Stability of discrete dynamical systems

Supplementary material

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1 Mathematical Modeling

Main idea of mathematical modeling:

1. Abstraction: real world problem (experimental data) is described by a mathematical formulation.
2. Aim: find an appropriate mathematical formulation and use the mathematical tools to investigate the real world phenomenon.
3. No model is THE right one, but only A right one.

However, a model should always be as **simple** as possible, as **detailed** as necessary.

There are two ways doing Biomathematics:

Qualitative theory	Quantitative theory
Modeling of the basic mechanisms in a simple way; parameter fitting and analysis of (concrete) data doesn't greatly matter. The results are qualitative. A rigorous analysis of the models is possible and the qualitative results can be compared with experimental results. Quantitative prediction of experimental results is not (main) goal of this approach.	Here, the model of the biological system is very detailed and parameters are taken from the experiments (e.g. by data fitting). The analysis of the system is less important than to get simulations of concrete situations. The results are qualitative, quantitative prediction should be possible. It is important to know a lot of details about the biological system.

Modeling approaches:

- Deterministic approach:
  1. Difference Equations: The time is discrete, the state (depending on time) can be discrete or continuous. Often used to describe seasonal events.
  2. Ordinary Differential Equations (ODEs): Time and state are continuous, space is a homogenous quantities. These approach is often used to describe the evolution of populations.

3. Partial Differential Equations (PDEs): Continuous time and further continuous variables, e.g. space. Used for example to model physical phenomena, like diffusion.

- Stochastic approach: Include stochasticity and probability theory in the model. Often used in the context of small populations (also when few data are available).

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In these notes we will only recap some properties of difference equations and focus on the stability of linear and nonlinear discrete models.

2 Discrete Linear Models

Time-discrete models means that the development of the system is observed only at discrete times $t_0, t_1, t_2, \dots$ and not in a continuous time course. Assume here that $t_{k+1} = t_k + h$ where $h > 0$ is a constant step.

An example for discrete linear models is the **Fibonacci equation** (1202), which you probably already know from the highschool. Fibonacci investigated how fast rabbits could breed, assuming that:

- Rabbits are able to mate at the age of one month and at the end of its second month the females can produce another pair of rabbits.
- The rabbits never die.
- The females produce one new pair every month from the second month on.

The Fibonacci sequence is defined by the following recursive formula:

$$x_{n+1} = x_n + x_{n-1}.$$

This equation can be formulated as a 2D discrete-linear system

$$\begin{aligned} x_{n+1} &= x_n + y_n \\ y_{n+1} &= x_n. \end{aligned}$$

Generally, a linear system in 2D can be written as

$$\begin{aligned} x_{n+1} &= a_{11}x_n + a_{12}y_n \\ y_{n+1} &= a_{21}x_n + a_{22}y_n \end{aligned}$$

or in matrix notation

$$\begin{pmatrix} x \\ y \end{pmatrix}_{n+1} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\text{Matrix } A} \begin{pmatrix} x \\ y \end{pmatrix}_n$$

We now look for the **stationary states** of the discrete system. The stationary state for a general discrete system $x_{n+1} = f(x_n)$ is a $\bar{x}$, such that $\bar{x} = f(\bar{x})$. Obviously, as long as we consider linear systems, $(\bar{x}, \bar{y}) = (0, 0)$ is a stationary state.

How to investigate stability.
Consider the system

$$u_{n+1} = Au_n. \tag{1}$$

Then $u_n = A^n u_0$, $n = 0, 1, 2, \ldots$ is the solution of (1) with initial condition u_0. Let λ an eigenvalue of A with the corresponding eigenvector u, then we have $A^n u = \lambda^n u$ and $u_n = \lambda^n u_0$ satisfies the difference equation (1).
For u_0 being a linear combination of eigenvectors of A, $u_0 = b_1 v_1 + \ldots + b_k v_k$, ($\lambda_i$ corresponding eigenvalue of the eigenvector u_i) we get as the solution of (1):

$$u_n = b_1 \lambda_1^n v_1 + \ldots + b_k \lambda_k^n v_k.$$

For the matrix A, the **spectral radius** $\rho(A)$ is defined by

$$\rho(A) := max\{|\lambda| : \lambda \text{ is eigenvalue of } A\}.$$

Theorem 1 *Let A be a $m \times m$ matrix with $\rho(A) < 1$. Then every solution u_n of (1) satisfies*

$$\lim_{n \to \infty} u_n = 0.$$

Moreover, if $\rho(A) < \delta < 1$, then there is a constant $C > 0$ such that

$$\|u_n\| \leq C \|u_0\| \delta^n$$

for all $n \in \mathbb{N}_0$ and any solution of (1).

Remark: If $\rho(A) \geq 1$, then there are solutions u_n of (1) which do not tend to zero for $n \to \infty$. E.g., let λ be an eigenvalue with $|\lambda| \geq 1$ and u the corresponding eigenvector, then $u_n = \lambda^n u$ is a solution of (1) and $\|u_n\| = |\lambda|^n \|u\|$ does not converge to zero for $n \to \infty$.

What happens, if the spectral radius reaches the 1?

Theorem 2 *Let A be a $m \times m$ matrix with $\rho(A) \leq 1$ and assume that each eigenvalue of A with $|\lambda| = 1$ is simple. Then there is a constant $C > 0$ such that*

$$\|u_n\| \leq C \|u_0\|$$

for every $n \in \mathbb{N}$ and $u_0 \in \mathbb{R}^m$, where u_n is solution of (1).

From now on, we consider a **linear two-dimensional discrete system**,

$$u_{t+1} = Au_t, \tag{2}$$

where u_t is a two-dimensional vector and A a real 2×2 matrix (nonsingular).

A "fast formula" for the **computation of the eigenvalues** is

$$\lambda_{1,2} = \frac{1}{2} tr(A) \pm \frac{1}{2} \sqrt{tr(A)^2 - 4det(A)}$$

Let λ be an eigenvalue, then the corresponding eigenvector(s) v, is(are) defined by

$$Av = \lambda v \leftrightarrow (A - \lambda I)v = 0.$$

Theorem 3 *For any real 2×2 matrix A there exists a nonsingular real matrix P such that*

$$A = PJP^{-1},$$

where J is one of the following possibilities

1. $$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

   if A has two real (not necessarily distinct) eigenvalues λ_1, λ_2 with linearly independent eigenvectors.

2. $$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

   if A has a single eigenvalue λ (with a single eigenvector).

3. $$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

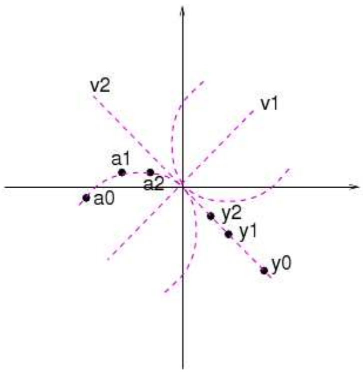
   if A has a pair of complex eigenvalues $\alpha \pm i\beta$ (with non-zero imaginary part)

For real eigenvalues:

Case 1a: $0 < \lambda_1 < \lambda_2 < 1 \Rightarrow$ (0,0) is a **stable node**

All solutions of equation (2) are of the form

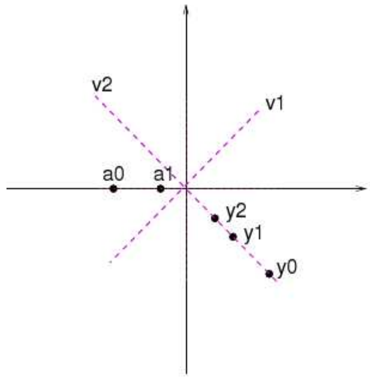
$$u_t = C_1 \lambda_1^t v_1 + C_2 \lambda_2^t v_2,$$



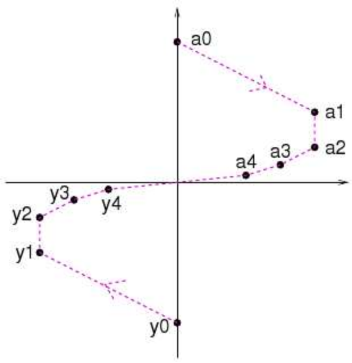
Case 1b: $0 < \lambda_1 = \lambda_2 < 1 \Rightarrow$ (0,0) is a **stable (one-tangent-)node**

There are two possibilities:

If A has one eigenvalue with two independent eigenvectors, case 1a can be slightly modified and the figure has the form

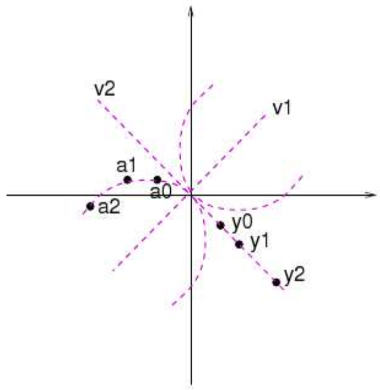


If A has a simple eigenvalue with only one independent eigenvector (and one generalized eigenvector v_2, i.e. $(A - \lambda I)^2 v_2 = 0$), then for $t \to \infty$, all solutions tend to 0.



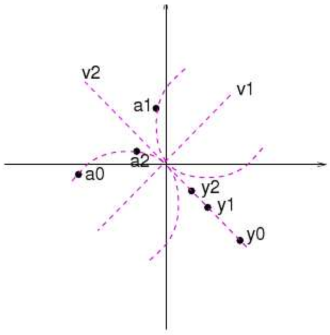
Case 2: $1 < \lambda_1 < \lambda_2 \Rightarrow$ (0,0) is a **unstable node**

The solutions go away from 0 for $t \to \infty$.



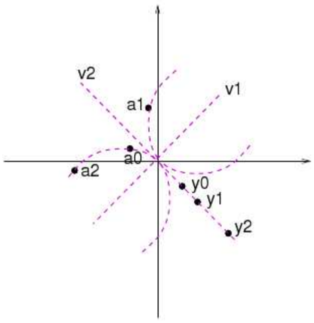
Case 3: $-1 < \lambda_1 < 0 < \lambda_2 < 1 Rightarrow$ (0,0) is a **stable node with reflection**.

Since λ_1^t has alternating signs, the solutions jump between the different branches (provided that $C_1 \neq 0$)

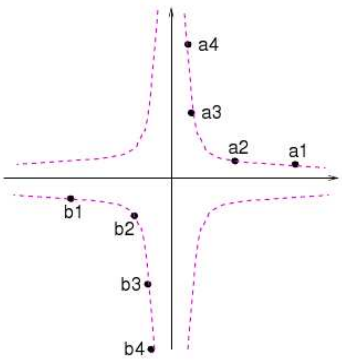


Case 4: $\lambda_1 < -1 < 1 < \lambda_2 \Rightarrow$ (0,0) is an **unstable node with reflection**

Unstable equilibrium, the solutions go away from (0,0), jumping in the direction of v_1.

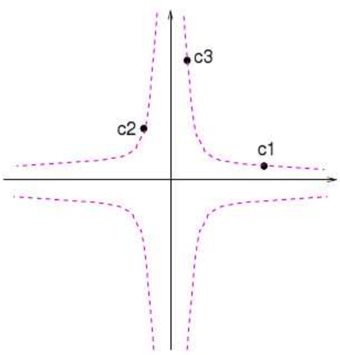


Case 5: $0 < \lambda_1 < 1 < \lambda_2 \Rightarrow$ (0,0) is a **saddle point**



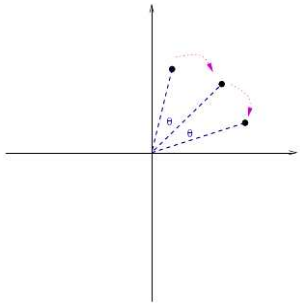
One direction (eigenvector) is stable, the other is unstable.

Case 6: $-1 < \lambda_1 < 0 < 1 < \lambda_2$ (0,0) is a **saddle point with reflection**



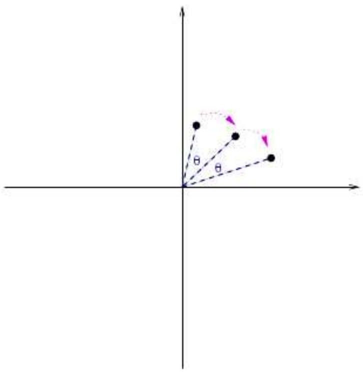
For complex eigenvalues:

Case 7: $\alpha^2 + \beta^2 = 1 \Rightarrow$ (0,0) is a **center**



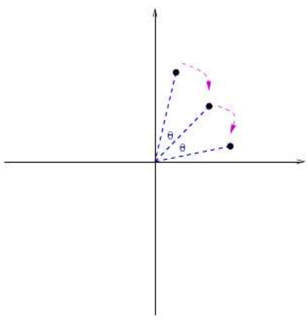
Each solution moves clockwise (with the angle θ) around a circle centred at the origin, which is called a center.

Case 8: $\alpha^2 + \beta^2 > 1 \Rightarrow$(0,0) is an **unstable spiral**



The solution moves away from the origin with each iteration, in clockwise direction.

Case 9: $\alpha^2 + \beta^2 < 1 \Rightarrow$(0,0) is a **stable spiral**



3 Nonlinear Discrete Models

The general nonlinear one-dimensional difference equation of first order is:

$$x_{n+1} = f(x_n). \tag{3}$$

The discrete linear equation (1) can be considered as a special case of (3).

We introduce now some general concepts:

Definition 1 *$\bar{x}$ is called* ***stationary point*** *of the system $x_{n+1} = f(x_n)$, if*

$$\bar{x} = f(\bar{x}).$$

$\bar{x}$ is also called fixed point or steady state.

An example:

$$x_{n+1} = ax_n + b \qquad \text{(i.e. } f(x_n) = ax_n + b), \tag{4}$$

where
a: constant reproduction rate; growth / decrease is proportional to x_n (Assumption: $a \neq 1$)
b: constant supply / removal.

Definition 2 *An* ***autonomous discrete nonlinear system*** *is given by*

$$u_{n+1} = f(u_n), \qquad n \in \mathbb{N}_0, \tag{5}$$

where $u_n \in \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m$ (or $f : D \to D,\ \ D \subseteq \mathbb{R}^m$).

If A is a $m \times m$ matrix, then the linear system $f(x) = Ax$ is a special case of (5).

3.0.1 Analysis of discrete nonlinear dynamical systems

One of our main tasks is the investigation of the behavior of x_n "in the long time run", i.e., for large n.
We look for stationary points of equation (4):

$$\begin{aligned} f(\bar{x}) = \bar{x} \quad &\Leftrightarrow \quad a\bar{x} + b = \bar{x} \\ &\Leftrightarrow \quad b = (1-a)\bar{x} \\ &\Leftrightarrow \quad \bar{x} = \frac{b}{1-a} \end{aligned}$$

Hence, there exists exactly one stationary state of (4).

Definition 3 *Let $\bar{x}$ be a stationary point of the system $x_{n+1} = f(x_n)$.*
$\bar{x}$ is called ***locally asymptotically stable*** *if there exists a neighborhood U of $\bar{x}$ such that for each starting value $x_0 \in U$ we get:*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

$\bar{x}$ is called ***unstable****, if $\bar{x}$ is not (locally asymptotically) stable.*

How to investigate stability of stationary points?
Consider a stationary point $\bar{x}$ of the difference equation $x_{n+1} = f(x_n)$. We are interested in the local behavior near $\bar{x}$. For this purpose, we consider the deviation of the elements of the sequence to the stationary point $\bar{x}$:

$$z_n := x_n - \bar{x}$$

z_n has the following property:

$$\begin{aligned} z_{n+1} &= x_{n+1} - \bar{x} \\ &= f(x_n) - \bar{x} \\ &= f(\bar{x} + z_n) - \bar{x}. \end{aligned}$$

Let the function f be differentiable in $\bar{x}$, thus we get $\lim_{h\to 0} \frac{f(\bar{x}+h)-f(\bar{x})}{h} = f'(\bar{x})$ and $f(\bar{x}+h) = f(\bar{x}) + h \cdot f'(\bar{x}) + O(h^2)$. This yields:

$$\begin{aligned} z_{n+1} &= f(\bar{x} + z_n) - \bar{x} \\ &= f(\bar{x} + z_n) - f(\bar{x}) \\ &= z_n \cdot f'(\bar{x}) + O(z_n^2). \end{aligned}$$

$O(z_n^2)$ is very small and can be neglected, i.e. we approximate the nonlinear system $x_{n+1} = f(x_n)$ by

$$z_{n+1} \approx z_n \cdot f'(\bar{x}),$$

which is again a linear difference equation, for which we already know the stability criteria.

Proposition 1 *Let f be differentiable. A stationary point $\bar{x}$ of $x_{n+1} = f(x_n)$ is*

- *locally asymptotically stable, if $|f'(\bar{x})| < 1$*
- *unstable, if $|f'(\bar{x})| > 1$*

Remark: These criteria are sufficient, but not necessary!

In case of the non-homogeneous, linear system we have $f'(\bar{x}) = a$, which means that the stationary point is locally asymptotically stable if $|a| < 1$ (respectively, unstable, if $|a| > 1$).

Definition 4 *Let $u_{n+1} = f(u_n)$ be an autonomous system, $f : D \to D$, $D \subseteq \mathbb{R}^m$. A vector $v \in D$ is called* ***equilibrium or steady state*** *or stationary point or fixed point of f, if $f(v) = v$ and $v \in D$ is called periodic point of f, if $f^p(v) = v$. p is a period of v.*

1. *Let $v \in D$ be a fixed point of f. Then v is called stable, if for each $\varepsilon > 0$ there is $\delta > 0$ such that*

   $$\|f^n(u) - v\| < \varepsilon \quad \textit{for all } u \in D \textit{ with } \|u - v\| < \delta \textit{ and all } n \in \mathbb{N}_0$$

   (i.e. $f^n(U_\delta(v)) \subseteq U_\varepsilon(v)$). If v is not stable, it is called unstable.

2. *If there is, additionally to 1., a neighborhood $U_r(v)$ such that $f^n(u) \to v$ as $n \to \infty$ for all $u \in U_r(v)$, then v is called asymptotically stable.*

3. *Let $w \in D$ be a periodic point of f with period $p \in \mathbb{N}$. Then w is called (asymptotically) stable, if w, $f(w), \ldots, f^{p-1}(w)$ are (asymptotically) stable fixed points of f^p.*

Remark: Intuitively, a fixed point v is stable, if points close to v do not move far from v. If additionally all solutions starting near v converge to v, v is asymptotically stable.

Theorem 4 *Let $u_{n+1} = f(u_n)$ be an autonomous system. Suppose $f : D \to D$, $D \subseteq \mathbb{R}^m$ open, is twice continuously differentiable in some neighborhood of a fixed point $v \in D$. Let J be the Jacobian matrix of f, evaluated at v. Then*

1. *v is asymptotically stable if all eigenvalues of J have magnitude less than 1.*

2. *v is unstable if at least one eigenvalue of J has magnitude greater than 1.*

Remark: If $max\{|\lambda| : \lambda \text{ eigenvalue of } K\} = 1$, then we cannot give a statement about the stability of the fixed point v by that criterion; the behavior then depends on higher order terms than linear ones.

3.0.2 The 2D case

Here we consider the 2D case more concrete. The discrete system can be formulated with the variables x and y:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned} \tag{6}$$

Stationary states $\bar{x}$ and $\bar{y}$ satisfy

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{y}) \\ \bar{y} &= g(\bar{x}, \bar{y}) \end{aligned}$$

We need the Jacobian matrix at a certain stationary point $(\bar{x}, \bar{y})$:

$$A = \begin{pmatrix} \frac{\partial f}{\partial x}|_{\bar{x},\bar{y}} & \frac{\partial f}{\partial y}|_{\bar{x},\bar{y}} \\ \frac{\partial g}{\partial x}|_{\bar{x},\bar{y}} & \frac{\partial g}{\partial y}|_{\bar{x},\bar{y}} \end{pmatrix}$$

The eigenvalues λ_1 and λ_2 of A yield the information about stability of the system.

Proposition 2 *Let $(\bar{x}, \bar{y})$ be a stationary state of the system*

$$x_{n+1} = f(x_n, y_n), \qquad y_{n+1} = g(x_n, y_n)$$

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be the Jacobian at the point $(\bar{x}, \bar{y})$ with eigenvalues λ_1, λ_2. Then:

- $|\lambda_{1,2}| < 1 \Rightarrow (\bar{x}, \bar{y})$ *is locally stable*
- $|\lambda_j| > 1$ *for one $j \in \{1, 2\}$ respectively.* $\Rightarrow (\bar{x}, \bar{y})$ *is unstable.*

In some cases, it is easier to handle the following (necessary and sufficient) condition: Both eigenvalues satisfy $|\lambda_i| < 1$ and the steady state $(\bar{x}, \bar{y})$ is stable, if

$$2 > 1 + det\, A > |tr\, A|. \tag{7}$$

This can be easily shown:
The characteristic equation reads

$$\lambda^2 - tr\, A\, \lambda + det\, A = 0$$

and has the roots

$$\lambda_{1,2} = \frac{tr\, A \pm \sqrt{tr^2 A - 4\, det\, A}}{2}$$

In case of real roots, they are equidistant from the value $\frac{tr\, A}{2}$. Thus, first has to be checked that this midpoint lies inside the interval $(-1, 1)$:

$$-1 < \frac{tr\, A}{2} < 1 \quad \Leftrightarrow \quad |tr\, A/2| < 1.$$

Furthermore, the distance from $tr\, A/2$ to either root has to be smaller than to an endpoint of the interval, i.e.

$$1 - |tr\, A/2| > \frac{\sqrt{tr^2 A - 4\, det\, A}}{2}.$$

Squaring leads to

$$1 - |tr\, A| + \frac{tr^2 A}{4} > \frac{tr^2 A}{4} - det\, A,$$

and this yields directly

$$1 + det\, A > |tr\, A|.$$

Advantage: It is not necessary to compute explicitely the eigenvalues

References

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[2] Holmgren, R. *A First Course in Discrete Dynamical Systems.* Springer-Verlag, New York (1994).

[3] Kuttler, C. *Lecture Notes on Mathematical Models in Biology*, TUM 2009