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Classification of critical phenomena in hierarchical small-world networks

S. Boettcher and C. T. Brunson

Department of Physics, Emory University - Atlanta, GA 30322, USA

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Abstract – A classification of critical behavior is provided in systems for which the renormalization group equations are control-parameter dependent. It describes phase transitions in networks with a recursive, hierarchical structure but appears to apply also to a wider class of systems, such as conformal field theories. Although these transitions generally do not exhibit universality, three distinct regimes of characteristic critical behavior can be discerned that combine an unusual mixture of finite- and infinite-order transitions. In the spirit of Landau's description of a phase transition, the problem can be reduced to the local analysis of a cubic recursion equation, here, for the renormalization group flow of some generalized coupling. Among other insights, this theory explains the often-noted prevalence of the so-called inverted Berezinskii-Kosterlitz-Thouless transitions in complex networks. As a demonstration, a one-parameter family of Ising models on hierarchical networks is considered.

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Introduction. – One of the most significant results of network science [1] is the realization that critical phenomena on complex networks behave differently from those found on a lattice geometry [2–4]. Before the advent of complex networks, random geometries were routinely modeled in terms of ordinary random graphs [5–7]. These are well understood and synonymous with the mean-field limit of ordinary lattices, often with little qualitative difference in their critical behavior [8]. Therefore, it came as a surprise that real-world networks would exhibit a dramatically distinct phenomenology, with a profound imprint of their geometry on the dynamics. What we now call complex networks, aside from being random, possess geometries dominated by small-world bonds and scale-free degree distributions [9,10]. These lead to novel, and often non-universal, scaling behaviors unknown for lattices, that have changed our appreciation, for example, of the risk of epidemics because scale-free networks possess a vanishing threshold for percolation [11,12]. In turn, the ability to conceive of *synthetic* phase transitions through the manipulation or *ab initio* design of network geometry is one of the promising targets for the emerging science of meta-materials [13,14]. In particular, the iterative structure of hierarchical networks may facilitate their realization in engineered devices to unlock and control their unconventional behaviors. Work on percolation [15–20], the Ising model [21–25], and the q-state Potts model [26–28] have shown that critical behavior, once thought to be exotic and model-specific [4], can be categorized with the renormalization group [26] for a large class of hierarchical networks with a hyperbolic structure.

The renormalization group (RG) [29,30] is a widely used method in statistical physics that is by now found in most textbooks [8,31,32]. It has allowed to categorize broad classes of equilibrium systems into enumerable sets of universality classes, each characterized by discrete features, such as their dimension and the symmetries adhered to by their Hamiltonians. Such universality is made possible through the property of *scaling* that is an inherent feature near critical points [33]. Scaling entails that system-specific details on the microscopic level become irrelevant, as the behavior over many orders in the range of the interactions becomes self-similar. In this framework, analogous behavior in a surprisingly wide set of phenomena, such as the condensation of fluids, spontaneous magnetization of materials, or the generation of particle masses in the early Universe, can be described with a few effective theories —a major intellectual accomplishment of modern physics [31].

Unlike the Euclidean arrangement of atoms in a lattice, agents in biological or social systems may exhibit complex

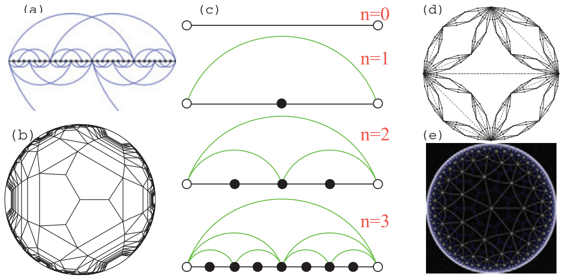

Fig. 1: (Colour on-line) Examples of hierarchical networks: (a) non-planar Hanoi network [15,37], (b) enhanced binary tree [17,25,38], (c) and (d) small-world versions of the Migdal-Kadanoff RG embedded in $d = 1$ [15,18] (shown for its first few recursions $n = 0, \ldots, 3$) and $d = 2$ [22,27], and (e) hyperbolic networks. After each recursion, small-world couplings access increasingly larger pools of variables, leading to "patchiness" [15].

networks of mutual interactions [3,4,9]. As the dependence on lattice dimensionality indicates, the study of critical phenomena is inseparable from the understanding of the geometry of the network [34]. It has been realized that many of the networks that are engineered by some natural or human activity themselves exhibit emergent complex properties, exemplified by the scale-free degree distribution of the internet. While these networks, and dynamical systems on them, may behave critical, those phenomena were soon found to be non-universal [10,35,36], *i.e.*, they are intimately tied to intricate details of the specific system. In this sense, it would seem unlikely that a sweeping classification could be devised. Here, we will categorize equilibrium phenomena observed on a large set of networks having hierarchical structure [17,22,25,35,37,38], as those in fig. 1. Our discussion pertains, for example, to the robustness of infinite-order transitions in distinct network models summarized in ref. [4], or in field theory, where it signals the loss of conformality [39]. However, it is most closely related to the recent observation of discontinuous ("explosive") transitions in ordinary percolation on hierarchical networks [18,20,28,40]. Our study shows that criticality in these models is generally non-universal but falls into three generic regimes. One of these regimes is an infinite-order transition reminiscent of that described by Berezinskii, Kosterlitz and Thouless (BKT) [32] but of very different origin. Reference [27] has provided a comprehensive scaling theory for this regimes. We find that it is flanked on one hand by a transition with a weaker, algebraic divergence, similar to a second-order transition (albeit non-universal), and, on the other, by a regime with an even stronger essential singularity, with percolation as a non-generic exception. Our approach also reveals the origin of the crossover between these regimes.

Renormalization of hierarchical networks. – To preface our discussion, consider RG for the probability κ_n of an end-to-end connection in fig. 1(c) [18]. Without the (arced) small-world bonds, recursively an infinite line is built up with $\kappa_{n+1} = \kappa_n^2$, entailing percolation ($\kappa_\infty = 1$) only for $\kappa_0 = 1$; any chance of missing a bond, *i.e.*, $\kappa_0 < 1$, loses the connection ($\kappa_\infty = 0$). If we now attribute a probability $p > 0$ to those arcs, then $\kappa_{n+1} = p + (1-p)\kappa_n^2$ and we must distinguish two possibilities:

1) If line and arc bonds vary independently, with $p \neq \kappa_0$ [16,41], then $\kappa_\infty = p/(1-p)$ for $0 \leq p \leq \frac{1}{2}$, while the unstable fixed point (FP) for $p < \frac{1}{2}$ at $\kappa_\infty = 1$ only becomes stable for $p > \frac{1}{2}$, both irrespective of κ_0. A non-trivial FP $\kappa_\infty(p)$ that is manipulated via an external parameter p but is attained *independent* of the control-parameter, *i.e.*, for any $\kappa_0 < 1$, is not uncommon [42], and can lead to interesting phenomena like the crossover between two interchanging FP [43].

2) If, however, all bonds, line and arc, are equivalent such that $\kappa_{n+1} = \kappa_0 + (1-\kappa_0)\,\kappa_n^2$ [15,18], then the FP $\kappa_\infty = \kappa_0/(1-\kappa_0)$ *explicitly* depends on the control-parameter κ_0. The consequences are dramatic: $\kappa_\infty(\kappa_0)$ becomes unphysical for $\kappa_0 > \frac{1}{2}$ where $\kappa_\infty = 1$ is now stable, a non-trivial critical point at $\kappa_0 = \frac{1}{2}$ ensues (that causes a *discontinuous* percolation transition [18]), and small-world bonds enforce a sub-extensive ("patchy") order even for $\kappa_0 < \frac{1}{2}$ [15].

Our classification pertains to the latter case, with control-parameter–dependent FP, $\kappa_\infty(\kappa_0)$. It conveniently applies to hierarchical networks on which RG is exact and transitions can be studied in detail. There, these regimes are characterized by the relative strength of small-world bonds [9]. A metric version of such networks, like the Migdal-Kadanoff RG [44] provides textbook examples for RG and universality [32]. But in the advent of complex networks, many hierarchical designs with non-metric (small-world or scale-free) properties, like those in fig. 1, have been devised and studied [17,22,25,35,37,38,45].

The central tenant of real-space RG consists of a procedure whereby the partition function of the original system is mapped recursively onto itself after tracing out a fraction $1 - 1/b$ of the dynamic variables, in some form of "blocking" together b variables. Prior couplings $\vec{\kappa}_n$ between them combine non-trivially to produce new, effective couplings $\vec{\kappa}_{n+1}$ between the remaining variables after the n-th RG-step while leaving the Hamiltonian form-invariant. This mapping constitutes the RG-flow

$$\vec{\kappa}_{n+1} = \mathcal{R}\,(\vec{\kappa}_n), \tag{1}$$

where $\mathcal{R}$ indicates a (typically non-linear) set of recursions. In the thermodynamic limit, $n \sim \log_b N \to \infty$, phase transitions are characterized purely by a local analysis for $\vec{\kappa}_n \sim \vec{\kappa}_{n+1} \sim \vec{\kappa}_\infty$ near FP of

$$\vec{\kappa}_\infty = \mathcal{R}\,(\vec{\kappa}_\infty)\,, \tag{2}$$

independent of $\vec{\kappa}_0$. Here, $\vec{\kappa}_0$ represents the "bare" (as of yet unrenormalized) couplings of the original system.

These carry the dependence on the system's control parameter $\mu \in [0,1]$. For example, in an Ising model it may refer to the temperature via the "activity" $\kappa_0 = \mu = e^{-\beta J}$ in units of $J = 1$, or in a percolation model it may refer to the bond percolation probability, $\kappa_0 = p = 1-\mu$. Since $\vec{\kappa}_0$ expresses microscopic details of a potentially large family of conceivable systems adhering to eq. (1), the insensitivity of $\vec{\kappa}_\infty$ on $\vec{\kappa}_0$ is an expression of universality: only certain symmetry properties of the original systems remain preserved by $\mathcal{R}$. Therefore, the linearized expansion, $\vec{\kappa}_n \sim \vec{\kappa}_\infty + \vec{\epsilon}_n$ with small $\vec{\epsilon}_n$ for large n, near the FP provides a full accounting of the macroscopically observable properties of any such system via the eigenvalue problem obtained from

$$\vec{\epsilon}_{n+1} = \frac{\partial \mathcal{R}}{\partial \vec{\kappa}}\left(\vec{\kappa}_\infty\right) \vec{\epsilon}_n. \tag{3}$$

The eigenvalues λ of the Jacobian $\frac{\partial \mathcal{R}}{\partial \vec{\kappa}}\left(\vec{\kappa}_\infty\right)$ and their eigenvectors $\vec{u}_\lambda$ provide the scaling exponents and scaling fields observed in the phase transition [8].

A much richer phenomenology arises when the RG-flow $\mathcal{R}$ itself becomes dependent on the control-parameter. In that case, eq. (1) generalizes to $\vec{\kappa}_{n+1} = \mathcal{R}\left(\vec{\kappa}_n; \mu\right)$ with

$$\vec{\kappa}_\infty = \mathcal{R}\left(\vec{\kappa}_\infty; \mu\right) \Longrightarrow \vec{\kappa}_\infty = \vec{\kappa}_\infty\left(\mu\right), \tag{4}$$

i.e., the FP $\vec{\kappa}_\infty$ becomes a non-trivial function of $\kappa_0 = \mu$. The consequences of such behavior (for a single coupling) are depicted in fig. 2. First, consider the case of constant FP shown in fig. 2(a). Drawing constant FP as a function of the control-parameter may seem redundant, however, it allows to illustrate the connection to the bare couplings $\kappa_0 = \mu$ (green dashed line). Below (above) the point where κ_0 intersects the unstable FP, the RG-flow evolves toward the stable FP at $\kappa_\infty = 0$ ($\kappa_\infty = 1$). Allowing for a non-linear choice of $\kappa_0\left(\mu\right)$ (like, $\kappa_0 = \mu^y$ for $y > 0$) reflects the universality in the family of systems obeying the same FP: no matter at which value of μ_c a system's bare coupling $\kappa_0\left(\mu\right)$ intersects the unstable FP, $\kappa_c = \kappa_0\left(\mu_c\right)$ is always the same, which guarantees identical (universal) critical behavior. This scenario also applies if the RG has parameters *independent* of μ [42,43]. Like the density of long-range bonds p in our introductory example, such a parameter merely shifts the horizontal line $\kappa_\infty(p)$ in fig. 2(a) up or down.

We claim that the remaining three panels in fig. 2 capture *all* generic features that can arise for μ-dependent FP. Variable FP $\kappa_\infty\left(\mu\right)$ can *collide*, either linearly or in a square-root branch point (BP); any other behavior (such as a higher-order BP) is exceptional. We can devise a simple theory[1] that reproduces these generic features. It thereby demonstrates the generality of this classification, not only accounting for hierarchical networks but for any physical system described by an RG-flow that explicitly depends on its control

[1]Our approach is similar in spirit to Landau's model of a phase transition found in many textbooks [8,31,32].

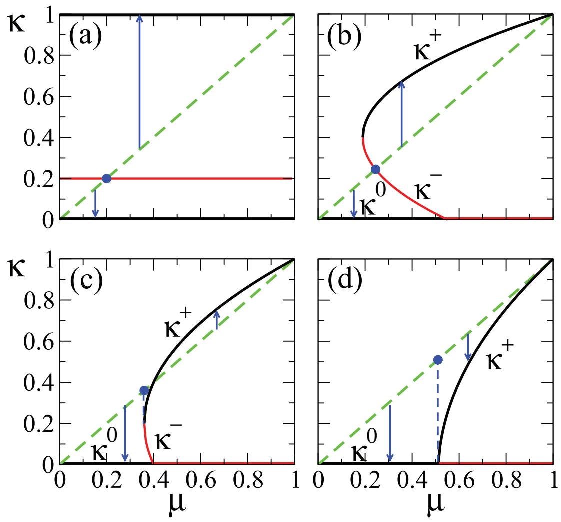

Fig. 2: (Colour on-line) Generic plots of the fixed points (FP) $\kappa_\infty^{0,\pm}$ as a function of μ (a) for conventional FP; (b), (c) for FP with a physical branch point (BP); and (d) for FP with BP outside the physical domain ($0 \le \mu, \kappa \le 1$). In (a), the μ-independence of FP ensures universal critical behavior wherever the bare couplings $\kappa_0\left(\mu\right)$ (green dashed line) intersects the unstable FP (red line); a blue dot and dashed line mark the critical point μ_c, and blue arrows indicate the RG-flow from $\kappa_0\left(\mu\right)$ toward the nearest stable FP (black lines). In (b), $\kappa_0\left(\mu\right)$ still intersects at an unstable FP *below* BP, such that the RG-flow does *not* pass BP, leading to quasi-conventional behavior but with μ_c-dependent critical exponents. In (c), $\kappa_0\left(\mu\right)$ is located above BP so that the RG-flow *must* pass near BP at μ_c, leading to BKT-like behavior. In (d), BP drops below the physical domain and only stable FP are accessible, resulting in an exponential divergence at the (marginally stable) intersection μ_c of two FP branches.

parameter [17,22,23,25,35,37–39,45]. For example, the networks in fig. 1 retain memory through ever longer non-renormalizing small-world bonds entering the flow at each level.

It is sufficient to consider the RG recursion for a single coupling κ_n with some control parameter μ. We argue that FP in a real, μ-dependent RG-flow $\mathcal{R}$ in eq. (4) will exhibit BP at some point (μ_B, κ_B). Near μ_B we express generically $\mathcal{R}\left(\kappa;\mu\right) \sim a\left(\mu\right)\kappa + b\left(\mu\right)\kappa^2 + c\left(\mu\right)\kappa^3$ because the need for a strong-coupling solution $\kappa_\infty^0 = 0$ prevents a constant term and requires at least a cubic form to achieve BP. With generically analytic coefficients at μ_B, we expect to leading order(s) $a\left(\mu\right) \sim a_0 + a_1\left(\mu - \mu_B\right)$, $b\left(\mu\right) \sim b_0$, $c\left(\mu\right) \sim c_0$ for $\mu \to \mu_B$. Locating BP at (μ_B, κ_B) fixes $a_0 = 1 + c_0\kappa_B^2$ and $b_0 = -2c_0\kappa_B$. To orient BP correctly requires $a_1/c_0 < 0$, and we set $a_1 = -c_0 A^2$ with $A > 0$. Finally, stability of the strong-coupling FP at κ_∞^0 demands $c_0 < 0$, and we may set $c_0 = -1$. This yields

$$\kappa_{n+1} - \kappa_n \sim \frac{\Delta\kappa}{\Delta n} \sim \left[-\kappa_B^2 + A^2\left(\mu - \mu_B\right)\right]\kappa_n + 2\kappa_B\kappa_n^2 - \kappa_n^3 \tag{5}$$

as a minimal model. After extracting $\kappa_\infty^0 \equiv 0$, the remaining FP equation indeed produces by design a BP at

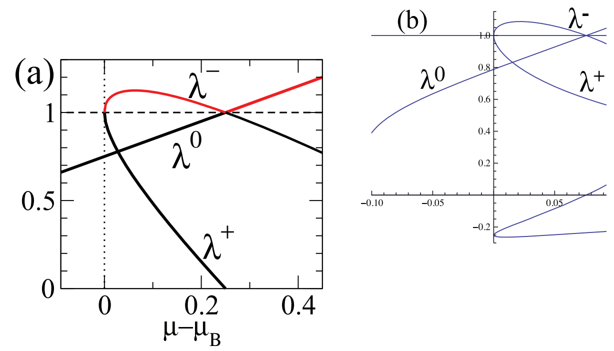

Fig. 3: (Colour on-line) Plot of the eigenvalues (a) in eqs. (7) for the RG-flow in eq. (5), here $A=1$, $\kappa_B=\frac{1}{2}$, and (b) for the RG-flow in ref. [23] for HN5 (at $y=0.1$). At BP ($\mu=\mu_B$), two conjugate eigenvalues emerge simultaneously with marginal stability, $\lambda^{\pm}(\mu_B)=1$, such that λ^- remains unstable (>1, marked in red) until the lower FP branch, κ_∞^-, drops below κ_∞^0, leading to an intersection of λ^0 and λ^- at some $\mu>\mu_B$. For physical $\kappa_B>0$, the critical point occurs at BP, $\mu_c=\mu_B$, while for unphysical $\kappa_B<0$, setting $\kappa_B\to-\kappa_B$ merely swaps $\lambda^\pm\to\lambda^\mp$ (see eq. (7)), and now $\lambda^0(\mu_c)=\lambda^+(\mu_c)=1$ intersect at $\mu_c>\mu_B$ with marginal stability.

(μ_B,κ_B) with FP branches

$$\kappa_\infty^\pm=\kappa_B\pm A\sqrt{\mu-\mu_B} \qquad (6)$$

for $\mu>\mu_B$. Local expansion near each FP as in eq. (3) provides the eigenvalues $\lambda(\mu)=\partial_\kappa\mathcal{R}(\kappa_\infty;\mu)$ depicted in fig. 3(a),

$$\begin{aligned}\lambda^0&=1-\kappa_B^2+A^2(\mu-\mu_B),\\ \lambda^\pm&=1\mp2A\kappa_B\sqrt{\mu-\mu_B}-2A^2(\mu-\mu_B).\end{aligned} \qquad (7)$$

Discussion of the RG-regimes. – In fig. 2, panels (b) and (c) correspond to cases where BP at (μ_B,κ_B) is in the physical domain ($0\le\mu,\kappa\le1$); panel (d) represents $\kappa_B\le0$. Within the domain of physical $\kappa_B>0$, the lower FP branch $\kappa_\infty^-(\mu)$ is unstable near μ_B ($\lambda^->1$, see fig. 3) while $\kappa_\infty^+(\mu)$ remains stable. Stable and unstable branches merge at BP, where particularly interesting phenomena arise. The decisive difference between panels (b) and (c) is the location of BP relative to the initial $\kappa_0(\mu)$.

For the case of panel (b) (*e.g.*, when long-range, hierarchical couplings are weakest [23]), μ_B is small and/or κ_B is closer to unity (or even above). Then, $\kappa_0(\mu)$ merely intersects the unstable branch $\kappa_\infty^-(\mu)$ at some critical point $\mu_c>\mu_B$ defined by $\kappa_\infty^-(\mu_c)=\kappa_0(\mu_c)$. The RG-flow (vertical blue arrows in fig. 2) for $0\le\mu<\mu_c$ advances toward strong coupling, κ_∞^0, while for $\mu_c<\mu\le1$ it flows toward $\kappa_\infty^+(\mu)$ (see footnote [2]). Near μ_c, the critical dynamics of the system is now determined by the local properties of the unstable FP $\kappa_\infty^-(\mu_c)$ that has been selected by the specific system via its bare coupling $\kappa_0(\mu)$. As for a conventional system in eq. (3), local analysis [8] of eq. (5) near $\kappa_\infty^-(\mu_c)$ yields the diverging correlation length,

$$\xi\sim|\mu-\mu_c|^{-\nu(\mu_c)},\qquad \mu\to\mu_c, \qquad (8)$$

but with a non-universal thermal exponent $y_t=\log_2\lambda^-(\mu_c)=1/\nu(\mu_c)$. For $\mu_c\searrow\mu_B$, λ^- becomes marginal and the exponent diverges as $\nu(\mu_c)\sim1/\sqrt{\mu_c-\mu_B}$. Yet, for $\mu_c>\mu_B$, the RG-flow *never* passes sufficiently near BP.

For the case of panel (c) where $\kappa_0(\mu)$ passes above BP (*e.g.*, for somewhat stronger long-range couplings [23]), the RG-flow *must* pass BP which now dominates criticality, *i.e.*, $\mu_c=\mu_B$, with an infinite-order divergence characterizing this regime. Well below (above) μ_B, the RG-flow evolves unperturbed to κ_∞^0 (to κ_∞^+), the closest stable FP. However, just below μ_B the RG-flow gets ever more impeded near BP before it can reach κ_∞^0. Asymptotically for $\mu\nearrow\mu_B$ near $\kappa_n\sim\kappa_B+\epsilon_n$ with small ϵ_n at large but intermediate n, eq. (5) provides

$$\epsilon_{n+1}-\epsilon_n=\frac{\Delta\epsilon_n}{\Delta n}\sim-\kappa_BA^2(\mu_B-\mu)-\kappa_B\epsilon_n^2. \qquad (9)$$

This relation exhibits a boundary layer, *i.e.*, in the limit $\mu\nearrow\mu_B$ the solution drastically changes behavior. With the methods of ref. [46], we rescale $\epsilon_n\to\gamma\epsilon_n$ and $n\to\delta n$ to obtain a balance for $\delta\sim1/\gamma\sim1/\sqrt{\mu_B-\mu}$. Accordingly, the characteristic width of the boundary layer scales with $n^*\sim1/\sqrt{\mu_B-\mu}$, which leads to the divergence in the correlation length characteristic of BKT,

$$\xi(\mu)\sim2^{n^*}\sim e^{\frac{\mathrm{const}}{\sqrt{\mu_B-\mu}}},\qquad \mu\nearrow\mu_B=\mu_c. \qquad (10)$$

Clearly, the physical origin of this singularity is not related to an actual BKT transition, with its formation of delicate topological structures, that requires a rare confluence of dimensionality and internal degrees of freedom for lattice models [32]. In fact, instead of being rare, it appears as one of three generic types of transition often found in hierarchical networks [4].

The most unconventional behavior is depicted in panel (d) of fig. 2, when $\kappa_B<0$ and BP has dropped below the physical regime, corresponding to the situation when long-range couplings dominate [23]). No unstable FP can be reached for any physical choice of $\kappa_0(\mu)$. The RG-flow always advances to the closest stable FP, either at strong coupling, κ_∞^0 for $0\le\mu<\mu_c$, or at patchy order, $\kappa_\infty^+(\mu)$ for $\mu_c<\mu\le1$. Both lines of FP cross at $\mu_c(>\mu_B)$, defined by the intersection $\kappa_\infty^+(\mu_c)=\kappa_\infty^0\equiv0$. This condition implies that both their eigenvalues are simultaneously equal *and* marginal, $\lambda^0(\mu_c)=\lambda^+(\mu_c)=1$, as κ_∞^0 must invert its stability at the intersection, making marginal stability inherent to *any* such system. In our model, $\kappa_\infty^+(\mu_c)=0$ in eq. (6) provides $-\kappa_B=A\sqrt{\mu_c-\mu_B}$, hence, eqs. (7) give $\lambda^{0,+}\sim1\pm A^2(\mu-\mu_c)$ for $\mu\to\mu_c$,

[2] Note that far away from μ_B, $\kappa_\infty^+(\mu)\to1$ only for some $\mu>\mu_c$, reflecting the physical phenomenon of "patchiness" [15,18]: hierarchical, long-range couplings enforce some semblance of order between otherwise uncorrelated (sub-extensive) patches of locally connected degrees of freedom even in the disordered regime; full disorder is often only reached at infinite temperature, dilution, etc. (*i.e.*, $\mu\to1$).

see fig. 3(a). Near $\kappa_\infty^{0,+}(\mu_c) = 0$, the local analysis on eq. (5) according to eq. (3) gives $\epsilon_{n+1} \sim \lambda^{0,+}\epsilon_n$ or $\epsilon_n \sim \epsilon_0 \exp\left(-nA^2 |\mu - \mu_c|\right)$ with a crossover at $n^* \sim 1/|\mu - \mu_c|$ that is generic when λ^0 and λ^+ intersect linearly. Thus, the divergence is

$$\xi(\mu) \sim 2^{n^*} = e^{\frac{\text{const}}{|\mu-\mu_c|}}, \qquad \mu \to \mu_c \tag{11}$$

for the correlation length. Again, the RG-flow does not pass BP, since it is located *below* the physical domain.

Behavior of the order parameter. – We can extend the discussion to include the effect of further control parameters, such as an external field $\eta_0 = \eta = e^{-\beta h}$. We find that its generic RG-flow can be expressed asymptotically as

$$\eta_{n+1} \sim \eta_n^{\lambda_h}, \qquad \lambda_h \sim 2 - C\kappa_\infty^+(\mu_c) \tag{12}$$

with some μ_c-dependent constant $C > 0$, near the critical point μ_c and for sufficiently small $\kappa_\infty^+(\mu_c)$. This satisfies the physical requirements on its FP, $\eta_\infty = 0, 1$; only for $h = 0$ the RG-flow remains at the unstable FP, *i.e.*, $\eta_n \equiv 1$ f. a. n, and for any $h > 0$ the stable strong-coupling FP at $\eta_\infty = 0$ is reached eventually. The eigenvalue near the unstable FP satisfies $\lambda_h \le 2$ such that the magnetic exponent becomes $y_h = \log_2 \lambda_h \le 1$, as shown in ref. [27]. There, a scaling theory is developed concerning the BKT regime based on the exponentially divergent correlation length in eq. (10), leading to an order-parameter (magnetization, fraction of sites on percolating cluster, etc.),

$$m \sim \xi^{-1} \sim \exp\left\{-\frac{\text{const}\,(1-y_h)}{(\mu_c-\mu)^{-x_t}}\right\}, \quad \mu \nearrow \mu_c, \tag{13}$$

when $y_t \to 0$. Of course, for $y_t > 0$ it is [8]

$$m \sim (\mu_c - \mu)^\beta, \qquad \beta = \frac{1-y_h}{y_t} \tag{14}$$

for small-world systems where N takes the role of L^d [27].

Our theory in eq. (5) not only explains the robustness of the value of $x_t = \frac{1}{2}$ conjectured in ref. [27], but also broadens the scope to a total of three generic regimes in the divergence of ξ, as we have explained. With the addition of eq. (12), we can account for the behavior of the order-parameter m. For the first two regimes where $\kappa_B > 0$, it is $\lambda_h < 2$ in eq. (12) so that $1 - y_h > 0$. In the regime with the weakest distortion of the FP, see fig. 2(b), we have shown in eq. (8) that $y_t = 1/\nu(\mu_c) > 0$, *i.e.*, it is $0 < \beta(\mu_c) < \infty$, similarly to an ordinary 2nd-order transition, except for its non-universal μ_c-dependence. In the BKT-regime, see fig. 2(c), it is $\mu_c = \mu_B$ and $\lambda^+(\mu_c) = 1$ in eq. (7) such that $y_t = 0$ and $\beta \to \infty$, which leads to eq. (13) described in ref. [27]. Finally, when $\kappa_B < 0$, see fig. 2(d), it is $\mu_c > \mu_B$ and $\kappa_\infty^+(\mu_c) = 0$. Then, $\lambda^+(\mu_c) \to 1$ such that $y_t = O(\mu_c - \mu)$ leads to the divergent correlation length in eq. (11); however, it is $\lambda_h \to 2$ in eq. (12) such that $y_t = 1 - o(\mu_c - \mu)$. In the most generic

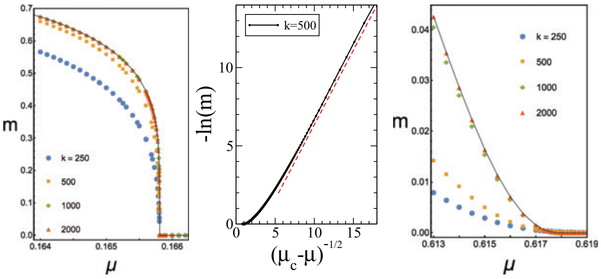

Fig. 4: (Colour on-line) Plot of the magnetization in the Ising model on the hierarchical network HN5 from ref. [23] with balance $y = K/L$ between nearest-neighbor and long-range couplings, K and L, (a) for $y = 0.1$, (b) for $y = 0.4$, and (c) for $y = 1$, which in sequence correspond to regimes (b) to (d) in figs. 2. The transition is (a) 2nd-order continuous with $\beta = 0.205\ldots$, (b) BKT-like, and (c) again continuous with $\beta = \left(3+\sqrt{5}\right)/4 = 1.30\ldots$. Strong finite-size effects ($N \sim 2^k < \infty$) remain throughout.

(analytic) case, we would expect that both, y_t and y_h, have linear corrections so that β in eq. (14) remains positive for $\mu \to \mu_c$ and the transition is continuous. This is indeed the observed phenomenology, for instance, for the one-parameter family of Ising models [47], first studied in ref. [23], that interpolates between $\kappa_B > 0$ and $\kappa_B < 0$. Surprisingly, percolation models on these hierarchical networks appear to provide quite common exceptions to this behavior [18,20,28,40], with $\beta = 0$, resulting in a remarkable *discontinuous* ("explosive") percolation transition [48]. In ref. [28], it was argued that such non-generic behavior, in form of merely a 2nd-order correction in y_h throughout these models, originates with the interplay of tree-like (hyperbolic) features superimposed on a geometric (1d-lattice) structure common to those networks.

As a demonstration for our theory, in ref. [47] we revisit the Ising model on the Hanoi network HN5 previously considered for $h = 0$ in ref. [23]. There, a one-parameter family of Ising models was conceived via the ratio between short-range and small-world coupling strengths that interpolates between all three regimes; fig. 11 in ref. [23] corresponds to fig. 2(b)–(d) here. That system is far more complex than our model here in that there are two couplings and three fields (when $h > 0$, for site-, bond-, and three-point magnetizations) to be renormalized. Yet, the same three regimes in the divergence of the correlation length ξ and the magnetization m ensue, as our theory predicts. Here, we only plot the magnetization of the Ising model on a hierarchical network, which follows 2nd-order behavior, eq. (14), in the first regime in fig. 4(a), it has an infinite-order transition, eq. (13), in the BKT regime in fig. 4(b), and it becomes again continuous in the regime of intersecting stable FP in fig. 4(c).

Conclusions. – We have introduced a simple RG-model to categorize the regimes of synthetic critical behaviors in hierarchical networks. The robustness of these regimes derives from the fact that branch points in control-parameter–dependent RG-flows are most generically a square-root singularity. Our theory specifically addresses the question [27] about the universality of the BKT result in eq. (10). The full exponential singularity in eq. (11) is even more robust, as it does not depend on the nature of the branch point singularity but merely on the fact that two intersecting lines of fixed points must switch stability. This *implies* marginally stable eigenvalues at the point of intersection. Those eigenvalues invariably scale linearly with the control parameter there. For the future, it would be interesting to explore our model prediction directly for hierarchical networks drawn from some ensemble, instead of using exactly renormalizable instance.

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