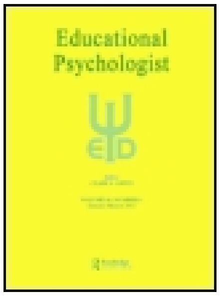

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Teaching and Learning Fraction and Rational Numbers: The Origins and Implications of Whole Number Bias

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Many researchers agree that children's difficulty with fraction and rational numbers is associated with their whole number knowledge, but they disagree on the origin of the whole number bias. This article reviews three explanations of the nature of the bias. These accounts diverge on the questions of whether or not early quantitative representation originates from a numerical-specific cognitive mechanism, and whether or not the early quantitative representation privileges discrete quantity. The review suggests that there does not yet appear to be sufficient evidence to decide among the competing accounts as to the nature of the whole number bias. Yet, in the search for the understanding, two important issues have been brought up with regard to learning and teaching fraction and rational numbers. One question is related to how learning about fraction and rational numbers may be organized in such a way to be less strained for taking advantage of the prior knowledge of whole numbers. The second issue is concerned with what causes the gap between learning number symbols and learning number concepts. These issues have been explored by drawing upon both the accounts of whole number bias and insights from developmental studies, neuropsychological studies, and teaching experiments. The present review of literature has shown that the issues of whole number bias reflect not merely a matter of interference between prior and new knowledge in children's construction of fraction concepts, but a constellation of more general questions with regard to the origin and development of numerical cognition. How the bias is conceptualized therefore has theoretical and instructional significance.

Among educators, fraction and rational numbers are notorious for the difficulty that elementary school students experience in learning them (e.g., Bright, Behr, Post, & Wachsmuth, 1988; Dufour-Janvier, Bednarz, & Belanger, 1987; Kerslake, 1986; Lesh, Behr, & Post, 1987; Mack, 1995; Ni, 2001; Novillis, 1979). Many people in the field of education and psychology agree that the difficulty is associated with children's whole number knowledge which represents numbers discretely and therefore may interfere with children's construction of the concept of fraction and rational numbers that are ordered and continuous (English & Halford, 1995; Gelman, 1991; Lamon, 1999; Post, Cramer, Behr, Lesh, & Harel, 1993; Streefland, 1991). However, many also disagree on the nature of the tendency in children to use the single-unit counting scheme applied to whole numbers to interpret instructional data on fractions. (This tendency will be called "whole number bias" hereafter in the article and the term is explained in the following paragraphs.) Debates about the origin of whole number bias are related to the very nature of human numerical cognition. These include whether the whole number bias is innate in infants or a late and sophisticated achievement of development (Feigenson, Carey, & Spelke, 2002; Gelman, 1991; Mix, Levine, & Huttenlocher, 2002a, 2002b; Sophian, 1998; Wynn, 1992a)

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and whether numerical cognition is derived from number-specific mechanism (Gallistel & Gelman, 1992; Wynn, 1992a) or from domain-general mechanism (Simon, 1997).

Different theoretical positions on the issues indicate different starting points and different constraints for the development of numerical cognition. They can have important implications for significant questions with regards to how fraction and rational number teaching and learning should be organized. For example, is there an intuitive developmental foundation for counting number as well as for fractions and rational numbers (Sophian, 2000)? What is an experiential base that can be an appropriate point of entry for children to expand whole numbers to fraction and rational numbers (e.g., Confrey, 1994; Steffe, 2002; Streefland, 1991)? How may prior learning about numbers affect later learning about fraction and rational numbers in children? Does certain architecture of numerical cognition exist in the brain that may explain the difficulty children experience in mapping from number symbols to number concepts?

We focus this article on the debate on the nature of the whole number bias. An attempt is made to sort out the current state of the evidence and to understand its implications for teaching and learning fraction and rational numbers. Although there are numerous issues relating to the debate, three aspects are reviewed for this article. The first is in regard to the origin of numerical knowledge and provides three different perspectives on the cause of the whole number bias. An innate constraint account assumes that the bias originates from an innate domain-specific mechanism for number concepts that privileges discrete quantity. The undifferentiated amount hypothesis for early quantitative representation assumes the whole number bias is not an innate endowment, but a developmental result of differentiation between discrete and continuous quantity in children. The learning account links the whole number bias to the instruction on whole numbers and fractions that fails to develop the conceptual ground in children for understanding fraction and rational numbers. Evidence that may support or question each of the accounts is evaluated.

The second theme is on instructional approaches that are intended to provide children the experiential and conceptual base in order to understand fraction and rational numbers, and thus to combat the whole number bias. A cognitive bias, regardless of whether it is innate or learned, provides both cognitive efficiency and inflexibility (Caverni, Fabre, & Gonzalez, 1990). The debate on the nature of whole number bias raises issues about cognitive efficiency versus inflexibility of learning in utilizing children's prior knowledge as the means to introduce new knowledge. This is not unique to mathematics instruction, but is particularly salient in teaching and learning about fractions and rational numbers. In light of this issue, the review evaluated the different instructional approaches (e.g., partitioning approach, measurement approach, and etc.).

The third theme of the article relates to the problem of discrepancies between learning number symbols and number concepts in children. Connecting meaning to different representation systems is one of the most significant aspects of learning about mathematics and presents a great challenge for many children. This is particularly evident when children are learning about fraction and rational numbers. A common observation is that for otherwise similar fraction tasks, children would show contrasting performances between those involving symbols and those not involving the symbols. The question then arises as to whether the gap reflects an innate limit to children's conceptual representation of amounts between whole numbers. Alternatively, it could be a manifestation of confusion between the fraction symbols and those of whole numbers. However, it has been very difficult to explain the behavior by merely looking into the behavior itself. Fortunately, recent brain research on human numerical cognition provides us a glimpse of the neuropsychological disassociation between different notation systems, between visual-spatial representation and language-dependent subsystems, that process numerical information. This line of research has shed some light on the matter, helping to explain why the mapping between number symbols and number concepts is not straightforward and why learning fraction and rational numbers is a recursive process.

The three themes identify the most significant issues, raising questions with regard to the origin and development of human numerical cognition on one hand; and to teaching and learning about fraction and rational numbers on the other hand. To explicate these matters research findings have been drawn from three fields. These areas are as follows: (a) developmental research with respect to the first theme, (b) instructional research relating mainly to the second theme, and finally (c) neuropsychological research relating mainly to the third theme. It is worth noting that the advantage of doing this kind of review is that the research from different fields represents different perspectives and can inform one another to converge insights into the common issues and problems. However, there is also a risk that this kind of review may result in being incoherent and simplifying the significant issues. To maximize the benefit and minimize the shortcomings, every effort has been made to relate the lines of specific research as far as possible to the focus of this article, that is, to understand the nature of the whole number bias and instructional implications that can be drawn from the findings.

At this stage, it is important to clarify the meanings of three key terms that are used throughout this article—fractions, rational numbers, and whole number bias—in order to minimize ambiguity or confusion of their frequent use in the text.

The term "bias" has been defined as being systematic and frequent deviation from a norm (Caverni et al., 1990). The whole number bias thus refers to a robust tendency to use the single-unit counting scheme to interpret instructional data on fractions. This bias causes children's difficulty to perceive whole numbers as decomposable units. Instances of the misapplications include confounding the number of pieces in a partition with the size of each piece, (e.g., 1/4 is bigger than

1/3 because 4 is bigger than 3), adding across numerators and denominators to add fractions (e.g., 1/2 + 1/3 = 2/5), counting noncongruent parts to name a fraction *one third* in a circle that is partitioned into a half and two fourths, and rejecting that there are any numbers between 0 and 1. Evans (1989) identified two kinds of bias. One type consists of failure to apply a concept or a principle that is understood (e.g., Wason, 1983). In other cases it appears that biases may reflect imperfect understanding of the concept or the principle involved (e.g., Weil-Barais & Vergnaud, 1990). Children's whole number bias belongs to the latter category. While they are learning about the ideas of fraction numbers, they are often overtaken by the aspects of fraction situations that appear to be consistent with what they have known about whole numbers.

As previously stated, a bias is systematic deviation from a norm. The norm usually refers to an adult model of something in developmental psychology[1] (e.g., the adult model of biology, Carey, 1985). A fundamental question in cognitive development concerns the manner in which children's behavior converges to adult models of the world. Therefore, many cognitive biases that have been observed in children are developmental phenomena (e.g., Markman, 1989). The whole number bias is one such example. Regardless of whether it is learned or innate, a bias, such as the whole number bias, will have significant implications for later learning and development. The present review demonstrates that the whole number bias, although it shows to be an interference of children's prior whole number knowledge with the new concept of fraction numbers, actually reflects a constellation of more general problems associated with the cognitive processes of number concept acquisition.

The term "fractions" and "rational numbers" are two genuinely associated, but nonexchangeable, terms from both a mathematical and a psychological point of view. In mathematics, the term "fraction" refers to anything written in the symbolic form a/b. It also refers specifically to the part-whole meaning of rational numbers, which appears to be the most accessible to school children. The part-whole interpretation is considered the conceptual base for the other interpretations of rational numbers, and operations on fractions—equivalence, addition, subtraction, multiplication, and division—form a basis for the formal symbolic computation in the field of rational numbers. Therefore, "fraction" is part of the rational number system because it represents the important part-whole meaning of rational numbers and it is the notation used for the computations of rational numbers (Lamon, 1999).

Researchers in mathematics education have different treatments in using the terms. Thompson and Saldanha (2003) make the distinction between "fraction," that refers to a "personally knowable system of ideas" (Kieren, 1993), and "rational number," that refers to a formal system built up by generations of mathematicians. They argue that the mathematical construction of rational numbers is very general and abstract and it is so far beyond the grasp of elementary and middle school students. As a result, they call the part of the school mathematics curriculum involving fraction numbers "fraction curriculum and instruction" (p. 95) to avoid confounding fractions as personal intuitive knowledge and rational numbers in a formal system. They suggest that textbook authors and users should do the same. Making this distinction reflects the researchers' concern that the elementary and middle school mathematics instruction overemphasizes symbolic manipulations and overlooks provision of the intuitive activities for students.

Other researchers do not make such a distinction between the terms. For example, the researchers of the Rational Number Project team (e.g., Behr, Harel, Post, & Lesh, 1992; Post et al., 1993) have used "rational number ideas" (Post et al., p. 342) or "rational number-related instruction" (Post et al., 1993, p. 337) to refer to students' thinking about fractions and instruction on fraction and rational numbers in general. However, these researchers do make the distinction between formal knowledge of numbers that is "explicit, apparently linearly ordered, and coherent with formal logic" (Kieren, 1993, p. 51) and personal conceptual knowledge that is "the interweaving of the intuitive and formal knowledge on personal basis" (p. 51). Nevertheless, they do not require that "formal knowledge" and "personal conceptual knowledge" correspond to "rational numbers" and "fraction" respectively because continuity exists between personal knowledge and formal knowledge (Kieren, 1993; Post et al., 1993; Carpenter, in Steffe, von Glasersfeld, Richards, & Cobb, 1983). However, this does not mean that rational numbers as a system is within the reach of children. Rather, it suggests that personal conceptual knowledge is a root of the individual's acquisition of the domain as a system. Besides, the correspondence does not seem mathematically defensible because the concept of fraction can be formal, for a mathematician, or intuitive for a developing child. The distinction and continuity between the two terms, as well as between personal and formal knowledge of mathematics, is a complex matter and its thorough explication would require a separate article (see Kieren, 1993, for an excellent discussion of personal and formal knowledge of mathematics). In the present article, the terms are used in the way that is consistent with Kieren's and the Rational Number Project team's. It is important to acknowledge the distinction, as well as the continuity, between the two terms at both the personal and the formal knowledge level. Therefore throughout the text the terms "fraction and rational numbers" are used to refer to learning and teaching

[1]Steffe, von Glasersfeld, Richards, and Cobb (1983) warned against the risk of using adult models as norms as they may interfere with the investigation into the child's conceptions. This warning should be well taken. However, it is probably unavoidable to use the adult models as a reference to study development (Mehler & Bonati, 2002). What is important for developmental study is that it should not only describe, but more importantly explain, how a baby turns into an adult. Also, adult models are not arbitrary and can be justified in terms of arguments concerning their suitability to the purpose (Baron, 1988).

fraction and rational numbers in general. However, the term "fraction" is used to describe children's thinking about situations that involve relative quantities.

Now, we present a discussion of the three themes of this review in the following sections.

THE ORIGIN OF THE WHOLE NUMBER BIAS

The controversy over the nature of the whole number bias hinges on two issues. The first is concerned with the origin of numerical competence. It asks whether early quantitative representation originates from a numerical-specific or general-purpose cognitive mechanism. The second issue is related to the nature of early quantitative representation. It asks whether or not the early quantitative representation is limited to discrete quantity. Understandably, the two issues are interconnected. To have a better understanding of the debate, a brief review of the evolving views is needed on the question of the origin of numerical cognition. We then consider each of the three accounts of the whole number bias.

Numerical abilities have been considered to derive from human linguistic competence (Chomsky, 1980; Hurford, 1987). The ability of calculation—a core of numerical competence—highly relies on the ability to read, write, produce, or comprehend numerals. Therefore, numerical abilities seem linked to the ability to mentally manipulate sequences of words or symbols according to the syntax of numerical notations and calculation algorithms. However, data on reaction time for normal adults to compare Arabic numerals, or to do a single-digit addition or multiplication, have suggested an internal analogical representation of quantity that is relatively independent of symbol manipulation. People take a shorter time to distinguish two distant numerosities, such as "3" and "9," than two closer ones, such as "3" and "4." That is, the time to decide which of two numbers is the larger or the smaller decreases with the numerical distance between them (Moyer & Landauer, 1967). It was explained as that the analogical magnitude representation of quantity led to greater confusion over nearby numbers and thus the discrimination became easier between two distant numbers than between two closer ones. The distance effect was found whether the comparison bears on Arabic numerals or on physical parameters such as line length, pitch, or numerosity (see Dehaene, 1992, 1997 for a comprehensive review of the research). People also take a longer time to calculate "5 + 8" than to calculate "2 + 3," that is, the time needed to solve a single-digit addition or multiplication increases with the size of the operants (Ashcraft, 1992), which is called "size effect." The distance effect and the size effect together suggest that digits are not compared or operated on at a symbolic level, but are processed as quantities, where we access and manipulate a representation of approximate quantities similar to a mental "number line," such as 1 as "__," 2 as "____," 3 as "______," 4 as "________," 5 as "__________," and so on.

The analog magnitude representation of numerical quantities in human adults is thought to have originated in the protonumerical abilities found in human infants. As in human adults, the distance effect was also observed with human infants, suggesting that infants have the access to an analogue representation of quantities. The preverbal representation of approximate numerical magnitudes is thought to have been inherited from our evolutionary past and to support the progressive emergence of language-dependent abilities such as verbal counting and symbolic calculation (Dehaene, 1992; Gallistel & Gelman, 1992). This has given rise to the assumption that there is a relatively independent "mental faculty" that is specialized for acquiring knowledge about quantity and numbers. The assumption entails certain inference about the nature of whole number bias although this was not originally intended by the assumption. This assumption has become pivotal for the current debates on the origin and development of human numerical cognition.

Three accounts have emerged with regard to the origin of the whole number bias. The innate constraint account hypothesizes an innate origin for the bias that directly links to certain properties of the preverbal representation of magnitudes. The undifferentiated amount account suggests that there is no innate bias privileging discrete quantity and that the bias has resulted from a developmental process of differentiation between discrete and continuous quantification. The learning account attributes the bias to the instruction that makes inappropriate use of children's prior whole number knowledge and fails to help children differentiate fraction and rational numbers from whole numbers. These accounts have implications for understanding how numerical cognition develops and how instruction on whole numbers and fraction numbers in elementary school classrooms may interact with what children might have known about quantity and number. In the following sections, each account is explained in turn with relevant evidence.

The Innate Constraint Hypothesis

In speech acquisition an innate structural base is assumed that is language-specific and determines the way that language input is processed in a certain manner (Chomsky, 1986). In the domain of numerical cognition, Gelman and other researchers (Gallistel & Gelman, 1992; Gelman, 1991; Gelman & Gallistel, 1978; Wynn, 1992a) assume a similar innate cognitive mechanism that generates mental magnitudes to represent numerosities.[2] The innate device is as-

[2]To escape the ambiguity of the word "number," Gelman and Gallistel (1978) use the term "numerosity" to refer specifically to a measurable numerical quantity. Quantification consists in grasping the numerosity of a perceived set and accessing the corresponding (possibly approximate) mental token that Gelman and Gallistel call the "numeron." Research found that the degree of surface similarity between sets of objects being compared affected children's performance on tasks to determine the numerical relation between the sets (Mix, 1999). This suggests that there may be a period when children can detect the numerosity of a particular set without seeing numerosity as the basis for categorization. Some researchers do not attribute "numerosity" to this lower level of counting (e.g., Steffe et al., 1983), but some do (e.g., Mix et al., 2002b; Carpenter, in Steffe et al., 1983).

sumed to work as that numerosities are represented internally by the continuous states of an analogue accumulator (Meck & Church, 1983). It is hypothesized that there is an endogenous pace maker that emits pulses at a constant rate. To begin counting, a switch is closed that has pulses enter into a container. When the accumulator is counting, the switch gates let in pulses one at a time. The resulting fullness of the container represents the total quantity.

According to this account, the children's initial representation of numerosity is assumed to be discrete blurs of mental magnitudes as the device for representing numerosities is rendered discrete by the discrete gating of bursts of impulses to the accumulator, one burst for each item in a set of discrete items. There is no provision in this system for generating representatives for fractional numerosities, despite the fact that the representation of numerosity by magnitudes in principle makes it possible to represent fractional numerosities. Gallistel and Gelman (1992) call the analog magnitude representation the preverbal counting mechanism as it is said isomorphic to that of verbal counting. The innate mechanism is therefore assumed to help children learn related verbal count words. But it is the discrete nature of the mental magnitude representation that is assumed to impede children's acquisition of fractional numbers that are very different from whole numbers. Hence, the whole number bias is assumed to be innately determined.

Among others, support for the innate constraint notion comes from two main sources.[3] One source is the finding of the prolonged difficulties children have encountered with fractions and rational numbers (e.g., Bialystok & Codd, 2000; Hartnett & Gelman, 1998; Hiebert, Wearne, & Taber, 1991; Kamii & Clark, 1995; Kerslake, 1986; Lesh et al., 1987). The other source is the reported early competence with discrete quantities, particularly, infants' ability to discriminate small sets of objects (Antell & Keating, 1983; Koechlin, Dehaene, & Mehler, 1997; Starkey, 1992; Starkey & Cooper, 1980; Starkey, Spelke, & Gelman, 1990; Strauss,& Curtis, 1981; Wynn, 1992a) and early acquisition of counting skills (Gelman & Gallistel, 1978; Gelman & Meck, 1983, 1986; Wynn, 1992b).

In several studies, infants of a few months old were found able to discriminate between small sets of objects (Antell & Keating, 1983; Starkey & Cooper, 1980; Strauss & Curtis, 1981; Koechlin et al., 1997; Wynn, 1992a). Habituation studies capitalize on the fact that when infants lose interest in a stimulus, they tend to look at it less. In these studies, infants were presented with repeated delivery of a fixed set of items and then were tested with a novel set of the items. An increasing looking time by the infants to the novel set, an indicator of preference for novelty, was used to suggest that the infants were able to discriminate the numerical differences between the small sets of objects (e.g., Starkey & Cooper, 1980). In the transformation studies that also make use of the habituation mechanism, but differ from the habituation studies in manipulating stimuli, small sets of objects were first shown to infants and then one or two more objects were either added or removed. For example, an original set contains two objects and one more object is added, then a set of the three objects is an expected result, whereas a set of the two objects is an unexpected result. Looking times by the infants to the expected and the unexpected outcome were then compared. A longer looking time to an unexpected outcome was used as an indicator that the infants were able to represent the numerosities as they seemed to realize that the number of objects in the sets changed when one object was added or taken away from it (Wynn, 1992a). These studies have been taken to support the innate constraint notion of the origin of numerical competence. That is, number competence starts with an innate domain-specific mechanism that privileges discrete quantity.

This assumption about the nature of early representation of quantity has important implications for understanding what develops with regard to the development of counting skills that are quite clearly relevant to number concept acquisition. According to the assumption, the preverbal representation of approximate numerical magnitudes is assumed to be the source of the implicit counting principles, such as the principle of one-to-one correspondence, stable order of a counting list, and abstraction (counting applied to any collections of entities), that may "bootstrap" the acquisition of verbal counting (Gallistel & Gelman, 1992; Gelman & Gallistel, 1978). The implicit (or innate) principles, however skeleton they may be, are assumed to guide the acquisition of the verbal counting procedure.

Empirical evidence was that 4-year-old children were shown to count appropriately when the number of a set to be counted was small and the objects were arranged in a straight line and at regular intervals from each other (Gelman & Meck, 1983, 1986). However, children of this age often failed to follow the correct counting procedure when counting a set of objects that were spread around a table, or were not movable (e.g., a set sticking on a sheet, Briars & Siegler, 1984; Fuson, 1988). Five-year-old children can usually count properly when they count a single set of objects (Nunes & Byrant, 1996). Moreover, the understanding of counting principles and the acquired counting skills facilitate children's learning of addition and subtraction with whole numbers (Baroody & Ginsburg, 1986; Carpenter & Moser, 1984; Fuson, 1988, 1992). However, whether or not the innate magnitude representation of quantity provides the conceptual primitives for the acquisition of counting skills is highly debatable (Carey, 2001; Steffe & Cobb, 1988; Steffe et al., 1983). This is discussed later in this section.

When comparing children's achievement in acquiring the counting skills to their experience with fraction and rational numbers, the outcome is less than positive. Many studies reported that children often apply the single-unit counting

[3]Animal research is also an important area that contributes to the understanding of the origin of numerical cognition (see Carey & Gelman, 1991).

scheme in situations involving fraction numbers (e.g., Ball & Wilson, 1996; Hartnett & Gelman, 1998; Kerslake, 1986; Mack, 1995). The whole number strategies appeared to be persistent and temporarily interfere with the development of the fraction and rational number concepts (Dufour-Janvier et al., 1987; Ni, 2001; Post et al., 1993; Streefland, 1991, 1993). Gelman attributed the difficulty to the innately specified mechanism which represents numerosities discretely and is incommensurate with the structural features of rational numbers that are ordered and continuous (Gallistel & Gelman, 1992; Hartentt & Gelman, 1998).

One example of evidence for the tendency toward the "discrete" representation over the "continuous" representation of numbers was that students were shown to be more responsive to the instructional manipulations that highlight the discrete character of decimal numbers than to the manipulations that highlight the continuous character of decimal numbers (Hiebert et al., 1991; Ni, 2000). In the study by Hiebert et al., one group of low-achieving fourth-grade students was introduced to decimal fractions during an 11-day unit that emphasized conceptual features of decimals. The instruction included some sessions that focused on connections between discrete physical representations of decimals (base-10 blocks) and written decimal symbols and some other sessions that focused on connections between continuous physical models (number line and a circle stopwatch) of decimals and written symbols. Post assessments indicated that the students showed significant improvement in their performance on tasks involving base-10 blocks, but had great difficulty with the tasks involving the continuous physical representations. That is, performance often was higher on common fraction tasks that were amenable to discrete counting strategies than on tasks that required scaling and measuring.

The preference for whole numbers as opposed to fraction and rational numbers also appeared in adults' thinking about numbers. Gigerenzer (1996, 1998) reported that when a problem requiring Bayesian inference was presented in terms of probability of a single event, most adult subjects failed to solve the problem. However when the same problem was presented in terms of natural frequencies by natural sampling, most adult subjects could solve the problem. Natural sampling is the sequential process of updating event frequencies from experience. Natural frequencies report the final tally of a natural sampling process. It was found that both laypersons and experts were more likely to use the Bayesian method when the information was presented in natural frequencies rather than in a probability format (Gigerenzer, 1996; Gigerenzer & Hoffrage, 1995). Gigerenzer (1998) argued that mental algorithms performing Bayesian-type inferences are designed by natural selection for natural frequencies acquired by natural sampling, and not for probabilities or percentages.

The innate constraint account posits that the child's learning about numbers is highly constrained by the innate numerical-specific device with highly refined operations. These operations enable the infants to focus attention on number-relevant inputs and to build up in memory number-relevant representations. These also constrain their subsequent learning, both enabling the acquisition of knowledge consistent with the early numerical representation, such as verbal counting, and rendering more difficult the acquisition of knowledge that does not fit the representation, such as fraction numbers.

However, this account has been challenged on several empirical grounds. A particularly vexing question is whether or not a numerical-specific module underlies infants' ability to discriminate small sets of objects reported in the studies using the habituation and the transformation research paradigm. Another question is whether or not infants and young children are able to represent continuous quantity and numbers between whole numbers although the innate constraint account assumes they are not. Also the link cannot be logically excluded between children's whole number bias and their learning experiences. These questions have been part of the driving force for the research that has led to alternative accounts of where the whole number bias is originated.

The Undifferentiated Amount Hypothesis of Early Quantitative Representation

The undifferentiated amount hypothesis (Mix et al., 2002a, 2002b) assumes that early representation of quantities is not number-based, but total-amount-based, for both a discrete quantity and a continuous quantity. The total-amount based representation uses information of continuous spatial dimensions (e.g., overall contour length or surface area or overall area) of objects to make the discriminations. The problem with the interpretation that a numerical-specific mechanism underlies the infants' ability was that number and the continuous spatial dimensions of objects covary in most of the habituation and transformation studies. It is methodologically difficult to isolate the numerical property and spatial dimensions of stimuli in the studies. It was therefore unclear whether or not infants' discriminating reactions to small sets of objects might have been based on a change in one or more spatial dimensions, such as area or contour, rather than a change in number.

Evidence is now available that casts doubt on the hypothesis that a numerical-specific mechanism underlies infants' ability to discriminate small sets of objects. In Clearfield and Mix's study (1999) they manipulated contour length of arrays and found that 6- to 8-month-old infants dishabituated to the novel contour length but not to the novel number. Feigenson, Carey, and Spelke (2002) replicated and extended the finding in a series of studies. They found that 6- to 7-month-old infants could discriminate arrays of small sets when number and total front surface were confounded. But infants dishabituated to a change in front area but not to a change in number when the two variables were competing against one another. Moreover, infants showed no sensitivity to a change in number when front surface was controlled.

This pattern of findings was extended to the transformation task (Wynn, 1992a). These results suggest a total-amount based representation, rather than a numerical-specific representation, underlying infants' ability to discriminate small sets of objects. Simon (1997) suggested that the infants' behavior to discriminate small sets of objects was nothing more than the competence to detect a same/not same discrimination. Such discrimination can be made with competencies that have no direct connection to the conceptual domain of number although it is consistent with arithmetical operations based on cardinal representations of quantity of small sets.

If the total-amount based representation underlies infants' ability to discriminate small sets of discrete objects, the same principle should apply for infants' ability to discriminate continuous amounts. According to Gao, Levine, and Huttenlocher (2000), infants of 6 months old could discriminate between a cup three-quarters full of liquid and a one-quarter full. Infants appear to be able to discriminate both discrete sets and continuous amounts. As it is not known whether or not there are two distinctive forms of early quantitative representation, the suggestion becomes more plausible that the finding of infants' discrimination of small discrete sets can be explained without posting exact number representation. In a recent comprehensive review of the studies of early quantitative representation in infants, Mix et al. (2002a) concluded that there was no convincing evidence of infants' discriminations of small sets based on numerosity but positive evidence of the discriminations based on nonnumerical information[4]—continuous spatial information.

Moreover, the findings of the infants' ability to discriminate discrete sets do not automatically suggest that they have no capacity to comprehend relational quantities or fractional amounts. However, any findings of such capacity in infants and preschoolers would falsify the proposal that early number representation privileges discrete quantities. In fact, notions of relative quantity may be more primitive than those of absolute quantity. According to the principle of Gestalt psychology, human perceptions of stimulus magnitude are inherently relative. The perception system's judgments of stimulus magnitude, such as perception of object size, line length, and weight, are affected by their context. New evidence of young children's competence in reasoning about relational quantity is converging (Mix, Levine, & Huttenlocher, 1999; Nunes & Bryant, 1996; Resnick & Singer, 1993; Sophian, 2000; Sophain, Garyantes, & Chang, 1997). Mix and her colleagues (Gao et al., 2000; Mix et al., 1999) examined young children's competence to understand continuous quantity or fractional amounts before the children received formal schooling. The researchers used nonverbal tasks in the study, in which a fractional amount (e.g., a quarter of circular sponge) was first displayed for a few seconds and then hidden. Subsequently, semicircles of sponge were shown being added to, or removed from, the hidden amount. The child's task was to indicate the resultant amount from among four pictures. They found an emergence of the ability to calculate small fraction amounts in children as young as 4 years old. The ability to calculate more complex mixed-numbers was found in 6- to 7-year-old children. Striking parallels were observed between the development of whole number and fraction calculation although fraction problems appeared to be more difficult than whole-number problems. These results suggest that the course of development for fraction concepts is not dramatically different from the course of development for whole numbers although the former consistently lags behind the latter. The evidence does not support the hypothesis that early representations of quantity promotes learning about whole numbers but interferes with learning about fractions.

There is also supporting evidence from the studies of young children's understanding of the consequences of the size of "n" in a n-split (Nunes & Bryant, 1996; Sophian et al., 1997). Correa (in Nunes & Bryant, 1996) used sharing tasks to examine levels of the understanding in 5-, 6-, and 7-year-old children, where the results of sharing sweets among different numbers of recipients were to be anticipated. In this study, two types of tasks were given to the children. In one situation, the number of recipients was the same between the two groups; in the other, the number was different. Results showed that there was a significant difference in the level of difficulty between the two types of problems. The children made very few mistakes when the number of recipients was the same in the two groups. The percentages of children performing above the chance level in the situations where the number of recipients was different were 30, 55, and 85, respectively, for the 5-, 6-, and 7-year-olds. Desli (in Nunes & Bryant, 1996) carried out a similar study with 6- to 8-year-olds to examine whether or not children understand the inverse relationship involving continuous quantities. As a group, the children in this particular study appeared to have done as well with continuous quantities as those children in Correa's study did when solving problems with discontinuous quantities.

These and other findings of early competence with continuous quantities and relational amounts directly challenge the notion of innate constraint that privileges discrete numbers. However, the question still remains unanswered as to why children continue to fail on tasks that involve fraction numbers. Some researchers (Mix et al., 1999; Sophian et al., 1997) have suggested that the children's difficulty with fractions and rational numbers might also be caused by the confusion of using the same written symbols for both whole numbers and fractional amounts. Many studies have indicated that there is an obvious gap between children's ability to understand properties of fraction numbers verbally and their ability to represent them symbolically (e.g., Saenz-Lud-

[4]We tend to think that "nonnumerical" does not equal "nonquantitative." Quantification based on total amount refers to the quantification by young children based on some mechanism other than application of a unit measure. Continuous quantity and discrete quantity are quantified in terms of the space the quantity occupies.

low, 1994; Mack, 1995). In Mack's study, Jane, a third-grade student, gave the correct answer "two eighths" to a verbally presented problem involving having one eighth of a pizza and getting one eighth more. She then wrote "1/8 + 1/8 = 2/16" and commented,

> I got the first one (meaning the one she answered verbally) wrong. It has to be two sixteenths because you have one whole pizza with eight pieces and you get another whole pizza with eight pieces, so there's two pizzas with sixteen pieces in all. (Mack, 1995, p. 432)[5]

This observation of the discrepancy was taken as evidence that children's problem with fraction and rational numbers lies mainly with their inability to understand what to do with the symbols, not with a limit to their quantitative representation (Mix et al., 1999; Sophian et al., 1997). However, we tend to think that the discrepancy itself does not suggest anything that necessarily contradicts the innate constraint notion. This is because a conceptual reference is the foundation for a symbol to acquire the symbolic status. Also, the acquisition of mathematical representation conventions does not merely mean applying labels to what one perceives. Learning conventions may act as the catalyst for some conceptual changes (Lehrer & Lesh, 2003). It has been shown that the acquisition of the counting conventions may contribute to children's understanding of cardinality and ordinality of numbers (Mix et al., 2002b). Therefore, the acquisition of the mathematical representation conventions may indicate important changes in knowledge representation at both the neuropsychological and conceptual level. This discrepancy issue is discussed further in the third part of this article.

Based on the findings of early sensitivity to both discrete and continuous quantity, Mix et al. (2002a, 2002b) hypothesize that early numerical representation begins with an undifferentiated notion of quantity, which is based on amount of substance for both discrete and continuous entities. The differentiation of the two kinds of quantity is a developmental achievement. The development is assumed to take place when children develop one-to-one correspondence and to continue when children begin to understand how counting determines cardinality and learn to quantify using different conventional measurement tools. Furthermore, the acquisition of the conventional counting skills to quantify discrete quantity has ramifications for children's representation of relative quantity. Presumably, the whole number bias is one of them. For example, children initially performed better on continuous quantity tasks but their performance began to decline once they were able to represent discrete quantities accurately by counting (Resnick & Singer, 1993). They were better able to handle tasks that involved changes in the target amount (i.e., the numerator) than those that involved changes in the reference (i.e., denominator). This suggests that they were using a discrete number strategy without taking the relative quantity aspect into consideration. This alternative account of the origin of early quantitative representation and its development suggest that children's whole number bias toward fraction and rational numbers is not an innate endowment but a developmental result of differentiation between two kinds of quantity in children.

The Learning Account

Before explicating the learning account, the meaning of "learning" needs to be clarified as the concept of human learning has changed significantly since the cognitive revolution in psychology (Gardner, 1985). First, learning is no longer seen as simply the acquisition of behavior. It also includes storing knowledge about relations in the world, as well as acquiring structural representations and mental models (Kuhn, 1995). Secondly, structural learning is possible because of learning tools that human beings are endowed with or can learn to acquire. These tools include imitation, contingent frequency computing, induction, analogy, metacognition, and re-representational and technological tools (i.e., man-made sign systems) (Halford, 1995; Lehrer & Lesh, 2003). Thirdly, structural learning characterizing human learning seeks knowledge that has predictive validity of different attributes about encountered experiences, projecting what one has known onto what has to be learned.

Therefore, the learning account does not necessarily suggest that the whole number bias be totally externally driven. Rather, it attributes the bias to the way in which children's prior knowledge about whole numbers interacts with the instruction on fractions that fails to build the conceptual ground needed for children to understand fraction and rational numbers (Post et al., 1993).

When children are introduced to the idea of fraction numbers in classroom, they have already had relatively solid ideas about what counts as "number," based on the single-unit counting based number theory. The ideas have served them well when they generate novel strategies for solving whole number addition and subtraction (Fuson, 1988; Steffe & Cobb, 1988) or conclude that there is always a next whole number (Hartnett & Gelman, 1998). It is just natural for children to project their number theory grounded in whole numbers onto fraction inputs. However, the result is not rewarding but frustrating for them. Unfortunately, instruction on fraction numbers in many classrooms neglects to pay adequate attention to the conceptual conflict that devel-

[5] A reviewer of this manuscript rightly pointed out that children's natural language is a symbol system too although the written notations may not have been construed as symbolic notion yet for the children when their natural language did. However, a distinction between verbal and written ways of making symbols may still need to be made because there may be separate neuropsychological subsystems for processing quantitative information represented by different notation systems as suggested by recent research. The convergence of represented information in the subsystems on a common semantic representation of quantity may need and represent a higher level of abstraction.

ops and the conceptual restructuring that is required for the transformation to take place from the concept of whole numbers to that of fraction numbers in children.

Initially, there was an ill-conceived start. The conventional "double-counts" treatment of fractions in classroom fails to help children differentiate fraction numbers from whole numbers (Davydov & Tsvetkovich, 1991; Kieren, 1994). This treatment, in a short run, makes it seem intuitive for children to assimilate their existing counting schemas. It also enables them to generate language about fractions. However, in the long-term, there is a risk of fraction numerators and denominators being seen by children in these situations as unrelated whole numbers representing two separate counts. A fraction is then regarded only as the outcome of a double-count, which runs counter to conceiving fractions as numbers representing a quantitative relation between two quantities. The method reinforces the tendency in children, regardless of its being innate or learned, to overgeneralize the counting scheme in learning about fractions and confines children's concept of fraction to merely that of "A thing out of B thing." This deficiency in children's concept for fractions has negative implications for what they subsequently can do and learn about advanced concepts and operations involving fraction and rational numbers (Ni, 2001; Thompson & Saldanha, 2003).

A similar problem occurs with the instruction on multiplication that defines the operation as repeated addition. This makes it easier for students to comprehend multiplication at that point, but it has negative consequence on children's later learning about the idea of multiplication as the relation between two measure dimensions, which is essential in the production of fraction multiplication. This is consistent with the constructivism wisdom that the relevancy of a stimulus with respect to one's prior knowledge has considerable influence on learning in either a positive or a negative way.

The initial ill-conceived introduction was accompanied by instruction that has not devoted sufficient time to develop children's conceptual understanding of the new kind of numbers. In many classrooms the acquisition of the concept of fraction is reduced to the limited part-whole scheme for generating language about fractions and to the rule "multiply or divide the numerator and denominator of a fraction by the same number" for calculating. This results in the separation of operations from their meanings, which makes the difficult content area even more difficult for students to assimilate. Children's number theory that is grounded in whole numbers is persisting even after a few years of instruction on fraction and rational numbers. This is reflected in such misconceptions about numbers held by some elementary school graduates and junior secondary school students as illustrated by the following quotes, "There is a hole between two numbers" (Dufour-Janvier et al., 1987), and "There is no number between 0 and 1" (Ni, 2001). Insightful instructional activities and sufficient time are a must in order for students to struggle, to experiment, and to discover the ideas underlying order and equivalence of fraction numbers. Without these factors, students will not be able to resolve the conflict between fraction numbers and their concept of whole numbers. In an interview with an eighth-grade student in the study by Post, Behr, and Lesh (1984), the student solved the problem: $3/4 = ?/8$. She explained her solution as finding a factor that changed 4 to 8 and then using the same factor to change 3 to 6. She was equally comfortable using an additive strategy on the very next task, $3/4 = ?/5$. She explained that she needed to add 1 to the numerator. When asked to explain why she multiplied in the first problem and added on the second one, she responded that "you first look for a whole number to multiple by and if you cannot find one then you look for a number to add."

In addition to the instruction on fraction that fails to take appropriate measures to adequately support the conceptual restructuring needed for children, the evidence of extraordinary sensitivity and the responsiveness of the human cognitive system to its internal and external environment, has added further credence to the learning account (e.g., Moore & Frye, 1986; Nunes, 1999; Siegler, 1994; Zohary, Celebrini, Britten, & Newsome, 1994). In this context, the research that concerns the relationship between external and internal representations in number concept development is pertinent. Although the nature of the relation is not yet resolved, there is convincing evidence to demonstrate that different external representations are variously effective in prompting internal representations that capture these different aspects. For example, decimal fractions represent a confluence of fractions and whole numbers (Hiebert et al., 1991). Decimals use base-10 notation to represent fractional quantities. Using base-10 notation to represent fractional quantity implies that the quantity has been measured by using very particular kinds of units. The amount that is not covered by whole units must be measured using tenths of units. Whatever amount remains must be measured using tenths of tenth units. This 10-to-1 relationship is just like that for whole-number quantities grouped by 10, by 100. Consequently, the notation used to express decimal quantities has many of the same features as the notation used to represent whole-number quantities. Clearly, decimal fractions have both discrete and continuous character. Instruction on decimal fractions relies heavily on the base-10 blocks to teach students decimal fractions. The base-10 blocks highlight the discrete character of decimals. While this technique helps students understand the structure of base-10 decimals, it may also foster a predisposition in students to represent numbers as being discrete, even if they are decimals. The effects of external representation on internal representation of the number concept suggest that children's whole number bias may be learned and is conceptual.

In sum, the previous discussion suggests that children's whole number bias at least in part has been induced by the present elementary school curriculum and by the type of instruction that privileges discrete numbers. This link can not be logically ruled out by other accounts.

An Evaluation of the Accounts

At issue in the debate is the very nature of the origin of number concept and its association with children's difficulty with fraction and rational numbers. These accounts suggest the different starting points for the development of quantitative concepts and the different causes for the whole number bias. There does not yet appear to be sufficient evidence to determine which of the competing accounts provides an adequate explanation about the nature of the whole number bias. However, the lines of research have improved our understanding about the development of number concepts and the delicate interplay between prior endowments, domain-specific or domain-general, and later learning in the development. In the following section, we will discuss the discrepancies in these accounts to enhance our understanding of theoretical and empirical questions being raised by the accounts.

The innate constraint account concerns the ontogeny of numerical cognition and the conceptual continuity between the origin and the development of numerical competence. The account assumes an innate domain-specific, number-specific endowment that directs an infant to perceive a stimulus in a number-specific way and therefore makes number concept possible. The innate capability provides an intuitive foundation for children to readily learn and acquire understanding of whole numbers but at the same time becomes a natural barrier to learning and acquiring the concepts of fraction numbers. This account assumes children's difficulty with fraction and rational numbers being caused by the lack of conceptual referent. As a result, the whole number bias in children's learning and acquisition of number concepts cannot be avoided. Children's difficulties with number concepts that are inconsistent with the innate bias will persist.

Delineating the initial state of quantitative or numerical cognition is critical for understanding just how these capacities develop. There are numerous observations that are consistent with the learning difficulties assumed by the innate constraint, but the problem of the account seems to affirm, rather than to explain and predict, the consequence. More significantly, there are several theoretical and empirical discrepancies with this perspective.

Empirically, the key evidence supporting infants' ability to discriminate small sets of objects for the account is called into question as to whether or not a number-specific module underlies the discrimination (Mix et al., 2002a, 2002b). This was addressed in the section on the undifferentiated amount hypothesis of early quantitative representation. Besides, the account assumes that the innately determined preverbal counting principles can be mapped onto the verbal counting. This implies that children are supposed to know "one, two, three" before they know what "some" and "many" mean. However, empirical evidence does not support that assumption. Children learn to count from the age of 2 and up to 4 years of age. From the beginning of learning to count, young children know what "one" means. They can pick one object from a pile when asked. In addition, 2- to 3-year-old children know that the other words in the count sequence contrast with "one." They always take a random number of objects greater than one, when asked to hand over "two, three,…" objects (Wynn, 1990). The findings suggest that children know "one" and "not one" (or "many"?) before they know "two, three,…"

More direct evidence for this account may be a shown association between number processing and a specific cerebral network. Human patients with lesions in the parietal and the prefrontal area of both hemispheres showed severe impairment with number processing (Fasotti, Eling, & Bremer, 1992; Luria, 1966). Several brain imaging studies with normal subjects collectively suggested that the parietal region was specifically activated during various number processing tasks (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Menon, Mackenzie, Rivera, & Reiss, 2002; Menon, Rivera, White, Glover, & Reiss, 2002). However, both parietal cortex and prefrontal cortex are high-level association cortices. The neural networks in these cortecis are not hard-wired but highly plastic, able to be rewired along with the experience of learning (Fuster, 1995). It is therefore difficult to infer that the neural networks for number processing in the parietal cortex may become a natural barrier for children to learn and acquire the concept of fraction and rational numbers. On the other hand, association cortex, especially the prefrontal cortex, undergoes late development in the course of ontogeny. In humans, the prefrontal cortex does not attain full maturity until adolescence (Fuster, 1997, 2000). The late development of these cortical areas is critical to number processing and may be one of the neural mechanisms that explains why children acquire the concepts of fraction and rational numbers at later ages. Obviously, this hypothesis is highly inferential because of the lack of support from current experimental studies.

Theoretically, the innate constraint account is questionable with regard to how the innately-determined magnitude representations or the preverbal counting system can be directly mapped onto the verbal counting system to represent whole numbers or positive integers in terms of form and computation device. Carey (2001) has pointed out two crucial properties of the assumed analog magnitude representation of numbers that do not support the claims about its conceptual primitives underlying children's verbal counting. First, the analog magnitude representations are not constructed by any iterative process. Noniterative processes for constructing analog magnitude representations include nothing that corresponds to the successor function of positive integers, the operation of "adding one." Second, analog magnitude representation of number has an upper limit, due to the capacity of the accumulator or the discriminability of the individuals in a set, or both, whereas the positive integer counting lists do not. Moreover, according to the accumulator model (Meck & Church, 1983), analog magnitude representations obscure the successor function of positive integers. Numerical values are compared by computing a ratio in the system, thus the difference between 1 and 2 is experienced as different from that between 2 and 3, which is again experienced as different from that between 3 and 4. Difference between 7 and 8 or any higher successive numerical values cannot be discriminated. Therefore, the analog magnitude representation may be one of starting points in human's numerical cognition. However, it seems that there is a lack of the conceptual primitives in the magnitude representation device that allows the preverbal counting system to be directly mapped onto the human verbal counting system to represent positive integers. Carey's analysis indicates a clear developmental discontinuity that takes place in the transcendence between the preverbal counting device and the verbal counting system.

Research on developmental changes in children's acquisition of counting procedure and construction of number concept further supports the discontinuity argument (Fuson, 1988; Steffe & Cobb, 1988; Steffe et al., 1983). Creation of the abstract and iterable counting unit is the core of the number concept, which is a result of a gradual acquisition and integration of ideas. Steffe et al. (1983) identified five distinct counting types according to the kind of unitary experiential item that the counter creates.[6] The progression in the construction of the counting types demonstrates how children build up the abstract concepts of unit and number in the context of counting activities. A parallel developmental change takes place in acquisition of unit compositions and decompositions. This includes the conceptual shift from the counted unit items being re-presented collections to composite units, from the counted unit items being embedded in a sequence to being disembedded, and then to make "one" become iterable and to establish part-whole operations in a number sequence. According to Steffe and Cobb (1988), the constructions of the abstract and iterable counting unit could take as long as a 2-year period for 6-year-old children.

The innate constraint account was proposed to explain how initially structured learning is possible, that is, how it is possible for an infant to attend to stimuli in a number-relevant way. The distance and the size effect observed both in human adults and infants suggest that the analog magnitude representation is innate and can be considered one of the beginning points in human numerical cognition. This innate endowment is supposed to put certain constraints on later development and learning. However, there are serious theoretical and empirical discrepancies to be explained before it is possible to establish the link between the innate endowment and the verbal counting skills and further between the endowment and the number concepts. Furthermore, as Sophain (1997) rightly pointed out

> The problem is not just one of overriding old constraints, because with each new level of generality, the complexities of mathematical learning increases, so that the problem of explaining how learning is possible is no less challenging at the more advanced levels than at the initial ones. The same learnability argument that led to the postulation of initial constraints on learning, therefore, implies that there must be constraints on later learning as well. What we need, then, is a theory that is not limited to initial, innate, constraints but that can account for the progressive emergence of new kinds of constraints that make possible new kinds of numerical learning as the child develops. (p. 295)

The undifferentiated amount hypothesis is partly a response to the inability of the innate constraint account to explain the available developmental data. This hypothesis assumes early quantitative representation is neither number-specific nor domain-specific. The early quantitative representation may be total amount-based, in which nonnumerical cues (or total spatial dimensions of objects) that covary with numerical cues can be used to discriminate both small sets of discrete objects and different amounts of continuous quantity. It is also possible that more than one cognitive mechanism may support early number-relevant computations (see Carey, 2001; Mix, 1999). Therefore, it is difficult to conclude that early numerical representation privileges discrete numbers. Developmentally, the acquisition of counting skills explicitly indicates the beginning of the differentiation between discrete quantities and continuous quantities. This would have ramifications for children's subsequent development and learning. One of these may be that the sense of relational quantity that is available to children is suppressed with the use of counting to determine exact numerosity of objects. The whole number bias is therefore not innately determined, but rather a result of the development. Consequently, there is no innately-determined disadvantage for individuals to learn about fraction and rational numbers.

Conversely, the undifferentiated amount hypothesis suggests that the characteristic of human perception of stimulus magnitude being inherently relative may be a potential perceptual foundation for children to understand the concept of relative quantity, such as ratios (Mix et al., 2002b; Resnick & Singer, 1993; Sophian, 2000). An anecdote from *Reader's Digest* clearly demonstrates what a 5-year-old child does know about relative magnitude.

[6]The types include perceptual unit items, figural unit items, motor unit items, verbal unit items, and abstract unit items. Counting of perceptual unit items refers to the counting behavior that requires actual perceptual items in order to establish units to be counted. Counting of figural unit items involves the counting behavior using a figural representation to re-present the items to be counted that are not perceptually available at the moment. Counting of motor unit items refers to the counting behavior which uses motor acts or movements (e.g., pointing acts, acts of putting up fingers) as a substitute for either the perceptual item or its figural re-presentation. Counting of verbal unit items links to the counting behavior in which each vocal production of a number word can be taken to stand for a countable perceptual item. Counting of abstract unit items is indicated in the counting behavior in which an utterance of a number word, such as "eight," can by itself be taken to imply the number word sequence "one, two, three, eight," as well as a collection of discrete unitary items that could be coordinated with that sequence of utterance (Steffe et al., 1983).

> A father tries to explain to the five-years-old daughter how much technology had changed. He pointed to his brand-new personal computer and told the girl that when he was in university, a computer with the same power would have been the size of a house. Wide-eyed, the girl asked, "How big was the mouse?" (Hinds, 2003, p. 47)

Children's intuitive understanding of relative quantity is not only demonstrated in the anecdote but has been reported in several recent studies (Nunes & Byrant, 1996; Sophian, 2000; Sophian et al., 1997). For example, Sophian's (2000) study showed that children as young as 4 and 5 years old made accurate spatial proportionality judgments. Moreover, their judgments were no less accurate when both response alternatives differed in spatial configuration from the sample than when all the configurations were alike, indicating that those judgments were based on relational information and not on the exact form of the stimuli. Sophian (2000) also suggested that instruction based on spatial ratios may help children attain a better understanding of fraction values, in that the visual properties of spatial ratios provide a way of conceptualizing the quantitative value of the ratios as an integral whole and a means of identifying the equivalence of fractions that are numerically different but proportionally the same (such as 1/3 and 4/6). It will be theoretically interesting and practically compelling to look for and make use of the plausible intuitive foundation for children to learn about fraction and rational numbers. This will help us to understand whether or not ratios, other than part-whole relation (English & Halford, 1995), are a more primitive base for establishing fraction and rational number concepts and processes. Teaching experiment research is needed to examine how instruction could facilitate the transformation of the visual properties of spatial ratios into numerical ratios in children.

The undifferentiated amount account of the early quantitative representation suggests no innate number-specific endowment, but rather the innate perceptual capacity, to detect overall spatial differences in objects, that allows infants to discriminate between discrete sets as well as continuous amounts. This implies that number concept development involves much learning to distinguish between relevant and irrelevant cues to numbers. This hypothesis thus leaves more room for the complexity of ontogenesis for the development of numerical cognition. One of the complexities of ontogenesis concerns the role of the cultural tools—number representation conventions—in the development of number concepts in children. Vygotsky's (1978) conceptualization of cognitive development points to the representation conventions as being part of the ontogenesis (if an individual is not deprived of the exposure to and application of cultural tools). Their acquisition and utility is therefore not merely a procedural matter. The undifferentiated amount hypothesis entails this position.

The numeration systems created by humans were to meet practical needs to determine quantities and to compensate the limits of human perception and memory. For example, the counting list and procedure reflect the principles to determine quantities and match closely to the limits of human perception and memory. The tool serves not only as the means for practical applications, but also as the object for cognitive reflections (Dehaene, 1997; Lehrer & Lesh, 2003; Saxe, 1991). Acquisition of the cultural tool increases an individual's computational potentials and at the same times also influences his representation of quantity and number. For a developing individual, the acquisition of the counting procedure signifies the cardinality of number and clearly contributes to children's ability to differentiate between discrete quantities and continuous quantities (Mix et al., 2002b). However, this development may also have an intriguing and unexpected consequence, namely the whole number bias. In fact, Debaene (1997) noted that children often lose their intuition about arithmetic in the progression in acquiring numerical knowledge, including the representation conventions or tools. This is partly due to the fact that mathematics instruction moves toward the formal approach emphasizing manipulation of symbols and axioms as children move to higher grades. In addition, the problem is also associated with the dual functions of mathematical notation systems. This will be discussed in more detail in subsequent sections on the discrepancy issue.

As the undifferentiated account adds to the complexity of ontogenesis, it makes the question of how the number concept originates harder to answer. More questions have arisen, such as how the number concept originates from the total-amount-based representation, what parameters (e.g., individuation, and one-to-one correspondence may be the plausible candidates) determine the emergence of the number concept, and whether representations of discrete and continuous quantities have different origins. These questions will open up a new research agenda in the field.

Commensurable to the undifferentiated account, the learning account links the whole number bias to the instruction on fraction numbers. The instruction does not take into serious consideration the fact that children's prior knowledge about whole numbers will have substantial impacts on their learning about fraction and rational numbers. The instruction fails to provide children with various opportunities to explicitly confront the conceptual conflict their prior knowledge about whole numbers has with fraction numbers. This may lead children to treating fractions as a natural extension of whole numbers. The problem may not just start with the instruction on fractions, rather, it may have its origins in the instruction on whole numbers that makes excessive use of singleton units and external representations of discrete nature. According to this account, the children's whole number bias, although perhaps unavoidable, can at least be alleviated, if the instruction can be corrected.

The early quantitative representation may be total amount-based. Some evidence suggests that infants do not quantify number of discrete entities at all but rather represent amount of substance (Feigenson et al., 2002; Gao et al., 2000; Mix et al., 2002a). At some point, children begin to quantify discrete sets of objects and continuous substances differently, both in encoding and transforming the amount. This difference is inevitable because there are conceptual and functional differences between the two types of quantity (Mix et al., 2002b). However, Piaget assumed the same cognitive mechanism was shared by the two types of quantification. He thought that subdivisions and ordering the subdivisions in the context of measuring continuous amounts were analogous to the addition of classes and seriation in the context of measuring discrete quantities (Piaget, Inhelder, & Szeminska, 1960). Nevertheless, developmentally, the acquisition of discrete quantification precedes the acquisition of continuous quantification and the acquisition of whole number precedes the acquisition of fractions. It has been suggested that children acquire a symbolic one-to-one representation of number prior to their learning the conventional counting skills and such a one-to-one mapping could only apply to discrete sets (Huttenlocher, Jordan, & Levine, 1994). Also, measuring discrete quantities is easier than measuring continuous quantities because discrete sets are already subdivided, one needs only to count the units. However, continuous amounts need to be segmented into equal units before they can be enumerated. Moreover, children's experience with class equivalence in counting may interfere with their recognizing the size equivalence of measurement units (Lehrer, Jenkins, & Osana, 1998). In quantifying discrete objects each individual counts the same as another, whether it is large or small (unless your goal is to count all items of the same size), this is not the case in quantifying continuous quantities.

The prior and later acquired knowledge or skills have complex interactions as the mind functions as a system, especially when they are genuinely related, such as whole numbers and fractions, addition and multiplication. The instances of whole number bias illustrate the complexity. A study of interference between multiplication and addition in children provides further evidence (Miller & Paredes, 1990). In third-grade, most pupils already knew many additions by heart. As they started to learn about multiplication, there was a dramatic increase in cross-operation errors, and substantial increase in the speed with which they were able to perform addition. The interference of multiplication to addition occurs in adults as well.

We tend to believe that the whole number bias is more a result of developmental and learning history rather than merely a product of the biased instruction. To fully understand what fractions represent, one needs the idea of equal units, counting, subsets, and so forth. In other words, the confusion between whole numbers and fraction concepts may be unavoidable precisely because the underlying concepts are inextricably linked. There are ways that instruction on whole numbers and fractions can be structured that may alleviate or reinforce the whole number bias, but the bias is unlikely to be avoided.

Certainly, given the learning tools that human beings are endowed with, it is theoretically and practically possible that one instructional approach is more effective than another in reinforcing or alleviating the whole number bias. The learning account, therefore, will always get support from this kind of evidence. However, it is probably also true that children develop competencies in the number domain in ways that cannot be predicted merely on the basis of environmental inputs and general learning mechanism because prior developmental and learning history will always influence the ways new learning takes place.

As the previous discussion has shown, the three accounts have provided different perspectives with which to examine the question of the origin of number concept and thus the cause of the whole number bias. Despite the differences in their theoretical positions, the accounts share some key implications for teaching and learning about fraction and rational numbers. First, they all point to the primacy of prior acquired knowledge/skills, whether or not they are supported by innate disposition, for the effectiveness or ineffectiveness of learning about fraction and rational numbers. Secondly, they all indicate the importance of differentiation of the two types of quantification in children's acquisition of fraction and rational numbers because the whole number bias does not make the differentiation.

These instructional implications of the accounts give rise to two dilemmas that are of theoretical and practical importance for understanding how children acquire fraction and rational number concepts and what instructional supports would optimize the progress toward desired outcomes. The focus of the first dilemma is about the trade-off between cognitive efficiency versus inflexibility of learning with regard to the utilization of children's prior knowledge as the means to introduce them new knowledge. Presumably, at a certain developmental stage, number representation privileges discrete quantities. In this case, the question is how learning fraction and rational numbers may be organized in such a way to be less strained for making use of children's prior knowledge of whole numbers.

The second dilemma is specific to mathematics learning and teaching, where a mathematical symbol, such as 1/2, takes its meaning from the situations to which it refers, while at the same time it derives its mathematical power by divorcing itself from those situations (Resnick, 1986). Children's difficulty with fraction numbers often occurs in their dealing with fraction symbols. There is a tension produced in children by the concrete and abstract representation of numbers. It remains unclear whether the problem lies merely in the difficulty in mastering the symbols or in the lack of conceptual referent. It is important to understand the internal processes underlying the tension that may cause the discrepancy between children's understanding number concepts and understanding their symbols.

The following two sections are devoted to an explication of the dilemmas by drawing upon the lines of research being discussed in this section as well as other research in mathematics education and neuropsychology. To address the first dilemma, different instructional approaches (e.g., partitioning approach, measurement approach, and etc.) to fraction instruction were reviewed to examine to what degree they were successful in dealing with the trade-off. To address the second dilemma, research on neuropsychological disassociation between different notation systems, between visual-spatial representation and language-dependent subsystems, was reviewed to look into plausible neuropsychological mechanism that may account for the discrepancy between learning number symbols and number concepts in children.

COGNITIVE EFFICIENCY VERSUS INFLEXIBILITY: INSTRUCTIONAL APPROACHES TO THE WHOLE NUMBER BIAS

A bias, regardless of whether it is innate or learned, provides both cognitive efficiency and inflexibility. When certain adaptive advantage of a bias is exploited, at the same time inflexibility brought about by the bias is reinforced. It is a challenge to manage the trade-off in instruction. The case in point is the relationship between the whole number and fraction and rational number instruction in the conventional elementary mathematics curriculum. Children use their knowledge about whole numbers to make sense about fraction numbers; and instruction on fractions makes use of the prior knowledge in children. The practice follows a common wisdom about learning and instruction, that is, to use what is already known to figure out what to be learned. There is nothing wrong with the wisdom because this is how our constructive minds work. The problem lies in that the instruction does not take measures accordingly to reduce the possible risk of reinforcing the whole number bias by taking advantage of the prior knowledge.

The whole number bias has to do with overgeneralizing the single-counting scheme to situations involving fractions or relative quantities. Facilitating the differentiation processes between quantification of discrete and continuous quantities therefore becomes prerequisite for children to develop the fraction concepts. Unfortunately, this is where the conventional "double-counts" treatment of fractions in classroom fails (Davydov & Tsvetkovich, 1991; Kieren, 1994). Instructional interventions have been attempted to look for an experiential or conceptual base that would help children understand the ideas of fraction numbers and make the conceptual transformation from whole numbers to fraction and rational numbers. In this direction, two distinctive approaches have emerged: One is to replace the static part-whole model with alternative instructional methods to introduce children to the concept of fraction and rational numbers, and the other to plan instruction on whole numbers and rational numbers simultaneously. In the following sections each of the approaches is examined to determine to what degree they were successful in addressing the trade-off dilemma by having children make use of their prior experience to explore new knowledge.

Equal Partitioning Approach Versus Measurement Approach

Two alternative instructional methods have been employed to replace the double-counting approach. They are the measurement-based approach (e.g., Davydov & Tsvetkovich, 1991; Moss & Case, 1999) and the equal-sharing approach (Empson, 1999; Streefland, 1991). More specifically, the equal sharing approach is based on partitioning in relation to multiplication and division. It emphasizes the equal-sized partitioned parts (or groups) in relation to a unit whole and the change in terms of ratio functions (in multiplication or division). The measurement approach emphasizes the concept of measurement units and unit hierarchies that are expressed in fractions or divisions. Conceptual analysis of properties of fraction and rational numbers and empirical studies of children's experience with quantities (Confrey, 1994; Davydov & Tsvetkovich, 1991; Nunes & Byrant, 1996; Streefland, 1991) suggests that both equal sharing and measurement experience provide an appropriate resource for teaching fraction and rational number concepts. Sharing and measuring activities require children to use the numerical concept of discrete structures as templates to partition continuous units on one hand (Steffe, 2002), on the other hand, they afford children the opportunity to experience something new about numbers that is different from what they have known about whole numbers by counting. For example, the principles about fraction and rational numbers can be developed and understood in the context of measurement activities or partitioning activities. These principles include that any quantity can be measured by some smaller quantity or partitioned into smaller quantities; and quantities can be compared, added, or subtracted only when they have the same units of measure.

To the authors' best knowledge, Streefland's work (Streefland, 1991, 1993) was the most representative and comprehensive of research on the equal sharing approach. The study entailed work with 13 fourth-grade students, examining their learning progression for 2 years. The rational for the approach was that equal sharing activities could naturally link concrete representations and the mathematical meaning of fraction. The activities would allow children to make use of self-generated partitioning schemes in the learning of fractions, making the learning processes to be generative and insightful for the children. Equal sharing contexts therefore were believed to be able to provide both the sources for children to form the concepts of fraction and the area for them to apply the concepts.

Streefland's approach consisted of five clusters of equal-sharing activities to facilitate children's learning about fractions. The first three clusters focused on the concrete contexts of equal sharing through which children explored the meaning of fraction and generated fraction terms. These included (a) serving-up activities that were designed to generate fraction terms and compare sizes of fractions; (b) seating arrangements and distribution activities to generate equivalent fractions and to interweave with ratio; and (c) activities that attached a length, weight, or price to a given unit for children to learn about the four computations with fractions. The next two clusters gave emphasis to solving fraction problems in a finer manner at the symbol level. These included (d) activities to decompose and compile fractions to "distance" fractions more from concrete sources and (e) activities to explore formal rules for the operations with fractions. The equal sharing approach demonstrated to have positive impacts on children's learning about fractions. The children showed a genuine understanding of the concept of fraction and ratio. They also were able to use models and application of diagrams generated from equal sharing situations to represent and to solve fraction problems. These benefits of the approach were also observed in first-grade children who were asked to solve equal sharing problems to explore the ideas of fraction numbers (Empson, 1999).

The measurement approach differs from the equal sharing approach in its treatment of how a fraction originates in specific reality. According to Davydov and Tsvetkovich (1991), fractions originate from the measurement of quantities with a chosen unit. A measured result is expressed in whole numbers that are usually approximate. To obtain a desired precision in the result of measurement, it is necessary to choose other, smaller units that have a definite relationship to the first unit of measure. Consequently, the measurement approach emphasizes the concept of measurement units and unit hierarchies that are expressed in fractions or divisions. In the studies of the approach by Davydov and Tsvetkovich (1991) with three classes of third-grade students in three consecutive years, they used the context of measuring length and weight of objects to introduce the children to the concept of fractions. Specific tasks included: (a) those to use given different measures to measure the same object, to compare the sizes of measure units in relation to the measured results; (b) those to establish the quantitative relation between the measuring units; and (c) those to choose a new measuring unit to measure the remainder that is related to the initial measure. The representation tools, provided by the researchers to the children, were paper strips and number lines. In their study, Davydov and Tsvetkovich (1991) reported that most of the students receiving the treatment performed well on the tasks requiring them to order fractions on a number line and to produce equivalent fractions.

As mentioned previously, these two approaches differ in the treatment of how a fraction originates in a situation. In addition to that, they also differ in the extent to which the content and contexts being used in the treatments are associated with children's prior experience. Streefland (1991) called the equal sharing approach a "realistic approach" because the task contexts involved in the approach appeared to have a closer association with children's life experience and their prior knowledge about numbers. Children were encouraged to create their own representation tools to represent and to solve the tasks. In contrast, Davydov and Tsvetkovich's measurement approach utilized a relatively formal method because it designated number lines as the representation tool for students to explore solutions to the measurement problems. The two tasks used in their respective studies, that are described below, reflect the contrast.

> A family, consisting of father, mother, Peter and Ann, have pizzas for lunch. The first one shared fairly. Mother divides the second one in four equal parts, too. Then, she said: "…I have had enough. You three can share this one." "No," said Ann, "one of these pieces is enough for me," and turning to Peter and her father, she added, "you two can share the rest." Peter and his father divided the pieces. How much do each of the family members get? (Streefland, 1993, p. 294)
>
> Children were asked to measure a certain volume of water in a jar with a small mug. They were asked to measure the same volume again with a different mug that was twice as small as the first one. After the measurement, the teachers asked the leading questions for students to discuss and to use number lines to answer them: (1) Why the second number was larger than the first, and by how many times, and (2) how these two measures were related. (Davydov & Tsvetkovich, 1991, pp. 108–109)

Streefland (1993) thought that the relatively formal treatment of the measurement activities in Davydov and Tsvetkovich's study was superficial due to its lack of experiential and intuitive foundation from children, which made children less likely to appreciate what the approach was going to offer. It is true that children have far greater richness of experience with equal sharing than with measuring. However, this does not mean that the measurement approach should be dismissed as less important. As Streefland (1991) pointed out

> Measuring with remainders, for instance, provokes refinement of the applied measuring-unit in a natural fashion; in the wake of such a process, decimal-fractions are introduced as well, where the applied measuring refinement happens to be decimal. But this, of course, is insufficient. This type of access can also be applied to do more than imitate the system in an insightful manner. The point, moreover, is to embed these accesses in a realistic context. The decisive factor is the way in which this takes place that is, the nature of the questions posed in order to pass beyond the access and penetrate the field of fractions. (p. 47)

Streefland's point was valid with respect to both the measurement and the equal sharing approach.

The equal sharing approach may have its own limitation too. The main advantage of the approach is that children can bring the partitioning schemes to the learning of fractions and use them to approach equal sharing tasks. When prior knowledge and experience can be called upon, new learning becomes more efficient. However, there may be a confusion of the prior concepts with the new concept given that equal sharing tasks appear to children to be commensurable with what they have already known about numbers. Mack (1995) found that children often solved sharing problems essentially by treating fractional units as wholes, which created confusion about the relationship between whole numbers and fractions. Also, the part-whole scheme associated with the equal sharing approach is less convenient to demonstrate the making of improper fractions (Steffe, 2002; Thompson & Saldanha, 2003; Tzur, 1999) as it is difficult to perceive within the scheme how it is possible that the number of parts (the numerator) can be larger than the whole being divided into the parts (the denominator).

On reflection, Streefland (1993) later discussed the issue that children's learning in his study showed to be quite context-dependent. Their constructions and productions of diagrams or models were limited to those they created and used with equal sharing contexts. The limitation became very obvious when the children were introduced to the number line representation of fractions. Eight out of the 13 experimental children had difficulty in using the number line representation because of the confusion of interval and point when constructing a number line and the difficulty in establishing the relative standpoint. Streefland (1993) suggested that producing representations of mathematical materials at an increasing distance from reality ought to be better guaranteed in instruction in order for the instructional treatment to function more systematically.

From a psychological point of view, the measurement approach is further away from children's prior experience with whole numbers than is the equal sharing approach. Indeed, counting is really just a special type of measurement. However, in counting, the units are given and the total is already divided into discrete items. In these other kinds of measurement, the total must be divided by imposing some other units; a chosen unit may be a fraction, instead of a "whole," as a measuring unit. Therefore, conventional measurement requires counting and more. Also, the equal sharing approach appeared more amendable to children because they could generate correct fraction terms by treating fraction units as wholes when solving equal sharing tasks even when the tasks involved continuous quantities (Mack, 1995). Consequently, on one hand children may more easily get engaged in the equal sharing tasks than in those using the measurement approach. However, on the other hand, the measurement approach may be more likely to elicit cognitive disturbance, and thus permit greater differentiation than does the partitioning approach in children between whole numbers and fractions. This is an open question that warrants further investigation. In Davydov and Tsvetkovich's study they did not use tasks that differed from the measurement tasks used in the experiment, such as equal sharing tasks, to assess the children's understanding of the concepts of fractions after the experiment. It was unclear to what degree that what the children had learned about fractions from the measurement contexts was context-dependent, as found in the study of the equal sharing approach.

Simultaneously Planning Instruction on Whole Numbers and Fraction Numbers

Indeed, replacing the static part-whole teaching model with the equal sharing or the measurement approach is one way to have instruction on fraction and rational number less strained for taking the advantage of children's whole number knowledge. An alternative method is to plan instruction on whole numbers and fraction and rational numbers simultaneously. To tackle the problem we need to trace a long way back to how we customarily teach whole number concepts and operations, which may later strain children from comprehending more advanced fraction number concepts. If instruction on whole numbers and fraction numbers is planned concurrently, early arithmetic instruction may be organized very differently. There have been two levels of thinking and ways to approach the problem in this regard. One operates at a relative local level by employing stimuli and activities that are amenable to linear measurement in early arithmetic learning to help alleviate the whole number dominance (Klein, Beishuizen, & Treffers, 1998; Siegal & Smith, 1997). The other approach is much more comprehensive and it identifies certain common conceptual links to unite whole number and fraction number learning and teaching (Behr, Harel, Post, & Lesh, 1992, 1993; Post et al., 1993).

At the local level, Klein et al.'s teaching experiment (1998) provided an excellent example. In their study, an empty number line (on which children could draw marks for themselves) was employed as a mental model to help second-grade children learn mental addition and subtraction and to develop greater flexibility in mental arithmetic. A structured bead string was used to introduce children how the empty number line works. The bead string has 100 beans, ordered following the 10-structure: 10 red beans followed by 10 white beans followed by 10 red beans and so on. The linear character of the number line was thought well suited to link up with informal procedures to solve addition and subtraction mentally. Problem representation is more transparent on the empty number line than on arithmetic blocks because the row of the numbers is not cut off at each 10. Students using empty number lines can be cognitively involved in their actions. In contrast, students who use materials such as arithmetic blocks sometimes tend to depend primarily on visualization, which results in a passive "reading off" behavior rather than cognitive involvement in the actions undertaken.

The teaching experiment demonstrated that the empty number line appeared to be a very powerful model for the learning of mental addition and subtraction. Compared to the control group who received the conventional arithmetic instruction emphasizing arithmetic computation, the children of the experimental group used a greater variety of solution procedures, but this flexibility did not influence their procedural competence. This specific method was shown to benefit not only average and better students but also weaker students. The study suggested that the use of an external representation of linear nature might facilitate flexible thinking in doing arithmetic and thus help alleviate problems with the whole number dominance.

The Rational Number Project group (Behr et al., 1992, 1993; Post et al., 1993) has been developing the comprehensive approach. One line of the team's research has been to identify cognitive links between additive and multiplicative structure in general, and between whole numbers and fraction numbers in particular, in the hope that the transition could be made easier for children. If such links did exist, according to Behr et al. (1992), multiplicative concepts could be learned earlier in the curriculum and be concurrent with learning about additive structures.

Identifying these links is a crucial component for implementing the proposal. The concept of unit has been identified as one of the links (Behr et al., 1993; Lamon, 1994; Post et al., 1993; Steffe, 2002). Behr et al. (1993) have demonstrated that the transformation involved in solving problems embodying the rational number subconstructs (i.e., part-whole, quotient, operator, rate, and measure) can be characterized in terms of compositions and recompositions of units. For example, the part-whole construct for 3/4 suggests two interpretations: three-fourths as parts of a whole is three one-fourth units, that is, 3(1/4-unit)s, or three-fourths as a composite part of a whole is one three-fourths unit, that is, 1(3/4-unit); the measure construct for 3/4 suggests an interpretation that three-fourths is three one-fourth units of a quantity with 1/4 being used as a measuring unit. Behr et al. suggested that a very limited variety of unit types used in whole number arithmetic instruction might be a factor contributing to children's rigid number concept. They proposed that children should be given more situations of whole number arithmetic that involve a variety of unit types and units of units (Behr et al., 1993, p. 17). Experience in representing and manipulating quantities that can be represented in these unit types is considered an important element for learning and understanding whole number arithmetic. This will not only provide a more adequate foundation, but also a cognitive bridge to learning and understanding fraction number concepts and operations.

Corroborating the works that identify unit concept as a cognitive bridge linking the whole number concept and the fraction and rational number concepts, Steffe (2002) has provided a detailed account of how children's concept of fraction may emerge from reorganizing their numerical counting schemes. The principal operations for the conceptual acquisition of numerical counting scheme are iterating a unit item and disembedding a numerical part from a numerical whole. A unit composite produced by the operations can be used as a template both to partition a continuous unit item into equal-sized parts and to iterate any of the parts to reconstruct the whole, which is considered basic of the scheme to be called a unit fractional scheme. Steffe (2002) has shown that children could use the operations to partition a continuous item (a stick) into equal parts, such as 10 equal parts, and understood the part-whole relation, 1 part as 1/10 and the whole composed of the part being iterated 10 times. However, the children's part-whole scheme that is applicable to unit fractions is still constrained by the partitioned whole being a specific collection of equal parts. The scheme makes it difficult for children to make sense of improper fractions, such as 11/10. At this point, the measurement tasks that ask for measuring a remainder that is related to the initial measure unit may be able to provide a cognitive scaffolding needed for children to make the conceptual shift from the part-whole scheme to understanding the invariant relation between the size of a measure unit and the size of the measured whole. The instructional potential of the measurement approach is worth being exploited.

The consideration to plan instruction on whole numbers and fraction numbers simultaneously suggests the possibility of introducing children to fractions and rational numbers at an earlier age. The question is then whether or not this is possible and also beneficial to children's development. Since measuring is considered a mental tool for learning about fraction and rational numbers, it thus may be useful to reframe the question into how is children's competence with measurement. The conceptual prerequisites underlying the act of measuring involve transitive inference and an understanding of units (Nunes & Bryant, 1996; Piaget et al., 1960). Empirical evidence suggests unmistakably that children have little difficulty with the transitive inference aspect of measurement (Bryant & Kopytynska, 1976; Miller, 1989). Young children show some understanding of units but the competence is still under development and most 6- to 7-year-old children can develop a understanding of the multiple relationships between measurement, units, and number (Nunes & Bryant, 1996).

The evidence of young children's competence with measurement suggests the possibility to expand children' number experiences with fraction numbers at an earlier age, rather than delaying it until they reach third or fourth grade. An instructional study showed that 5-year-old children were responsive to a brief training intervention to understand inverse relations (Sophian et al., 1997). In Sophian et al.'s study, the children were allowed to compare the amounts one recipient would receive in two settings where different numbers of recipients shared the same amount of cookies. The children were then told the correct answer that with the same amount to share, the more recipients were there, the fewer cookies each could get. They were then given a posttest consisting of four contrast-recipients sharing problems that were different from those used in the pretest. The 5-year-olds receiving the training showed substantial improvements in performance on the sharing problems. On the posttest 13 experimental children, versus only 4 control children out of 24 children for each group, responded correctly to all 4 of the sharing problems on the posttest; whereas on the pretest only 2 in each group did so. In another training study 6 of the 14 experimental children showed an undifferential response pattern (either choose-less-numerous or choose-more-numerous response pattern toward the sharing and nonsharing problems) on the pretest. These children shifted to the differentiated response patterns on the posttest; whereas none of the 15 control children who showed undifferential response pattern on the pretest shifted to the differentiated one.

In a study by Empson (1999) with a first-grade classroom, children's understanding of the fractions changed in important ways after a 5-week instruction that revolved around children's solutions and discussions of equal sharing tasks. By the end of instruction, 16 children out of 17 in the group were able to use strategies other than repeat halving to solve equal sharing tasks. Most of the children understood the reverse relation between number of share and size of potions. More significantly, about half of the class was able to use their knowledge to solve novel problems. The results of the few instructional studies suggest that children younger than third grade are able to learn and to reason about relational quantity with familiar equal sharing activities. But it should be borne in mind that the possibility to introduce children at an earlier age to the concepts about fraction numbers does not suggest having them exposed to fraction symbols at a younger age. Rather, it is possible and also fruitful to expose them to various situations that stimulate their relational reasoning (Lamon, 1999).

One important aim, consistent for all the approaches previously discussed, was to find the experiential base for children to construct the concept of fraction. The equal sharing approach views partitioning as the basis. The children's actions, such as sharing, folding, and dividing, are considered to be the experiential basis for multiplicative reasoning because they produce simultaneously multiple versions of an original (Confrey, 1994). This makes them distinctively different from the actions, such as affixing, joining, and removing, which are associated with addition, where the change is determined through identifying a unit and then counting consecutive instances of that unit. Empirical studies (Emerson, 1999; Sophian et al. 1997; Streefland, 1991) demonstrated positive effects of the equal sharing approach on children's making sense of some properties of fraction numbers, supporting the theoretical analysis of the conceptual connection between the actions of splitting and the development of multiplicative reasoning. On the other hand, the insights that children might have gained from solving equal sharing problems were intersected with the whole number concept and the counting routine. Children may not be able to inhibit treating fractional units as wholes, which generated some confusion about the relation between whole numbers and fractions (Mack, 1995). This indicates a considerable conceptual gap for children to overcome from "division of an object" to "division of a number."

Davydov and Tsvetkovich (1991) considered that the measurement approach would help alleviate the gap between division of an object to division of a number as the approach emphasizes the concept of equal measurement units and unit hierarchies that are expressed in fractions or divisions. They believed that the measurement approach was more likely than the equal sharing approach to accent the epistemological origin of fraction concept, division of number. Lehrer's (2003) review of the literature on understanding of measurement in children has also shown that conceptual changes about measurement essentially involves the change "in a network or web of ideas related to unit" (p. 181), such as equal units, iteration of unit, unit hierarchy, and etc. These ideas are crucial for subsequent development of the understanding of rational numbers and multiplicative structures as the work of Rational Number Project team has suggested. However, the suggested long-term benefit from the measurement approach has yet to be demonstrated in comparison to that of the equal sharing approach. It will be theoretically and instructionally interesting to examine relative effectiveness of each approach in narrowing the conceptual gap between "division of an object" to "division of a number."

The Rational Number Project team (Behr et al., 1992, 1993; Post et al., 1993) examined the connection that would connect the whole number and the fraction and rational number concepts. They have identified the concept of unit as the most feasible bridge to link the two conceptual fields. There have been suggestions that a flexible unit concept would provide children with a sound conceptual base for them to learn about fraction and rational numbers. This approach points to the problems with fraction instruction that lie not only in the "double-count" way to introduce children to the concept of fraction, but also in the conventional whole number instruction. The proposal of planning instruction on whole number and fraction and rational numbers simultaneously is theoretically promising. The plan is more proactive than preventive, compared to the other approaches that are limited to the change of the instruction on fraction and rational numbers.

In the searching for an experiential and conceptual base for children to learn about fraction and rational numbers, the alternative instructional approaches may differ in their relative effectiveness in terms of the trade-off between cognitive efficiency and flexibility. Probably there is no single approach that would be the "best" method for helping children "penetrate the concepts of fraction" given the fact that most learning following a particular instructional approach has shown to be more or less context-dependent (Mack, 1995; Mix, 2002; Streefland, 1991, 1993). Therefore, multiple, complementary, approaches are required. However, before that is possible, research-based knowledge is needed to determine relative effectiveness of each of the approaches (Ni & Saxe, 2003). Obviously, different approaches have different emphasis on the origin of fraction and therefore they would have different affordance for children's cognitive and mathematical exploration.

CONCRETE VERSUS ABSTRACT REPRESENTATION: UNDERSTANDING THE DISCREPANCIES BETWEEN LEARNING NUMBER SYMBOLS AND NUMBER CONCEPTS

Children often discover and use mathematical ideas that they are not necessarily able to express with symbols (e.g., Mack, 1995; Mix et al., 1999). This intuitive knowledge is the root of mathematics as a rational enterprise. On the other hand, intuition must be analyzed and the analysis must be made in mathematical terms. Therefore, learning mathematics also means learning to acquire the language of mathematics. Consequently, connecting meaning to different representation systems is one of the most significant aspects of learning about mathematics and presents a great challenge for many children. This is particularly evident in learning about fraction and rational numbers.

It is a common observation that for otherwise similar fraction tasks, children show contrasting performance between those involving symbols and those not involving the symbols. Studies have shown that most children can correctly answer the fraction problem "how much is one third plus one third" as "two thirds" verbally. However, when the same problem is represented symbolically or graphically, they reach the answer as 1/3 + 1/3 = 2/6. They even believe both answers are correct (Ball & Wilson, 1996; Mack, 1995). The contrast is also reflected in children's making association of fraction symbols to graphical representations. Children exhibited ceiling-level performance on regional area representations but floor-level performance on number line tasks (Ni, 2001).

The accounts of the origin of the number concept offer different interpretations of the observed discrepancy. The innate constraint account considers that the gap reflects a limit to children's conceptual representation of amounts between whole numbers. The undifferentiated amount account ascribes the gap to children's confusion of the fraction symbols with those of whole numbers. The explanations seem plausible but hardly falsifiable. The instructional approaches previously reviewed are supposed to help alleviate the problem because the conceptual foundations intended by the approaches should facilitate establishing the connection between mathematical meanings and mathematical notation systems. Nevertheless, the approaches themselves do not explain anything about what causes the discrepancy. Fortunately, recent brain research on human numerical cognition provides some glimpse of neuropsychological disassociation between different notation systems, between visual-spatial representation and language-dependent subsystems, that process numerical information. This line of research has shed some light on our understanding of the discrepancy phenomenon. In the following sections, we first present findings that suggest the separate subsystems for processing numerical information. We then look into developmental research for some clues about how the eventual translation between the subsystems may become possible. Finally, we discuss the notion of the dual functions of mathematical notations to further illuminate the discrepancy phenomenon. The research suggests that the gap has more to do with the conceptual representation of quantities than with the confusion of the symbols.

Architecture of Internal Representation of Numerical Quantities

Recent neuropsychological research on numerical cognition suggests the existence of certain architecture of internal representation of numerical quantities (Dehaene & Cohen, 1991; Dehaene, Dehaene-Lambertz, & Cohen, 1998; Dehaene et al., 1999). That is, there may be separate subsystems responsible for processing information represented by different notation systems. Dehaene et al.' study (1999) reveals separate linguistic competence and visuo-spatial representations underlying exact calculation and approximate estimation of quantities respectively. In their study, English and Russian bilingual subjects were first taught a set of exact or approximate sums in either language. When tested on the trained exact problems (e.g., "What is two plus two?"), the subjects performed faster on the exact computation tasks presented in the language used in the training session than on those presented in the nontraining language, regardless of whether they were trained in English or Russian. For approximate tasks (e.g., "Is six closer to four or to nine?"), in contrast, performance was equivalent in the two languages. The performance pattern was observed with another group of bilingual subjects who were asked to do more complex exact and approximate calculation problems. The differentiated performance patterns showed a language-switching cost for carrying out the exact calculation tasks, whereas the approximate tasks were shown to be language-independent. Using the brain imaging technique, the researchers reported greater activation in the bilateral parietal area for approximation than for exact calculation and greater activation in the left inferior prefrontal region for exact calculation although both the bilateral parietal and the prefrontal areas were activated for both types of calculation.

Research also showed that the processing of Arabic numerals could be dissociated neuropsychologically from the processing of verbal numerals in brain-lesioned patients (Dehaene, 1992; McCloskey, 1992). Under the conditions of speeded processing in normal adults, quantity priming[7] was observed when a prime and a target number were presented in the same notation system (Arabic, or verbal, or dot pat-

[7]"Quantity prime" refers to the phenomenon that when two numbers are presented in close temporal succession, responses to the second number are facilitated to a variable extent depending on its numerical relation to the first number. For example, in comparing sizes of two numbers, one's response would be faster to a target number "8" than to "6" for a given prime number such as "5" because of the distance effect (also see page 30).

terns) but not observed when the two were presented in different notations. In a less speeded condition, quantity prime was observed both within and between verbal and Arabic notation system but not between verbal and dot patterns (Koechlin, Naccache, Block, & Dehaene, 1999).

These findings suggest that the internal representation of numerical quantities seems to initially dissociate into multiple notation-specific subsystems and/or into visual-spatial representation and language-dependent subsystems. Their convergence on a common semantic representation of quantity may occur at a later stage of processing for a particular stimulus or at a later stage of development for a developing child. The findings provide a neuropsychological explanation of why learning number concepts and number notation systems appear to be disassociated in children.

Fodor (1983) called each of the subsystems that processes information in a particular way as "modules of mind." He assumes that the human mind consists of special-purpose "modules" or input systems with highly constrained innate specifications.[8] One property of these modules is that they are informationally encapsulated, that is, other parts of the mind have no access to the internal workings of a module, only to its outputs. According to this module hypothesis, each module for processing each of the notation systems, or for processing language-dependent or language-independent system, may be self-contained and therefore may have no direct access to the information elsewhere in the mind in its early development. This explains why young children find nothing contradictory when they explain verbally "one third plus one third equals two thirds" and calculate symbolically "1/3 + 1/3 = 2/6." But a logical question then follows of how the translation between the subsystems would become possible, which does occur in later development.

Karmiloff-Smith (1992) proposed a developmental mechanism, described as "representational redescription"[9] to explain how communication between modules becomes possible. "Representation redescription" refers to the mechanism by which information that is processed in a self-encapsulated system is "re-written" in such a way that the information becomes accessible to other systems. She argues that a fundamental aspect of human development is the process by which information that is *in* a cognitive system becomes knowledge *to* that system. One example in language acquisition that is well studied involves the child's learning of rules for inflecting nouns and verbs, with the use of morphemes such as plural *–s* and past tense *–ed*. When children first produce correct instances of plural and past tense forms, some of these forms conform to an inflectional pattern shared by a large number of forms, others belong to minor patterns, and still others are irregular, not predictable by rule. At the next stage, the correct but irregular and "minor patterns" forms are partially or totally eclipsed by incorrect forms that conform to the general patterns. For example, children would be inclined to use "goed" instead of "went" as the past tense of "go." Eventually, the correct forms reassert themselves. According to Karmiloff-Smith (1986), these "errors" indicate that children come to make explicit marks on the subunits that have both semantic and phonetic substance. They are aware of the internal structures of subunits. More significantly, language itself becomes a complex "object" in the child's mind for him or her to explore.

What a mind has acquired may undergo constant representational changes of this kind, by which communication between different subsystems becomes possible. It is probably also by the representational changes that an entity acquires conceptual status and thus becomes explicit enough for cognitive manipulations. Children's evolving concept of fraction illustrates these changes (Steffe, 2002; Tzur, 1999). Children generate unit fractions by using a partitioning action. Initially, they think of a unit fraction merely as a part of a whole, which does not afford any numerical operations, such as generating nonunit fractions. This only becomes possible when children are able to think of a unit fraction as a symbolized, *iterable* part the magnitude of which is based on the numerosity of the partitioned whole. However, the researchers observed that once children iterated a unit fraction more than the number of parts in the partitioned whole, for example iterating 1/8 nine times, they simply changed their thinking about the part from 1/8 to 1/9. Even though children can think of a unit fraction as an iterable part, their thinking about unit fractions is still constrained by the partitioned whole being a specific collection of equal parts (i.e., its numerosity), not on an invariant relation between the size of a part and the size of the whole. Tzur (1999) termed the conception of a unit fraction as the partitive fraction scheme, which is characterized by the children's ability to iterate and operate numerically on unit and nonunit fractions as long as those fractions do not exceed the partitioned whole. For children to generate and understand improper fractions using the iteration operation, they need to conceive an iterable part as an invariant, multiplicative relation with respect to the reference whole. Children's understanding of unit fraction is "re-described" from a part contained in a partitioned whole, to a *symbolized iterable part* used in conjunction with number knowledge, then to a *symbolized iterable relation* between the size of any two units, one of which is regarded as the whole.

Another example is the development of children's partitioning strategies. Partitioning is a fundamental activity from which children generate their initial ideas about fractions. The earliest viable strategy that most children use for partitioning an object is halving and repeated halving. The strategy helps children generate particular fraction terms, such as 1/2, 1/4, 1/8 and so on but also confines them to the fractions

[8]It is often said that what makes something a module is that it is domain-specific. But Fodor's (2000) use of the term "module" refers primarily to a property of processes, not a property of information in the sense of Chomsky's innate language database (e.g., grammar). The use of the term "module" here does not entail the meaning of "domain-specificity."

[9]It remains unknown how the mind *re-describes* stored information in order to make it accessible to the "central system" for cognitive manipulation.

of special status (e.g., Spinillo & Bryant, 1991; Pothier & Sawada, 1983). Nonhalving partitioning strategies enable children to become familiar with 1/n fractions beyond those confined to the halving partitioning strategy. Use of nonhalving strategies requires the understanding about the ways in which a given number of partitions can fit into the unit. Piaget et al. (1960) described the scheme as an anticipatory scheme. It represents more internal control by children over partitioning activities in order to meet task demands.

A third example involves the mastery and manipulation of various representations of fractions. A regional area, a set, or a number line may be used to represent a fraction, such as 1/2. These graphical representations differ not only in visual presentation, but also in their emphasis on particular aspects of the meanings represented by a fraction number. In particular, regional area or set representations appear to make the part-whole relationship represented by a fraction more salient, whereas number lines tend to accentuate the measure meaning of a fraction (e.g., English & Halford, 1995; Behr, Lesh, Post, & Silver, 1983; Novillis, 1979). Moreover, the set representation involves discrete quantity, whereas the number line and regional area involve continuous quantity. Translation among the graphical representations may become possible for an individual only when each of the representations is "re-described" in the cognitive system as standing for the common quantitative relation.

The previous examples of how a number concept may evolve or be "re-described" in a child's mind show that "symbolizing" means more than mapping mathematical symbols to mathematical ideas. The foundation that underlies the mapping is the construction of mathematical ideas, such as "fraction," or "partitioning," with the status of *conceptual* entity, that is, the ideas are explicitly represented in the mind and they are capable of possible cognitive manipulations. The revealing architecture of internal representation of numerical quantities suggests that the conceptual changes in numerical representations will be slow, laborious, and recursive, the point we will elaborate in a moment.

Dual Functions of Mathematical Notations

The gap between learning about number concepts and learning their notation systems is further complicated by a paradox that is central to mathematical thinking (Lehrer & Lesh, 2003; Resnick, 1986). On one hand, a fraction such as 1/2 takes its meaning from the situations to which it refers; on the other hand, it derives its mathematical power by divorcing itself from those situations. Several studies have demonstrated children's difficulty in recognizing the dual functions of mathematical notations (Ball & Wilson, 1996; Empson, 1999; Nik Pa, 1987 in Steffe, 2002). For example, Nik Pa, in interviewing nine 10- and 11-year-old children, found that they could not find 1/5 of ten items because "one-fifth" referred to one in five single items. The children separated a collection of 10 items into two collections of five and then designated one item in a collection of five as "one-fifth." Children's concept of fraction relies on and is also constrained by clearly evident referent.

The restriction can also be characterized in terms of the distinction made by Gallistel and Gelman (1992) between the conceptual and categorical uses of number. According to Gallistel and Gelman, a categorical use of number refers to all sets of a given numerosity, such as three apples, three animals, or three squares. By contrast, a conceptual use of number is not by what it refers to, but rather in terms of a system of mental operations that are isomorphic to some of the arithmetic operations. "A number smaller than four but bigger than two uniquely identifies "three" without any referent of "three." As the tension between the conceptual and categorical use of number was a driving force in the history of mathematics, the same kind of tension may also play a significant role in the individual construction of the knowledge system about numbers. For example, the development in children from conserving number identity via counting concrete objects to conserving equivalence of nonspecified quantities is of a similar type of abstraction (Karmiloff-Smith, 1992; Piaget, 1965).

Therefore, the disassociation between notation systems and number concepts in children's understanding of number concepts is related to the nature of mathematics knowledge, as reflected in the paradox just discussed. It also mirrors the particular architecture of internal representation of numerical quantities. This antinomy appears to be particularly acute for children in learning about fraction numbers when the new concept of number has conflicts with their whole number scheme. The previous discussion clearly shows that children's confusion of fraction symbols with whole numbers is not merely a matter of not mastering the notations but it has more to do with the internal processes of conceptual restructuring. Furthermore, the whole number bias, showing to be an interference of children's prior whole number scheme with the new concept of fraction numbers, also reflects a set of more general problems associated with the cognitive processes of abstraction and concretization of mathematical ideas. Therefore, a broader perspective is needed to view and to approach the whole number bias in teaching and learning fraction and rational numbers. In this connection, we return to the point of recursive process that was raised a moment ago.

The notion of the separate subsystems for internal representation of numerical quantities and that of the representation redescription mechanism to channel the subsystems, highlight the primacy of the recursive process in the development of fraction and rational number concepts in children. Kieren (1976, 1993) has already made this point. The proposition becomes even more significant in light of the new understanding of human numerical cognition.

Conceptually, whole numbers and rational numbers represent two distinctively different conceptual fields in terms of both developmental analysis (Clark & Kamii, 1996; Kamii & Clark, 1995) and formal mathematics analysis (Vergnaud, 1994). Conceptual restructuring is required for children to make the transition from the understanding of the concept of whole numbers to that of the concepts of fraction and rational numbers. Vergnaud (1994) has argued and demonstrated that mathematical concepts draw their meanings from a variety of situations whose analysis and treatment require several kinds of concepts, procedures, and symbolic representations that are different but interconnected. Vergnaud (1994) termed the networks of mathematical concepts as conceptual fields. He described the conceptual field of multiplicative structures as a network of distinct but interconnected concepts such as multiplication, division, fractions, ratios, rational number, and linear and nonlinear functions. The notion of conceptual fields stresses that situations and problems that substantiate the different aspects of mathematical concepts shape the acquisition of multiplicative structures. Given the nature of the multiplicative field, no concept acquisition in the field will be a single acquisition. Even for the same concept or strategy, it needs multiple mental representations, by which children's initial ideas that are associated with particular actions and operations subsequently become explicit to the mind, thus become progressively more manipulable and flexible (Kamiloff-Smith, 1992).

Neuropsychologically, the recent findings suggest the existence of separate subsystems for representations of quantities (Dehaene et al., 1999; Koechlin et al., 1999). The brain needs to "re-describe" the representations in subsystems in order to have them converge on a common semantic representation of quantity. The neuropsychological constraint is consistent with the findings of the context-specific nature of early number learning. For example, various experiences children have with one-to-one correspondence appear initially to be disconnected. A child's competence in one context can be highly encapsulated and does not generalize at all to similar laboratory tasks or a related naturalistic task (Mix, 2002). Some simple variations on the standard number conservation paradigm have revealed substantial limits to children's understanding of conservation (Gelman & Bailargeon, 1983; Miller, 1989). Similar developmental changes have been observed in children with the acquisition of counting skills (Fuson, 1988; Wynn, 1992a), solving addition and subtraction word problems (Hiebert, 1986; Siegler, 1994), and the acquisition of concept of fraction equivalence (Ni, 2001). Children may master the same concept in one context after another, only gradually are they able to put the diverse experiences into a coherent conceptual structure.

The recursive process of the acquisition of mathematical concepts and skills entails that establishing connections between number concepts and numerical symbols is in nature driven by the internal processes of representational changes, probably at both the conceptual and the neuropsychological levels. Numerical symbols will become a processing burden, rather than a cognitive "amplifier" (Vygotsky, 1978), to an individual if he or she does not develop the required conceptual foundation. One reason for the failure of fraction and rational number instruction is that the recursive nature of the conceptual restructuring from whole numbers to fraction and rational numbers is not recognized in our elementary school mathematics curriculum and instruction. Unless the recursive nature of mathematical learning and development receives the due respect, it is unrealistic to expect major change and improvement to take place in fraction and rational number learning and instruction in our elementary schools.

FINAL REMARKS

This review has presented three accounts of the cause of whole number bias and examined the effectiveness of the different approaches to fraction and rational number instruction. While there does not yet appear to be sufficient evidence to decide among the competing accounts as to the nature of whole number bias, and among the approaches as to their ultimate effectiveness, the lines of research have provided us a wealth of complex results on both nature and nurture in the acquisition of number concepts. In particular, they have advanced our understanding in several aspects related to the development of human numerical cognition. They have shown that there is considerable prior experience and knowledge that infants and young children bring to the number acquisition tasks, much richer than previously assumed, and there are powerful and enduring effects of the prior experience and knowledge on children's later number concept acquisitions. They have suggested that the characteristic of human perception, being inherently relative, may be an intuitive developmental foundation for fractions and rational numbers. However, the potential has not yet been explored. More importantly, the sensitivity to relative relation may become less sharp because of an individual's experience with the predominant counting numbers. The research has also demonstrated that the primacy of the recursive process in number concept acquisition has neuropsychological underpinning.

Several questions for future research with regard to teaching and learning fraction and rational numbers have been suggested in the different sections of the article. Here, we highlight two of the research questions that directly concern the reorganization of elementary school fraction instruction.

One is that we may want to consider instruction on fraction numbers at an earlier grade. Quantitative development in early childhood takes place on multiple fronts; children's informal understanding of quantitative relation before they enter school is not limited to discrete sets, but also includes continuous amount and relative quantity (Goswami, 1989; Mix et al., 2002b; Sophian, 2000; Sophian et al., 1997). Also, early learning experience has demonstrated a powerful influence on later learning. Therefore, it does not seem to make sense to delay instruction on fraction numbers until third grade. The delay may not mean merely a delay, but as a result extra constraint may be brought on subsequent learning about fraction numbers as the single-counting scheme is taking a deeper root in children. The consideration of instruction on fractions at earlier grades certainly requires planning instruction on whole numbers and fraction numbers simultaneously. Empson's (1999) study was one of the first attempts in this direction. More research, especially longitudinal studies, are needed to investigate the long-term learning effects of the earlier exposure to situations that cultivate relational reasoning in children.

In association with expanding number experience of children at an earlier age, we need a better understanding of how the different instructional approaches would help children learn in differentiated ways. The equal sharing approach has been identified as an appropriate entry point for children to explore ideas about fraction numbers (Empson, 1999; Sophian et al., 1997; Streefland, 1991) because equal sharing activities have an intimate relation with children's prior learning experience. However, the part-whole scheme associated with the equal sharing approach appears to be less generalizable to the part-whole relations beyond the equal sharing context as the scheme is confined to the partitioned whole being a specific collection of equal parts (Mack, 1995; Steffe, 2002; Streefland, 1991; Tzur, 1999). In this regard, the measurement approach has been suggested as having greater affordance for students to develop a more sophisticated concept of unit and thus to enhance the concept that conceives an iterable part as invariant, multiplicative relation with respect to the reference whole (Davydov & Tsyetkovich, 1991; Lehrer, 2003). The comparison does not mean one approach is necessarily superior or more important than the other. Rather, given the complex constraints prior and new learning would have on each other, it calls for systematic, concerted, long-term teaching experiments in order to understand how the sequence of instructional measures would facilitate the learning about fraction and rational numbers.

Among others, the extension of the whole numbers to fraction numbers is a significant development in children. As this review has demonstrated, the issue of the whole number bias is not merely about a matter of interference between prior and new knowledge in children's construction of fraction concepts, but about a set of more general questions with regard to the origin and development of numerical cognition. How the bias is conceptualized therefore has broad theoretical and instructional significance. Important breakthroughs will probably come from joint research efforts in neuroscience, developmental psychology, and instructional experiments. We hope that this review will provide the catalyst for future dialogue on this contentious topic in the field.

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