CS 565 Spring 2022 Homework 5 (Axiomatic Semantics Again)

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April 6, 2022

Problem 1 (2 points). What weakest precondition does the wp function compute for each of the following programs and postconditions:

a. $\mathsf{wp}(\ \mathsf{X} := \mathsf{X} + 1,\ \{\mathsf{X} > 5\})$
$\equiv \{\mathsf{X} + 1 > 5\} = \{\mathsf{X} > 4\}$

b. $\mathsf{wp}(\ \mathsf{X} := \mathsf{Y},\ \{\mathsf{X} \neq \mathsf{Y}\})$
$\equiv \{\mathsf{Y} \neq \mathsf{Y}\} = \{\mathsf{False}\}$

c. $\mathsf{wp}(\ \mathsf{X} := \mathsf{X} + 1;\ \mathsf{Y} := \mathsf{Z},\ \{\mathsf{X} = \mathsf{Y}\})$
$\equiv \{\mathsf{X} + 1 = \mathsf{Z}\}$

d. $\mathsf{wp}(\ \textbf{if}\ (\mathsf{X} < 10)\textbf{then}\ \mathsf{Z} := \mathsf{X}\ \textbf{else}\ \mathsf{Z} := \mathsf{Y},\ \{\mathsf{Z} < 10\})$
$\equiv \mathsf{X} < 10 \rightarrow \{\mathsf{X} < 10\}\ \wedge \mathsf{X} \not< 10 \rightarrow \{\mathsf{Y} < 10\}$
$= \mathsf{X} < 10 \rightarrow \{\mathsf{X} < 10\}\ \wedge 10 \leq \mathsf{X} \rightarrow \{\mathsf{Y} < 10\}$

Problem 2 (1 point). Using the typing rules from Figure 1, provide a derivation tree showing that the following program has the indicated type:

$$
\begin{array}{c}
\dfrac{[f : \text{Bool} \to \text{Bool}](f) = \text{Bool} \to \text{Bool}}{f : \text{Bool} \to \text{Bool} \vdash f : \text{Bool} \to \text{Bool}}\ \textsc{T-Var} \\
\dfrac{}{f : \text{Bool} \to \text{Bool} \vdash \text{false} : \text{Bool}}\ \textsc{T-False} \\
\dfrac{\dfrac{\textsc{T-True}\ \dfrac{}{f : \text{Bool} \to \text{Bool} \vdash \text{true} : \text{Bool}} \qquad \dfrac{}{f : \text{Bool} \to \text{Bool} \vdash \text{false} : \text{Bool}}\ \textsc{T-False}}{f : \text{Bool} \to \text{Bool} \vdash \mathbf{if}\ \text{false}\ \mathbf{then}\ \text{true}\ \mathbf{else}\ \text{false} : \text{Bool}}\ \textsc{T-If}}{f : \text{Bool} \to \text{Bool} \vdash f(\mathbf{if}\ \text{false}\ \mathbf{then}\ \text{true}\ \mathbf{else}\ \text{false}) : \text{Bool}}\ \textsc{T-App}
\end{array}
$$

Problem 3 (1 point). Using the typing rules from Figure 1, provide a derivation tree showing that the following program has the indicated type:

$$
\begin{array}{c}
\dfrac{[f : \text{Bool} \to \text{Bool}\ ; x : \text{Bool}](f) = \text{Bool} \to \text{Bool}}{f : \text{Bool} \to \text{Bool}\ ; x : \text{Bool} \vdash f : \text{Bool} \to \text{Bool}}\ \textsc{T-Var} \\
\dfrac{[f : \text{Bool} \to \text{Bool}\ ; x : \text{Bool}](x) = \text{Bool}}{f : \text{Bool} \to \text{Bool}\ ; x : \text{Bool} \vdash x : \text{Bool}}\ \textsc{T-Var} \\
\dfrac{\dfrac{\dfrac{\textsc{T-True}\ \dfrac{}{f : \text{Bool} \to \text{Bool}\ ; x : \text{Bool} \vdash \text{true} : \text{Bool}} \qquad \dfrac{}{f : \text{Bool} \to \text{Bool}\ ; x : \text{Bool} \vdash \text{false} : \text{Bool}}\ \textsc{T-False}}{f : \text{Bool} \to \text{Bool}\ ; x : \text{Bool} \vdash \mathbf{if}\ x\ \mathbf{then}\ \text{true}\ \mathbf{else}\ \text{false} : \text{Bool}}\ \textsc{T-If}}{f : \text{Bool} \to \text{Bool}\ ; x : \text{Bool} \vdash f(\mathbf{if}\ x\ \mathbf{then}\ \text{false}\ \mathbf{else}\ \text{true}) : \text{Bool}}\ \textsc{T-App}}{f : \text{Bool} \to \text{Bool} \vdash \lambda\ x : \text{Bool}.\ f(\mathbf{if}\ x\ \mathbf{then}\ \text{false}\ \mathbf{else}\ \text{true}) : \text{Bool} \to \text{Bool}}\ \textsc{T-Abs}
\end{array}
$$

Problem 4 (1 point). Using the typing rules from Figure 1, provide a context Γ under which the term `f x y` has type bool. Can you give a simple and informal description of the set of *all* such contexts?

$$f : \text{nat} \to \text{nat} \to \text{bool}\ ;\ x : \text{nat}\ ;\ y : \text{nat}$$

As long as the last type (return type) of f is bool, other types are unrestricted. Any context fit this predicate falls into the set.

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T} \tag{T-Var}$$

$$\frac{\Gamma, x : T_1 \vdash t \ : \ T_2}{\Gamma \vdash \lambda\, x : T_1.\ t \ : \ T_1 \to T_2} \tag{T-Abs}$$

$$\frac{\Gamma \vdash t_1 \ : \ T_2 \to T_1 \qquad \Gamma \vdash t_2 \ : \ T_2}{\Gamma \vdash t_1\ t_2 \ : \ T_1} \tag{T-Abs}$$

$$\frac{}{\Gamma \vdash \mathsf{true} \ : \ \mathsf{Bool}} \tag{T-True}$$

$$\frac{}{\Gamma \vdash \mathsf{false} \ : \ \mathsf{Bool}} \tag{T-False}$$

$$\frac{\Gamma \vdash t_1 \ : \ \mathsf{Bool} \qquad \Gamma \vdash t_2 \ : \ T \qquad \Gamma \vdash t_3 \ : \ T}{\Gamma \vdash \textsf{\textbf{if}}\ t_1\ \textsf{\textbf{then}}\ t_2\ \textsf{\textbf{else}}\ t_3 \ : \ T} \tag{T-If}$$

Figure 1: Typing rules for the simply typed lambda calculus with booleans from the STLC chapter of PLF.