The Maximum Intensity of Hurricanes

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ABSTRACT

An exact equation governing the maximum possible pressure fall in steady tropical cyclones is developed, accounting for the full effects of gaseous and condensed water on density and thermodynamics. The equation is also derived from Carnot's principle. We demonstrate the existence of critical conditions beyond which no solution for the minimum central pressure exists and speculate on the nature of hurricanes in the supercritical regime.

1. Introduction

A complete understanding of the physics of hurricanes requires, among other things, knowledge of the factors that set an upper bound on their intensity. In Emanuel (1986, hereafter referred to as I) it was argued that such an upper bound is determined by the product of the maximum possible latent heat input from ocean to atmosphere and a thermodynamic efficiency proportional to the temperature difference between the sea surface and lower stratosphere. This upper bound was directly related to the maximum possible pressure deficit in the eye. Predictions of minimum sustainable pressure compare very well with values obtained from a nonhydrostatic time-dependent axisymmetric primitive equation model described in Rotunno and Emanuel (1987), and appear to correlate quite well with the intensities of the strongest tropical cyclones observed (see Anthes, 1982). The interpretation of the upper bound on intensity in terms of the Carnot cycle is quite general and assumes only that mechanical dissipation is limited to the boundary layer and the outflow at large radii.

In I the effects of water substance on density and heat capacity were neglected. Our present purpose is to derive an exact equation for the maximum pressure drop, which accounts for fully reversible thermodynamics and the effects of water substance on density and to point out that the equation has no solution under certain conditions. In section 2 the pressure equation is derived from the thermal wind relation for steady axisymmetric flow and a single graph showing solutions to that equation under all conditions is presented. An interpretation of the breakdown of the equation under certain conditions is offered in section 3. Section 4 contains concluding remarks.

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2. The relation for minimum central pressure

Consider a steady-state axisymmetric hurricane over an ocean with uniform temperature. We shall assume that outside a frictional boundary layer, and except at large radii in the outflow, three properties of the flow are conserved: Angular momentum per unit mass (M), total entropy (s) and total water (liquid plus vapor, Q). We shall also assume hydrostatic and gradient balance (except in regions of dissipation) and ice phase physics will be ignored. The constraints of axisymmetry and hydrostatic and gradient balance are relaxed in the rederivation from Carnot's principle presented in appendix C.

The conserved quantities are defined

$$M \equiv rV + \frac{1}{2} fr^2, \tag{1}$$

$$s \equiv (C_{pd} + QC_l)\ln T + \frac{L_v w}{T} - R_d \ln p_d - wR_v \ln(\mathrm{RH}), \tag{2}$$

$$Q \equiv w + l, \tag{3}$$

where r is the radius, V the azimuthal velocity, f the Coriolis parameter (assumed constant), C_{pd} the heat capacity of dry air at constant pressure, C_l the heat capacity of liquid water, L_v the latent heat of vaporization (a function of temperature T), R_d the gas constant of dry air, R_v that of water vapor, p_d the partial pressure of dry air, w the mixing ratio, RH the relative humidity and l the mass of liquid per unit mass of dry air. The reader is referred to Iribarne and Godson (1973) for a derivation of (2) under saturated conditions; the last term makes s conserved under unsaturated conditions as well, as shown in appendix A.

a. Constraints on the distributions of entropy, total water and angular momentum

The assumption that conserved variables do not vary along angular momentum surfaces, when coupled with the thermal wind balance approximation, places a powerful constraint on the distributions of the conserved variables with respect to each other. We here derive that constraint. As an aside, we note that while in the following derivation we assume that s and Q take on their boundary layer values along each M surface in the hurricane, this is equivalent only to asserting that the density of air above the boundary layer has the same value as the density of parcels lifted reversibly from the boundary layer and does not require that the air be everywhere saturated and filled with cloud. That this condition on density is actually met in the tropical atmosphere has been beautifully demonstrated by Betts (1982).

The thermal wind equation for axisymmetric flow is (see I):

$$\frac{1}{r^3}\left(\frac{\partial M^2}{\partial p}\right)_r = -\left(\frac{\partial \alpha}{\partial r}\right)_p, \tag{4}$$

where α is the specific volume, which we regard as a function of the three variables p, s and Q (i.e., two state variables and a measure of the total water content). Using the chain rule, then, (4) becomes

$$\frac{1}{r^3}\left(\frac{\partial M^2}{\partial p}\right)_r = -\left(\frac{\partial \alpha}{\partial s}\right)_{p,Q}\frac{\partial s}{\partial r} - \left(\frac{\partial \alpha}{\partial Q}\right)_{p,s}\frac{\partial Q}{\partial r}. \tag{5}$$

We next assume that s and Q are functions of M alone. This may be regarded as a requirement of neutrality to slantwise convection (see I) or alternatively, where there is strong circulation, as the constraint that all conserved quantities are constant along streamlines and are therefore invariant with respect to each other. With this assumption, (5) can be written

$$\frac{1}{r^3}\left(\frac{\partial M^2}{\partial p}\right)_r = -\frac{\partial M}{\partial r}\left[\left(\frac{\partial \alpha}{\partial s}\right)_{p,Q}\frac{ds}{dM} + \left(\frac{\partial \alpha}{\partial Q}\right)_{p,s}\frac{dQ}{dM}\right]. \tag{6}$$

Dividing through by $\partial M/\partial r$, we obtain

$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial p}\right)_M = \left(\frac{\partial \alpha}{\partial s}\right)_{p,Q}\frac{ds}{dM} + \left(\frac{\partial \alpha}{\partial Q}\right)_{p,s}\frac{dQ}{dM}. \tag{7}$$

Now the partial derivatives on the right-hand side of (7) may be rewritten using the Maxwell relations derived in appendix B. These are

$$\left(\frac{\partial \alpha}{\partial s}\right)_{p,Q} = \frac{1}{1+Q}\left(\frac{\partial T}{\partial p}\right)_{s,Q}, \tag{8}$$

$$\left(\frac{\partial \alpha}{\partial Q}\right)_{p,s} = \frac{1}{1+Q}\left[\left(\frac{\partial \Phi}{\partial p}\right)_M + C_l\frac{\partial}{\partial p}[T(1-\ln T)]_{s,Q}\right], \tag{9}$$

where $\Phi \equiv gz + \frac{1}{2}V^2$. Substitution of (8) and (9) into (7) results in an equation which can be exactly integrated along M surfaces. The result of this integration is

$$-\frac{M}{r^2} = \frac{1}{1+Q}\left\langle (T-T_0)\frac{ds}{dM} + \frac{dQ}{dM}\{\Phi - \Phi_0 + C_l[T(1-\ln T) - T_0(1-\ln T_0)]\}\right\rangle, \tag{10}$$

where the subscript 0 represents evaluation as $r \rightarrow \infty$. This constraint on the distributions of s and Q with respect to M is a direct consequence of the thermal wind relation and the specification that s and Q are functions of M alone.

b. Constraint on the variation of entropy at the surface

We next show that (10) places a constraint on the variation of entropy along constant altitude surfaces. Specifically, we evaluate (10) along $z = 0$. In using (10) in the mixed layer we need not assume gradient balance there as long as we define M as the angular momentum that *would* obtain if there were gradient balance. The assumptions of no variation of s and Q along M so defined is still equivalent to the assumption of slantwise neutrality.

We begin by multiplying (10) through by $\partial M/\partial r$ and also relating V to M using (1). The result is

$$-\frac{1+Q}{2r^2}\left(\frac{\partial M^2}{\partial r}\right)_z = (T_s - T_0)\left(\frac{\partial s}{\partial r}\right)_z + \left(\frac{\partial Q}{\partial r}\right)_z\left\{-gz_0 + \frac{1}{2}\left[\frac{M^2}{r^2} + f^2(r^2 - r_1^2)\right] + C_l[T_s(1-\ln T_s) - T_0(1-\ln T_0)]\right\}, \tag{11}$$

where the subscript s refers to evaluation at the surface ($z = 0$). The quantity r_1 is a very large radius (technically infinite) although in practice it can be regarded as the radius at which the temperature and height approach T_0 and z_0 in the outflow. (It comes from relating V to M in the limit of $r \rightarrow \infty$.) In appendix C we show that this term is energetically interpretable as the kinetic energy that must be used to spin up water mass in the anticyclonic outflow. In any real situation, the water will have fallen out before large radii are achieved. Even if an appreciable amount of condensate flows out to radii as large as 10 000 km, the term r_1 in (11) can be shown to have little influence on the result. We retain it here for generality.

The terms involving M^2 in (11) can be combined into a single term, so that the left-hand side becomes

$$-\frac{1}{2r^2}\left(\frac{\partial}{\partial r}\right)_z[(1+Q)M^2]. \tag{12}$$

From the gradient wind equation (see I), we have

$$(1+Q)M^2 = r^3(1+Q)\alpha\left(\frac{\partial p}{\partial r}\right)_z + \frac{1}{4}f^2r^4(1+Q)$$

$$= r^3\alpha_d\left(\frac{\partial p}{\partial r}\right)_z + \frac{1}{4}f^2r^4(1+Q), \qquad (13)$$

where α_d is the volume per unit mass of dry air. Using (13) and (12), (11) can be put in the form

$$-\alpha_d\left(\frac{\partial p}{\partial r}\right)_z = \left(\frac{\partial}{\partial r}\right)_z\left[\frac{1}{2}r\alpha_d\frac{\partial p}{\partial r} + \frac{1}{4}f^2r^2(1+Q)\right]$$

$$-\frac{1}{8}f^2r_1^2\left(\frac{\partial Q}{\partial r}\right)_z + (T_s - T_0)\left(\frac{\partial s}{\partial r}\right)_z + \left(\frac{\partial Q}{\partial r}\right)_z$$

$$\times\{-gz_0 + C_l[T_s(1-\ln T_s) - T_0(1-\ln T_0)]\}. \qquad (14)$$

By differentiating (2) and making use of the Clausius-Clapeyron equation, the first law of thermodynamics can be written (for saturated *and* unsaturated processes)

$$-\alpha_d dp = Tds - (C_{pd} + C_lQ)dT$$

$$- d[L_v w] - C_lT\ln T dQ. \qquad (15)$$

If (15) is substituted for the left-hand side of (14), the result is

$$T_0\frac{\partial s}{\partial r} = \frac{\partial}{\partial r}\left[\frac{1}{2}r\alpha_d\frac{\partial p}{\partial r} + \frac{1}{4}f^2r^2(1+Q) + L_v w + (C_{pd} + C_lQ)T_s\right]$$

$$+ \frac{\partial Q}{\partial r}\left[-gz_0 + C_l(T_0\ln T_0 - T_0) - \frac{1}{8}f^2r_1^2\right]. \qquad (16)$$

We next define an "outer radius," r_a where $\partial p/\partial r$ vanishes. This may be identified as the radius of the outermost closed isobar. Integrating (16) from this outer radius to the storm center along $z = 0$, we obtain

$$\overline{T_0}(s_c - s_a) = (w_c - w_a)\left[C_lT_s + L_v + C_l\overline{T_0}^*(\ln\overline{T_0}^* - 1) - \frac{1}{8}f^2\overline{r_1^2}^* - g\overline{z_0}^*\right] - \frac{1}{4}f^2r_a^2(1 + w_a). \qquad (17)$$

where the subscript c stands for "storm center". The overbarred quantities are defined

$$\overline{T_0} \equiv \frac{1}{s_c - s_a}\int_a^c T_0 ds,$$

$$\overline{(\;)}^* \equiv \frac{1}{Q_a - Q_c}\int_a^c (\;)dQ.$$

Here $\overline{T_0}$ is the entropy-weighted mean "outflow" temperature which the moist isentropic surfaces asymptote to at large radii. The quantities $\overline{z_0}^*$, $\overline{r_1^2}^*$ and $\overline{T_0}^*$ are similarly defined, but the weighting is proportional to the total water content Q rather than to the moist entropy. For simplicity, we hereafter assume that the entropy-weighted average temperature is equal to the total water-weighted average; i.e., $\overline{T_0}^* = \overline{T_0}$. Given $\overline{T_0}$, $\overline{r_1^2}^*$ and $\overline{z_0}^*$ the above amounts to a specific relationship between the entropy change along the sea surface and the change in water vapor mixing ratio.

c. The central pressure equation and its interpretation

Using the definition of moist entropy, (2), the relation (17) can be expressed as an equation for the minimum central pressure. Specifically, we use (2) to evaluate s_c and s_a, where both are evaluated at $z = 0$ and it is assumed that air is saturated at $r = 0$. The result can be expressed in the form

$$\ln x = -A\left[\frac{1}{x} - B\right], \qquad (18)$$

where

$$x \equiv \frac{p_{dc}}{p_{da}}$$

$$A \equiv \frac{\epsilon}{1-\epsilon}\frac{L_v}{R_vT_s}\frac{e_s}{p_{da}}\left[1 - \underbrace{\frac{g\overline{z_0}^*}{\epsilon L_v}}_{\text{(a)}} - \underbrace{\frac{1}{8}\frac{f^2\overline{r_1^2}^*}{\epsilon L_v}}_{\text{(b)}} + \underbrace{\frac{C_lT_s}{\epsilon L_v}(\epsilon + (1-\epsilon)\ln(1-\epsilon))}_{\text{(c)}}\right]$$

$$B \equiv RH\left[1 + \frac{e_s}{p_{da}}\frac{\ln(\mathrm{RH})}{A}\right] + \frac{1}{4}\frac{f^2r_a^2\left(1 + \mathrm{RH}\frac{e_s}{p_{da}}\right)}{R_dT_s(1-\epsilon)A}. \qquad (19)$$

In the above expression, p_{dc} and p_{da} are the central and ambient surface values of the partial pressure of dry air, RH is the ambient surface relative humidity and ϵ is a thermodynamic efficiency, defined

$$\epsilon \equiv \frac{T_s - \overline{T_0}}{T_s}. \qquad (20)$$

(This efficiency may be as large as ⅓ in the present tropical atmosphere.) The relation (18) is the desired result.

In appendix C, (18) is rederived from a Carnot cycle argument which does not restrict the flow to be hydrostatic or in gradient balance, and for which the assumption of axisymmetry is pertinent only to term (b) and the last term in the definition of B in (19). Moreover, that argument shows that term (a) in (19) is simply the potential energy used to lift water substance to the outflow level; term (b) is the energy required to move the water substance against the radial pressure gradient, and term (c) represents the contribution of water substance to the heat capacity.

Term (b) is technically infinite; this shows that if all the water substance added by the ocean to the air really had to be carried to infinite radius in the outflow (and thus accelerated to infinite anticyclonic velocity), no truly steady hurricane could exist. In practice, air cannot flow out to infinity in the finite lifetime of a hurricane, and most of the water falls out. Moreover, asymmetry and dissipation remove the constraint of angular momentum conservation at relatively small radii. For typical values of f and L_v on earth, r_1 would have to be of order 20 000 km to make term (b) of order unity when Q is conserved. We therefore neglect (b) hereafter.

Since (18) is the focus of this paper, it is worthwhile to discuss the various terms that comprise it. Before doing so, we first rewrite (18) in the form

$$\ln(x) - \frac{e_a}{p_{da}}\ln(\mathrm{RH}) = -A\left[\frac{1}{x} - \mathrm{RH}\right] + \frac{1}{4}\frac{f^2 r_a^2\left(1 + \dfrac{e_a}{p_{da}}\right)}{R_d T_s(1-\epsilon)}. \qquad (21)$$

The left-hand side of (21) represents a combination of the total entropy change due to isothermal expansion and the work done against friction in the boundary layer which, by appendix C, Eq. (C8), is proportional to the change of $\ln(p)$. The quantity enclosed by brackets in the first term on the right-hand side of (21) is proportional to the total increase in mixing ratio from the ambient environment to the storm center. The $1/x$ term represents the pressure dependence of the saturation mixing ratio while RH represents the thermodynamic disequilibrium of the air–sea system which is the energy source of the hurricane. The smaller the relative humidity, the greater the air–sea entropy difference.

The increase in mixing ratio is multiplied by A as defined by (19). The first term of A represents the actual addition of latent heat to the system while term (c) describes the increase of entropy due to addition of water mass. As previously discussed, terms (a) and (b) represent work done on water substance to lift it against gravity and accelerate it in the outflow.

Finally, the last term in (21) represents the work done to restore the angular momentum of the outflow to its ambient value.

The relation for the minimum central pressure, (18), has two roots, one root, or no roots, depending on A and B. To see this, first define y such that $y \equiv 1/x = p_{da}/p_{dc}$. Then (18) becomes

$$y = \exp[A(y - B)]. \qquad (22)$$

The left- and right-hand sides of (22) are individually plotted against y in Fig. 1 for the case $A = 0.5$, $B = 0.8$. The two intersection points correspond to two solutions. Reference to the derivation of (18) shows that the left-hand side is the maximum pressure drop that can be sustained by a given inward increase in vapor mixing ratio, while the right-hand side includes the pressure dependence of the core mixing ratio itself. This leads naturally to the following interpretation of Fig. 1:

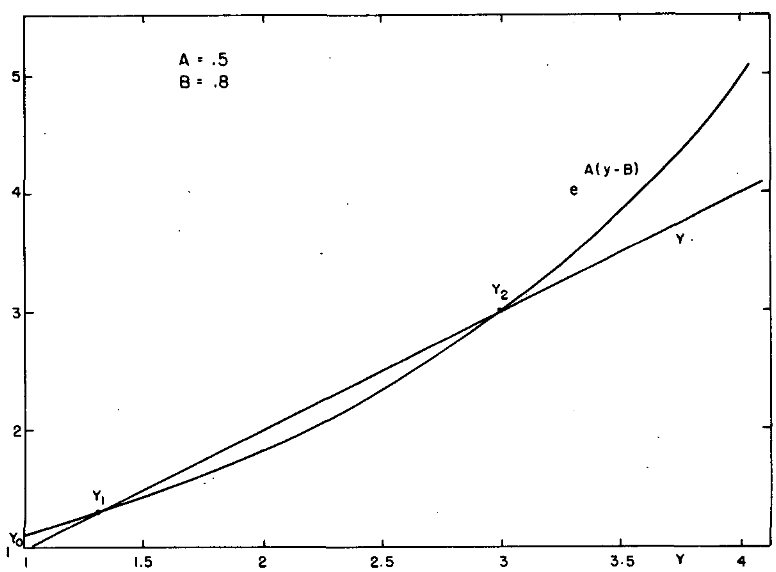

FIG. 1. The left- and right-hand sides of (22) as a function of y, for $A = 0.5$ and $B = 0.8$. The central pressure deficit would be given by Y_0 if there were no isothermal expansion effects. Y_1 and Y_2 denote the equilibrium solutions of (22).

At ambient pressure ($y = 1$), the pressure drop that could be sustained energetically merely by increasing the relative humidity from its ambient value to 100% at constant pressure is a finite value reflected by y_0. If such a pressure drop were realized, however, the actual core mixing ratio would increase further since $w_s = 0.622e_s/p_d$. This is reflected by moving to the right along the exponential curve. This additional water vapor leads to a further decrease in the core pressure (increase in y), and so on until the first intersection point, y_1, is reached. The fact that $y_1 > y_0$ reflects the additional input of heat energy from isothermal expansion. Now if the system is forced to move beyond y_1, the core saturation mixing ratio (reflected by the exponential curve) becomes less than that necessary to sustain the central pressure drop (y) and the system would have to spin down. In this sense, y_1 is a stable equilibrium point of the system.

The solution y_2 (which in this case occurs at the very small value of ps_{dc}/p_{da} of 1/3) is unstable, however. If one moves to the right of y_2, the core mixing ratio (exponential curve) is always larger than that necessary to sustain the pressure drop (y) and so the system would continue to intensify indefinitely. On the other hand, moving to the left of y_2 leads to core mixing ratios less than those necessary to sustain the pressure drop and the system spins down. Thus, y_2 is an unstable, and therefore unobservable, solution; the solution of interest is y_1.

Suppose, however, that we either increase A or decrease B. This causes the exponential curve in Fig. 1 to move upward, so that solutions y_1 and y_2 move in toward each other. Eventually, they coalesce into a single solution and for larger A or smaller B no solution exists at all. By the preceding argument, the system is always unstable in this event, and moves inexorably toward lower pressure. This may be thought of as a "runaway" Carnot engine in which central pressure falls lead, through isothermal expansion, to increased heat content, which drives further pressure falls, and so on. If a new equilibrium is achieved, its energetics must be quite different from those of ordinary hurricanes; presumably the excess energy generation is ultimately balanced by large *internal* dissipation; otherwise, the central pressure would literally approach the saturation vapor pressure of the sea water. We apply the term *hypercane* to any mature storm that might exist beyond the parameter range where equilibrium solutions are possible.

The threshold values of A or B beyond which no solutions exist can be found by requiring that the two curves in Fig. 1 have the same slope at their intersection point. In addition to (22) being satisfied, then, we have

$$1 = A \exp[A(y - B)] = Ay,$$

so that

$$x\,(=1/y) = A,$$

$$A = e^{BA-1}. \tag{23}$$

The latter can be solved explicitly for a critical value of B for a given value of A:

$$B_c = \frac{1 + \ln A}{A}. \tag{24}$$

No solutions exist for $B < B_c$.

General solutions of (18) for $x\,(=p_{dc}/p_{da})$ are shown in Fig. 2. The equilibrium central pressures decrease very rapidly close to the critical curve, along which $x = A$.

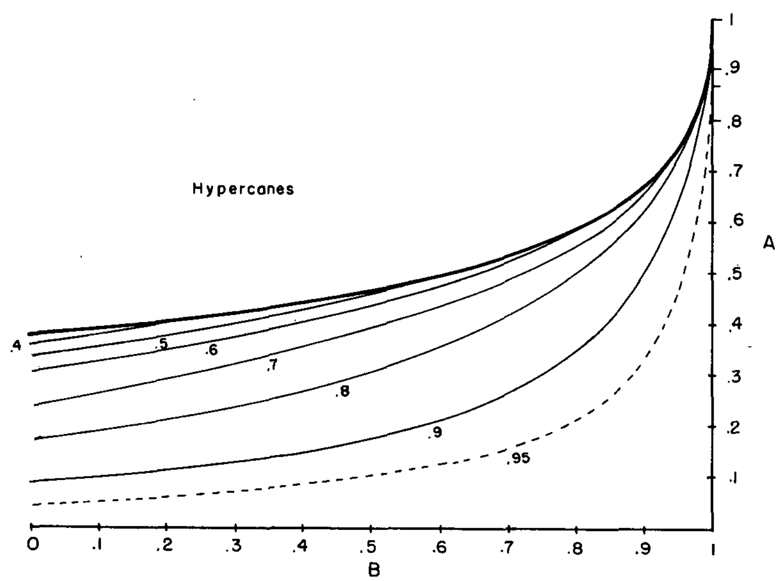

FIG. 2. General solutions of (18) for $x\,(=p_{dc}/p_{da})$, as a function of A and B. The heavy solid curve denotes the critical condition beyond which no solutions exist.

Actual minimum pressures are shown in Fig. 3a as a function of $\overline{T_0}$ and T_s for an ambient pressure of 1013 mb and humidity of 80%. These have been calculated from (19) using the appropriate gas constants and temperature-dependent L_v, with e_s given by Bolton (1980). The last term in the definition of B (19) is ne-

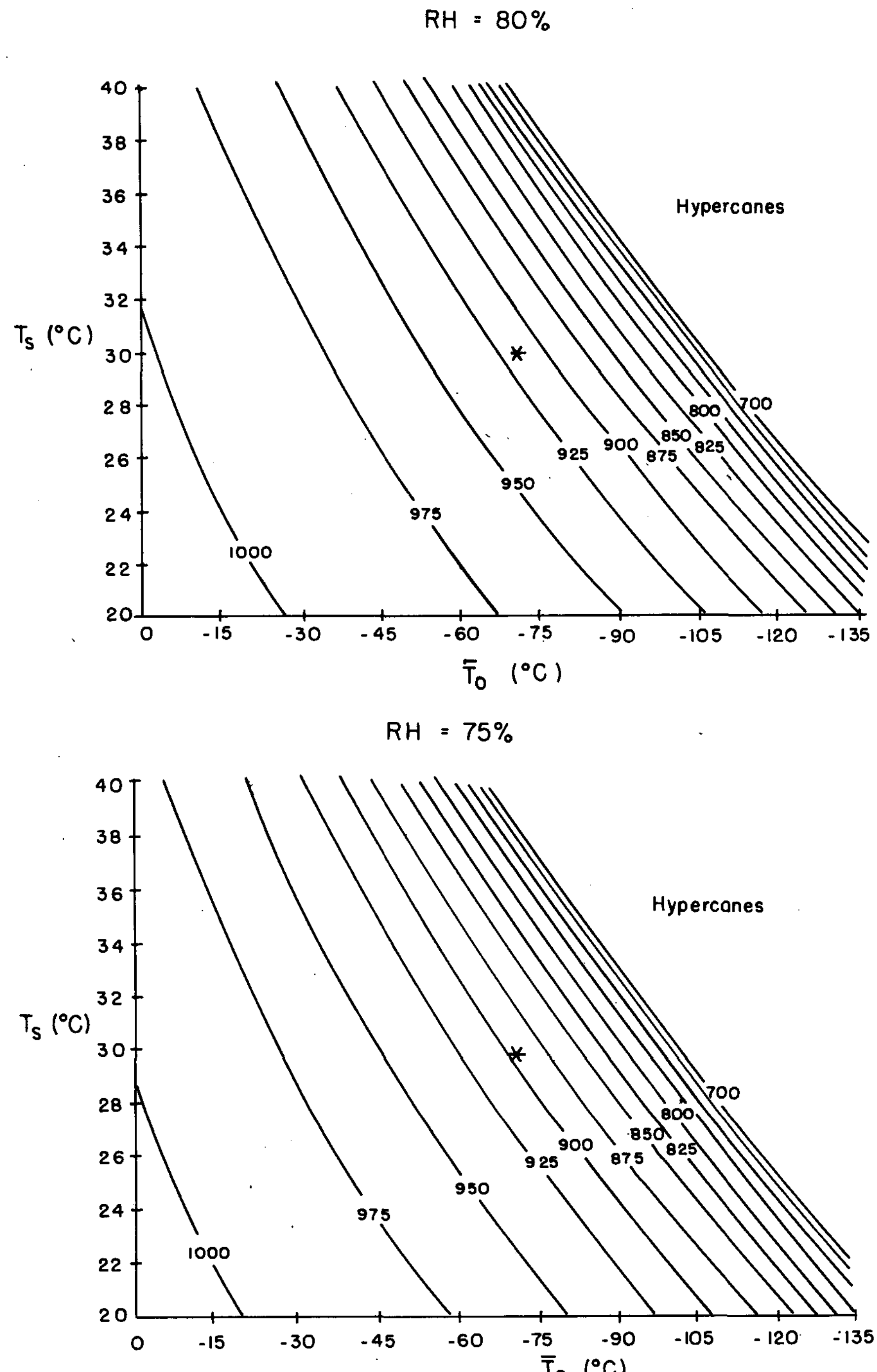

FIG. 3. (a) Minimum sustainable central pressure (mb) of tropical cyclones as a function of sea surface temperature (T_s) and entropy-weighted mean outflow temperature ($\overline{T_o}$), assuming reversible thermodynamics and an ambient surface relative humidity of 80%. The asterisk denotes mean August conditions in the near-equatorial western North Pacific. (b) As in (a) but for a surface relative humidity of 75%.

glected; it is generally very small unless r_a exceeds about 1000 km. Rotunno and Emanuel (1987) showed that the intensity of numerically simulated hurricanes was insensitive to r_a as long as it was within an upper bound. The following procedure was used to estimate $\overline{z_0}^*$: From the conservation of h (appendix A) we have

$$gz_0 = \frac{(C_{pd} + QC_l)(T_s - T_0) + L_vQ - L_{vo}w_0}{1 + Q} = \frac{(C_{pd} + QC_l)(T_s - T_0) + L_vQ[1 - (L_{vo}/L_v)w_0/Q]}{1 + Q}. \quad (25)$$

Were it not for the last term in the numerator of (25), gz_0 would be a function of Q, T_s and T_0 alone. Since w_0 will be small at low temperatures, the last term within brackets will be small provided the saturation mixing ratio at the outflow level is small compared to the boundary layer mixing ratio. Thus w_0/Q is related to ϵ. Rather than calculate w_0 explicitly, we approximate it by curve fitting its dependence on ϵ from tephigrams. We find that the approximation

$$F(\epsilon) \equiv 1 - \frac{L_{vo}}{L_v}\frac{w_0}{Q} \approx \begin{cases} 8.7\epsilon - 19.33\epsilon^2, & \epsilon < 0.23 \\ 1, & \epsilon > 0.23 \end{cases} \quad (26)$$

produces errors less than 5% over a range of surface temperatures between 20 and 40°C. Using this and averaging (25) over the range of Q between w_c and w_a yields

$$g\overline{z_0}^* \approx \epsilon T_sC_l + L_vF(\epsilon) + \frac{1}{w_c - w_a}\ln\frac{1 + w_c}{1 + w_a} \times [\epsilon T_s(C_{pd} - C_l) - L_vF(\epsilon)]. \quad (27)$$

As this depends on w_c, which in turn depends on p_c, (18) must be solved iteratively. We estimate that the errors arising from approximating $1 - w_0/Q$ by (26) are less than 2 mb.

Figure 3b shows the minimum central pressures for reversible ascent when the ambient humidity is 75%. Total pressure drops are about 25% greater, indicating that the pressure drop is nearly proportional to 1 − RH, which is in turn proportional to the entropy input from the ocean.

Note in Fig. 3 that at current lower stratospheric temperatures, the sea surface temperature would have to be greater than 38°–40°C to permit hypercanes. Likewise, at present sea surface temperatures hypercanes could only occur if the lower stratosphere cooled to less than about −105°C.

Real hurricanes do not, of course, operate on reversible thermodynamics. The largest irreversible effect is the fallout of precipitation, which reduces the water loading term (a) in (19) as well as the effect of the heat capacity of water substance, represented by term (c). It is possible to define a pseudo-adiabatic hurricane by removing all condensate instantaneously. Then, according to appendix C, $\overline{z_0}^*$ is a mean altitude, weighted by the water vapor content, and term (c) in (19) is somewhat altered. As calculation of these is rather involved, we shall instead obtain an upper limit on A in (19) by ignoring the water loading and heat capacity terms (a) and (c). This will in general lead to estimates of the pressure drop that are about 5% too large.

Figures 4a and 4b show estimates of the minimum pressure obtained by ignoring the water loading and heat capacity terms. These represent slight underestimates of the minimum pressure of pseudo-adiabatic hurricanes. At high surface temperatures, the difference between the reversible and pseudo-adiabatic minimum pressures is very substantial, reflecting the importance of water loading in limiting the hurricane intensity in the reversible case.

The asterisks in Figs. 3 and 4 denote the most extreme conditions that exist in the western North Pacific Ocean, where the surface humidity varies between about 75% and 80%. The pseudoadiabatic case accurately reflects the record minimum pressures of 870–900 mb in that region. Apparently, the most intense hurricanes approach the actual thermodynamic limit on their intensity.

3. Hurricanes in the supercritical regime

The absence of stationary solutions of (18) for sufficiently large values of A and sufficiently small B implies that if steady hurricanes are possible at all, their energetics must involve processes not accounted for in this analysis. As pointed out previously, the breakdown of (18) may be regarded as arising from a runaway isothermal expansion in which falling pressure leads to a heat input that is more than sufficient to drive further pressure falls. In view of this interpretation and the assumptions made in the Carnot cycle development of (18) (see appendix C), we may speculate on the nature of processes that might actually limit hurricanes in the supercritical regime:

1) Inflow in supercritical hurricanes is not approximately isothermal, as observed in subcritical storms. This possibility would appear to depend on the details of the boundary layer physics of hurricanes; the inflow might become so strong that surface fluxes are unable to keep up with adiabatic expansion.

2) The vortex may become supercritical in the classical sense; i.e., the outflow may not match up with the ambient environment. In I it was assumed that outflow is weak enough that gravity waves can propagate inward from infinity, so that the outflow temperature $\overline{T}_0$ can be calculated from the ambient sounding. This was shown to be the case in the numerical simulations by Rotunno and Emanuel (1987). Were the outflow sufficiently strong, however, gravity waves

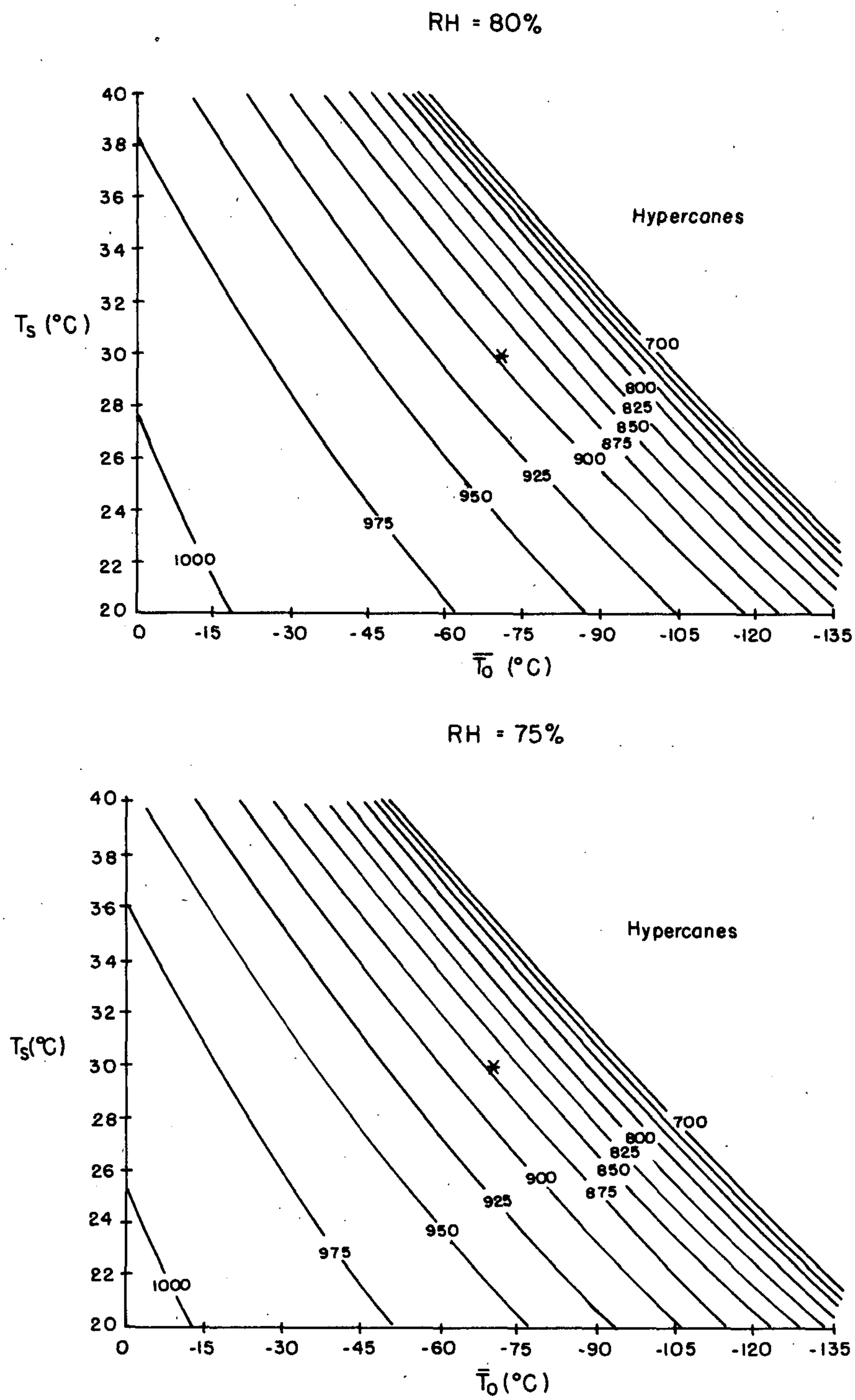

FIG. 4. As in Fig. 3 but for solutions of (18) ignoring condensate loading and heat capacity effects.

would not be able to penetrate inward of a critical radius at which a shock would presumably form, and $\bar{T}_0$ might be controlled by the vortex itself rather than by the environment. This possibility also depends on the boundary layer physics since the secondary flow in a steady vortex is frictionally driven.

3) The assumption that all of the dissipation occurs in the inflow layer and at large radii in the outflow

breaks down and internal dissipation becomes important. This implies a turbulent breakdown of the vortex, presumably at very high intensity.

To these considerations we add that hurricanes occurring over oceanic mixed layers of finite depth are known to cool the sea surface temperatures by upwelling and mixing; this may limit hurricane intensity under some conditions.

Let us suppose for the present that whatever process limits the intensity of hypercanes nevertheless results in storms of extraordinary intensity. We can speculate on several aspects of the structure of such storms based simply on their intensity and on some suppositions about their boundary layer structure.

In the first place, hypercanes would penetrate large distances into the stratosphere due to the very high core values of the moist entropy. The distance above the tropopause of the highest outflow from the eyewall is approximately

$$\delta z \approx \left(\frac{dz}{ds}\right)_{st} \delta s, \quad (28)$$

where $(ds/dz)_{st}$ is the rate of increase of moist entropy with height in the stratosphere. We have neglected water loading of the updraft. If we approximate moist entropy by dry entropy in the ambient stratosphere and assume that the latter is isothermal with a temperature $\bar{T}_0$, we have

$$\delta z \approx \frac{\bar{T}_0}{g} \delta s. \quad (29)$$

The entropy increase δs is the increase between the ambient environment and the core along the surface. Neglecting effects due to the heat capacity of water, this is [from (2)]

$$\delta s = \frac{L_v}{T_s} \delta w - R_d \delta \ln p_d.$$

Using the definition of mixing ratio in the above, (29) becomes

$$\delta z = \frac{R_d \bar{T}_0}{g} \left[\frac{L_v}{R_v T_s} \frac{e_s}{p_{da}} \left(\frac{1}{x} - \mathrm{RH}\right) - \ln x\right], \quad (30)$$

where $x \equiv p_{dc}/p_{da}$.

For $T_s = 40°\mathrm{C}$, $T_0 = -73°\mathrm{C}$, RH = 0.8 and $x = 0.5$ this gives a penetration of about 13 km into the stratosphere, compared to a penetration of about 2 km for $T_s = 30°\mathrm{C}$, $T_0 = -73°\mathrm{C}$, RH = 0.8 and $x = 0.9$. Thus it appears that hypercanes would extend much further into the stratosphere than present-day hurricanes.

It is also possible to show that the ratio of the radius of maximum winds to the outer radius would be relatively small in hypercanes. In order to demonstrate this, it is necessary to close the zero moist potential vorticity model with a second condition on the radial distribution of boundary layer pressure or moist entropy. Equation (16) may be considered a first such relationship. If the steps leading from (16) to (17) are repeated but while allowing the radial integral to be evaluated between two arbitrary radii, the result is

$$\overline{T_0}(s_1 - s_2) = \frac{1}{2}\left(r\alpha_d \frac{\partial p}{\partial r}\right)_1 - \frac{1}{2}\left(r\alpha_d \frac{\partial p}{\partial r}\right)_2 + \frac{1}{4} f^2[r_1^2(1 + w_1) - r_2^2(1 + w_2)] + (w_1 - w_2)\left[C_l T_s + L_v + C_l \overline{T_0}^*(\ln \overline{T_0}^* - 1) - \frac{1}{8} f^2 \overline{r_1^2}^* - g\overline{z_0}^*\right], \quad (31)$$

where the subscripts 1 and 2 refer to the two arbitrary radii and the overbar represents averages defined over that interval.

To obtain approximate relationships for maximum wind speed and radius of maximum wind we make an ad hoc closure assumption based on observations. Inside the radius of maximum winds, we assume that the gradient wind field is in solid body rotation, while outside that radius we assume constant relative humidity. The first condition is a direct condition on the pressure field while the second ties variations of moist entropy to variations in pressure.

As we are only interested in the qualitative behavior of the maximum wind and the radius of maximum wind, we shall make several further simplifying assumptions, viz. 1) neglect water loading and the effect of the heat capacity of water substance in (31) and in the definition of s, (2); 2) assume that $\overline{T_0}$ does not depend on the averaging interval; 3) assume that the radial pressure gradient is equal to the radial gradient of the partial pressure of dry air; and 4) neglect w where it multiplies f^2 in (31). With these conditions, (31) becomes

$$\overline{T_0}(s_1 - s_2) = \frac{1}{2} R_d T_s \left[r_1\left(\frac{\partial \ln p_d}{\partial r}\right)_1 - r_2\left(\frac{\partial \ln p_d}{\partial r}\right)_2\right] + \frac{1}{4} f^2(r_1^2 - r_2^2) + L_v(w_1 - w_2). \quad (32)$$

Within the radius of maximum winds, the gradient wind equation assumes the form

$$R_d T_s \frac{\partial \ln p_d}{\partial r} = \frac{V^2}{r} + fV = V_m\left[V_m \frac{r}{r_m^2} + f\frac{r}{r_m}\right],$$

where V_m is the maximum gradient wind, occurring at radius r_m. Integrating this from $r = 0$ to $r = r_m$ results in

$$R_d T_s \ln \frac{p_{dm}}{p_{dc}} = \frac{1}{2} V_m[V_m + f r_m] = \frac{1}{2} r_m R_d T_s \left(\frac{\partial \ln p_d}{\partial r}\right)_{r=r_m}. \quad (33)$$

Using (33) and evaluating (32) between $r = 0$ and $r = r_m$ results in

$$-R_d T_s \ln \frac{p_{dm}}{p_{dc}} = \overline{T_0}(s_c - s_m) - L_v(w_c - w_m), \quad (34)$$

where we have neglected the Coriolis terms in this interval. Substituting the definitions of moist entropy (2) and mixing ratio, we obtain

$$(2 - \epsilon) \ln \frac{p_{dm}}{p_{dc}} = \frac{\epsilon L_v}{R_v T_s} \frac{e_s}{p_{dc}} \left[1 - \mathrm{RH} \frac{p_{dc}}{p_{dm}}\right]. \quad (35)$$

Here p_{dm} is p_d evaluated at $r = r_m$ and we have assumed a relative humidity RH at $r = r_m$. Given the central pressure from (18), (35) may be used to obtain p_{dm} and (33) then gives V_m (neglecting the Coriolis term).

The radius of maximum winds may be found by matching (35) to the result of integrating (32) inward from r_a to r_m under the assumption of constant RH. The differential equation for p_d derived from (32) is then

$$\frac{1}{2} r \frac{\partial \ln p_d}{\partial r} + (1 - \epsilon) \ln \frac{p_d}{p_{da}} + \frac{\epsilon L_v}{R_v T_s} \mathrm{RH} \frac{e_s}{p_{da}} \left[\frac{p_{da}}{p_d} - 1\right] = \frac{1}{4} \frac{f^2 r_a^2}{R_d T_s} \left(1 - \frac{r^2}{r_a^2}\right). \quad (36)$$

To solve (36), we make the further assumption that $(p_{da} - p_d)/p_{da} \ll 1$ in the range of pressures between p_{da} and p_{dm}. Then

$$\frac{p_{da}}{p_d} - 1 \approx \ln \frac{p_{da}}{p_d}.$$

Using this, the solution of (36) evaluated at $r = r_m$ is

$$\ln \frac{p_{da}}{p_d} = \frac{1}{4} \frac{f^2 r_a^2}{R_d T_s} \left\{ \frac{1}{C+1} \left[\left(\frac{r_m}{r_a}\right)^2 - \left(\frac{r_a}{r_m}\right)^{2C} \right] - \frac{1}{C} \left[1 - \left(\frac{r_a}{r_m}\right)^{2C} \right] \right\}, \quad (37)$$

where

$$C \equiv 1 - \epsilon\left(1 + \frac{L_v w_a}{R_d T_s}\right). \quad (38)$$

Under the approximation that $r_m \ll r_a$, (37) can be solved for r_m as a function of r_a and p_{dm}:

$$r_m \approx r_a \left[\frac{4 R_d T_s}{f^2 r_a^2} C(1 + C) \ln \frac{p_{da}}{p_{dm}} \right]^{-1/2C}. \quad (39)$$

Given r_a, p_{dm} (from (35)) and p_{dc}, (39) yields an estimate of r_m. Clearly, as w_a increases (i.e., surface temperature increases) and as the outflow temperature decreases (ϵ increases), C becomes smaller and so does r_m.

Examples of estimates of r_m and V_m calculated from (18), (33), (35) and (39) are shown in Table 1 for two different sea surface temperatures one of which is typical of present conditions while the other nears, but does not exceed, the hypercane threshold. Estimates of the relative values of r_m and r_a and of V_m for extreme hurricanes under present conditions are reasonable. When the sea surface temperature is increased to near the hypercane threshold, the radius of maximum winds becomes very small unless the outer radius is increased. From this we may conclude that hypercanes would either have very small eyes or very large outer radii.

TABLE 1. Estimates of r_m and V_m for two different sea surface temperatures and for $p_a = 1013$ mb, RH = 80%, $T = -73°$C and $f = 5 \times 10^{-5}$ s^{-1}.

T_s (°C)	r_a (km)	p_c (mb)	p_m (mb)	V_m (m s^{-1})	r_m (km)
30	700	894	917	80	26
35	700	762	788	96	2
35	1500	762	788	96	64

4. Conclusions

We have derived an exact equation (18) governing the minimum sustainable central pressure of hurricanes. The Carnot cycle derivation presented in appendix C shows that the only approximations necessary in deriving (18) are 1) no radial temperature gradient in the mixed layer, and 2) no dissipation except within the inflow and at large radii in the outflow. The assumption of axisymmetry has a relatively small effect on the steady state central pressure.

Both the reversible and pseudo-adiabatic values of the parameters A and B in (18) yield regimes under which no solution to (18) exists. Under these conditions, the Carnot engine experiences a runaway isothermal expansion which drives the central pressure ever lower, unless the expansion ceases to be isothermal or unless internal dissipation becomes large, implying very high intensity. We call mature storms that might occur in the supercritical regime *hypercanes,* and show that they would extend very high into the stratosphere and have either very large outer radii or very small eyes.

Holding the temperature of the lower stratosphere constant, sea surface temperatures would have to be 6° to 10°C warmer than present values to sustain hypercanes. It is very unlikely that this has happened in the recent geologic past or will happen in the near future. Some estimates based on oxygen isotope determinations from fossil foraminifera (e.g., see Frakes, 1986) indicate middle Cretaceous tropical sea surface temperatures as much as 7°C warmer than at present, suggesting that hypercanes might have been possible at that time, unless the lower stratosphere was also substantially warmer. Nor can one rule out the existence of supercritical regimes very early in the earth's history, when temperatures may have been substantially warmer than at present.

The existence of hypercane regimes in the geological past aside, the question of whether a runaway Carnot engine is dynamically possible remains. The solutions to (18) under present conditions give quite reasonable estimates of the central pressures of the most intense storms on record (see Emanuel, 1987), indicating that the upper bound provided by (18) is actually achieved in a small number of storms. Whether a tropical cyclone of extraordinary intensity and large internal dissipation is actually possible in the supercritical regime constitutes a challenging question that might be answered through numerical modeling.

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APPENDIX A

Expressions for Moist Entropy and Moist Static Energy Valid in Saturated and Unsaturated Air

The following quantity can be shown to be conserved during reversible moist or dry adiabatic expansion:

$$s = (C_{pd} + QC_l)\ln T + \frac{L_v w}{T} - R_d \ln p_d - wR_v \ln(\mathrm{RH}), \quad \text{(A1)}$$

where C_{pd} and C_l are the heat capacities of dry air and liquid water, respectively, Q is the total water content, L_v is the heat of vaporization (a function of temperature), w the vapor mixing ratio, R_d and R_v are the gas constants for dry air and water vapor, respectively, and RH is the relative humidity.

Differentiation of (A1) yields

$$Tds = (C_{pd} + QC_l)dT + L_v dw - \frac{R_d T}{p_d} dp_d + wdL_v - \frac{L_v w}{T} dT - wR_v Td\ln(\mathrm{RH}) - R_v T \ln(\mathrm{RH})dw. \quad \text{(A2)}$$

The last term of (A2) vanishes since reversible changes in w can only occur at RH = 1. The temperature dependence of L_v is given by

$$dL_v = (C_{pv} - C_l)dT,$$

where C_{pv} is the heat capacity of water vapor at constant pressure. Also, the Clausius-Clapeyron equation may be written:

$$L_v \frac{dT}{T} = R_v T \frac{de_s}{e_s} = R_v T\left(\frac{de}{e} - \frac{d\mathrm{RH}}{\mathrm{RH}}\right),$$

since $e_s = e/\mathrm{RH}$. Using these two expressions in (A2) gives

$$Tds = (C_{pd} + wC_{pv} + lC_l)dT + L_v dw - \frac{R_d T}{p_d} dp_d - wR_v T \frac{de}{e}. \quad \text{(A3)}$$

Finally, we note that since $w = \alpha_d/\alpha_v$, where α_d and α_v are the specific volumes of dry air and water vapor, respectively, (A3) can be written

$$Tds = (C_{pd} + wC_{pv} + lC_l)dT + L_v dw - \alpha_d dp, \quad \text{(A4)}$$

which is a direct statement of the first law of thermodynamics written as entropy changes per unit mass of dry air. This proves that (A1) is a uniformly valid expression for entropy of moist air.

A uniformly conserved moist static energy can also be derived from (15). Using the hydrostatic equation, this is

$$h \equiv (C_{pd} + QC_l)T + L_v w + (1 + Q)gz. \quad \text{(A5)}$$

It follows from (A5) that for reversible processes

$$\begin{aligned} dh &= (C_{pd} + QC_l)dT + L_v dw + wdL_v + (1 + Q)gdz \\ &= (C_{pd} + QC_l)dT + L_v dw + w(C_{pv} - C_l)dT + (1 + Q)gdz \\ &= (C_{pd} + wC_{pv} + lC_l)dT + L_v dw - \alpha_d dp = 0. \end{aligned} \quad \text{(A6)}$$

The last line of the above is simply a statement of the first law of thermodynamics. This shows that h is conserved for hydrostatic reversible displacements.

APPENDIX B

Maxwell's Relations for Reversible Moist Processes

We begin with the first law of thermodynamics, which can be obtained by differentiating (2) and making use of the Clausius-Clapeyron equation. The result is

$$Tds = -\alpha_d dp + (C_{pd} + QC_l)dT + d[L_v w] + C_l T \ln TdQ. \quad \text{(B1)}$$

This is valid in both saturated and unsaturated air. The last term on the right arises because the system is not closed to exchange of water mass. Before proceeding further, we rewrite the first term on the right as follows:

$$\begin{aligned} -\alpha_d dp &= -d[\alpha_d p] + pd(\alpha_d) \\ &= -d\left[\alpha_d p_d + \frac{\alpha_d}{\alpha_v}\alpha_v e\right] + pd(\alpha_d) \\ &= -d[R_d T + wR_v T] + pd(\alpha_d). \end{aligned} \quad \text{(B2)}$$

Combining (B2) with (B1) yields

$$Tds = pd(\alpha_d) + (C_{vd} + QC_l)dT + d[(L_v - R_v T)w] + C_l T \ln TdQ, \quad \text{(B3)}$$

where C_{vd} is the heat capacity of dry air at constant volume.

We next define a moist enthalpy h:

$$h \equiv p\alpha_d + (C_{vd} + C_l Q)T + (L_v - R_v T)w. \quad \text{(B4)}$$

It then follows from (B3) and (B4) that

$$dh = Tds + \alpha_d dp + C_l T(1 - \ln T)dQ. \quad \text{(B5)}$$

Using (B5) we then obtain the relations

$$\left(\frac{\partial h}{\partial s}\right)_{p,Q} = T,$$

$$\left(\frac{\partial h}{\partial p}\right)_{s,Q} = \alpha_d,$$

$$\left(\frac{\partial h}{\partial Q}\right)_{s,p} = C_l T[1 - \ln T], \quad \text{(B6)}$$

from which it follows, by cross-differentiation, that

$$\left(\frac{\partial \alpha_d}{\partial s}\right)_{p,Q} = \left(\frac{\partial T}{\partial p}\right)_{s,Q},$$

$$\left(\frac{\partial \alpha_d}{\partial Q}\right)_{s,p} = \left(\frac{\partial}{\partial p}[C_l T(1 - \ln T)]\right)_{s,Q}. \quad \text{(B7)}$$

Furthermore, since $\alpha = \alpha_d/(1 + Q)$, (B7) can be written

$$\left(\frac{\partial \alpha}{\partial s}\right)_{p,Q} = \frac{1}{(1+Q)}\left(\frac{\partial T}{\partial p}\right)_{s,Q}, \quad \text{(B8)}$$

$$\left(\frac{\partial \alpha}{\partial Q}\right)_{s,p} = \frac{1}{1+Q}\frac{\partial}{\partial p}[C_l T(1 - \ln T)]_{s,Q} - \frac{\alpha}{1+Q}. \quad \text{(B9)}$$

Finally, the last term on the right of (B9) can be reexpressed using the hydrostatic equation

$$-\alpha = \frac{\partial \phi}{\partial p}, \quad \text{(B10)}$$

where $\phi = gz$ is the geopotential. Using the chain rule and gradient wind balance it is possible to show that

$$\left(\frac{\partial \phi}{\partial p}\right)_r = \left(\frac{\partial \Phi}{\partial p}\right)_M, \quad \text{(B11)}$$

where $\Phi = \phi + \frac{1}{2}V^2$ and V is the gradient wind. Using (B11) and (B10), (B9) becomes

$$\left(\frac{\partial \alpha}{\partial Q}\right)_{s,p} = \frac{1}{1+Q}\frac{\partial}{\partial p}[C_l T(1 - \ln T)]_{s,Q} + \frac{1}{1+Q}\left(\frac{\partial \Phi}{\partial p}\right)_M. \quad \text{(B12)}$$

The relations (B8) and (B12) are the desired Maxwell's relations.

APPENDIX C

Carnot Cycle Derivation of (18)

We begin by writing a Bernoulli Equation for steady flow:

$$d\left(\frac{1}{2}V^2\right) + \alpha dp + gdz - \mathbf{F}\cdot d\mathbf{l} = 0, \quad \text{(C1)}$$

where V is the magnitude of the total velocity vector, $\mathbf{F}$ is the vector friction force and $\mathbf{l}$ is a unit vector parallel to streamlines. It is understood that the derivatives of (C1) are everywhere along streamlines. We next integrate (C1) around a closed streamline as indicated in Fig. C1. The result is

$$\oint \alpha dp = \oint \mathbf{F}\cdot d\mathbf{l}, \quad \text{(C2)}$$

which simply expresses a balance between pressure work and dissipation in steady flow. We also integrate (C1) between points a and c in Fig. C1. Since $z = 0$ and V vanishes at both points the result is

$$\int_a^c \alpha dp = \int_a^c \mathbf{F}\cdot d\mathbf{l}. \quad \text{(C3)}$$

We next assume that all frictional dissipation occurs only between points a and c and between points o and o' in Fig. C1. Thus

$$\oint \mathbf{F}\cdot d\mathbf{l} = \int_a^c \mathbf{F}\cdot d\mathbf{l} + \int_o^{o'} \mathbf{F}\cdot d\mathbf{l}. \quad \text{(C4)}$$

Combining (C2), (C3) and (C4) results in

$$\int_a^c \alpha dp = \oint \alpha dp - \int_o^{o'} \mathbf{F}\cdot d\mathbf{l}. \quad \text{(C5)}$$

Since $\alpha = \alpha_d/(1 + Q)$, it follows that

$$\alpha = \alpha_d - \alpha Q. \quad \text{(C6)}$$

Using (C6) in (C5) we obtain

$$\int_a^c \alpha_d dp = \oint \alpha_d dp - \int_o^{o'} \mathbf{F}\cdot d\mathbf{l} + \int_a^c \alpha Q dp - \oint \alpha Q dp. \quad \text{(C7)}$$

The last two terms of (C7) can be written as

$$-\int_c^a Q\alpha dp,$$

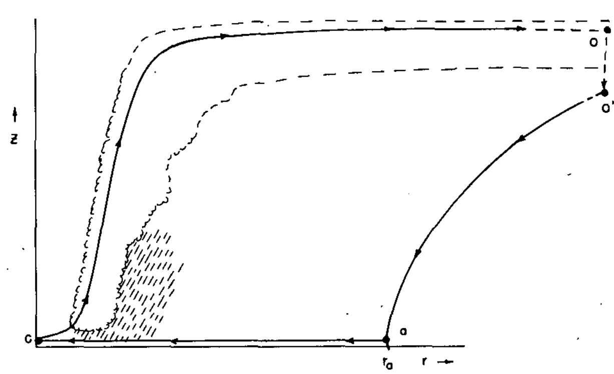

FIG. C1. Illustrating the path integral for the Carnot cycle. Points o and o' are taken to lie at very large radius.

where it is understood that the path of integration is c–o–o'–a. Once again using (C1) to replace αdp in the above, the last two terms in (C7) can be written

$$-\int_c^a Q\alpha dp = \int_c^a Q\left[d\left(\frac{1}{2}V^2 + gz\right) - \mathbf{F}\cdot d\mathbf{l}\right]$$
$$= -\int_o^{o'}\left[\left(\frac{1}{2}V^2 + gz\right)dQ + Q\mathbf{F}\cdot d\mathbf{l}\right], \quad \text{(C8)}$$

where we have made use of the vanishing of z and V at the end points and the assumption that irreversible processes act only between o and o' along the integration path c–o–o'–a.

We once again use (C1) to estimate the frictional dissipation necessary to close the Carnot cycle. Under the plausible assumption that the flow is hydrostatic between o and o', (C1) shows that $\mathbf{F}\cdot d\mathbf{l} = d(\frac{1}{2}V^2)$ in this region. Using this estimate in (C8) and (C7) the two relations may be combined to give

$$\int_a^c \alpha_d dp$$
$$= \oint \alpha_d \mathrm{dp} - \int_o^{o'}\left[(gz)dQ + d\left((1+Q)\frac{1}{2}V^2\right)\right]$$
$$= \oint \alpha_d dp - g\overline{z_0}^* Q\Big|_o^{o'} - \left[\frac{1}{2}(1+Q)V^2\right]\Big|_o^{o'}, \quad \text{(C9)}$$

where $\overline{z_0}^*$ is defined as before.

Finally, we use (15) to eliminate $\alpha_d dp$ from (C9). After some integrations by parts, and noting that entropy only changes at the surface and between o and o', we obtain

$$\overline{T_0}(s_c - s_a) = (w_c - w_a)[L_v + C_l T_s + C_l \overline{T_0}^*(\ln \overline{T_0}^* - 1) - g\overline{z_0}^*] + \left[\frac{1}{2}(1+Q)V^2\right]\Big|_o^{o'}. \quad \text{(C10)}$$

Using (1) to relate V to r and M and noting that $M_a = \frac{1}{2}fr_a^2$, it can easily be seen that (C10) is equivalent to (17).

In the pseudo-adiabatic case we allow all condensate to fall out of the system immediately upon forming. To derive a pressure equation in this case, we start with (C7) and this time insist that $Q = w$ everywhere. Then (C8) is instead written

$$-\int_c^a Q\alpha dp = \int_c^a w\left[d\left(\frac{1}{2}V^2 + gz\right) - \mathbf{F}\cdot d\mathbf{l}\right]$$
$$= \left(\frac{1}{2}V_0^2 + gz_o\right)(\overline{w_u} - \overline{w_d}) - \int_o^{o'} w\mathbf{F}\cdot d\mathbf{l}, \quad \text{(C11)}$$

where $\frac{1}{2}V_o^2 + gz_o$ is evaluated at point o and $\overline{w_u}$ and $\overline{w_d}$ are defined

$$\overline{w_u} \equiv \frac{1}{\frac{1}{2}V_o^2 + gz_o}\int_c^o wd\left(\frac{1}{2}V^2 + gz\right),$$
$$\overline{w_d} \equiv \frac{1}{\frac{1}{2}V_o^2 + gz_o}\int_a^o wd\left(\frac{1}{2}V^2 + gz\right). \quad \text{(C12)}$$

Combining (C12) with (C7) gives

$$\int_a^c \alpha_d dp = \oint \alpha_d dp - \int_o^{o'}(1+w)\mathbf{F}\cdot d\mathbf{l} + \left(\frac{1}{2}V_o^2 + gz_o\right)(\overline{w_u} - \overline{w_d}). \quad \text{(C13)}$$

As before, we use the first law to eliminate $\alpha_d dp$ from (C13). In the pseudoadiabatic case, the first law can be written (see Iribarne and Godson, 1973)

$$-\alpha_d dp = Tds' - (C_{pd} + wC_l)dT - d[L_v w]. \quad \text{(C14)}$$

Unlike the reversible case, s' is a state variable as well as a conserved quantity so that its material derivative is the same as its general derivative. Substituting (C14) into (C13) yields

$$\overline{T_0}(s'_c - s'_a) = L_v(w_c - w_a) + C_l(T_s - T_0)(\widetilde{w_u} - \widetilde{w_d}) - \left(\frac{1}{2}V_o^2 + gz_o\right)(\overline{w_u} - \overline{w_d}) + \int_o^{o'}(1+w)d\left(\frac{1}{2}V^2\right), \quad \text{(C15)}$$

where

$$\widetilde{w_u} \equiv \frac{1}{T_s - T_o}\int_c^a wdT,$$
$$\widetilde{w_d} \equiv \frac{1}{T_s - T_o}\int_a^o wdT.$$

This can be directly compared to (C10) which applies to the reversible case. The main differences are the different definitions of s and s' and the replacement of $w_c - w_a$ by weighted vertical averages of $w_u - w_d$ in the heat capacity and gravitational terms. This means that the effects of heat capacity and weight of water substance are much less in the pseudo-adiabatic than in the reversible case.

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