Anti-ferromagnetic Ising Model in Hierarchical Networks

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The Ising antiferromagnet is a convenient model of glassy dynamics. It can introduce geometric frustrations and may give rise to a spin glass phase and glassy relaxation at low temperatures. We apply the antiferromagnetic Ising model to 3 hierarchical networks which share features of both small world networks and regular lattices. Their recursive and fixed structures make them suitable for exact renormalization group analysis as well as numerical simulations. We first explore the dynamical behaviors using simulated annealing and discover an extremely slow relaxation at low temperatures. Then we employ the Wang-Landau algorithm to investigate the energy landscape and the corresponding equilibrium behaviors for different system sizes. Besides the Monte Carlo methods, renormalization group is used to study the equilibrium properties in the thermodynamic limit and to compare with the results from simulated annealing and Wang-Landau sampling.

I. INTRODUCTION

The Ising antiferromagnet can introduce geometric frustrations and may give rise to interesting dynamics and phases. What may make it more interesting is applying the antiferromagnetic model to hierarchical networks which have a lattice backbone and small-world structure. These networks could produce different critical phenomena from that found on lattice geometries [1].

II. RG CALCULATIONS

The setup of renormalization is started by separating the Ising Hamiltonian into hierarchies

$$-\beta\mathcal{H} = \sum_{n=1}^{k-2}(-\beta\mathcal{H}_n) + \mathcal{R}(K_2, K_3, \cdots) \tag{1}$$

where $\mathcal{R}$ is the coupling beyond $\mathcal{H}_n$ of levels $k > 2$. $\mathcal{H}_n$ depends on the interactions K_0 on the backbone and L_0, L_1, $K_1, \cdots$ among the long range couplings. We describe the RG procedure network by network.

A. Ising antiferromagnet on HN3 and HN5

The Hamiltonian with magnetic field is

$$\begin{aligned}-\beta\mathcal{H}_n = {} & K_0\left(x_{n-2}x_{n-1} + x_{n-1}x_n + x_nx_{n+1} + x_{n+1}x_{n+2}\right)\\ & +K_1(x_{n-1}x_{n+1}) + yL_1(x_{n-2}x_{n+2})\\ & +L_0(x_{n-2}x_n + x_nx_{n+2}) + 4I\\ & +\frac{H_K}{2}(x_{n-2} + 2x_{n-1} + 2x_n + 2x_{n+1} + x_{n+2})\\ & +\frac{H_L}{2}(x_{n-2} + 2x_n + x_{n+2}) + \frac{H}{2}(x_{n-1} + x_{n+1})\\ & +\frac{T}{2}(x_{n-2}x_{n-1}x_n + x_nx_{n+1}x_{n+2})\end{aligned} \tag{2}$$

where $y = 0$ for HN3 and $y = 1$ for HN5. In the RG process of antiferromagnetic Ising model, one of the major differences is definition of μ. The reduced temperature is defined as

$$\mu = \exp[\frac{2}{T}] \tag{3}$$

where T is the temperature. Thus, $\mu \geq 1$, and

$$T \to 0 : \mu \to \infty : \frac{1}{\mu} \to 0_+$$

$$T \to \infty : \mu \to 1_+ : \frac{1}{\mu} \to 1_-$$

1. Internal Energy of HN3

The ground state energy is

$$GS_{HN3} = -1 \tag{4}$$

See the Figure 1 and 2 for results.

2. Internal Energy of HN5

The ground state energy for $N \to \infty$ is

$$GS_{HN5} = -\frac{7}{6} = -1.166666666\ldots \tag{5}$$

See the Figure 3 and 4 for results.

3. Magnetization of HN3

Magnetization vs. big H
This part is to learn how $< m >$ changes with H, which shows a different phenomena as shown in Figure 5.

4. Magnetization of HN5

Magnetization vs. big H
This part is to learn how $< m >$ changes with H, which shows a different phenomena as shown in Figure 5.

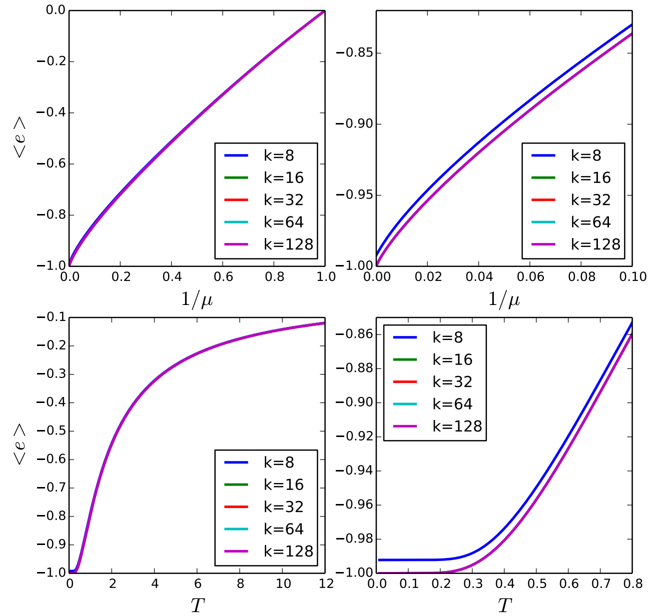

FIG. 1. HN3 Ising AFM Internal Energy vs temperature.

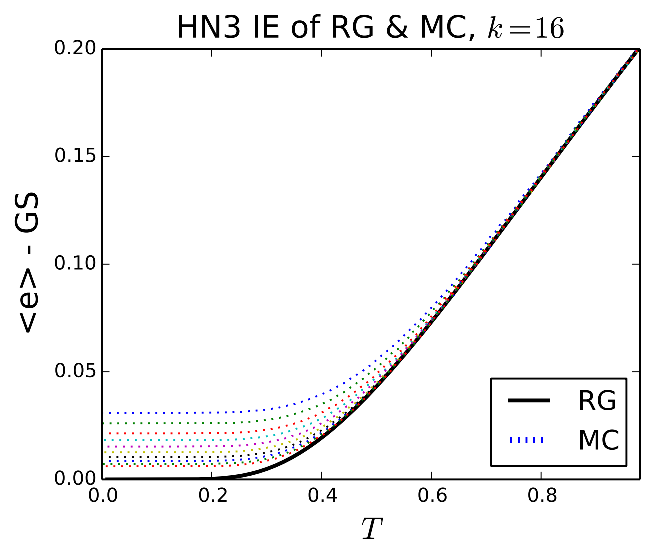

FIG. 2. HN3 Ising AFM Internal Energy from RG and MC.

5. Specific Heat of HN3

See Figure 7 for results.

6. Specific Heat of HN5

See Figure 8 for results.

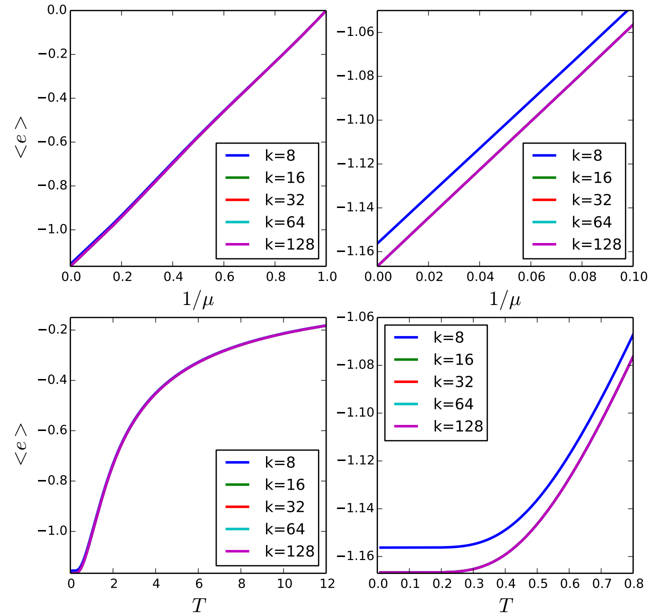

FIG. 3. HN5 Ising AFM Internal Energy vs temperature.

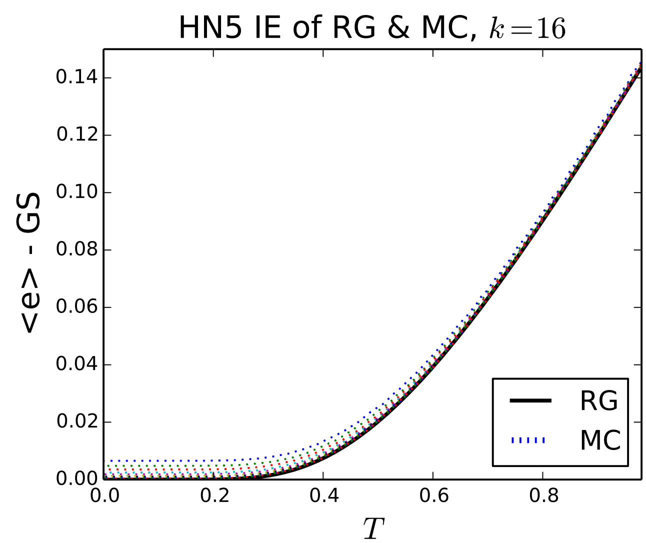

FIG. 4. HN5 Ising AFM Internal Energy from RG and MC.

7. Susceptibility of HN3

See Figure 9 for results.

8. Susceptibility of HN5

See Figure 10 for results.

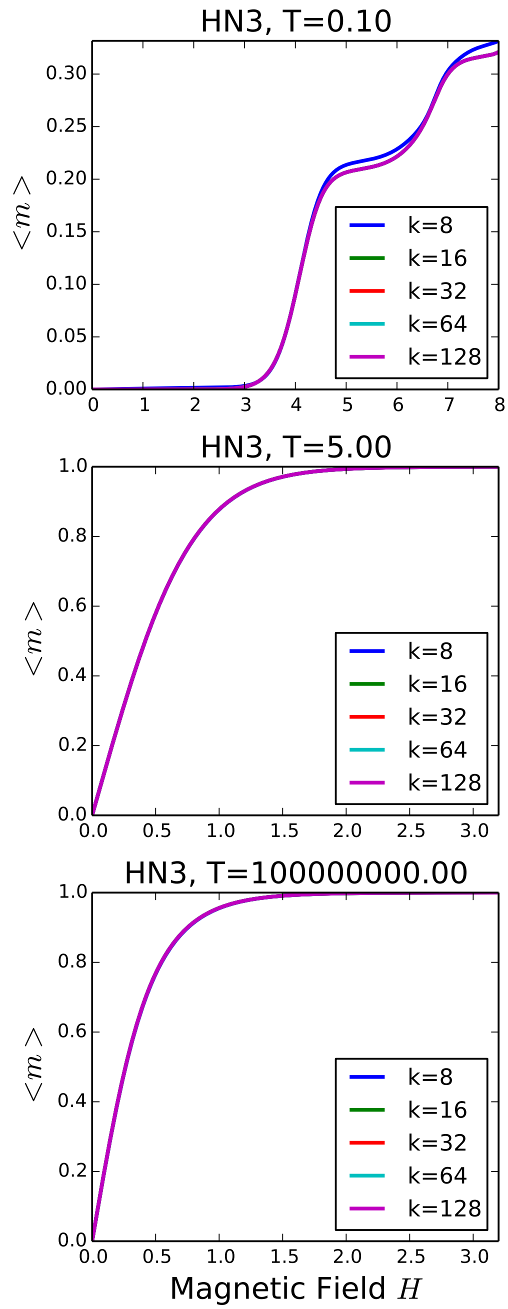

FIG. 5. HN3 Ising AFM magnetization vs magnetic field from RG. H in the RG is actually reduced magnetic field, i.e. H/T.

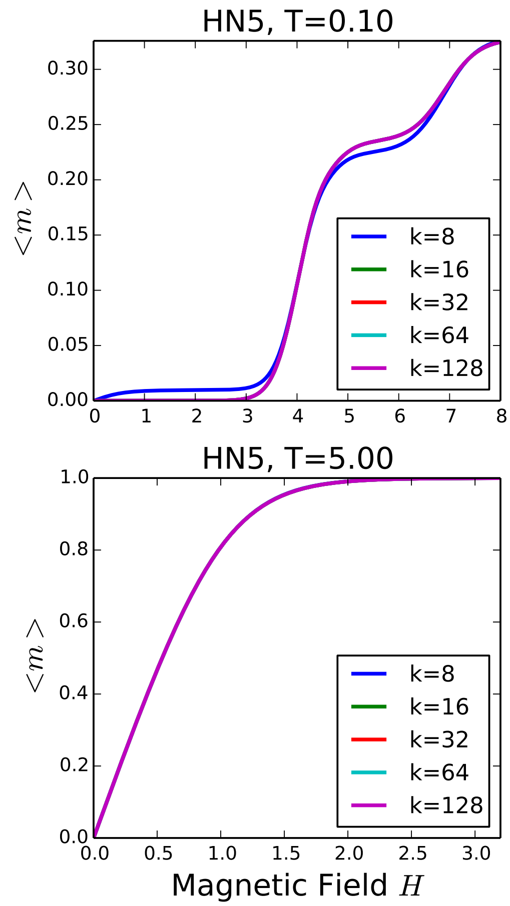

FIG. 6. HN5 Ising AFM magnetization vs magnetic field from RG. H in the RG is actually reduced magnetic field, i.e. H/T.

B. Ising ferromagnet on HNNP

The Hamiltonian with magnetic field is

$$\begin{aligned}-\beta\mathcal{H}_n = & K_0\left(x_{n-2}x_{n-1} + x_{n-1}x_n + x_n x_{n+1} + x_{n+1}x_{n+2}\right) \\ & + K_1(x_{n-2}x_{n+1} + x_{n-1}x_{n+2}) + yL_1(x_{n-2}x_{n+2}) \\ & + L_0(x_{n-2}x_n + x_n x_{n+2}) + 4I \\ & + \frac{H_K}{2}(x_{n-2} + 2x_{n-1} + 2x_n + 2x_{n+1} + x_{n+2}) \\ & + \frac{H_L}{2}(x_{n-2} + 2x_n + x_{n+2}) + \frac{H}{2}(x_{n-1} + x_{n+1}) \\ & + \frac{T}{2}(x_{n-2}x_{n-1}x_n + x_n x_{n+1}x_{n+2}) \qquad (6)\end{aligned}$$

where $y = 0$ is for HNNP, and $y = 1$ is for HN6. Here we discuss HNNP first, so $y = 0$.

Recall that, by fixed point analysis, the transition temperature from stable fixed point to chaotic fixed is $\mu = 3$ or $T = 1.8205$. We will try to study this temperature more.

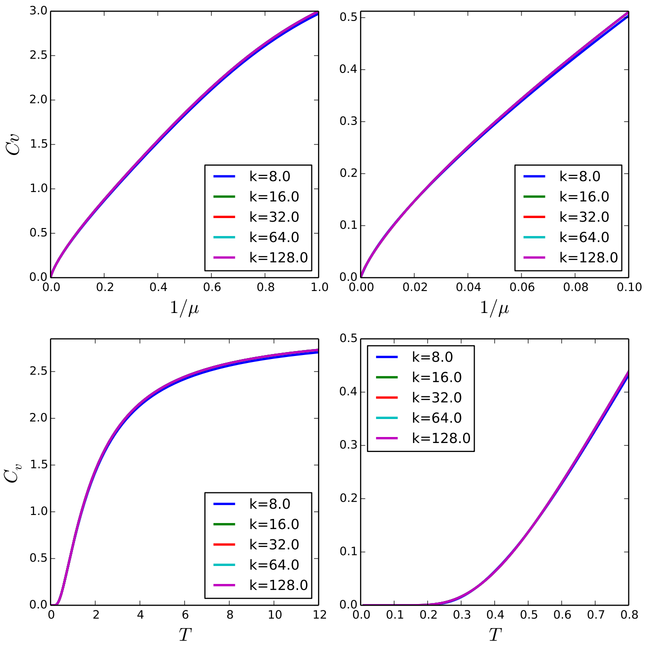

FIG. 7. HN3 Ising AFM Specific Heat from RG.

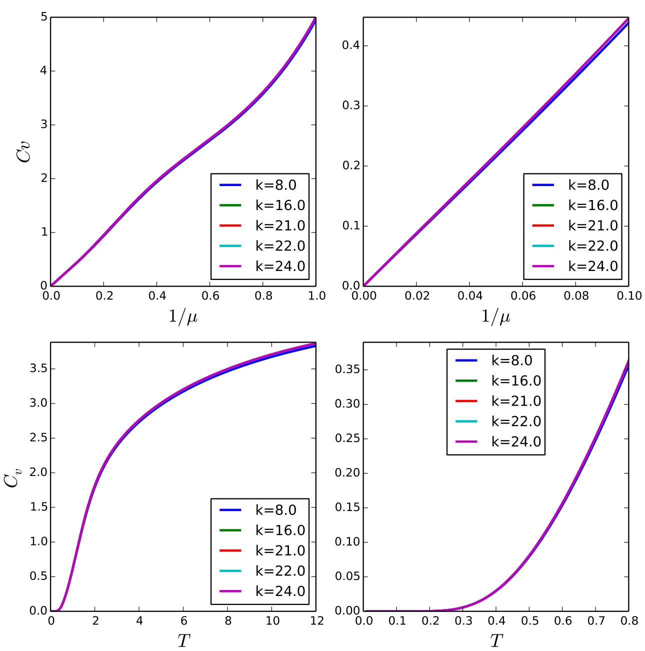

FIG. 8. HN5 Ising AFM Specific Heat from RG.

1. *Internal Energy of HNNP*

The ground state energy for $N \to \infty$ is

$$GS_{HNNP} = -1.48031496062992\ldots = -\frac{188}{127} \qquad (7)$$

See the Figure 11 and 12 for results.

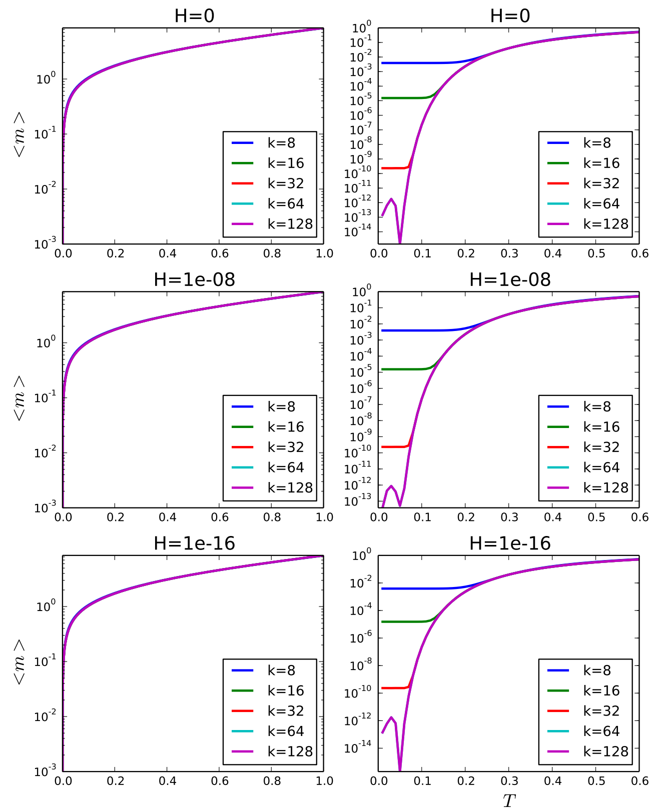

FIG. 9. HN3 Ising AFM Susceptibility from RG. The precision of RG is already N[1000]. I may try smaller T step size at low T to improve the accuracy at low T.

2. *Internal Energy of HN6*

The ground state energy for $N \to \infty$ is

$$GS_{HNNP} = -1.28571428571428\ldots = -\frac{9}{7} \qquad (8)$$

See the Figure 13 and 14 for results.

3. *Magnetization of HNNP*

Magnetization vs. big H
This part is to learn how $<m>$ changes with H, which shows a different phenomena as shown in Figure 15.

Magnetization vs. small H
This part is to learn how $<m>$ changes with small H and how it may break the śymmetry. See Figure 16 for results.

Magnetization vs. T
This part is to learn how $<m>$ changes with small H and how it may break the śymmetry. See Figure 17 for results.

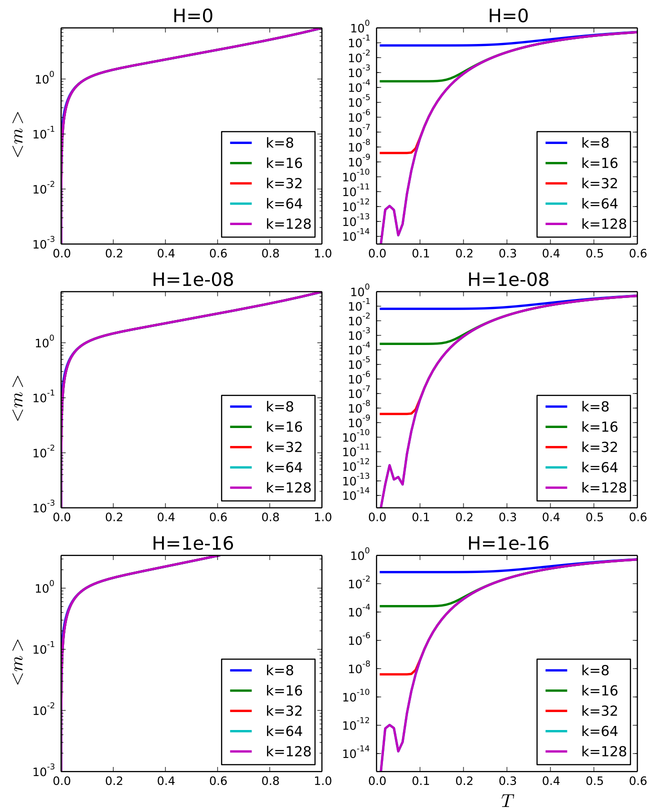

FIG. 10. HN5 Ising AFM Susceptibility from RG. The precision of RG is already N[1000]. I may try smaller T step size at low T to improve the accuracy at low T.

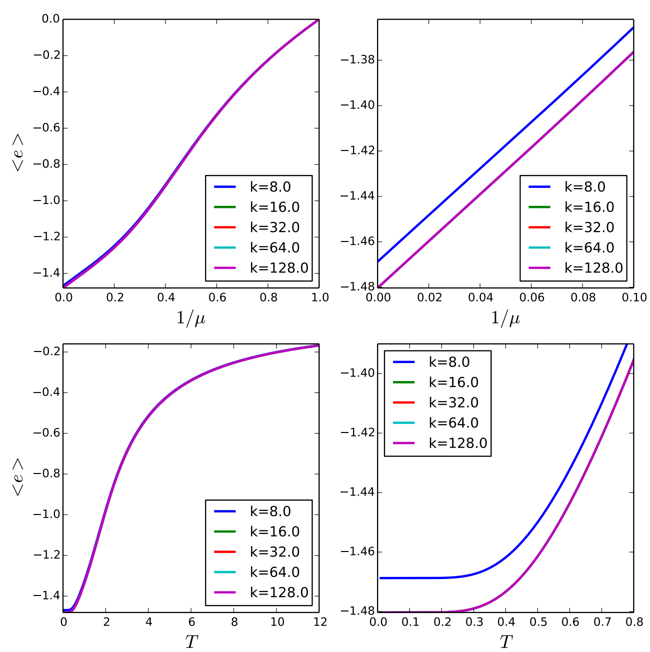

FIG. 11. HNNP Ising AFM Internal Energy vs temperature.

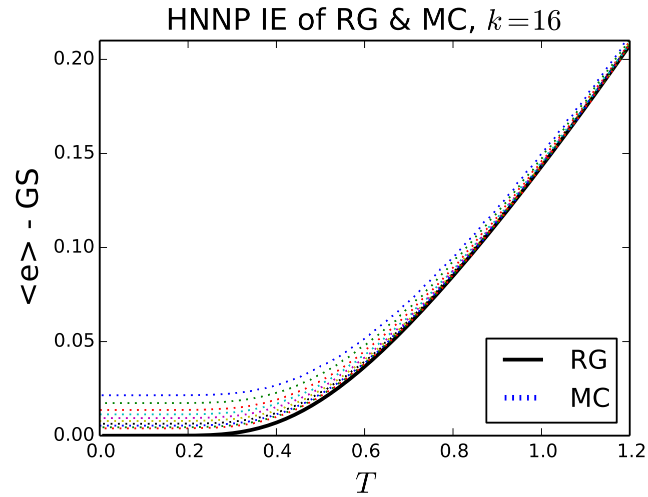

FIG. 12. HNNP Ising AFM Internal Energy from RG and MC.

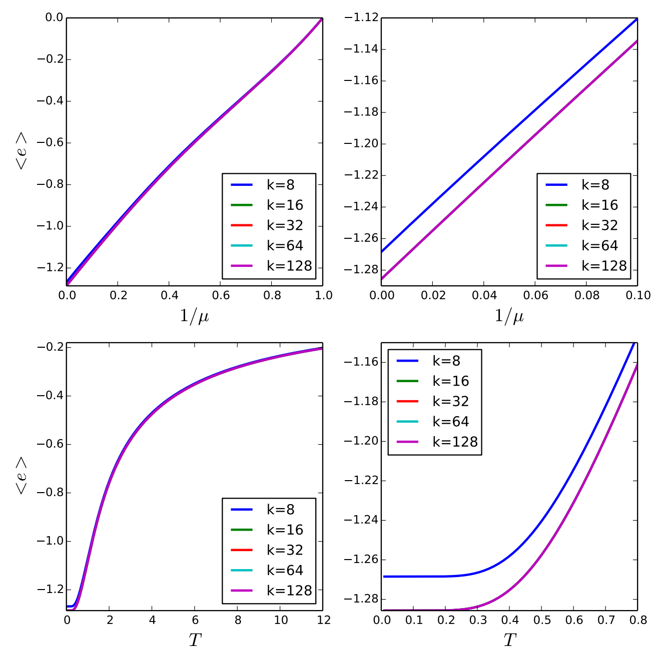

FIG. 13. HN6 Ising AFM Internal Energy vs temperature.

4. *Specific Heat of HNNP*

See Figure 18 for results.

5. *Specific Heat of HN6*

See Figure 19 for results.

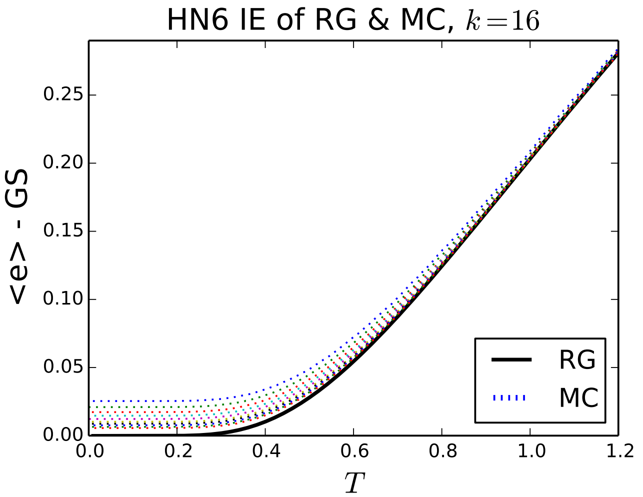

FIG. 14. HN6 Ising AFM Internal Energy from RG and MC.

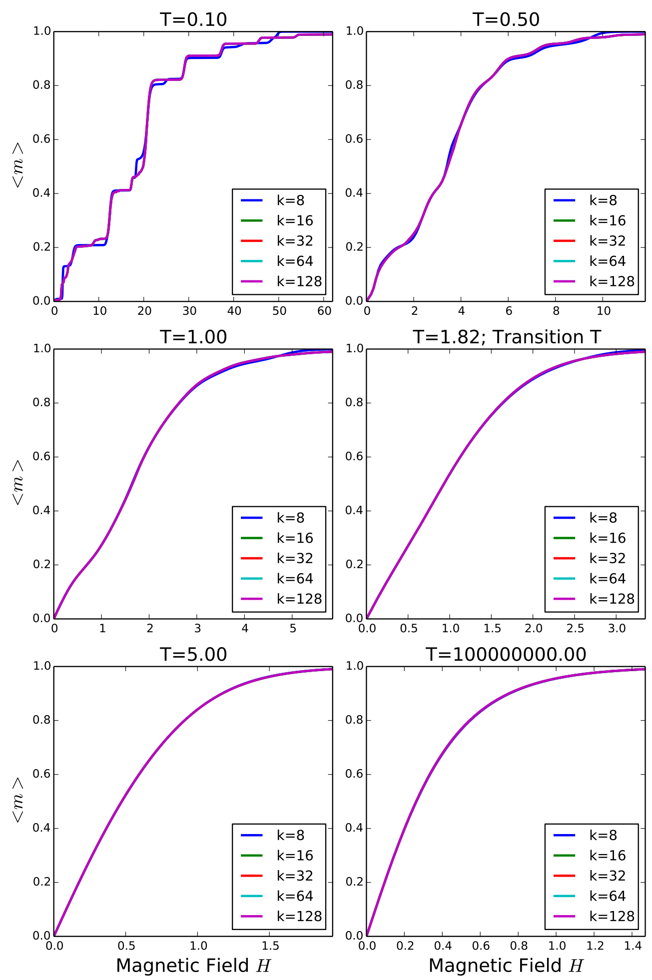

FIG. 15. HNNP Ising AFM magnetization vs magnetic field from RG. H in the RG is actually reduced magnetic field, i.e. H/T.

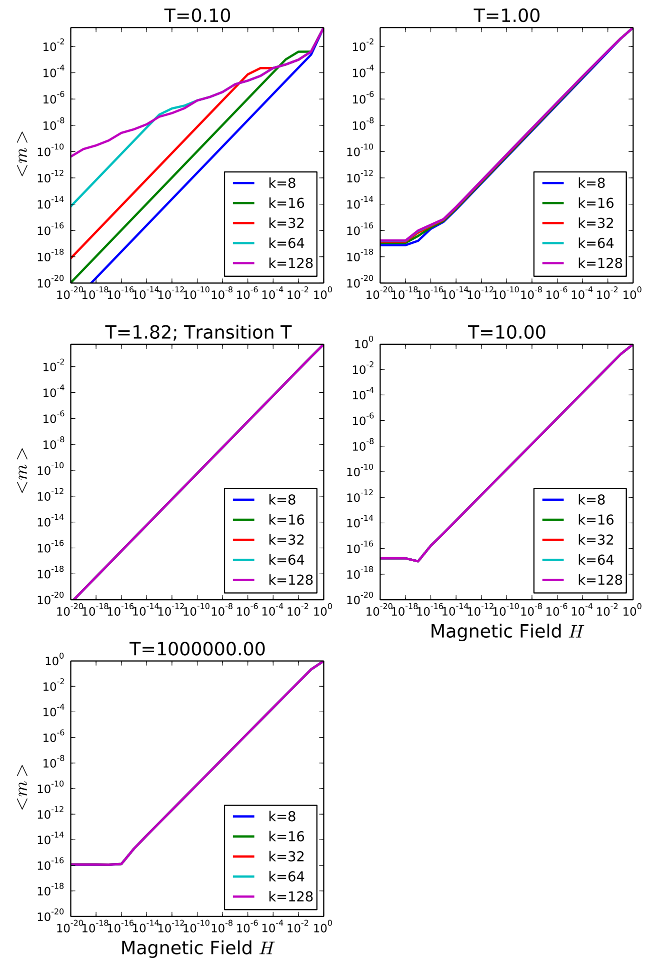

FIG. 16. HNNP Ising AFM magnetization vs magnetic field from RG. This part is to learn how $< m >$ changes with small H and how it may break the symmetry: H in the RG is actually reduced magnetic field, i.e. H/T.

6. *Susceptibility of HNNP*

See Figure 20 for results.

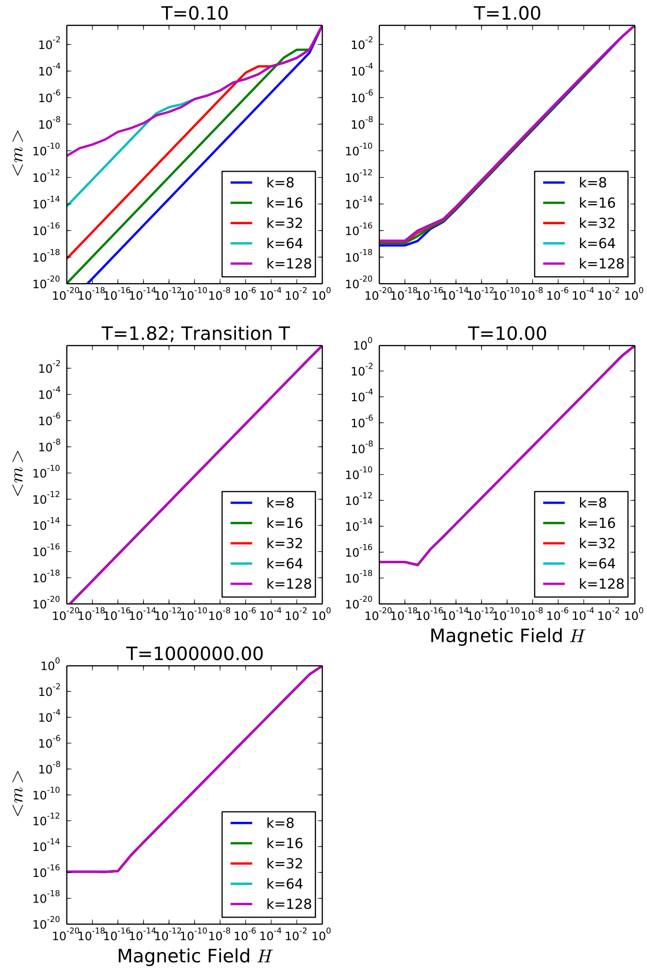

FIG. 17. HNNP Ising AFM magnetization vs temperature from RG at different H.

[1] S. Boettcher and C. Brunson, EPL (Europhysics Letters) **110**, 26005 (2015).

[2] S. Boettcher and C. Brunson, Physical Review E **83**, 021103 (2011).

[3] T. Nogawa, T. Hasegawa, and K. Nemoto, Physical Review E **86**, 030102 (2012).

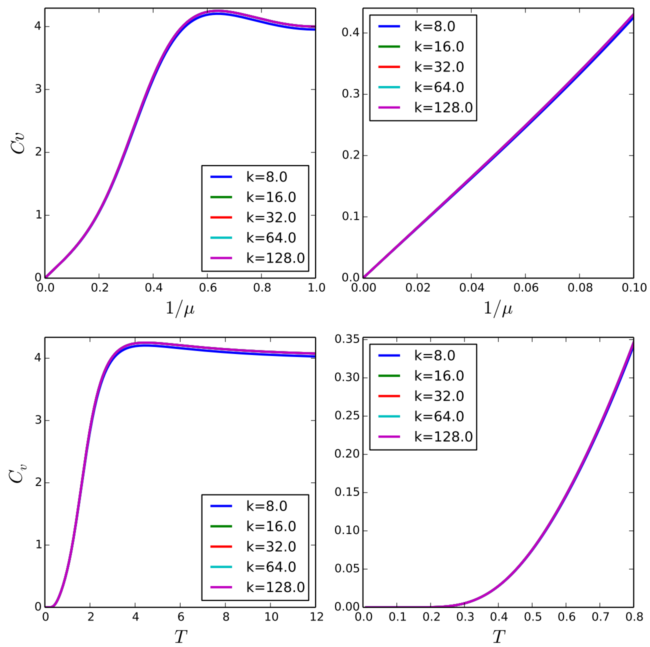

FIG. 18. HNNP Ising AFM Specific Heat from RG.

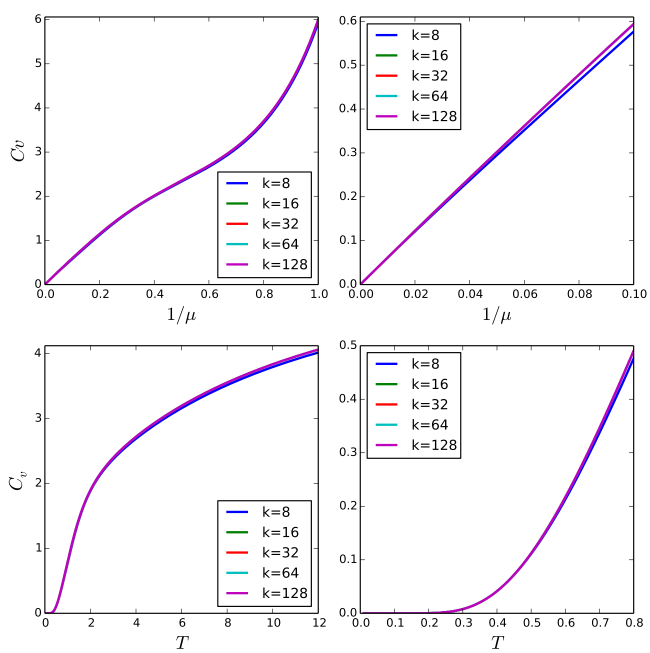

FIG. 19. HN6 Ising AFM Specific Heat from RG.

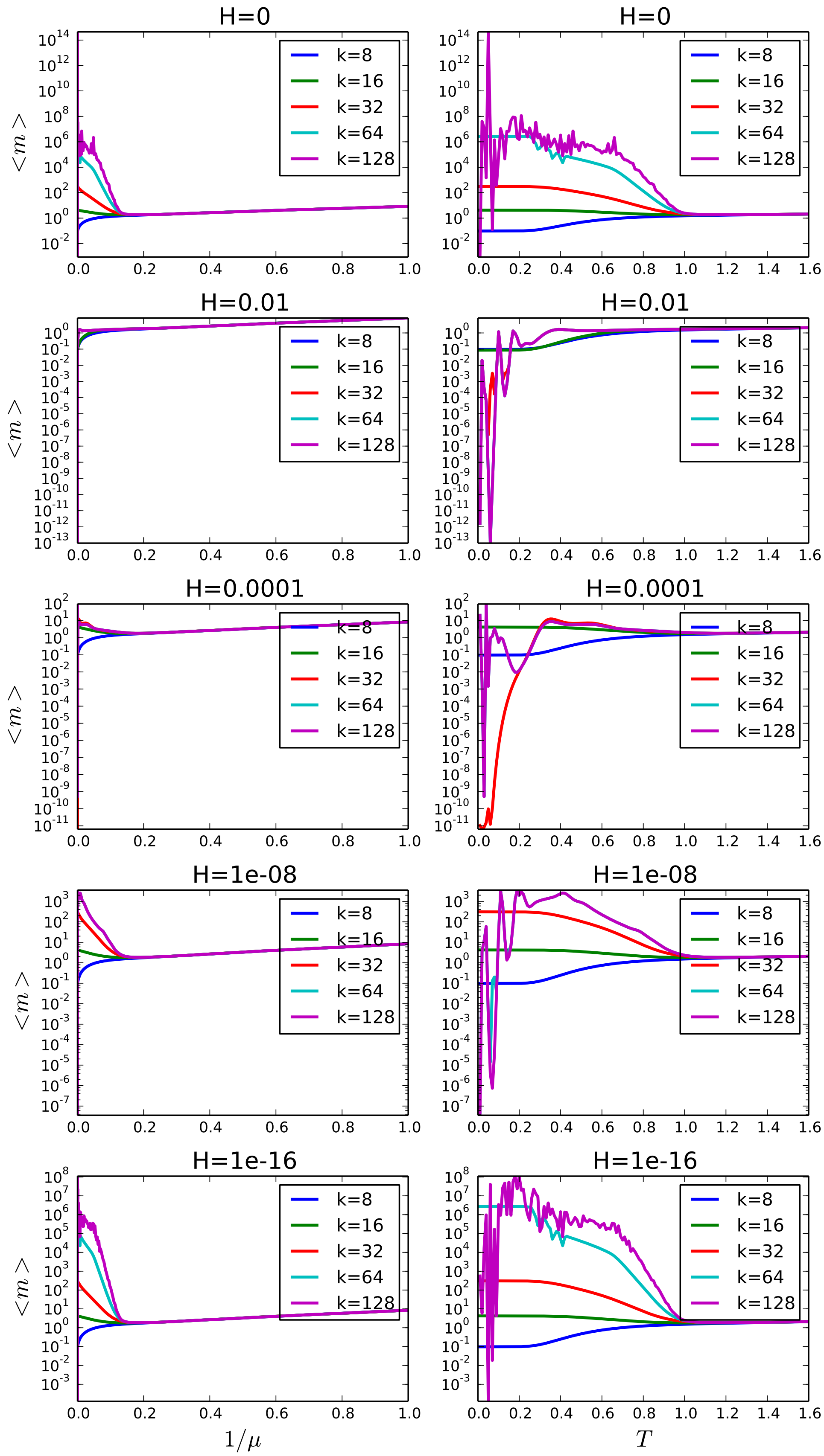

FIG. 20. HNNP Ising AFM Susceptibility from RG. The precision of RG is already N[1000]. I may try smaller T step size at low T to improve the accuracy at low T.