

## Migdal-Kadanoff RG Approach

*One of the simplest implementation of the RG program is found in the "decimation" approach, first discussed by Migdal in his study of lattice gauge theories. This method, which is exact for one dimensional models, can also be used in an approximate way to examine the critical behavior in higher dimensions, where finite temperature transitions exist. When hand-tailored to the specific problem at hand (as later developed by Kadanoff), this real-space RG approach often provides surprisingly accurate results for the critical exponents. The price to pay for its simplicity is a somewhat uncontrolled character of this approximation, which also does not provide a complete explanation for the remarkable universality of the critical phenomena.*

### Decimation in $d = 1$

As our first encounter with an RG calculation, consider a one-dimensional Ising model with the partition function

$$Z = \sum_{\{S_i = \pm 1\}} \exp \left\{ \sum_i [K S_i S_{i+1} + h S_i] \right\}.$$

Instead of introducing block-spin variables as imagined by Kadanoff, we perform a "decimation" procedure to eliminate every second spin in the lattice. That is, we perform a partial sum over spins in one sublattice. As we will now show, this transformation preserves exactly the form of the partition function, while producing new values of the coupling constants. At the same time, it halves the number of degrees of freedom, while doubling the unit cell size. Thus, it accomplishes precisely the task outlined by Kadanoff, corresponding to  $b = 2$ .

Let us see how this is done in practice. Consider three neighboring spins (call them 1, 2, and 3), and perform in the partition function a summation over the spin 2 only. This will generate an effective interaction between spins 1 and 3, as well as a correction to the external field acting on these sites. To do this, we use the following notation

$$w(i, j) = K S_i S_j + \frac{1}{2} h (S_i + S_j),$$

so that we can write

$$-\beta H = \sum_i w(i, i+1).$$

[Note that we have assigned a "half" of the external field  $h$  to each dimer to avoid double-counting.]

Now we perform the summation over spins on one sublattice, and the renormalized Hamiltonian takes the same form, only with the renormalized dimer interaction of the form

$$\begin{aligned} \tilde{w}(i, j) &= \frac{1}{2}h(S_i + S_j) + \ln \sum_{\{S_k=\pm 1\}} \exp \{[K S_i S_k + K S_k S_j + h S_k]\} \\ &= \frac{1}{2}h(S_1 + S_3) + \ln 2 + \ln \cosh \{[K (S_1 + S_3) + h]\}. \end{aligned}$$

At first glance, this expression for  $\tilde{w}(i, j)$  looks nothing like the form of  $w(i, j)$ . The algebra simplifies further if we concentrate on small field, where we can retain only the terms linear in  $h$

$$\tilde{w}(i, j) \approx \frac{1}{2}h(S_i + S_j) + \ln 2 + \ln \cosh \{K (S_i + S_j)\} + h \tanh \{K (S_i + S_j)\}.$$

But then we notice that any function of spin variables can at most be a linear function of each spin, since  $S_i^2 = 1$ . To see this, we examine an even and odd powers of the sum of two spins

$$\begin{aligned} (S_i + S_j)^{2n} &= 2^{2n-1}(1 + S_i S_j), \\ (S_i + S_j)^{2n+1} &= 2^{2n} (S_i + S_j). \end{aligned}$$

Any even function can be written as

$$f_{\text{even}}\{K (S_i + S_j)\} = \sum_{n=0} \frac{a_n K^{2n} (S_i + S_j)^{2n}}{n!} = \sum_{n=0} \frac{a_n K^{2n} 2^{2n-1}}{n!} (1 + S_i S_j) = \frac{1}{2} f_{\text{even}}(2K) (1 + S_i S_j).$$

Similarly, for any odd function

$$f_{\text{odd}}\{K (S_i + S_j)\} = \sum_{n=0} \frac{b_n K^{2n+1} (S_i + S_j)^{2n+1}}{n!} = \sum_{n=0} \frac{b_n K^{2n+1} 2^{2n}}{n!} (S_i + S_j) = \frac{1}{2} f_{\text{odd}}(2K) (S_i + S_j).$$

Using these expressions, we get

$$\begin{aligned} \tilde{w}(i, j) &= \frac{1}{2} \ln \cosh(2K) S_i S_j + \frac{1}{2} h [1 + \tanh(2K)] (S_i + S_j) + \ln 2 + \frac{1}{2} \ln \cosh(2K) \\ &= K' S_i S_j + \frac{1}{2} h' (S_i + S_j), \end{aligned}$$

where the renormalized coupling constants  $K'$  and  $h'$  are

$$K' = \frac{1}{2} \ln \cosh(2K),$$

$$h' = h [1 + \tanh(2K)] + O(h^2),$$

and we dropped the spin-independent terms, as they do not contribute to any expectation values, etc.

### Thermodynamic behavior of the $d = 1$ Ising model

Now we are in a position to analyze the behavior of the RG equations. If we have a phase transition, it should exist only at  $h = 0$ , which we focus on. To look for a phase transition, we look for a fixed point

$$K^* = \frac{1}{2} \ln \cosh(2K^*).$$

Unfortunately, this equation has no "nontrivial" fixed point at finite  $K^*$ , since the function  $f(x) = \frac{1}{2} \ln \cosh(2x) \leq x$ , for any  $0 < x < \infty$ . In fact, precisely because of this condition, the coupling constant decreases under iteration, i.e.  $K \rightarrow K^* = 0$ . As we have seen from general arguments of Kadanoff, stable fixed points describe phases, while it is the unstable fixed point that represent the critical point where  $\xi \rightarrow \infty$ . In this  $d = 1$  example, we have one stable fixed point at  $K = 0$ , corresponding to  $T = \infty$ , i.e. the high-temperature (paramagnetic) phase. We have also found that this fixed point is reached no matter which initial (bare) value of  $K$  we chose. We conclude that no phase transition exists at any finite temperature, in agreement with what we already know for the  $d = 1$  Ising model.

But is there a critical point anywhere? Yes there is! We do find an unstable fixed point at  $K^* = +\infty$  (i.e. corresponding to  $T = 0$ ). We can now examine the RG flows in the vicinity of this  $T = 0$  fixed point, in order to determine the scaling of the correlation length in this regime. We concentrate on the RG equation for  $K$  in the regime of  $K \gg 1$ . Here

$$\begin{aligned} K' &= \frac{1}{2} \ln \{ \exp(2K) + \exp(-2K) \} - \frac{\ln 2}{2} \\ &= K + \frac{1}{2} \ln \{ 1 + \exp(-4K) \} - \frac{\ln 2}{2} \\ &\approx K - \frac{\ln 2}{2} + O(\exp\{-4K\}). \end{aligned}$$

If we start at a "bare" value  $K_o = J/T \gg 1$  (i.e.  $T \ll J$ ),  $K$  decreases under iterations, but it does that **very slowly**! How does  $K$  depend on the renormalized length scale after many

iterations? Well, after  $n$  iterations, the new scale is  $b = 2^n$ , or  $\ln b = n \ln 2$ . Under a single iteration, (assume  $n \gg 1$ ), the change of  $\ln b$  is

$$d(\ln b) = \ln 2.$$

And the change of the coupling constant is

$$dK = -\frac{\ln 2}{2}.$$

Using these results, the  $\beta$ -function is

$$\beta(K) = \frac{dK}{d \ln b} = -\frac{1}{2}.$$

Note that the  $\beta$ -function in this particular case is not linear in the coupling constant as at a standard (finite  $K^*$ ) fixed point. As a result, we do not obtain powerlaw, but rather exponential behavior, as follows. A similar situation is found, for example, at the Berezinskii-Kosterlitz-Thouless transition and its descendents (Caldeira-Legett problem, the Kondo problem), which we'll discuss by the end of the course. Quite generally, such behavior is often found at the **lower critical dimension**; the  $d = 1$  Ising model indeed falls into that category.

To obtain the explicit scale dependence  $K(b)$  of the coupling constant we integrate the above differential equation and find

$$K(b) = K_o - \frac{1}{2} \ln b.$$

But how can we calculate the correlation length? Well, we use the Kadanoff scaling expression (we focus at  $h = 0$ )

$$\xi(K) = b \xi(K(b)).$$

Under renormalization ( $b \gg 1$ )  $K(b) \rightarrow 0$ , which corresponds to  $T \rightarrow \infty$ , where  $\xi \approx 1$ . In this regime  $b \approx \exp\{2K_o\} = \exp(2J/T)$ , and we find

$$\xi \approx \exp\{2J/T\}.$$

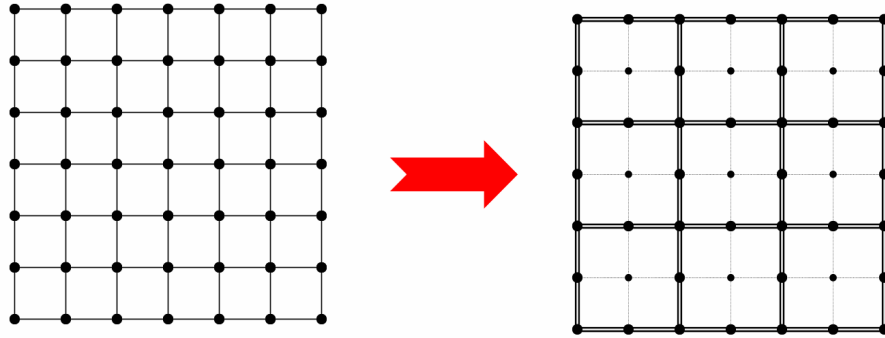
We recall that the same result was obtained from the domain wall argument, since the "activation" energy for creating two domain walls was  $\varepsilon = 2J$  (one cannot create a single domain wall; to create any droplet, one must create one domain wall and one "anti" domain wall).

Similar arguments can be used to calculate the full scaling expressions for all physical quantities (**Problem 3.2**).

### Critical behavior in $d = 2$ in the Migdal-Kadanoff approach

The extreme simplicity of the  $d = 1$  example resulted from the fact that each site had only two nearest neighbors, and that decimation over any one given spin (site) simply generated an effective interaction between second neighbors, but the new effective Hamiltonian retained the same form. In higher dimensions this is not longer true. Here, a simple calculation (**Problem 3.3**) shows that new multiple-spin interactions terms are generated, and the scale-invariance is ruined! Is the decimation approach now completely useless? Well...yes and no. In a rigorous sense it cannot be used, since the applicability of the RG ideas of Kadanoff require preserving the scale invariance.

Nevertheless, a clever approximate (if rather uncontrolled) RG transformation can be still used, which is known as the "potential-moving" method or the **Migdal-Kadanoff approximation**. Consider as an example an Ising model on a square lattice, where we examine a particularly simplistic scheme. The idea is that the energetics or the long-distance behavior will not be much affected if we (rather arbitrarily) "move" some of the bonds as shown in the figure. The particular scheme we use here is chosen for simplicity, as the



approximation is rather uncontrolled anyway. Now the spins at the center of the new unit cells (of size  $b = 2$ ) become disconnected from the rest, so they become irrelevant. After moving the bonds, each side of the new cells has bonds of strength  $2J$ . As we can see from the figure, the spins at the center of each cell side are now **connected only to two neighbors**, just as in the  $d = 1$  case we have just studied. We can immediately perform the decimation over these spins, and obtain the renormalized Hamiltonian precisely **of the same form** as before! We can immediately write the recursion relations

$$K' = \frac{1}{2} \ln \cosh(4K),$$

$$h' = h [1 + \tanh(4K)] + O(h^2).$$

These look almost exactly the same as in  $d = 1$ , except with one time but crucial difference: now  $K \rightarrow 2K$ . How much difference can this make? Well...a lot! To see this, let us look for a nontrivial fixed point at  $h = 0$ , which solves the equation

$$K^* = \frac{1}{2} \ln \cosh(4K^*).$$

The solution is now **completely different!** To see this, note that

$$\frac{1}{2} \ln \cosh(4K) \approx \begin{cases} 4K^2 \ll K, & K \ll 1 \\ 2K > K, & K \gg 1 \end{cases}.$$

Thus, the function  $K - \frac{1}{2} \ln \cosh(4K)$  must change sign at some finite  $K = K^*$ , i.e. **we have a finite temperature transition.**

We can be more specific. A numerical solution finds  $K^* \approx 0.30469$ , or  $(T_c/J)^{RG} = 1/K^* = 3.282$ . This can be compared to the mean-field prediction  $(T_c/J)^{MF} = 4$  (the coordination number for a square lattice is  $z = 4$ ), and the exact solution of Lars Onsager, giving  $(T_c/J)^{exact} = 2.2692$ . As we can see, our simplistic RG scheme provides an improvement over mean-field theory, but is still far from accurate. As we mentioned before, more accurate estimates for  $T_c$  can be obtained even from improved mean-field approaches. In contrast, no mean-field approach can produce critical exponents different from Landau theory, and this is where the advantage of the RG approach becomes significant.

To compute the critical exponents, we expand the recursion relation around the fixed point. Write  $K = K^* + t$ , and expand in  $t$  to get

$$t' = 1.6786t = b^{\lambda_t}t,$$

where we have used the fact that in our the present scheme  $b = 2$ , and

$$\lambda_t = \frac{\ln(1.678)}{\ln 2} = 0.74674.$$

The correlation length exponent  $\nu = 1/\lambda_t = 1.3392$ . This can be compared to the exact solution in  $d = 2$  (due to Onsager)  $\nu_{exact} = 1$ , and the mean-field result  $\nu_{mf} = 0.5$ . As we can see, even this very rough RG calculation is able to predict a value for the correlation length exponent which is quite different then the mean-field prediction, and within 30% of the exact value.

It is also interesting to calculate the other exponents. To do this, we need to compute  $\lambda_h$  from the recursion relation for  $h$

$$h' \approx h [1 + \tanh(4K)].$$

Since the fixed point is at  $h = 0$ , and we have already expanded the expressions to leading (linear) order in  $h$ , we can replace  $K$  in this expression by its fixed-point value, and write

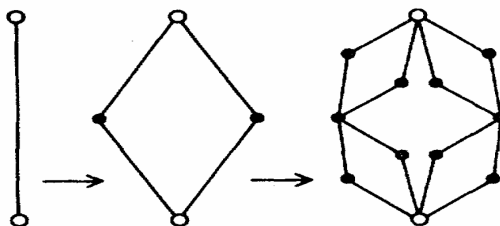
$$h' \approx h [1 + \tanh(4K^*)] = 1.8393 h = b^{\lambda_h} h,$$

where

$$\lambda_h = \frac{\ln(1.8393)}{\ln 2} = 0.87916.$$

We can now calculate all critical exponents following the Kadanoff scaling procedure described in the previous lecture (**Problem 3.3**).

What we did here is a particularly simple, if rather arbitrary, application of the Migdal-Kadanoff (MK) procedure. The method was exact in  $d = 1$ , but does it make sense in higher dimensions, or for that matter, in **any problem** where a finite temperature transition exists? And how about universality? One thing is clear: should we have chosen another bond-moving scheme (e.g. on another lattice), the precise values of the critical exponents would not be the same. This is, unfortunately, the price we paid for the simplicity of the MK approximation - it simply does not answer these questions. To really understand the origin of universality we'll have to explore a more controlled RG procedure, which we will explore it in the next few lectures.



We end by a more encouraging comment about the MK approach. While really not well justified on regular lattices, this method proves to actually be **exact** on a special class of **"hierarchical" lattices** that correspond to a fractal spatial dimension (see for example: R. B. Griffiths and M. Kaufman, Phys. Rev. B **26**, 5022 (1982) and references therein).