there are forces of constraint that are determined by their effect on the motion of the objects and are not known beforehand or describable by some force law. For example: an object sliding down an inclined plane is constrained to move along the surface of the inclined plane (Figure 8.7(a)) and the surface exerts a contact force on the object; an object that slides down the surface of a sphere until it falls off experiences a contact force until it loses contact with the surface (Figure 8.7(b)); gas particles in a sealed vessel are constrained to remain inside the vessel and therefore the wall must exert force on the gas molecules to keep them inside the vessel (Figure 8.7(c)); and a bead constrained to slide outward along a rotating rod is acted on by time dependent forces of the rod on the bead (Figure 8.7(d)). We shall develop methods to determine these constraint forces although there are many examples in which the constraint forces cannot be determined.

8.3.1 Contact Forces

Pushing, lifting and pulling are *contact forces* that we experience in the everyday world. Rest your hand on a table; the atoms that form the molecules that make up the table and

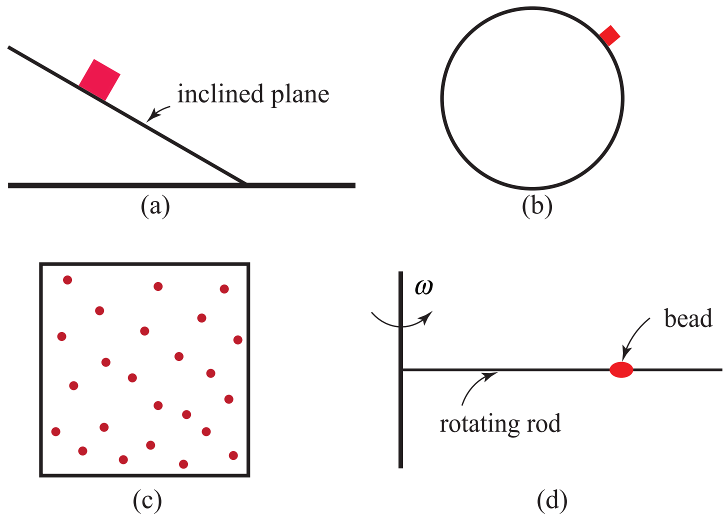

Figure 8.7: Constrained motions: (a) particle sliding down inclined plane, (b) particles sliding down surface of sphere, (c) gas molecules in a sealed vessel, and (d) bead sliding on a rotating rod.

your hand are in contact with each other. If you press harder, the atoms are also pressed closer together. The electrons in the atoms begin to repel each other and your hand is pushed in the opposite direction by the table.

According to Newton's Third Law, the force of your hand on the table is equal in magnitude and opposite in direction to the force of the table on your hand. Clearly, if you push harder the force increases. Try it! If you push your hand straight down on the table, the table pushes back in a direction perpendicular (normal) to the surface. Slide your hand gently forward along the surface of the table. You barely feel the table pushing upward, but you do feel the friction acting as a resistive force to the motion of your hand. This force acts tangential to the surface and opposite to the motion of your hand. Push downward and forward. Try to estimate the magnitude of the force acting on your hand.

The force of the table acting on your hand, $\vec{\mathbf{F}}^C \equiv \vec{\mathbf{C}}$, is called the *contact force*. This force has both a normal component to the surface, $\vec{\mathbf{C}}_\perp \equiv \vec{\mathbf{N}}$, called the *normal force*, and a tangential component to the surface, $\vec{\mathbf{C}}_\parallel \equiv \vec{\mathbf{N}}$, called the *friction force* (Figure 8.8).

The contact force, written in terms of its component forces, is therefore

$$\vec{\mathbf{C}} = \vec{\mathbf{C}}_\perp + \vec{\mathbf{C}}_\parallel \equiv \vec{\mathbf{N}} + \vec{\mathbf{f}}. \tag{8.10}$$

Any force can be decomposed into component vectors so the normal component, $\vec{\mathbf{N}}$, and the tangential component, $\vec{\mathbf{f}}$, are not independent forces but the vector components

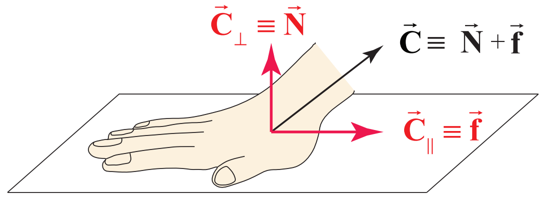

Figure 8.8: Normal and tangential components of the contact force.

of the contact force, perpendicular and parallel to the surface of contact. The contact force is a *distributed force* acting over all the points of contact between your hand and the surface. For most applications we shall treat the contact force as acting at single point but precaution must be taken when the distributed nature of the contact force plays a key role in constraining the motion of a rigid body.

In Figure 8.7, the forces acting on your hand are shown. These forces include the contact force, $\vec{\mathbf{C}}$, of the table acting on your hand, the force of your forearm, $\vec{\mathbf{F}}_{\mathrm{forearm}}$, acting on your hand (which is drawn at an angle indicating that you are pushing down on your hand as well as forward), and the gravitational interaction, $\vec{\mathbf{F}}^g$, between the earth and your hand.

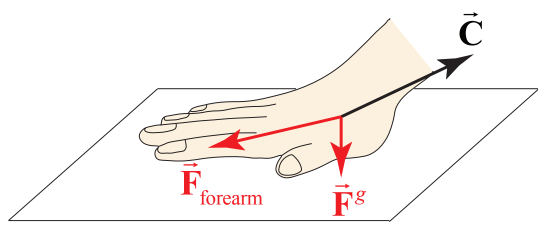

Figure 8.9: Forces on hand when moving towards the left.

One point to keep in mind is that the magnitudes of the two components of the contact force depend on how hard you push or pull your hand and in what direction, a characteristic of constraint forces, in which the components are not specified by a force law but dependent on the particular motion of the hand.

8.3.2 Example:Normal component of the contact force and weight

Hold a block in your hand such that your hand is at rest (Figure 8.10(a)). You can feel the "weight" of the block against your palm. But what exactly do we mean by "weight"?

There are two forces acting on the block as shown in Figure 8.10(B). One force is the gravitational force between the earth and the block, and is denoted by $\vec{\mathbf{F}}^g = m\vec{\mathbf{g}}$

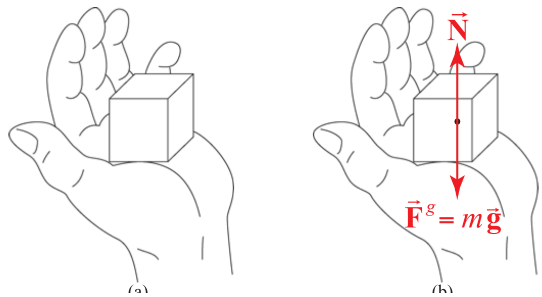

Figure 8.10: (a) Block resting in hand; (b) Forces on block.

. The other force acting on the block is the contact force between your hand and the block. Because your hand is at rest, this contact force on the block points perpendicular to the surface, and hence has only a normal component, $\vec{\mathbf{N}}$. Let N denote the magnitude of the normal force. Because the object is at rest in your hand, the vertical acceleration is zero. Therefore Newton's Second Law states that

$$\vec{\mathbf{N}} + \vec{\mathbf{F}}^g = \vec{\mathbf{0}}. \tag{8.11}$$

Choose the positive direction to be upwards and then in terms of vertical components we have that

$$N - mg = 0, \tag{8.12}$$

which can be solved for the magnitude of the normal force

$$N = mg, \tag{8.13}$$

When we use the expression "weight of the block", we often are referring to the effect the block has on a scale or on the feeling we have when we hold the block. These effects are actually effects of the normal force. We say that a block "feels lighter" if there is an additional force holding the block up. For example, you can rest the block in your hand, but use your other hand to apply a force upwards on the block to make it feel lighter in your supporting hand.

The word "weight," is often used to describe the gravitational force that Earth exerts on an object. We shall always refer to this force as the *gravitational force* instead of "weight." When you jump in the air, you feel "weightless" because there is no normal force acting on you, even though Earth is still exerting a gravitational force on you; clearly, when you jump, you do not turn gravity off!

This example may also give rise to a misconception that the normal force is always equal to the mass of the object times the magnitude of the gravitational acceleration at the surface of the earth. The normal force and the gravitational force are two completely different forces. In this particular example, the normal force is equal in magnitude to the gravitational force and directed in the opposite direction because the object is at rest. The normal force and the gravitational force do not form a Third Law interaction pair of forces. In this example, our system is just the block; the normal force and gravitational force are external forces acting on the block.

Let's redefine our system as the block, your hand, and Earth. Then the normal force and gravitational force are now internal forces in the system and we can now identify the various interaction pairs of forces. We explicitly introduce our interaction pair notation to enable us to identify these interaction pairs: for example, let $\vec{\mathbf{F}}^{g}_{E,B}$ denote the gravitational force on the block due to the interaction with Earth. The gravitational force on Earth due to the interaction with the block is denoted by $\vec{\mathbf{F}}^{g}_{B,E}$, and these two forces form an interaction pair. By Newton's Third Law, $\vec{\mathbf{F}}^{g}_{E,B} = -\vec{\mathbf{F}}^{g}_{B,E}$ Note that these two forces are acting on different objects, the block and Earth. The contact force on the block due to the interaction between the hand and the block is then denoted by $\vec{\mathbf{N}}_{H,B}$. The force of the block on the hand, which we denote by $\vec{\mathbf{N}}_{B,H}$, satisfies $\vec{\mathbf{N}}_{H,B} = -\vec{\mathbf{N}}_{B,H}$. Because we are including your hand as part of the system, there are two additional forces acting on the hand. There is the gravitational force on your hand $\vec{\mathbf{F}}^{g}_{E,H}$, satisfying $\vec{\mathbf{F}}^{g}_{E,H} = -\vec{\mathbf{F}}^{g}_{H,E}$, where $\vec{\mathbf{F}}^{g}_{H,E}$ is the gravitational force on Earth due to your hand. Finally there is the force of your forearm holding your hand up, which we denote $\vec{\mathbf{F}}_{F,H}$. Because we are not including the forearm in our system, this force is an external force to the system. The forces acting on your hand are shown in Figure 8.11.

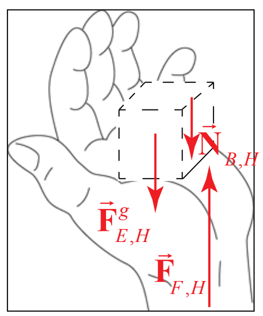

Figure 8.11: Forces on hand hand.

8.3.3 Kinetic and static friction

When a block is pulled along a horizontal surface or sliding down an inclined plane there is a lateral force resisting the motion. If the block is at rest on the inclined plane, there is still a lateral force resisting the motion. This resistive force is known as dry friction, and there are two distinguishing types when surfaces are in contact with each other. The first type occurs when the two objects are moving relative to each other; the friction in that case is called *kinetic friction* or *sliding friction*. When the two surfaces are non-moving but there is still a lateral force as in the example of the block at rest on an inclined plane, the force is called *static friction*.

Leonardo da Vinci was the first to record the results of measurements on kinetic friction over a twenty-year period between 1493–4 and about 1515. Based on his measurements, he identified two key properties of the force of kinetic friction, $\vec{\mathbf{f}}^k$, between two surfaces: The magnitude of kinetic friction is proportional to the normal force between the two surfaces,

$$f_k = \mu_k \, N, \tag{8.14}$$

where μ_k is called the *coefficient of kinetic friction*. The second result is rather surprising in that the magnitude of the force is independent of the contact surface. Consider two blocks of the same mass, but different surface areas. The force necessary to move the blocks at a constant speed is the same. The block in Figure 8.12(a) has twice the contact area as the block shown in Figure 8.12(b), but when the same external force is applied to either block, the blocks move at constant speed. These results of da Vinci were rediscovered by Guillaume Amontons and published in 1699. The third property that kinetic friction is independent of the speed of moving objects (for ordinary sliding speeds) was discovered by Charles Augustin Coulomb.

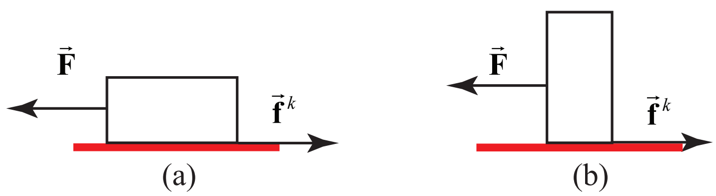

Figure 8.12: (a) and (b): kinetic friction is independent of the contact area.

The kinetic friction force on surface 2 moving relative to surface 1 is denoted by, $\vec{\mathbf{f}}^k_{1,2}$. The direction of the force is always opposed to the relative direction of motion of surface 2 relative to the surface 1. When one surface is at rest relative to our choice of reference frame we will denote the friction force on the moving object by $\vec{\mathbf{f}}^k$.

The second type of dry friction, static friction occurs when two surfaces are static relative to each other. Because the static friction force between two surfaces forms a third law interaction pair, we will use the notation $\vec{\mathbf{f}}^s_{1,2}$ to denote the static friction force on surface 2 due to the interaction between surfaces 1 and 2. Push your hand forward along a surface; as you increase your pushing force, the frictional force feels stronger and stronger. Try this! Your hand will at first stick until you push hard enough, then your hand slides forward. The magnitude of the static frictional force, f_s, depends on how hard you push.

If you rest your hand on a table without pushing horizontally, the static friction is zero. As you increase your push, the static friction increases until you push hard enough that your hand slips and starts to slide along the surface. Thus the magnitude of static friction can vary from zero to some maximum value, $(f_s)_{max}$, when the pushed object begins to slip,

$$0 \leq f_s \leq (f_s)_{max}. \tag{8.15}$$

Is there a mathematical model for the magnitude of the maximum value of static friction between two surfaces? Through experimentation, we find that this magnitude is, like kinetic friction, proportional to the magnitude of the normal force

$$(f_s)_{max} = \mu_s N \tag{8.16}$$

Here the constant of proportionality, μ_s, is the *coefficient of static friction*. This constant is slightly greater than the constant mu_k associated with kinetic friction, $\mu_s > \mu_k$. This small difference accounts for the slipping and catching of chalk on a blackboard, fingernails on glass, or a violin bow on a string.

The direction of static friction on an object is always opposed to the direction of the applied force (as long as the two surfaces are not accelerating). In Figure 8.13(a), an external force, $\vec{\mathbf{F}}$, is applied the left, and the static friction, $\vec{\mathbf{f}}^s$, is directed to the right opposing the external force. In Figure 8.13(b), the external force, $\vec{\mathbf{F}}$, is applied to the right, and the static friction, $\vec{\mathbf{f}}^s$, is now pointing to the left.

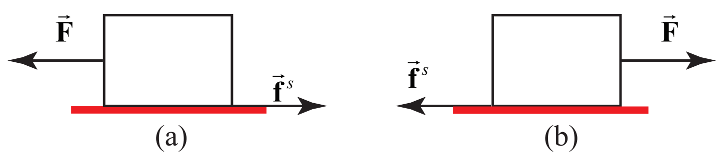

Figure 8.13: (a) and (b): External forces and the direction of static friction.

Although the force law for the maximum magnitude of static friction resembles the force law for sliding friction, there are important differences:

1. The direction and magnitude of static friction on an object always depends on the direction and magnitude of the applied forces acting on the object, where the magnitude of kinetic friction for a sliding object is fixed.

2. The magnitude of static friction has a maximum possible value. If the magnitude of the applied force along the direction of the contact surface exceeds the magnitude of the maximum value of static friction, then the object will start to slip (and be subject to kinetic friction.) We call this the *just slipping condition*.

8.4 Free-body Force Diagrams

When we try to describe forces acting on a collection of objects we must first take care to specifically define the collection of objects that we are interested in, which define our *system*. Often the system is a single isolated object but it can consist of multiple objects. Because force is a vector, the force acting on the system is a vector sum of the individual forces acting on the system

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots. \tag{8.17}$$

A *free-body force diagram* is a representation of the sum of all the forces that act on a single system. We denote the system by a large circular dot, a "point". (Later on in the course we shall see that the "point" represents the center of mass of the system.) We represent each force that acts on the system by an arrow (indicating the direction of that force). We draw the arrow at the "point" representing the system. For example, the forces that regularly appear in free-body diagram are contact forces, tension, gravitation, friction, pressure forces, spring forces, electric and magnetic forces, which we shall introduce below. Sometimes we will draw the arrow representing the actual point in the system where the force is acting. When we do that, we will not represent the system by a "point" in the free-body diagram. Suppose we choose a Cartesian coordinate system, then we can resolve the force into its component vectors Because force is a vector, the force acting on the system is a vector sum of the individual forces acting on the system

$$\vec{\mathbf{F}} = F_x\ \hat{\mathbf{i}} + F_y\ \hat{\mathbf{j}} + F_z\ \hat{\mathbf{k}}. \tag{8.18}$$

Each one of the component vectors is itself a vector sum of the individual component vectors from each contributing force. We can use the free-body force diagram to make these vector decompositions of the individual forces. For example, the - component of the force is

$$F_x = F_{1,x} + F_{2,x} + \cdots. \tag{8.19}$$

8.4.1 Modeling physical systems

One of the most central and yet most difficult tasks in analyzing a physical interaction is developing a physical model. A physical model for the interaction consists of a description of the forces acting on all the objects. The difficulty arises in deciding which forces to include. For example in describing almost all planetary motions, the Universal Law of Gravitation was the only force law that was needed. There were anomalies, for example the small shift in Mercury's orbit. These anomalies are interesting because they may lead to new physics. Einstein corrected Newton's Law of Gravitation by introducing General Relativity and one of the first successful predictions of the new theory was the perihelion precession of Mercury's orbit. On the other hand, the anomalies may simply be due to the complications introduced by forces that are well understood but complicated to model. When objects are in motion there is always some type of friction present. Air friction is often neglected because the mathematical models for air resistance are fairly complicated even though the force of air resistance substantially changes the motion. Static or kinetic friction between surfaces is sometimes ignored but not always. The mathematical description of the friction between surfaces has a simple expression so it can be included without making the description mathematically intractable. A good way to start thinking about the problem is to make a simple model, excluding complications that are small order effects. Then we can check the predictions of the model. Once we are satisfied that we are on the right track, we can include more complicated effects.

8.5 Tensile Forces

Try this as an exercise. Grip the ends of your fingers together and pull outward as shown in Figure 8.14. Your muscles and tendons are exerting a force that you feel on your finger tips and throughout your arms, wrists, hands and fingers.

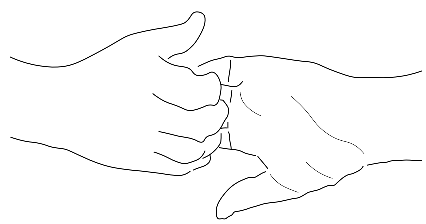

Figure 8.14: Tensile force in wrist, hands and fingers.

This type of pulling force or an extension force is called a *tensile stress force*. If you push your palms together, the force you experience is called a *tensile compression force*. Both extension and compression forces are directed normal to a surface. Extension forces point outward while compression forces point inward.

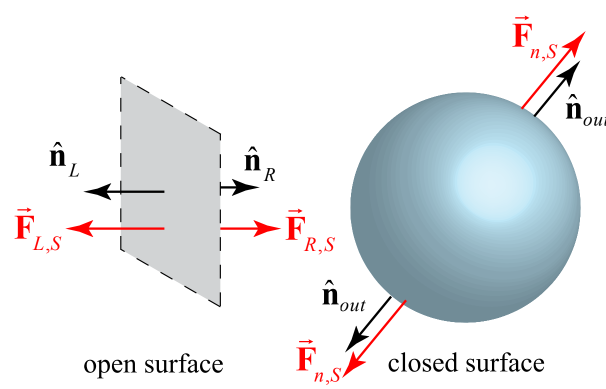

Figure 8.15: (a) Open surface with tensile forces pointing outward; (b) closed surface with tensile forces pointing outward.

Consider an open surface as shown in Figure 8.15 (a), with two unit normal vectors, $\hat{\mathbf{n}}_{\mathbf{L,S}}$ and $\hat{\mathbf{n}}_{\mathbf{R,S}}$. The two applied forces $\vec{\mathbf{F}}_{L,S}$ and $\vec{\mathbf{F}}_{R,S}$ are perpendicular to the surface. Suppose the surface does not deform, which means that the magnitude of the two forces are equal. Then the tension T is defined to be the scalar product $T = \vec{\mathbf{F}}_{L,S} \cdot \hat{\mathbf{n}}_{\mathbf{L,S}} = \mathbf{F}_{\mathbf{R,S}} \cdot \hat{\mathbf{n}}_{\mathbf{R,S}}$.

Examples of objects that can come under tension are stretched rubber bands, ropes that are pulled or bear a load, and muscles that may lengthen, shorten, or remain the same due to locomotor activity Figure 8.16 [3]

Now press your hands together as shown in Figure 8.17, Each hand is exerting a normal force on the other hand, called a *tensile compression force*. Because this force is distributed over the surface area of the hands, the *pressure* is defined as the magnitude of the force divided by the contact area, which we shall return to in more detail in Chapter 9.

The compressive forces on the open and closed surfaces shown in Figure 8.18 are perpendicular to the surface and now point inward.

8.6 Tension in a Rope

Consider a block (object 1) that is attached to a very light rope (object 2) at the point B. The other end of the rope is pulled by an applied force $\vec{\mathbf{F}}_{2,A}$ at the point A, (Figure 8.18a). At first we will assume that the rope is massless, with $m_2 \approx 0$.

Choose a coordinate system with an origin at point B, the $\hat{\mathbf{i}}$-unit vector pointing in the positive x-direction and the $\hat{\mathbf{j}}$-unit vector pointing upward in the direction normal to the surface (Figure 8.20). The force diagrams for the system consisting of the rope and block is shown in Figure 8.20, and for the rope and block separately in Figure 8.21, where $\vec{\mathbf{F}}_{2,1} = F_{2,1}\hat{\mathbf{i}}$ is the force on the block (object 1) due to the rope (object 2), $\vec{\mathbf{F}}_{1,2} = -F_{1,2}\hat{\mathbf{i}}$ is the force on the rope due to the block, and $\vec{\mathbf{F}}_{2,A} = F_{A,2}\hat{\mathbf{i}}$.

The forces on the rope and the block must each sum to zero. Because the rope is not accelerating, Newton's Second Law applied to the rope requires that

$$\vec{\mathbf{F}}_{2,A} + \vec{\mathbf{F}}_{1,2} = m_2\vec{\mathbf{a}}_2 \approx \vec{\mathbf{0}}. \tag{8.20}$$

In terms of magnitudes, Newton's Second Law on the rope is then

$$F_{A,2} - F_{1,2} \approx 0 \quad (\text{massless rope}). \tag{8.21}$$

If we now consider the case that the rope is very light but not massless, $m_2 \neq 0$ but $m_2\vec{\mathbf{g}} \approx 0$ then the forces acting at the ends of the rope are nearly horizontal. If the rope-block system is moving at constant speed or at rest, with zero acceleration $\vec{\mathbf{a}}_2 = \vec{\mathbf{0}}$, then in terms of magnitudes Newton's Second Law is

$$F_{A,2} - F_{1,2} = 0 \quad (\text{rope moving at constant speed or at rest}). \tag{8.22}$$

[3] OpenStax - https://cnx.org/contents/FPtK1zmh@8.25:fEI3C8Ot@10/Preface, Version 8.25 from the Textbook OpenStax Anatomy and Physiology Published May 18, 2016

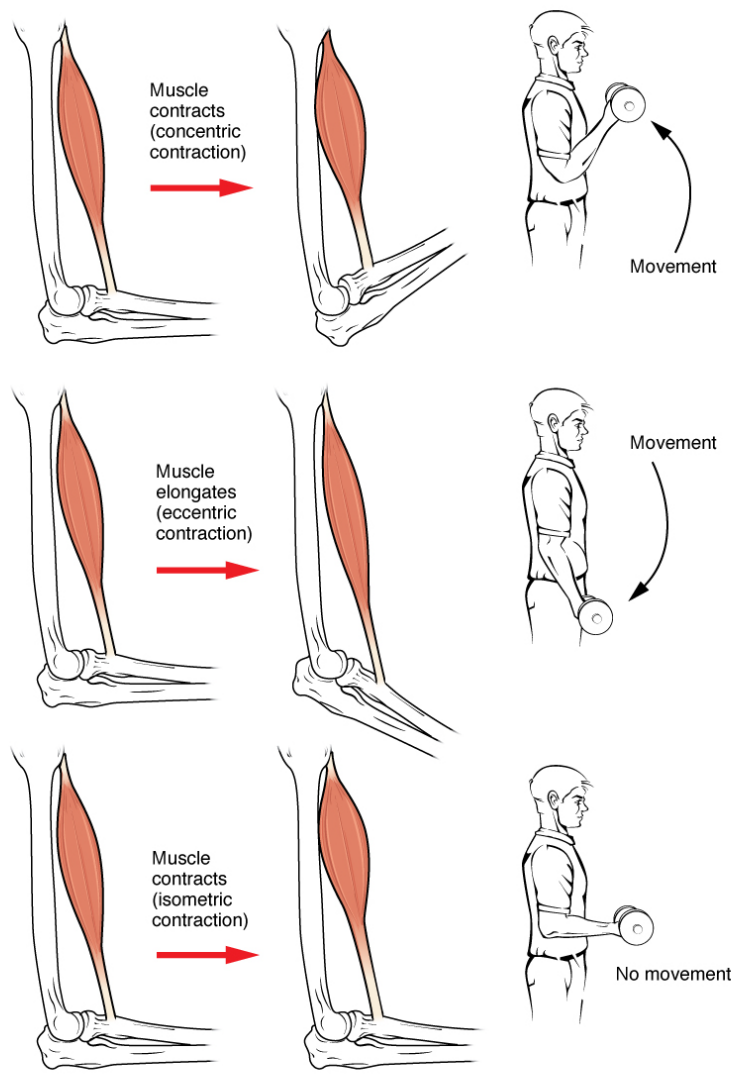

Figure 8.16: Concentric, eccentric and isometric muscle contraction.

Newton's Second Law applied to the block in the $\hat{\mathbf{i}}$-direction requires that $F_{2,1} - f = 0$. Newton's Third Law, applied to the block-rope interaction pair requires that $F_{1,2} = f_{2,1}$, (note this is in terms of magnitudes). Therefore

$$F_{\mathrm{A},2} = F_{1,2} = F_{2,1} = f. \tag{8.23}$$

Thus the applied pulling force is transmitted through the rope to the block since it has the same magnitude as the force of the rope on the block. In addition, the applied pulling force is also equal to the friction force on the block.

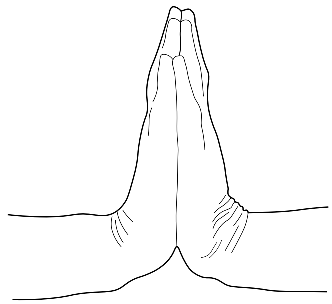

Figure 8.17: Compressive forces applied to the hands.

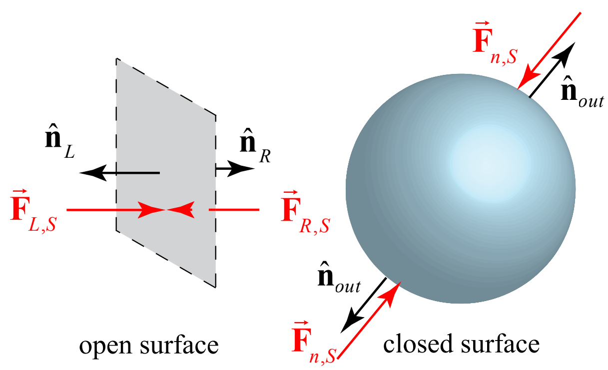

Figure 8.18: Compressive forces applied to (a) an open surface; (b) a closed surface.

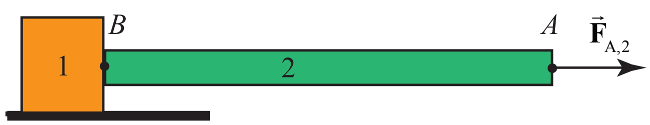

Figure 8.19: Rope pulling a block.

How do we define "tension” at some point in a rope? Suppose make an imaginary slice of the rope at a point P, a distance x_P from the origin at the point B where the rope is attached to the block. The imaginary slice divides the rope into two sections, labeled L (left) and R (right), as shown in Figure 8.20.

There is now a Third Law pair of forces acting between the left and right sections of the rope. Denote the force acting on the left section by $\vec{\mathbf{F}}_{R,L}(x_P)$, and the force acting on the right section by $\vec{\mathbf{F}}_{R,L}(x_P)$. Newton’s Third Law requires that the forces in this

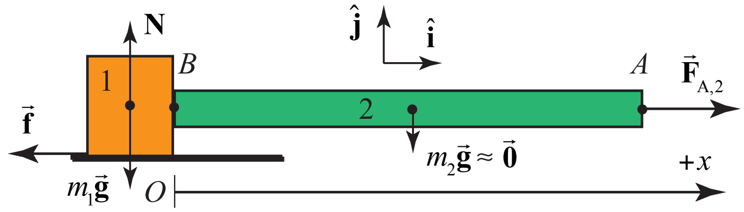

Figure 8.20: Forces on system of rope and block.

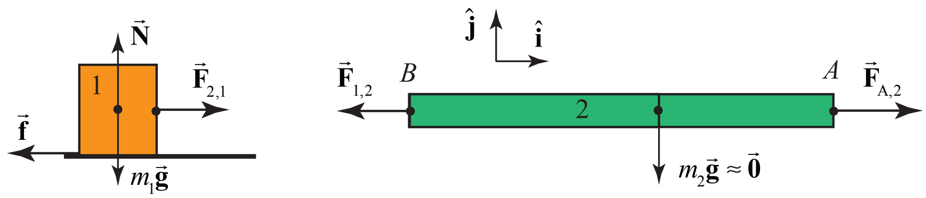

Figure 8.21: Rope pulling a block.

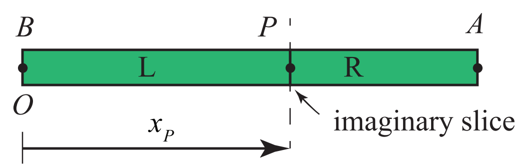

Figure 8.22: Imaginary slice through the rope.

interaction pair are equal in magnitude and opposite in direction.

$$\vec{\mathbf{F}}_{R,L}(x_P) = -\vec{\mathbf{F}}_{R,L}(x_P) \tag{8.24}$$

The force diagram for the left and right sections are shown in Figure 8.23 where $\vec{\mathbf{F}}_{1,L}$ is the force on the left section of the rope due to the block-rope interaction. (We had previously denoted that force by $\vec{\mathbf{F}}_{1,2}$). Now denote the force on the right section of the rope side due to the pulling force at the point A by $\vec{\mathbf{F}}_{A,R}$, (which we had previously denoted by $\vec{\mathbf{F}}_{A,2}$).

The *tension* $T(x_P)$ at a point P in the rope lying a distance x_P from the left end of the rope, is the magnitude of the action-reaction pair of forces acting at the point P,

$$T(x_P) = |\vec{\mathbf{F}}_{R,L}(x_P)| = |\vec{\mathbf{F}}_{L,R}(x_P)| \tag{8.25}$$

For a rope of negligible mass, under tension, as in the above case, (even if the rope is accelerating) the vector sum of the horizontal forces applied to the left section and the

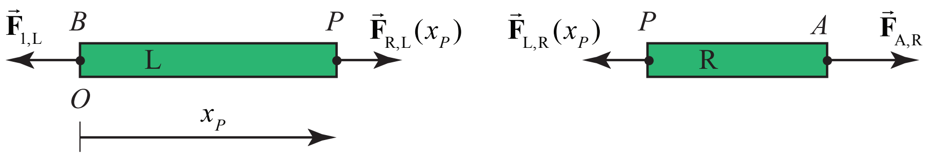

Figure 8.23: Force diagram for the left and right sections of rope.

right section at the point P of the rope are zero, $\vec{\mathbf{F}}_{R,L}(x_P) + \vec{\mathbf{F}}_{L,R}(x_P) = \vec{\mathbf{0}}$. Therefore the tension is uniform and is equal to the applied pulling force,

$$T = F_{A,R} \tag{8.26}$$

8.6.1 Example: Tension in a massive rope

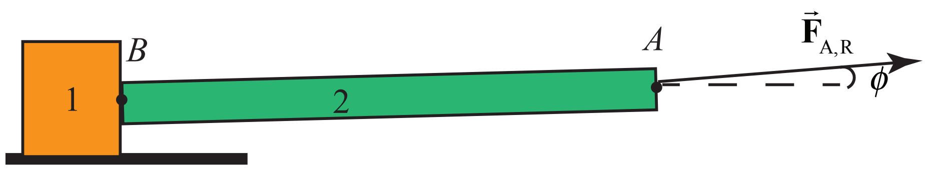

Figure 8.24: Massive rope pulling a block.

Consider a block of mass m_1 that is lying on a horizontal surface. The coefficient of kinetic friction between the block and the surface is μ_k. A uniform rope of mass m_2 and length d is attached to the block. The rope is pulled from the side opposite the block with an applied force of magnitude $|\vec{\mathbf{F}}_{A,2}| \equiv F_{A,2}$. Because the rope is now massive, the pulling force makes an angle ϕ with respect to the horizontal in order to balance the gravitational force on the rope, (Figure 8.24). Determine the tension in the rope as a function of distance x from the block.
Answer

In the following analysis, we shall assume that the angle ϕ is very small and depict the pulling and tension forces as essentially acting in the horizontal direction even though there must be some small vertical component to balance the gravitational forces. The key point to realize is that the rope is now massive and we must take in to account the inertia of the rope when applying Newton's Second Law. Consider an imaginary slice through the rope at a distance x from the block (Figure 8.25), dividing the rope into two sections. The right section has length $d - x$ and mass $m_R = m_2((d - x)/d)$ The left section has length x and mass $m_L = m_2(x/d)$.

The free body force diagrams for the two sections of the rope are shown in Figure 8.26, where $T(x)$ is the tension in the rope at a distance x from the block, and $F_{1,L} =$

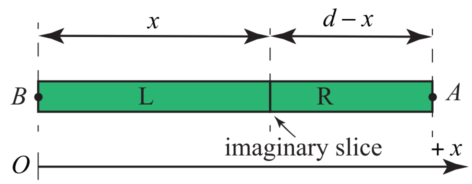

Figure 8.25: Imaginary slice through the rope.

$|\vec{\mathbf{F}}_{1,L}|$ is the magnitude of the force on the left-section of the rope due to the rope-block interaction. Apply Newton's Second Law to the right section of the rope yielding

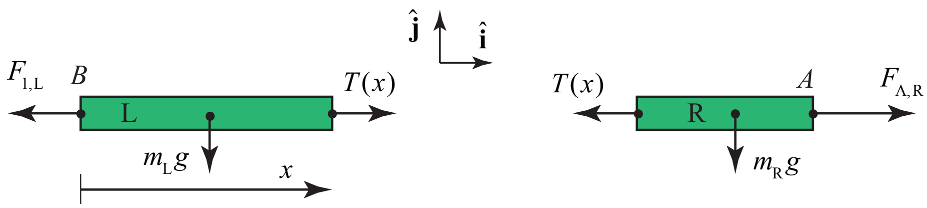

Figure 8.26: Force diagram for the left and right sections of rope.

$$F_{A,R} - T(x) = m_R a_R = m_2 \frac{(d-x)}{d} a_R, \tag{8.27}$$

where a_R is the x-component of the acceleration of the right section of the rope. Apply Newton's Second Law to the left slice of the rope yielding

$$T(x) - F_{1,L} = m_L a_L = m_2 (x/d) a_L, \tag{8.28}$$

where a_L is the x-component of the acceleration of the left piece of the rope. The force

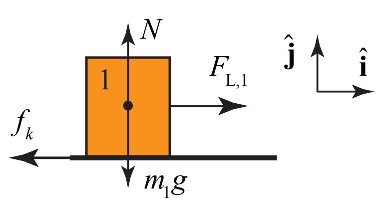

Figure 8.27: Force diagram on sliding block.

diagram on the block is shown in Figure 8.27. Newton's Second Law on the block in the $+\hat{\mathbf{i}}$-direction is $F_{L,1} - f_k = m_1 a_1$ and in the $+\hat{\mathbf{j}}$-direction is $N - m_1 g = 0$. The kinetic friction force acting on the block is $f_k = \mu_k m_1 g$. Newton's Second Law on the block in the $+\hat{\mathbf{i}}$-direction becomes

$$F_{L,1} - \mu_k m_1 g = m_1 a_1, \tag{8.29}$$

Newton's Third Law for the block-rope interaction is given by $F_{L,1} = F_{1,L}$. Equation 8.28 then becomes

$$T(x) - (\mu_k m_1 g + m_1 a_1) = m_2(x/d)a_{\mathrm{L}}. \tag{8.30}$$

Because the rope and block move together, the accelerations are equal which we denote by the symbol $a \equiv a_1 = a_L$. Then Equation 8.31 becomes

$$T(x) = \mu_k m_1 g + (m_1 + m_2(x/d))a. \tag{8.31}$$

This result is not unexpected because the tension is accelerating both the block and the left section and is opposed by the frictional force.

Alternatively, the force diagram on the system consisting of the rope and block is shown in Figure 8.28. Newton's Second Law becomes

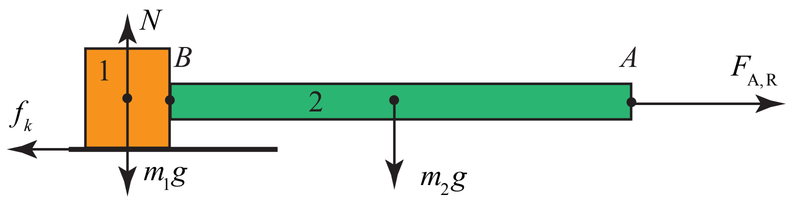

Figure 8.28: Force diagram on block-rope system.

$$F_{A,R} - \mu_k m_1 g = (m_1 + m_2)a. \tag{8.32}$$

Solve Equation for $F_{A,R}$ and substitute into Equation 8.8.3, and solve for the tension $T(x)$ yielding Equation 8.31.

8.6.2 Example: Tension in a suspended rope

A uniform rope of mass M and length L is suspended from a ceiling (Figure 8.6.3(a). The magnitude of the acceleration due to gravity is g.

(a) Find the tension in the rope at the upper end where the rope is fixed to the ceiling.

(b) Find the tension in the rope as a function of the distance from the ceiling.

(c) Find an equation for the rate of change of the tension with respect to distance from the ceiling in terms of M, L, and g. **Answer**

(a) Begin by choosing a coordinate system with the origin at the ceiling and the positive

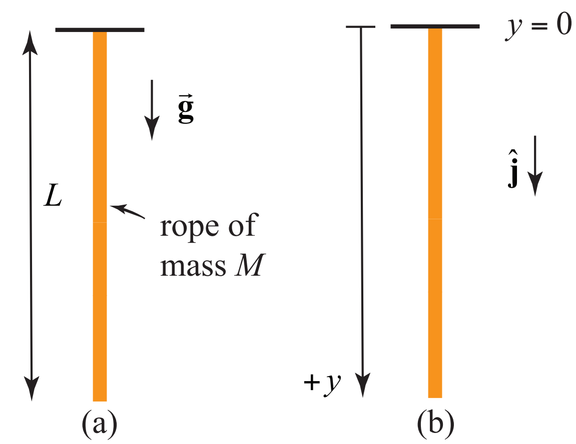

Figure 8.29: (a) Rope suspended from ceiling; (b)Coordinate system for suspended rope.

y-direction pointing downward (Figure 8.6.3(b)). In order to find the tension at the upper end of the rope, choose as a system the entire rope. The forces acting on the rope are the force at $y = 0$ holding the rope up, $T(y = 0)$, and the gravitational force on the entire rope, $M\vec{\mathbf{g}} = Mg\hat{\mathbf{j}}$. The free-body force diagram is shown in Figure 8.30. Because

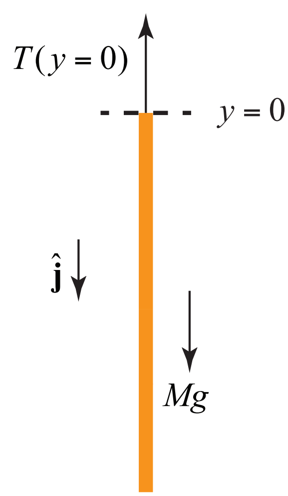

Figure 8.30: Force diagram on rope.

the acceleration is zero, Newton's Second Law on the rope is $Mg - T(y = 0) = 0$. Therefore the tension at the upper end is $T(y = 0) = Mg$.

(b) Recall that the tension at a point is the magnitude of the action-reaction pair of forces acting at that point. Make an imaginary slice in the rope a distance y from the ceiling separating the rope into an upper segment 1, and lower segment 2 (Figure 8.31(a)). Choose the upper segment as a system with mass $m_1 = M(y/L)$. The forces acting on the upper segment are the gravitational force, the force holding the rope up $T(y=0)$, and the tension $T(y)$ at the distance y from the top of the rope that is pulling the upper segment 1 down. The free-body force diagram is shown in Figure 8.31(b).

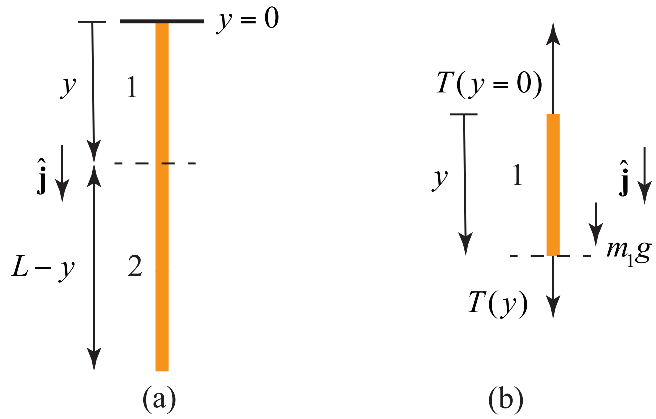

Figure 8.31: (a) Imaginary slice separates rope into two pieces. (b) Free-body force diagram on upper piece of rope.

Apply Newton's Second Law to the upper segment: $m_1 g + T(y) - T(y = 0) = 0$. Therefore the tension at a distance y from the ceiling is $T(y) = T(y = 0) - m_1 g$. Because $m_1 = M(y/L)$ is the mass of the segment 1 and $T(y = 0) = Mg$ is the tension at the upper end, Newton's Second Law becomes

$$T(y) = Mg(1 - (y/L)). \tag{8.33}$$

As a check, we note that when $y = L$, the tension $T(y = L) = 0$, which is what we expect because there is no force acting at the lower end of the rope.

(c) Differentiate Equation 8.33 with respect to y:

$$\frac{dT}{dy} = -\frac{Mg}{L}. \tag{8.34}$$

The rate that the tension is changing at a constant rate with respect to distance from the top of the rope.

8.6.3 Continuous systems and Newton's Second Law as a differential equation

We can determine the tension at a distance y from the ceiling in the Example 8.6.2, by an alternative method, a technique that will generalize to many types of "continuous systems". Choose a coordinate system with the origin at the ceiling and the positive y-direction pointing downward as in Figure (b). Consider as the system a small element of the rope between the points y and $y+\Delta y$. This small element has length Δy and mass $\Delta m = M(\Delta y/L)$ and is shown in Figure 8.32. The forces acting on the small element

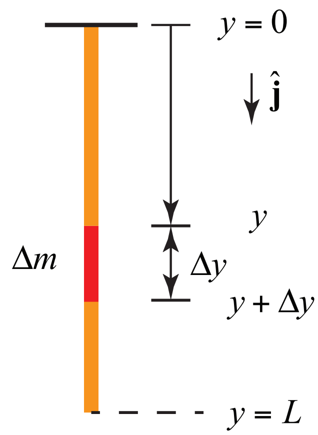

Figure 8.32: Small mass element of the rope.

are the tension, $T(y)$ at y directed upward, the tension $T(y+\Delta y)$ at $y+\Delta y$ directed downward, and the gravitational force Δmg directed downward. The tension $T(y+\Delta y)$ is equal to the tension $T(y)$ plus a small difference ΔT,

$$T(y+\Delta y) = T(y) + \Delta T. \tag{8.35}$$

The small difference in general can be positive, zero, or negative. The free body force diagram is shown in Figure 8.33. Now apply Newton's Second Law to the small element

$$\Delta mg + T(y) + \Delta T - T(y) = 0 \tag{8.36}$$

The difference in the tension is then $\Delta T = -\Delta mg$. We now substitute our result for the mass of the element $\Delta m = M(\Delta y/L)$, and find that

$$\Delta T = -M(\Delta y/L)g. \tag{8.37}$$

Divide through by Δy, yielding $\Delta T/\Delta y = -(M/L)g$. Now take the limit in which the length of the small element goes to zero, $\Delta y \to 0$,

$$\lim_{\Delta y \to 0} \frac{\Delta T}{\Delta y} = -(M/L)g. \tag{8.38}$$

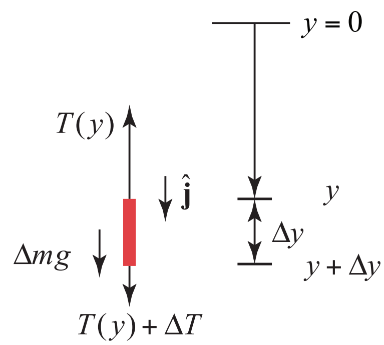

Figure 8.33: Free body force diagram on small mass element.

The left hand side of Equation is the definition of the derivative of the tension with respect to y, and so the differential equation describing how the tension varies with position along the rope is given by

$$\frac{dT}{dy} = -\frac{M}{L}g.$$

in agreement with 8.34.

We can solve this differential equation by a technique called *separation of variables*. We rewrite the equation as $dT = -(M/L)g dy$ and integrate both sides. Our integral will be a definite integral in which we integrate a 'dummy' integration variable y' from $y' = 0$ to $y' = y$ and the corresponding tension integration variable T' from $T' = T(y = 0)$ to $T' = T(y)$,

$$\int_{T'=T(y=0)}^{T'=T(y)} dT' = -(M/L)g \int_{y'=0}^{y'=y} dy'. \tag{8.39}$$

After integration and substitution of the limits, we have that

$$T(y) - T(y = 0) = -(M/L)gy. \tag{8.40}$$

Use the fact that tension at the top of the rope is $T(y = 0) = Mg$ and find that

$$T(y) = Mg(1 - (y/L)),$$

in agreement with our earlier result 8.33.

8.7 Simple Harmonic Motion

Our first example of a system that demonstrates simple harmonic motion is a spring-object system on a frictionless surface, shown in Figure 8.34. The object is attached

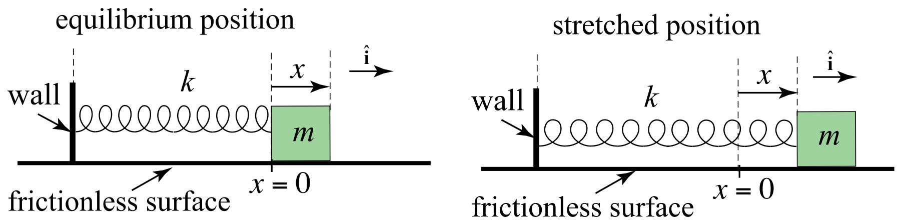

Figure 8.34: (a) Equilibrium position of a spring-object system; (b) stretched position of spring-object system.

to one end of a spring. The other end of the spring is attached to a wall at the left in Figure 8.34. Assume that the object undergoes one-dimensional motion. The spring has a spring constant k and equilibrium length l_{eq}. Choose the origin at the equilibrium position and choose the positive x-direction to the right in the Figure 8.34. In the figure, $x > 0$ corresponds to an extended spring, and $x < 0$ to a compressed spring. Define $x(t)$ to be the position of the object with respect to the equilibrium position.

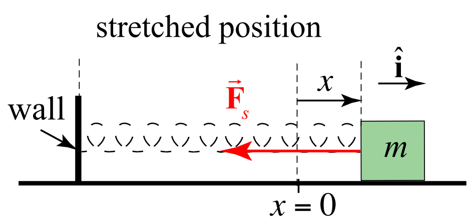

Figure 8.35: (Free-body force diagram on object.

The force acting on the object is a linear restoring force, $\vec{\mathbf{F}}_s = F_{x,s}\hat{\mathbf{i}} = -kx\hat{\mathbf{i}}$. The free body force diagram is shown in (Figure 8.35). At time t, Newton's Second Law is

$$-kx = md^2x/dt^2. \tag{8.41}$$

This equation of motion, Equation 8.41, is called the *simple harmonic oscillator equation* (SHO). Because the spring force depends on the stretched (or compressed) distance form equilibrium $x(t)$ the acceleration is not constant. Equation 8.41 is a second order linear differential equation, in which the second derivative of the dependent variable is proportional to the negative of the dependent variable,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{8.42}$$

In this case, the constant of proportionality is k/m. Equation 8.42 can be solved exactly using energy considerations or other advanced techniques but instead we shall first guess the solution and then verify that the guess satisfies the SHO differential equation.

We are looking for a position function $x(t)$ such that the second time derivative position function is proportional to the negative of the position function. Because the sine and cosine functions both satisfy this property, we make a preliminary *ansatz* (educated guess) that our position function is given by

$$x(t) = A\cos(\omega_0\, t), \tag{8.43}$$

where ω_0 and A are as of yet an undetermined constants.

We shall now find the condition that the constant ω_0 must satisfy in order to insure that the function in Equation 8.43 solves the simple harmonic oscillator equation, Equation 8.41. The first and second derivatives of the position function are given by

$$\begin{aligned} \tfrac{dx}{dt} &= -\omega_0 A\sin(\omega_0 t) \\ \tfrac{d^2x}{dt^2} &= -\omega_0{}^2 A\cos(\omega_0\, t) = -\omega_0{}^2 x. \end{aligned} \tag{8.44}$$

Substitute the second derivative, the second expression in Equation 8.44, and the position function, Equation 8.43, into the SHO Equation 8.41, yielding

$$-\,\omega_0{}^2 A\cos(\omega_0\, t) = -\frac{k}{m}A\cos(\omega_0\, t). \tag{8.45}$$

Equation 8.45 is valid for all times provided that

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{8.46}$$

One possible solution for the position of the block is then

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\, t\right). \tag{8.47}$$

We will now determine the constant A for an spring-object system that is stretched and then released form rest. At $t = 0$, the position of the object is $x_0 = l_0 - l_{eq}$. Evaluating Equation 8.47 at $t = 0$ yields

$$x(0) = A\cos\left(\sqrt{\frac{k}{m}}\, 0\right) = A, \tag{8.48}$$

because $\cos 0 = 0$. Therefore the position function for the spring-object system is then

$$x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}\, 0\right), \tag{8.49}$$

where $x_0 = l_0 - l_{eq}$.

8.8 Worked Examples

8.8.1 Example: Block falling down a staircase

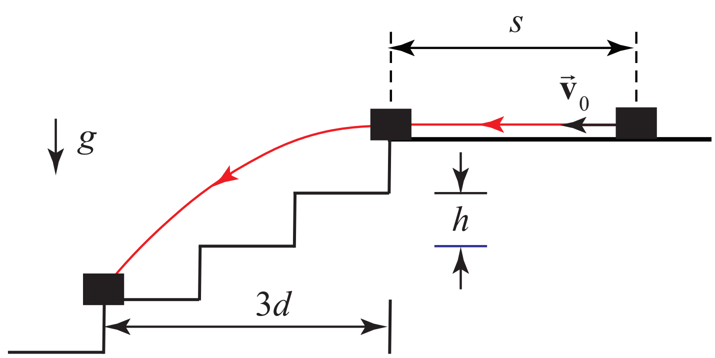

Figure 8.36: Block falling down staircase.

A block of mass m at time $t = 0$ has velocity $\vec{\mathbf{v}}_0$, (speed v_0),. It slides a distance s along a horizontal floor and then falls off the top of a staircase (Figure 8.36). The coefficient of kinetic friction between the block and the floor is μ_k. The block strikes at the far end of the third stair, a horizontal distance $3d$ from the edge of the floor. Each stair has a rise of h and a run of d. Neglect air resistance and use g for the gravitational constant. What is the distance s that the block slides along the floor?

Answer

Overview and strategy: There are two distinct stages to the block's motion, the initial horizontal motion and then free fall. We can describe the horizontal motion using Newton's Second Law and use the equations fo motion for two-dimensional free fall..The given final position of the block, at the far end of the third stair, will determine the horizontal component of the velocity at the instant the block left the top of the stairs. This in turn can be used to determine the time the block decelerated along the floor, and hence the distance traveled on the floor. The given quantities are m, v_0, μ_k, g, h and d.

For the horizontal motion, choose coordinates with the origin at the initial position of the block. Choose $\hat{\mathbf{i}}$ to be horizontal, directed to the left in Figure 8.37, and $\hat{\mathbf{j}}$ to be vertical (up). At time t, the position function of the block is $\vec{\mathbf{r}}(t)$. The forces on the object are gravity $\vec{\mathbf{g}}$, the normal force $\vec{mathbfN}$ and the kinetic frictional force $\vec{\mathbf{f}}_k$. The vector components of Newton's Second Law are

$$\begin{aligned} -f_k &= m\,a_x \\ N - m\,g &= 0 \end{aligned} \tag{8.50}$$

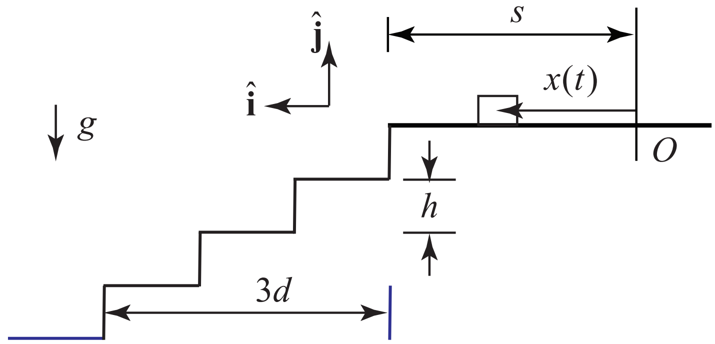

Figure 8.37: Block falling down staircase.

The magnitude of the frictional force is $f_k = \mu N = \mu mg$, therefore

$$-\mu_k mg = m\, a_x. \tag{8.51}$$

Hence the x-component of acceleration is

$$a_x = -\mu_k g. \tag{8.52}$$

Because the acceleration is constant the x-component of the velocity is given by

$$v_x(t) = v_0 - \mu_k g\, t. \tag{8.53}$$

The initial position at $x(t=0) = 0$, hence the displacement is given by

$$x(t) = v_0\, t - \frac{1}{2}\mu_k g\, t^2. \tag{8.54}$$

Let t_1 denote the time the block just leaves the landing at $x(t_1) = s$, and denote the speed just when it reaches the landing by $v_x(t_1) = v_1$. Then at time t_1, Equation 8.54 becomes

$$s = v_0\, t_1 - \frac{1}{2}\mu_k g\, {t_1}^2, \tag{8.55}$$

and Equation 8.53 becomes

$$v_1 = v_0 - \mu_k g\, t_1. \tag{8.56}$$

The time t_1 when the block reaches the edge of the landing depends on the yet to be determined v_1:

$$t_1 = \frac{v_0 - v_1}{\mu_\mathrm{k} g}. \tag{8.57}$$

SubstituteEquation 8.57 into Equation 8.55 to find an expression for the distance s in terms of the speed v_1:

$$s = v_0\left(\frac{v_0 - v_1}{\mu_\mathrm{k} g}\right) - \frac{1}{2}\mu_k g\left(\frac{v_0 - v_1}{\mu_\mathrm{k} g}\right)^2 = \frac{{v_0}^2 - {v_1}^2}{2\mu_\mathrm{k} g}. \tag{8.58}$$

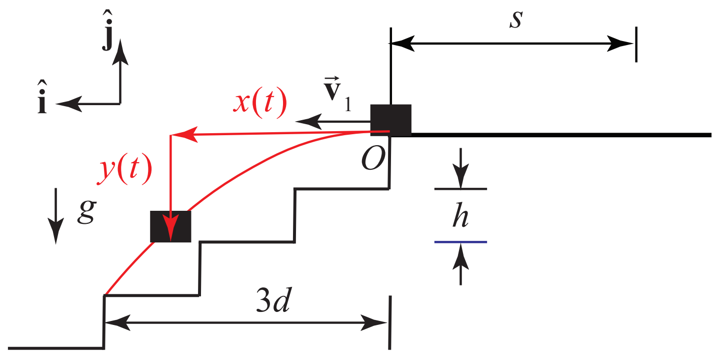

Figure 8.38: Coordinate system for block in free fall.

The block is in free fall from the top of the stair to the far end of the third stair. We will use the fact hat we know where the block hots the stairs to determine the time of fall and the speed v_1. For this motion, keep the same unit vectors but now choose the origin at the top of the stairs, and set $t = 0$ when the object first goes into free fall. The components of acceleration are $a_x = 0$ and $a_y = -g$. The initial x-component of velocity is v_1, the initial y-component of velocity $v_{y,0} = 0$. the initial position components are $x(t = 0) = 0$ and $y(t = 0) = 0$. Let t_2 denote the instant the object hits the stair, where $y(t_2) = -3h$ and $x(t_2) = 3d$. The equations describing the components of velocity at time t_2 are

$$\begin{aligned} v_x(t_2) &= v_{x,1}\,, \\ v_y(t_2) &= -gt_2. \end{aligned} \tag{8.59}$$

The equations describing the components of position at time t_2 are

$$\begin{aligned} x(t_2) &= 3d = v_1\,t_2, \\ y(t_2) &= -3h = -\tfrac{1}{2}g{t_2}^2. \end{aligned} \tag{8.60}$$

The time t_2 is related to the speed v_1 by the horizontal position equation (Equation 8.60)

$$t_2 = \frac{3d}{v_1}. \tag{8.61}$$

Now substitute this expression for t_2 into the vertical position equation (Equation 8.60) and solve for ${v_1}^2$

$${v_1}^2 = \frac{3gd^2}{2h}. \tag{8.62}$$

Finally substitute ${v_1}^2$ from Equation 8.62 into Equation 8.58 and solve for the desired distance s

$$s = \frac{v_0^2 - (3gd^2/2h)}{2\mu_k g}. \tag{8.63}$$

8.8.2 Example: Cart moving on a track

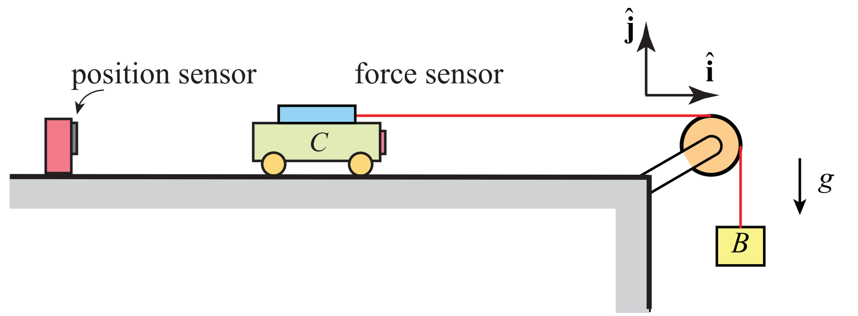

Figure 8.39: A falling block will accelerate a cart on a track via the pulling force of the string. The force sensor measures the tension in the string.

Consider a cart C that is free to slide along a horizontal track (Figure 8.39). A force is applied to the cart via a string that is attached to a force sensor mounted on the cart, wrapped around a pulley and attached to a block B on the other end. When the block is released the cart will begin to accelerate. The force sensor and cart together have a mass m_c, and the suspended block has mass m_B. Neglect the small mass of the string and pulley, and assume the string is inextensible. The coefficient of kinetic friction between the cart and the track is μ_k.

(a) What is the acceleration of the cart?

(b) What is the tension in the string?

Answer

In general, we would like to draw free-body diagrams on all the individual objects (cart, sensor, pulley, rope, and block) but we can also choose a system consisting of two (or more) objects knowing that the forces of interaction between any two objects will cancel in pairs by Newton's Third Law. In this example, we shall choose the sensor/cart as one body, and the block as the other body. The free-body force diagram for the sensor/cart with a choice of unit vectors is shown in Figure 8.40.

There are three forces acting on the sensor/cart: the gravitational force $\vec{\mathbf{g}} = -m_C g\hat{\mathbf{j}}$, the pulling force $\vec{\mathbf{T}}_{R,C} = T\hat{\mathbf{i}}$ of the rope on the force sensor, and the contact force between the track and the cart. In Figure 8.40, we decompose the contact force into its two components, the kinetic frictional force $\vec{\mathbf{f}}_k = -f_k\hat{\mathbf{i}}$ and the normal force, $\vec{\mathbf{N}} = N\hat{\mathbf{j}}$.

The cart is only accelerating in the horizontal direction with $\vec{\mathbf{a}}_C = a_C\hat{\mathbf{i}}$. The component of the force in the vertical direction must be zero. We can now apply Newton's

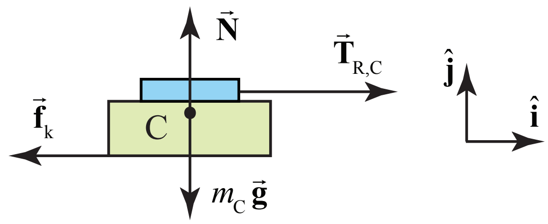

Figure 8.40: Force diagram on sensor/cart with a vector decomposition of the contact force into horizontal and vertical components.

Second Law in the horizontal and vertical directions and find that

$$\begin{aligned} \hat{\mathbf{i}}: &\quad T_{\mathrm{R,C}} - f_{\mathrm{k}} = m_{\mathrm{C}} a_{\mathrm{C}}, \\ \hat{\mathbf{j}}: &\quad N - m_{\mathrm{C}} g = 0. \end{aligned} \tag{8.64}$$

The magnitude of the kinetic frictional force is

$$f_k = \mu_k N = \mu_k m_{\mathrm{C}} g. \tag{8.65}$$

Thus the horizontal component of Newton's Second Law (Equation 8.64) becomes

$$T_{\mathrm{R,C}} - \mu_{\mathrm{k}} m_{\mathrm{C}} g = m_{\mathrm{C}} a_{\mathrm{C}}. \tag{8.66}$$

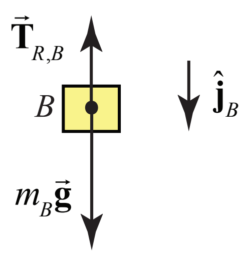

Figure 8.41: Forces acting on the block.

The force diagram for the block is shown in Figure 8.41. Note that we chose the unit vector $\hat{\mathbf{j}}_B$ pointing downward. Note that we made a different choice of direction for the unit vector in the vertical direction in the free-body diagram for the block shown in Figure 8.40. Each free-body diagram has an independent set of unit vectors that define a sign convention for vector decomposition of the forces acting on the free-body and the acceleration of the free-body. In our example, with the unit vector pointing downwards in Figure 8.41, if we solve for the component of the acceleration and it is positive, then we know that the direction of the acceleration is downwards. The two forces acting on the block are the pulling force of the string, $\vec{\mathbf{T}}_{R,B} = -T\hat{\mathbf{j}}_B$ and the gravitational force $m_B\vec{\mathbf{g}} = m_B g\hat{\mathbf{j}}_B$. We now apply Newton's Second Law to the block where the acceleration vector of the block is $\vec{\mathbf{a}}_B = a_B\hat{\mathbf{j}}_B$:

$$\hat{\mathbf{j}}_B : m_B g - T_{R,B} = m_B a_B. \tag{8.67}$$

There is a second subtle way that signs are introduced with respect to the forces acting on a free-body. In our example, the force between the string and the block acting on the block points upwards, so in the vector decomposition of the forces acting on the block that appears on the left-hand side of Equation 8.66, this force has a minus sign and the quantity $T_{R,B}$ is assumed positive.

Our assumption that the mass of the rope and the mass of the pulley are negligible enables us to assert that the tension in the rope is uniform and equal in magnitude to the forces at each end of the rope,

$$T_{R,B} = T_{R,C} \equiv T. \tag{8.68}$$

We also assumed that the string is inextensible (does not stretch). This implies that the rope, block, and sensor/cart all have the same magnitude of acceleration,

$$a_C = a_B \equiv a. \tag{8.69}$$

We can now rewrite the horizontal equation of motion for the sensor/cart, Equation 8.64, as

$$T - \mu_k m_C g = m_C a, \tag{8.70}$$

and the equation of motion (Equation 8.67 for the block as

$$m_B g - T = m_B a. \tag{8.71}$$

We have only two unknowns T and a, so we can now solve the two equations (Equation 8.70) and (Equation 8.71) simultaneously for the acceleration of the sensor/cart and the tension in the rope. We first solve Equation 8.70 for the tension

$$T = \mu_k m_C g + m_C a. \tag{8.72}$$

and then substitute T into Equation 8.71 and solve for the acceleration

$$a = \frac{m_B g - \mu_k m_C g}{m_C + m_B}. \tag{8.73}$$

Now substitute a into Equation 8.70 and solve for the tension in the rope:

$$T = (\mu_k + 1)\frac{m_C m_B}{m_C + m_B} g. \tag{8.74}$$

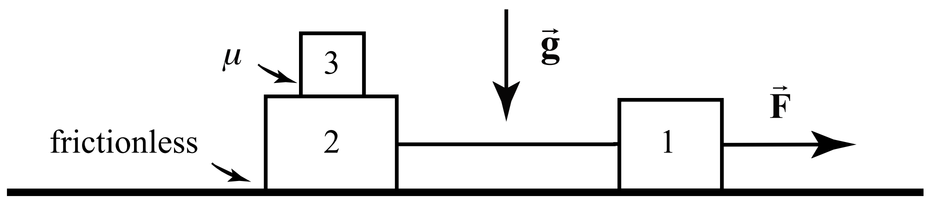

Figure 8.42: Rope pulling lower block 2.

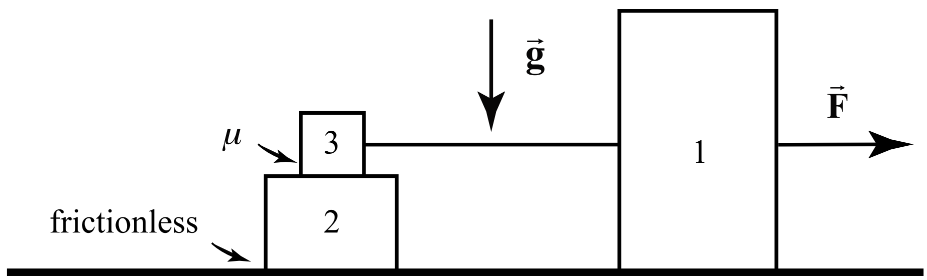

Figure 8.43: Rope pulling upper block 3.

8.8.3 Pulling blocks

(a) A force of magnitude $F \equiv |\vec{\mathbf{F}}|$ is applied to block 1 as shown in Figure 8.42. Block 1 and block 2 are connected by an inextensible massless rope. Block 3 is resting on top of block 2. The masses of blocks 1,2 and 3 are m_1, m_2 and m_3 respectively, where $m_3 < m_2$. The coefficient of static friction between block 2 and block 3 is μ_s. The table is frictionless. The gravitational acceleration g is directed downward. What is the magnitude of the applied force F such that block 3 just slips? The jus slipping condition means that blocks 2 and 3 are still at rest relative to each other. Any stronger applied force F would result in block 2 moving forward with respect to block 3. Express the answer in terms of the quantities m_1, m_2, m_3, μ_s and g .

(b) Now suppose a string is attached between block 1 and block 3 (Figure 8.43). What is the magnitude of the applied force F such that block 3 just slips?

Answer

Strategy: We shall apply Newton's Second Law to each block. The just slipping condition is actually two conditions: one, the friction force between the two blocks is maximum; and two, the blocks are at rest with respect to each other so they have the same acceleration.

Choose positive $\hat{\mathbf{i}}$-direction to the right and positive $\hat{\mathbf{j}}$-direction up. The free body force

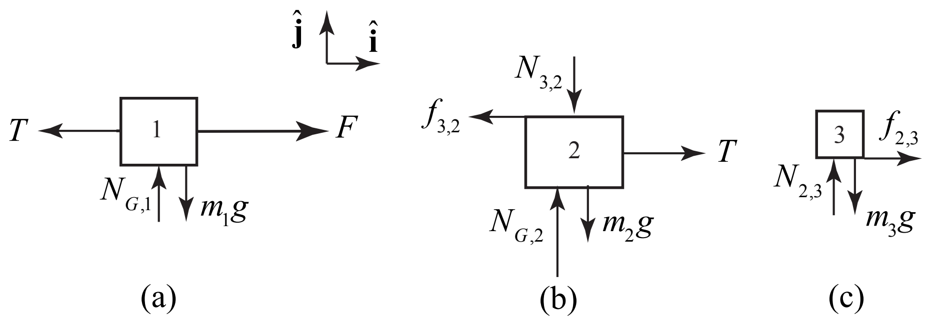

Figure 8.44: Free body force diagrams on (a) block 1, (b) block 2, (c) block 3, for the case when the rope pulls the lower block 2.

diagrams on the three blocks are shown in Figure 8.44. When the magnitude of the applied force is greater than a maximum magnitude, block 3 just slips. Therefore at that maximum value all three blocks have the same acceleration a.

Let T denote the tension in the string connecting blocks 1 and 2. Newton's Second Law applied to block 1 in the horizontal direction is

$$\hat{\mathbf{i}}: \; F - T = m_1 a \tag{8.75}$$

Newton's Second Law applied to block 1 in the vertical direction is

$$\hat{\mathbf{j}}: \; N_{G,1} - m_1 g = 0, \tag{8.76}$$

where $N_{G,1}$ is the normal force between the block 1 and the ground acting on block 1.

Newton's Second Law applied to block 2 in the horizontal direction is

$$\hat{\mathbf{i}}: \; T - f_{3,2} = m_2 a, \tag{8.77}$$

where $f_{3,2}$ is the magnitude of the friction force between blocks 2 and 3 acting on block 2. Newton's Second Law applied to block 2 in the vertical direction is

$$\hat{\mathbf{j}}: \; N_{G,2} - N_{3,2} - m_2 g = 0, \tag{8.78}$$

where $N_{G,2}$ is the normal force between block 2 and the ground acting on block 2 and $N_{3,2}$ is the normal force between blocks 2 and 3 acting on block 2.

Newton's Second Law applied to block 3 in the horizontal direction is

$$\hat{\mathbf{i}}: \; f_{2,3} = m_3 a. \tag{8.79}$$

where $f_{2,3}$ is the magnitude fo the friction force between blocks 2 and 3 acting on block 3. The two friction forces between blocks 2 and 3 form a Third Law pair and their magnitude are equals, $f \equiv f_{2,3} = f_{3,2}$. Newton's Second Law applied to block 3 in the vertical direction is

$$\hat{\mathbf{j}}: \; N_{2,3} - m_3 g = 0 \tag{8.80}$$

where $N_{2,3}$ is the normal force between blocks 2 and 3 acting on block 3. The two normal forces between blocks 2 and 3 form a Third Law pair and their magnitude are equal, $N \equiv N_{2,3} = N_{3,2}$.

The condition that the blocks just slip, is the that the friction force $f_{2,3}$ between blocks 2 and 3 is equal to

$$f_{2,3} = \mu_s N_{2,3} = \mu_s m_3 g. \tag{8.81}$$

We can now solve for the acceleration of the blocks by substituting Equation 8.81 into Equation 8.79. Thus

$$a = \mu_s g \tag{8.82}$$

We can solve for the tension in the rope by combining Equations 8.77, 8.79 and 8.82, with the result that

$$T = (m_2 + m_3)\mu_s g. \tag{8.83}$$

We can now solve for the maximum value of the applied force by substituting Equation 8.83 and the value $a = \mu_s g$ into Equation 8.75:

$$F = (m_1 + m_2 + m_3)\mu_s g. \tag{8.84}$$

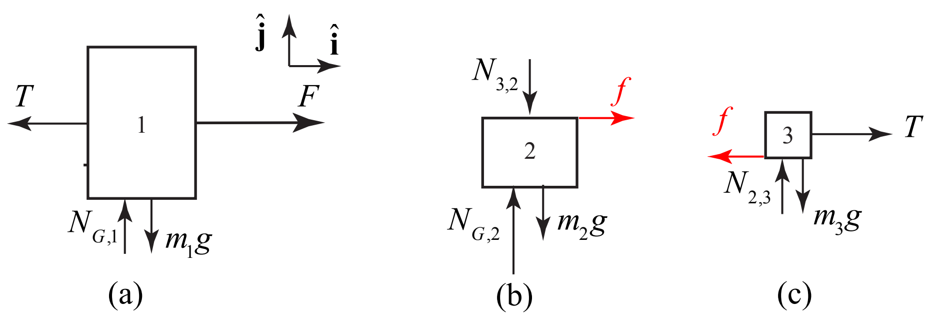

Figure 8.45: Free body force diagrams on (a) block 1, (b) block 2, (c) block 3, for the case when the rope pulls the upper block 3.

(b) When the rope pulls the upper block 2, the free body force diagrams are shown in Figure 8.45. Notice that the directions for the friction forces between blocks 2 and 3 are now switched. from the case in part (a).

The equations describing Newton's Second Law for block 1 have not changed and the condition describing the maximum value for the friction force has also not changed, $f = \mu_s m_3 g$. What has changed from part (a) are the horizontal equations for block 2 and block 3.

For block 2:

$$f = \mu_s m_3 g = m_2 a. \tag{8.85}$$

Thus the acceleration a is now

$$a = \frac{m_3}{m_2}\mu_s g. \tag{8.86}$$

Because $m_3 < m_2$, the acceleration at the just slipping condition in part (b) is less than in part (a). This is not that surprising. If you try to pull the lighter block 3, it will slip more quickly than if you try to pull the heavier block 2.

For block 3:

$$T - f = m_3 a. \tag{8.87}$$

The tension and maximum force have the same form but both are smaller due to the change in the acceleration. The tension is

$$T = (m_2 + m_3)a = \frac{(m_2 + m_3)(m_3)}{m_2}\mu_s g. \tag{8.88}$$

The maximal force is

$$F = (m_1 + m_2 + m_3)a = \frac{(m_1 + m_2 + m_3)(m_3)}{m_2}\mu_s g. \tag{8.89}$$

Check the result

In the limit that the lower block is much more massive than the upper block, $m_3 << m_2$, the just-slipping acceleration goes to zero, $a = (m_3/m_2)\mu_s g \rightarrow 0$. The lower block will not move at all, and the maximum force

$$F = ((m_1 + m_2 + m_3)(m_3)/m_2)\mu_s g \rightarrow m_3\mu_s g,$$

which is the maximum static friction force.

8.8.4 Example: Two oscillating springs

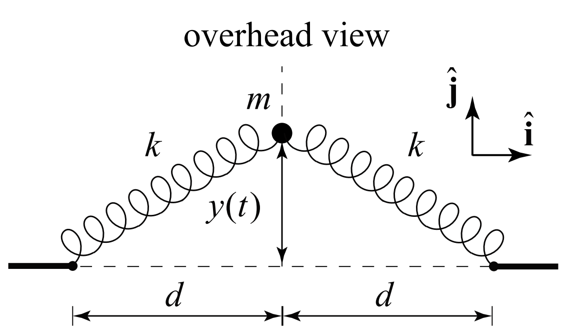

Figure 8.46: Overhead view of two oscillating springs on a frictionless table.

Two identical massless springs of spring constant k are fixed in place on a horizontal frictionless table. The equilibrium length of each spring is l_{eq}. The fixed points are separated by a distance $2d$. An object of mass m is attached to the free ends of both springs and is free to move on the table. At time t, the object lies on the perpendicular bisector of the line connecting the two fixed points at a distance $y(t)$ above the midpoint of that line. Find a differential equation that the acceleration $d^2/y/dt^2$ of the object satisfies.

Answer

Strategy: We shall first apply Newton's Second Law to each block. The just slipping condition is that the friction force between the two

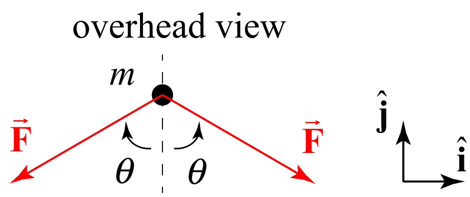

Figure 8.47: Free-body force diagram on object attached to two springs.

The free-body force diagram on the object is shown in Figure 8.47. Because the equilibrium length of the spring is l_0 , each spring has been stretched by an amount

$$\Delta l = (d^2 + y^2)^{1/2} - l_{eq}. \tag{8.90}$$

The magnitude of each spring force is

$$F \equiv \left|\vec{\mathbf{F}}\right| = k\Delta l = k((d^2 + y^2)^{1/2} - l_{eq}) \tag{8.91}$$

Newton's Second Law in the $bf\,\hat{j}$-direction is

$$-2F\cos\theta = m\frac{d^2y}{dt^2} \tag{8.92}$$

From the geometry of the arrangement

$$\cos\theta = \frac{y}{(d^2 + y^2)^{1/2}} \tag{8.93}$$

Substituting nthe expression for F (Equation 8.91) and $\cos\theta$ (Equation 8.93) into Newton;'s Second Law (Equation 8.92) yields

$$-\frac{2k((d^2 + y^2)^{1/2} - l_{eq})}{(d^2 + y^2)^{1/2}}y = m\frac{d^2y}{dt^2}. \tag{8.94}$$

The motion is *periodic* in that the object oscillates back and forth but it is not a simple harmonic oscillation. In the limit that the unstretched length of the springs goes to zero, $l_0 \to 0$, then Equation 8.94 becomes

$$-2ky = m\frac{d^2y}{dt^2}.. \tag{8.95}$$

This is the equation for simple harmonic motion that is equivalent to the oscillation of a single object-spring with *effective spring constant* $2k$.

8.8.5 Example: Constraint conditions and analysis of motion for system consisting of pulleys, blocks and ropes

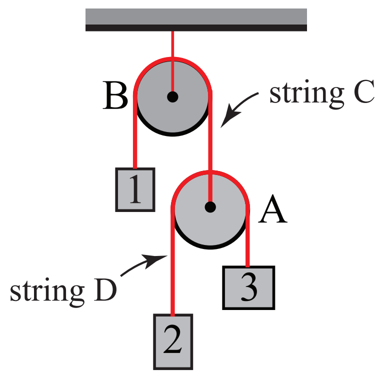

Figure 8.48: Constrained pulley system .

Consider the arrangement of pulleys and blocks shown in Figure 8.48. The pulleys are assumed massless and frictionless and the connecting strings are massless and inextensible. Denote the respective masses of the blocks as m_1, m_2 and m_3. The upper pulley in the figure is free to rotate but its center of mass does not move. Both pulleys have the same radius R.

(a) How are the accelerations of the objects related?

(b) Draw force diagrams on each moving object.

(c) Solve for the accelerations of the objects and the tensions in the ropes.

Answer

Strategy: We will separate the system into individual masses and pulleys. For each object draw free body force diagrams and apply Newton's Second Law. In order to find the constraint conditions between accelerations of various objects, we will use the fact that the lengths of the strings C and D are constant. We shall express these lengths in terms of the coordinate functions for each of the moving objects. Because the first and second derivatives of the lengths are zero, setting the second derivative equal to zero will establish relationships between the various accelerations of the objects. Recall that the acceleration $a_y(t)$ of an object is related to the second derivative of the position function $y(t)$ according to $a_y(t) = d^2y(t)/dt^2$

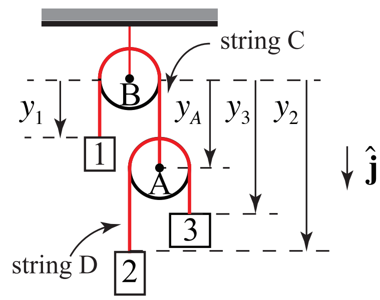

Figure 8.49: Coordinate system for pulleys and blocks pulley.

(a) Choose an origin at the center of the upper pulley. Introduce coordinate functions for the three moving blocks, y_1, y_2 and y_3. Introduce a coordinate function for the moving pulley A, y_A. Choose downward for the unit vector $\hat{\mathbf{J}}$; the coordinate system is shown in Figure.8.49.

The length of string C is given by

$$l_C = y_1 + y_A + \pi R. \tag{8.96}$$

where πR is the arc length of the rope that is in contact with the pulley. Because the rope is assumed to be inextensible, this length l_C is constant and therefore the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_C}{dt^2} = \frac{d^2 y_1}{dt^2} + \frac{d^2 y_A}{dt^2} = a_1 + a_A. \tag{8.97}$$

Thus block 1 and the moving pulley's components of acceleration are equal in magnitude but opposite in sign,

$$a_A = -a_1. \tag{8.98}$$

The length of string D is given by

$$l_D = (y_3 - y_A) + (y_2 - y_A) + \pi R = y_3 + y_2 - 2y_A + \pi R. \tag{8.99}$$

The length of the rope l_D is constant and so the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_D}{dt^2} = \frac{d^2 y_2}{dt^2} + \frac{d^2 y_3}{dt^2} - 2\frac{d^2 y_A}{dt^2} = a_2 + a_3 - 2a_A. \tag{8.100}$$

Substitute the acceleration a_A (Equation 8.98) into Equation 8.98 to find the constraint condition between the accelerations of the three blocks

$$0 = a_2 + a_3 + 2a_1. \tag{8.101}$$

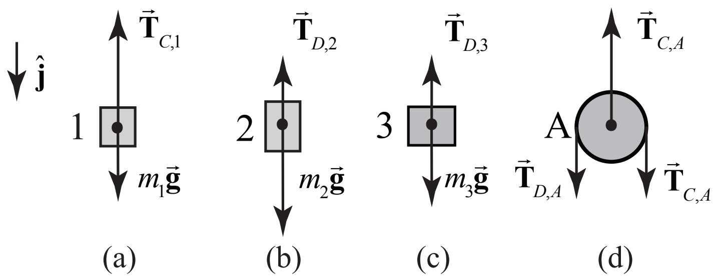

Figure 8.50: Free-body force diagram on (a) block 1; (b) block 2; (c) block 3; (d) pulley.

b) Free-body Force diagrams: the forces acting on block 1 are: the gravitational force $m_1\vec{\mathbf{g}} = m_1 g\hat{\mathbf{j}}$ and the pulling force $\vec{\mathbf{T}}_{C,1} = -T_C\hat{\mathbf{j}}$ of string C acting on the block 1, where the magnitude of this force is T_C. Because the string is assumed to be massless and the pulley is assumed to be massless and frictionless, the tension in the string is uniform and equal in magnitude to the pulling force of the string on the block. The free-body diagram on block 1 is shown in Figure 8.50(a).

Newton's Second Law applied to block 1 is then

$$\hat{\mathbf{j}}: \quad m_1 g - T_C = m_1\, a_1. \tag{8.102}$$

The forces on block 2 are the gravitational force $m_2\vec{\mathbf{g}} = m_2 g\hat{\mathbf{j}}$ and the string supporting the block $\vec{\mathbf{T}}_{D,2} = -T_D\hat{\mathbf{j}}$, with magnitude T_D. The free-body diagram for the forces acting on block 2 is shown in Figure 8.50(b). Newton's Second Law applied to block 2 is

Newton's Second Law applied to block 2 is then

$$\hat{\mathbf{j}}: \quad m_2 g - T_D = m_2\, a_2. \tag{8.103}$$

The forces on block 3 are the gravitational force $m_3\vec{\mathbf{g}} = m_3 g\hat{\mathbf{j}}$ and the force of the string on the block, $\vec{\mathbf{T}}_{D,3} = -T_D\hat{\mathbf{j}}$, with magnitudes equal to T_D because pulley A has been assumed to be both frictionless and massless. The free-body diagram for the forces acting on block 3 is shown in Figure 8.50(c). Newton's Second Law applied to block 3 is then Newton's Second Law applied to block 3 is then

$$\hat{\mathbf{j}}: \quad m_3 g - T_D = m_3\, a_3. \tag{8.104}$$

The forces on the moving pulley A are the string forces $\vec{\mathbf{T}}_{D,A} = T_D\hat{\mathbf{j}}$ and $\vec{\mathbf{T}}_{D,A} = T_D\hat{\mathbf{j}}$ that pulls down on the pulley on each side with magnitude T_D. String C holds the pulley up with a force $\vec{\mathbf{T}}_{C,A} = -T_C\hat{\mathbf{j}}$ with the magnitude T_C equal to the tension in string C. The free-body diagram for the forces acting on the moving pulley is shown in Figure 8.50(d). We have assumed that the mass of the pulley is negligible as is the gravitational force. Newton's Second Law applied to the pulley A is

$$\hat{\mathbf{j}}: \quad 2T_D - T_C = m_A\, a_A = 0. \tag{8.105}$$

Thus the tension in the two strings must satisfy,

$$2T_D = T_C. \tag{8.106}$$

We are now in position to determine the accelerations of the blocks and the tension in the two strings. We record the relevant equations as a summary:

$$\begin{aligned}
&0 = a_2 + a_3 + 2a_1,\\
&m_1 g - T_C = m_1\, a_1,\\
&m_2 g - T_D = m_2\, a_2,\\
&m_3 g - T_D = m_3\, a_3,\\
&2T_D = T_C
\end{aligned}$$

There are five equations with five unknowns, so we can solve this system. Our strategy will be to use the constraint condition as our backbone for our algebraic solutions. We use Equation 8.106, to replace T_C with $2T_D$ in each of the Second Law equations, and they solve for the accelerations and subsstitute each one into the constraint condition (Equation) and then solve for the tension T_D. The Second Law equations in terms of the accelerations are

$$a_1 = g - \frac{2T_D}{m_1},$$

$$a_2 = g - \frac{T_D}{m_2},$$

$$a_3 = g - \frac{T_D}{m_3}.$$

The constraint consition now becomes

$$0 = g - \frac{T_D}{m_2} + g - \frac{T_D}{m_3} + 2g - \frac{4T_D}{m_1} = 4g - T_D\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{4}{m_1}\right), \tag{8.107}$$

which we can solve for T_D

$$T_D = \frac{4g}{\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{4}{m_1}\right)} = \frac{4g\, m_1\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.108}$$

Therefore the tension in string C is

$$T_C = 2T_D = \frac{8g\, m_1\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.109}$$

The acceleration for block 1 is then

$$a_1 = g - \frac{2T_D}{m_1} = g - \frac{8g\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} = g\frac{m_1\, m_3 + m_1\, m_2 - 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.110}$$

The acceleration for block 2 is then

$$a_2 = g - \frac{T_D}{m_2} = g - \frac{4g\, m_1\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} = g\frac{-3\, m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.111}$$

The acceleration for block 3 is then

$$a_3 = g - \frac{T_D}{m_3} = g - \frac{4\, g\, m_1\, m_2}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} = g\frac{m_1\, m_3 - 3\, m_1\, m_2 + 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.112}$$

As a check on our algebra we note that

$$\begin{aligned}
&2\, a_1 + a_2 + a_3 = \\
&2g\frac{m_1\, m_3 + m_1\, m_2 - 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} \\
&+g\frac{-3\, m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} \\
&+g\frac{m_1\, m_3 - 3\, m_1\, m_2 + 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} \\
&= 0.
\end{aligned}$$

8.8.6 Example: Accelerating wedge

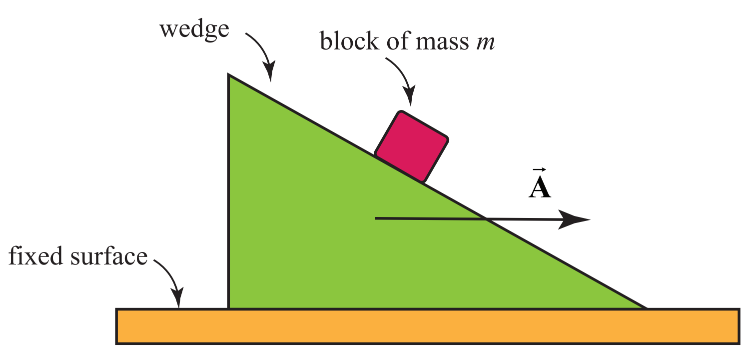

Figure 8.51: Accelerating wedge.

A 45° wedge is pushed along a fixed surface with constant horizontal acceleration $\vec{\mathbf{A}}$ according to an observer at rest with respect to the fixed surface. A block of mass$m$

slides without friction down the wedge (Figure 8.51). Find the acceleration of the block with respect to an observer at rest with respect to the surface. Write down a plan for finding the magnitude of the acceleration of the block. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any free-body force diagrams or sketches that you plan to use.

Answer

Strategy: First choose a coordinate system for the block and wedge as shown in

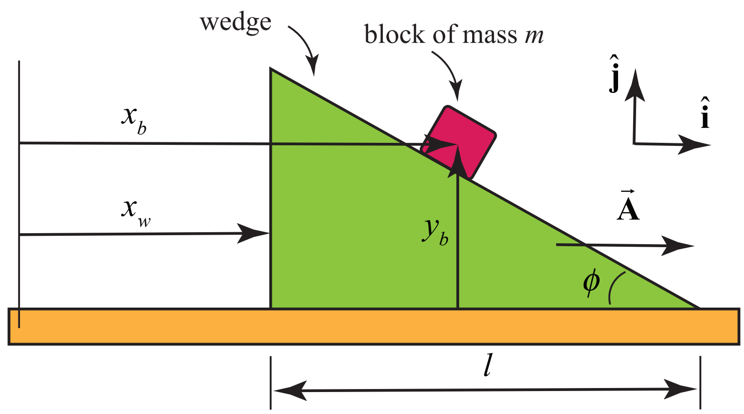

Figure 8.52: Coordinate system for the accelerating wedge.

Figure 8.43. Then the acceleration is $\vec{\mathbf{A}} = A_{x,w}\hat{\mathbf{i}}$ where $A_{x,w} = d^2x_w/dt^2$ is the x-component of the acceleration of the wedge. We shall apply Newton's Second Law to the block sliding down the wedge. Because the wedge is accelerating, there is a constraint relation between the x- and y- components of the acceleration of the block. We shall find the constraint relationship between the components of the accelerations of the block and wedge by a geometric argument. From Figure 8.52, we see that

$$\tan\phi = \frac{y_b}{l - (x_b - x_w)}. \tag{8.113}$$

Therefore

$$y_b = (l - (x_b - x_w))\tan\phi. \tag{8.114}$$

Differentiate Equation 8.114 twice with respect to time noting that $d^2l/dt^2 = 0$:

$$\frac{d^2y_b}{dt^2} = -\left(\frac{d^2x_b}{dt^2} - \frac{d^2x_w}{dt^2}\right)\tan\phi. \tag{8.115}$$

Recall that the definition of acceleration is the second derivative of the position function so the constraint condition for the two components of the acceleration of the block is

$$a_{b,y} = -(a_{b,x} - A_{x,w})\tan\phi. \tag{8.116}$$

We now draw a free-body force diagram for the block (Figure 8.53).

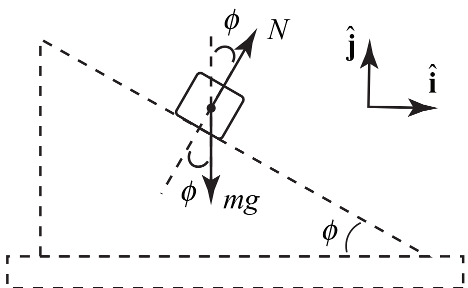

Figure 8.53: Free-body force diagram for block.

Newton's Second Law in the $\hat{\mathbf{i}}$-direction is

$$N \sin\phi = ma_{b,x}. \tag{8.117}$$

Therefore the normal force is

$$N = \frac{ma_{b,x}}{\sin\phi}. \tag{8.118}$$

Newton's Second Law in the $\hat{\mathbf{j}}$-direction is

$$N\cos\phi - mg = ma_{b,y}. \tag{8.119}$$

Substituting our result for the normal force and rearranging terms yields

$$ma_{b,x}(\text{cotan}\ \phi + \tan\phi) = m(g + A_{w,x}\tan\phi), \tag{8.120}$$

which we can solve for the x-component of the acceleration of the block

$$a_{b,x} = \frac{g + A_{w,x}\tan\phi}{\text{cotan}\ \phi + \tan\phi}. \tag{8.121}$$

Now substitute our expression for $a_{b,x}$ into Equation 8.116 and hence the y-component of the acceleration of the block is

$$\begin{aligned} a_{b,y} &= -(a_{b,x} - A_{w,x})\tan\phi = -\left(\frac{g + A_{w,x}\tan\phi}{\text{cotan}\ \phi + \tan\phi} - A_{w,x}\right)\tan\phi \\ &= \frac{A_{w,x} - g\tan\phi}{\text{cotan}\ \phi + \tan\phi} \end{aligned}. \tag{8.122}$$

We can now substitute the given values for $\phi = 45°$, then $\text{cotan}\ 45° = \tan 45° = 1$, and so Equations 8.121 and 8.122 are then

$$a_{b,x} = \frac{g + A_{w,x}}{2} \tag{8.123}$$

and

$$a_{b,y} = \frac{A_{w,x} - g}{2}. \tag{8.124}$$

The magnitude of the acceleration of the block is then

$$\begin{aligned} a = \sqrt{a_{b,x}{}^2 + a_{b,y}{}^2} &= \sqrt{\left(\frac{g + A_{w,x}}{2}\right)^2 + \left(\frac{A_{w,x} - g}{2}\right)^2} \\ &= \sqrt{\left(\frac{g^2 + A_{w,x}{}^2}{2}\right)}. \end{aligned} \tag{8.125}$$

8.8.7 Catenary: shape of a suspended rope

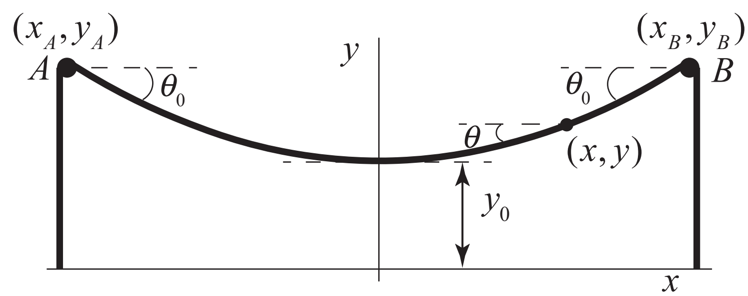

Figure 8.54: Suspended rope.

Suppose a uniform rope is suspended above two rollers. Let λ denote the mass per unit length of the rope. The rope makes an angle θ_0 with respect to the horizontal at the points A and B where the rope just contacts the rollers. Choose a coordinate system as shown in Figure 8.54. The ends of the rope are located at the points $(x_A, 0)$ and $(x_B, 0)$. The center point of the rope is located at the point $(0, y_0)$ and the rollers are located at the points (x_A, y_A) and (x_B, y_B). (Note that $y_A = y_B$, and $x_A = -x_B$., where $x_B > 0$). The two endpoints of the reach are located at $(x_A, 0$ and $(x_B, 0$ Let the function $y(x)$ describe the shape of the rope between the two rollers. Let $T(\theta) \equiv T(x, y)$ denote the tension in the rope at the point (x, y) where the rope makes an angle θ with respect to the horizontal. Let $T_h = T(\theta)\cos(\theta)$. '

(a) Find a differential equation satisfied by the function $y(x)$, in terms of d^2y/dx^2, dy/dx, λ, g and T_h.

(b) Solve the differential equation in part a) to find an expression for $y(x)$ describing the shape of the rope, .

(c) Find the tension $T(x, y\,)$in the rope at an arbitrary point (x, y) in the rope.

(d) Suppose the left roller is at a height y_A , the right roller is a height y_B, and the two hanging pieces reach down to $y = 0$ (Figure). Suppose the tension at the low point of the rope is $T_h = y_0\lambda g$. Will the rope slip over the rollers?

(e) What is the difference in tension $\Delta T = T(B) - T(A)$ between the points B and A?

Answer

(a) Apply Newton's Second Law to the small section of the suspended rope with

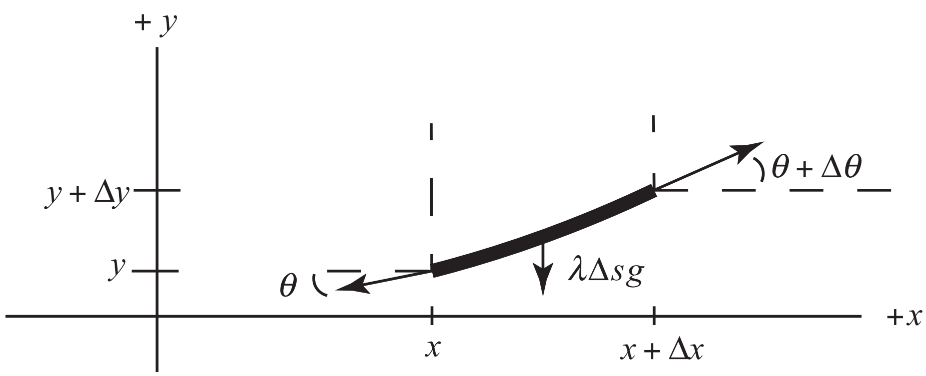

Figure 8.55: Free-body force diagram on small section of rope.

arc length $\Delta s = (\Delta x^2 + \Delta y^2)^{1/2} = \Delta x(1 + (\Delta y/\Delta x)^2))^{1/2}$, in which the two ends of the small piece make angles θ and $\theta + \Delta\theta$ with respect to the horizontal (see Figure 8.55). There are two tensile forces acting at the ends of the rope $T(\theta)$ and $T(\theta + \Delta\theta)$ and a gravitational force $\Delta m = \lambda\Delta s g$. Because the rope is not accelerating the equation for the balance of forces in the vertical direction is given by

$$T(\theta + \Delta\theta)\sin(\theta + \Delta\theta) = T(\theta)\sin(\theta) + \lambda\Delta x(1 + (\Delta y/\Delta x)^2))^{1/2} g \tag{8.126}$$

Newton's Second Law applied in the horizontal direction yields

$$T(\theta + \Delta\theta)\cos(\theta + \Delta\theta) - T(\theta)\cos(\theta) = 0. \tag{8.127}$$

Note that the quantity

$$T(\theta + \Delta\theta)\cos(\theta + \Delta\theta) = T(\theta)\cos(\theta) \equiv T_h \tag{8.128}$$

is a constant. Divide Equation 8.126 by Equation and then rearrange as

$$\frac{\tan(\theta + \Delta\theta) - \tan(\theta)}{\Delta x} = \frac{(\lambda)((1 + (\Delta y/\Delta x)^2)^{1/2} g}{T_h}. \tag{8.129}$$

Now take the limit as $\Delta x \to 0$.

$$\lim_{\Delta x \to 0} \frac{\tan(\theta + \Delta\theta) - \tan(\theta)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\lambda((1 + (\Delta y/\Delta x)^2)^{1/2} g}{T_h} \tag{8.130}$$

In the limit $\Delta x \to 0$ then $\Delta s \to dx(1 + (dy/dx)^2)^{1/2}$ where $dy/dx = \tan\theta$. By definition, the second derivative of $y(x)$ at the angle θ is given by

$$\frac{d^2 y}{dx^2}(\theta) = \lim_{\Delta x \to 0} \frac{(dy/dx)\big|_{(\theta+\Delta\theta)} - (dy/dx)\big|_{(\theta)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\tan(\theta + \Delta\theta) - \tan(\theta)}{\Delta x}. \tag{8.131}$$

Using this definition of the second derivative, Equation 8.130 can be written as a differential equation describing the shape of the rope

$$\frac{d^2 y}{dx^2} = \frac{(\lambda g)((1 + (dy/dx)^2)^{1/2}}{T_h}. \tag{8.132}$$

(b) We can integrate Equation 8.133 as follows. Let $f(x) = dy/dx$. Then Equation 8.133 becomes

$$\frac{df}{dx} = \frac{(\lambda g)((1 + (f(x)^2)^{1/2}}{T_h}. \tag{8.133}$$

Separating variables yields an integral equation

$$\int \frac{df}{((1 + (f(x)^2)^{1/2}} = \int \frac{(\lambda g)dx}{T_h}. \tag{8.134}$$

The result of integration is

$$\sinh^{-1}(f) = \frac{(\lambda g)x}{T_h} + c_1. \tag{8.135}$$

The constant can be determined by noting that at the bottom of the rope where $dy/dx = 0$, $f(0) = 0$ and that $\sinh^{-1}(0) = 0$, therefore the constant of integration $c_1 = 0$. Hence

$$f(x) = \frac{dy}{dx} = \sinh\left(\frac{\lambda g}{T_h}x\right). \tag{8.136}$$

Equation 8.136 can be written as the integral equation

$$\int dy = \int \sinh\left(\frac{\lambda g}{T_h}x\right) dx. \tag{8.137}$$

After integration the equation describing the shape of the rope is then

$$y(x) = \frac{T_h}{\lambda g}\cosh\left(\frac{\lambda g}{T_h}x\right) + c_2. \tag{8.138}$$

The constant c_2 can be determined by the condition that at $y(x = 0) = y_0$. (Recall that $\cosh(0) = 1$). Therefore

$$c_2 = y_0 - \frac{T_0}{\lambda g}. \tag{8.139}$$

The solution for $y(x)$ is then

$$y(x) = \frac{T_h}{\lambda g}\cosh\left(\frac{\lambda g}{T_0}x\right) + y_0 - \frac{T_h}{\lambda g}. \tag{8.140}$$

This equation describes a curve called a *catenary* (derived from the ancient greek word for chain).

(c) At the left roller (point A), the rope makes an angle θ_0 just before it is in contact with the roller:

$$\tan\theta_0 = \frac{dy}{dx}(A) = \sinh\left(\frac{\lambda g}{T_h}x_A\right). \tag{8.141}$$

Therefore T_h is given by

$$T_h = \frac{\lambda g x_A}{\sinh^{-1}(\tan\theta_A)}. \tag{8.142}$$

The tension $T(x, y)$ in the rope at a point (x, y), where the rope makes an angle θ with respect to the horizontal, is given by

$$\begin{array}{l} T(x,y) = \dfrac{T_h}{\cos(\theta)} = \dfrac{T_h}{\cos(\theta)}(\cos^2(\theta) + \sin^2(\theta))^{1/2} = T_h(1+\tan^2(\theta))^{1/2} \\ = T_h(1+(dy/dx)^2)^{1/2}. \end{array}. \tag{8.143}$$

(Note that $\dfrac{1}{\cos(\theta)} = \dfrac{ds}{dx} = (1+(dy/dx)^2)^{1/2}$.) Recall that Equation 8.133 implies that

$$(1+(dy/dx)^2)^{1/2} = \frac{T_h}{\lambda g}\frac{d^2y}{dx^2}. \tag{8.144}$$

Therefore the tension can be expressed in terms of the second derivative as

$$T(x,y) = \frac{{T_h}^2}{\lambda g}\frac{d^2y}{dx^2}. \tag{8.145}$$

The second derivative of 8.140 is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\sinh\left(\frac{\lambda g}{T_h}x\right)\right) = \frac{\lambda g}{T_h}\cosh\left(\frac{\lambda g}{T_h}x\right). \tag{8.146}$$

Rewrite Equation 8.140 as

$$\cosh\left(\frac{\lambda g}{T_h}x\right) = \frac{\lambda g}{T_h}(y(x) - y_0) + 1. \tag{8.147}$$

Then the tension in the rope at the point (x, y) is

$$\begin{array}{l} T(x,y) = \dfrac{{T_h}^2}{\lambda g}\dfrac{d^2y}{dx^2} = \dfrac{{T_h}^2}{\lambda g}\dfrac{\lambda g}{T_h}\cosh\left(\dfrac{\lambda g}{T_0}x\right) \\ = \lambda g(y(x) - y_0) + T_0. \end{array}. \tag{8.148}$$

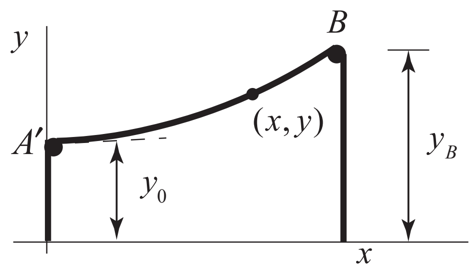

Figure 8.56: Static configuration of rope.

At the low point of the rope $0, y_0$, the tension is $T(A') = T_h = \lambda g y_0$. If the left roller were located at this point (A'), and the length of the rope hanging over the left roller were equal to y_0, then the rope will not slip because the tension is balanced by the weight of the hanging end, $\lambda y_0 g$.

If the length of the rope hanging over the right roller is equal to y_B, then the tension at right roller is

$$T(B) = \lambda g(y_B - y_0) + \lambda g y_0 = \lambda g y_B, \tag{8.149}$$

which is equal to the weight of the rope hanging over the right roller and so the rope will also not slip. A similar argument requires that in order for the rope not to slip, the length of the rope hanging over the left roller must be equal to y_A, where the tension is $T(A) = \lambda g y_B$. Therefore the difference in tension is

$$\Delta T = T(B) - T(A) = \lambda g(y_B - y_A). \tag{8.150}$$