Antiferromagnetic Ising Model in Hierarchical Networks

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Introduction

The antiferromagnetic (AFM) Ising model is a convenient, yet powerful model of glassy dynamics. This model can introduce geometric frustrations which may cause spin glass phases and glassy relaxation at low T [1, 2].

We apply the AFM Ising model to 4 hierarchical networks (HNs) which share features of both *small-world networks* and *regular lattices*. By studying them, we try to gain insights to the following questions:

- Are there phase transitions and/or spin glass transitions?
- At low temperatures, is there an slow glassy relaxation?
- How do different structures affect the transitions?

The model and HNs allow us to use both computational methods and a theoretical method to explore these questions (see **Methods** section).

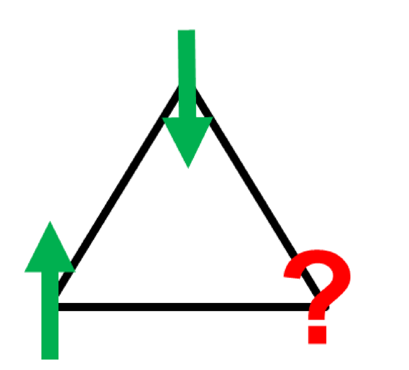
Model & Hierarchical Netowrks

Antiferromagnetic Ising model:

The Hamiltonian is
$$H = -\sum_{\{i,j\}} J\, S_i S_j$$
where $\{i,j\}$ are neighbors; the spin $S_i = \pm 1$; and the interaction $J < 0$.

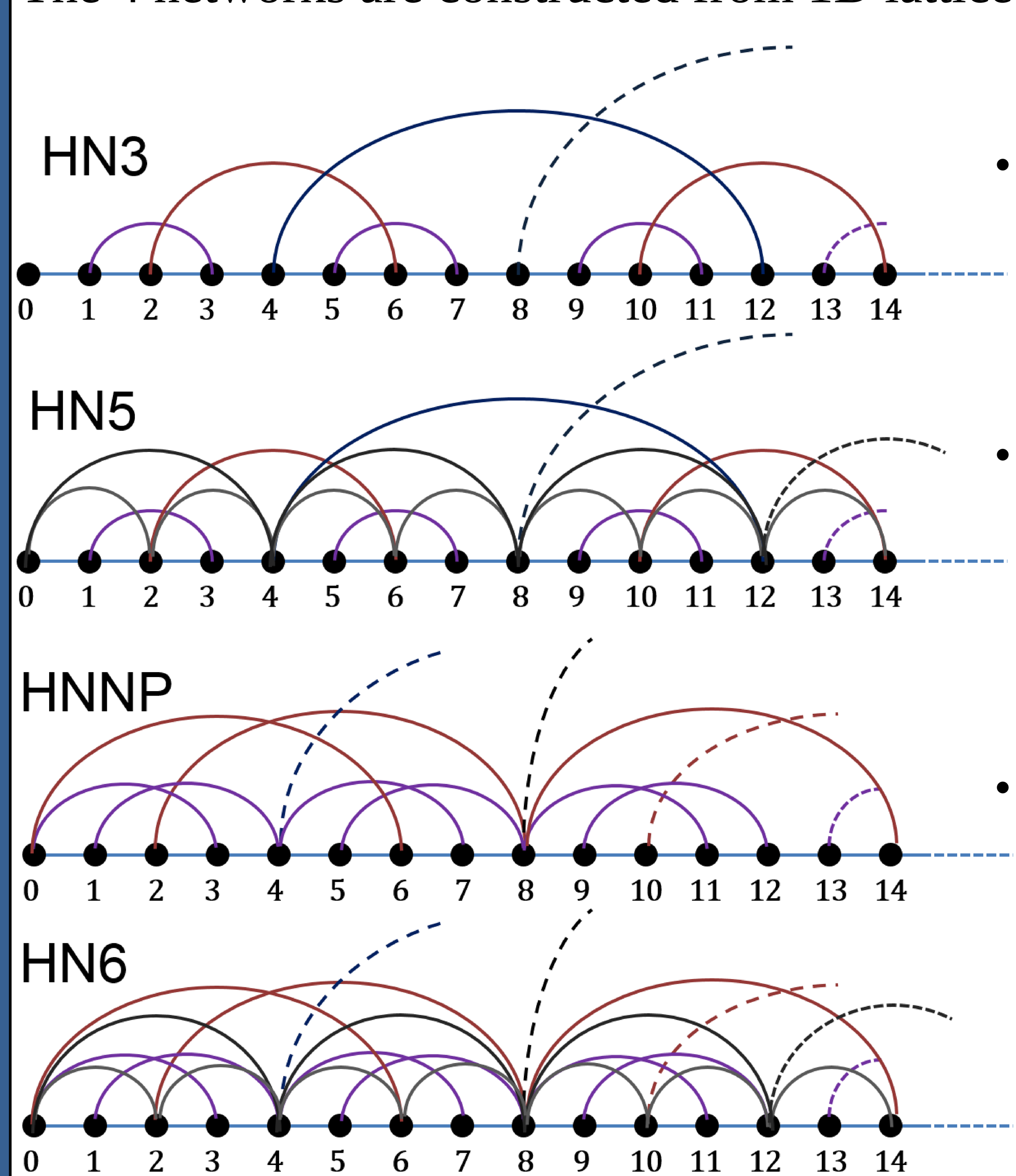
Frustration in AFM model:

At low T, physical systems stay near the ground states. Geometric frustrations can introduce degenerate ground states and thereafter interesting phase transitions.



Hierarchical Networks [3]

The 4 HNs are constructed from a 1D lattice. Each site n $(0 < n < N)$ can be uniquely parameterized by two integers (i, j)
$$n(i,j) = 2^{i-1}(2j+1)$$
where i denotes the level in the hierarchy, and j labels consecutive sites. The 4 networks are constructed from 1D lattice in the following ways.



- *HN3*: degree 3, planar
  for each level i, sites $j = 1$ are connected to $j = 3$; then, 5 to 7; $\cdots$.
- *HN5*: degree 5, planar
  besides links in HN3, sites n of level $i \geq 2$ are connected to sites $n \pm 2^{i-1}$.
- *HNNP*: degree 5, non-planar
  for each level i, sites n with odd (even) j are connected to sites $n + 3 \times 2^{i-1}$ (sties $n - 3 \times 2^{i-1}$).
- *HN6*: average degree 6, non-planar
  besides in links in HNNP, sites n of level $i \geq 2$ are connected to sites $n \pm 2^{i-1}$.

Reasons of using HNs:

1. The HNs can be solved using renormalization group (RG), which shows us the equilibrium properties in the thermodynamic limit;
2. The HNs share more features with regular lattices than mean-field models, which may contribute more insights to real-world systems.
3. They have different structures. Specifically, they have different average degrees; HN3 and HN5 are planar while HNNP and HN6 are not.

Methods

Simulated Annealing (SA) can be considered as a cooling experiment to learn the dynamical behavior of the model. The simulational procedure is:
1. randomly pick a spin; 2. flip it with $P = \min[1, \exp(-\Delta E/T)]$; 3. update $T \leftarrow (T - dT)$ every sweep (N random steps); 4. Stop until $T = 0$.

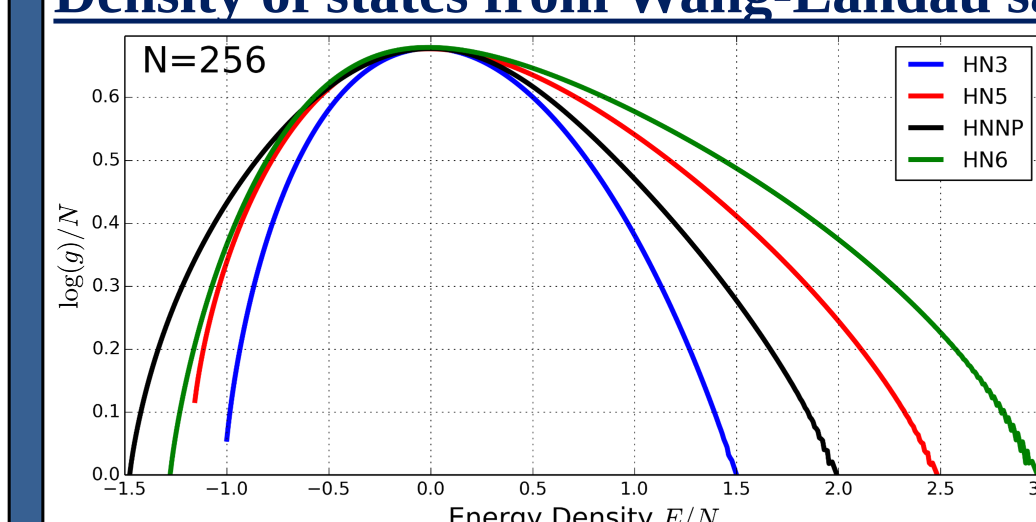
Wang-Landau (WL) sampling has been shown as an efficient method [4] to find density of states (DOS) in the Ising model. The procedure is:
1. Set all DOS $g(E) = 1$ and $H(E) = 0$ (histogram); 2. randomly pick a spin; 3. flip the spin with $P = \min\left[1, \exp\left(g(E)/g(E_{\text{new}})\right)\right]$; 4. When state E is visited: $g(E) \leftarrow g(E) * f$ and $H_i \leftarrow (H_i + 1)$; 5. update $f \leftarrow \sqrt{f}$ if H is flat, and repeat steps 2, 3, and 4; 6. Stop until $f < 1 + 10^{-8}$ (initially, $f = e$).

Renormalization Group (RG) can be applied to these recursively built HNs to find the equilibrium solutions in the thermodynamic limit. We can

1. analyze its fixed point solutions to uncover possible phase transitions.
2. derive the equilibrium properties, such as magnetization, susceptibility, etc.

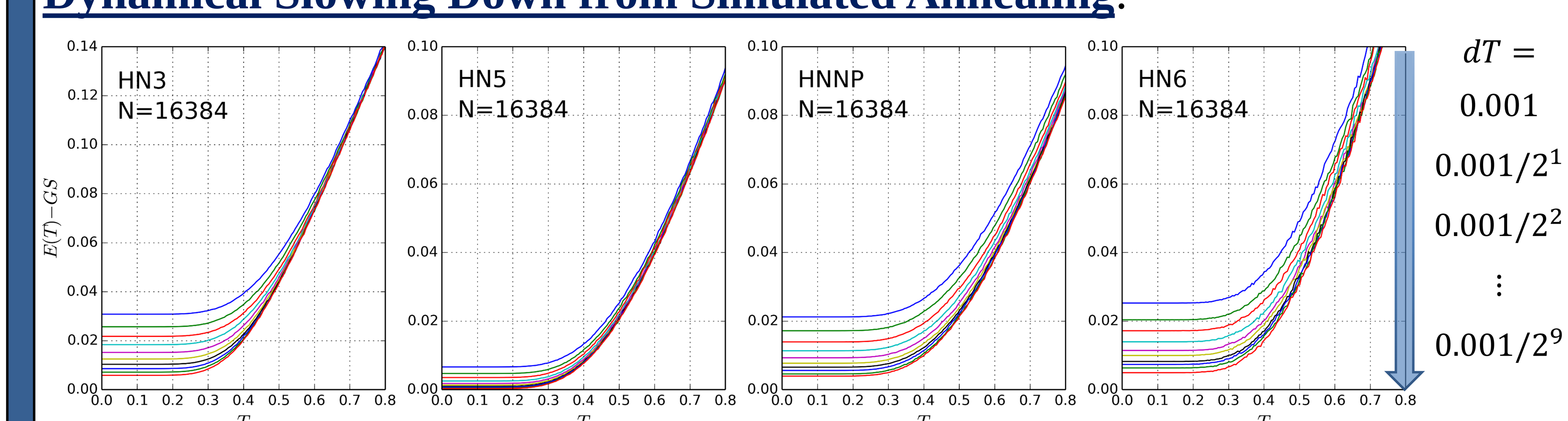
Results

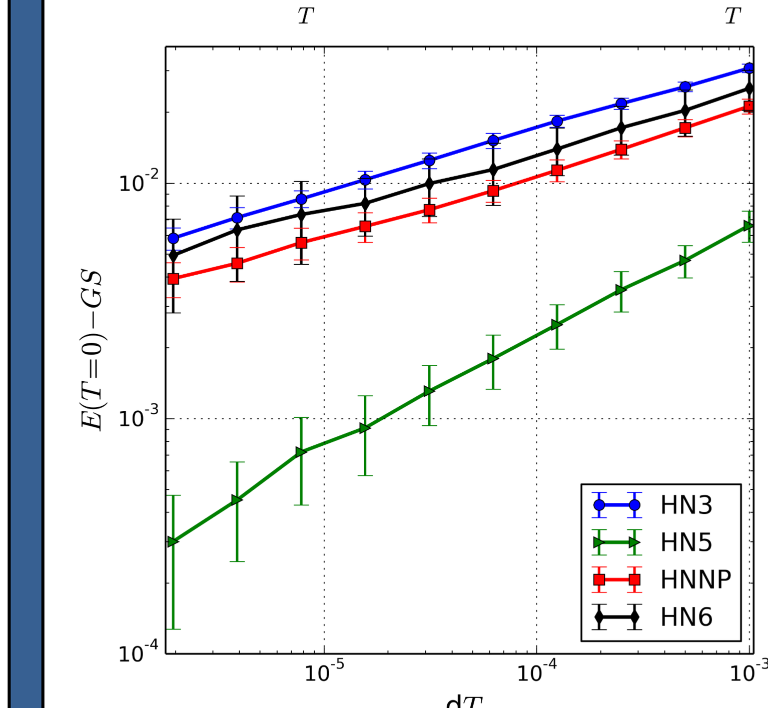
Density of states from Wang-Landau sampling:



- HN3 & HN5: many ground states (GS), HNNP&HN6: unique GS;
- Due to geometric frustrations, WL cannot converge for $N > 1024$;
- No phase transitions are detected in the equilibrium solutions from WL.

Dynamical Slowing Down from Simulated Annealing:



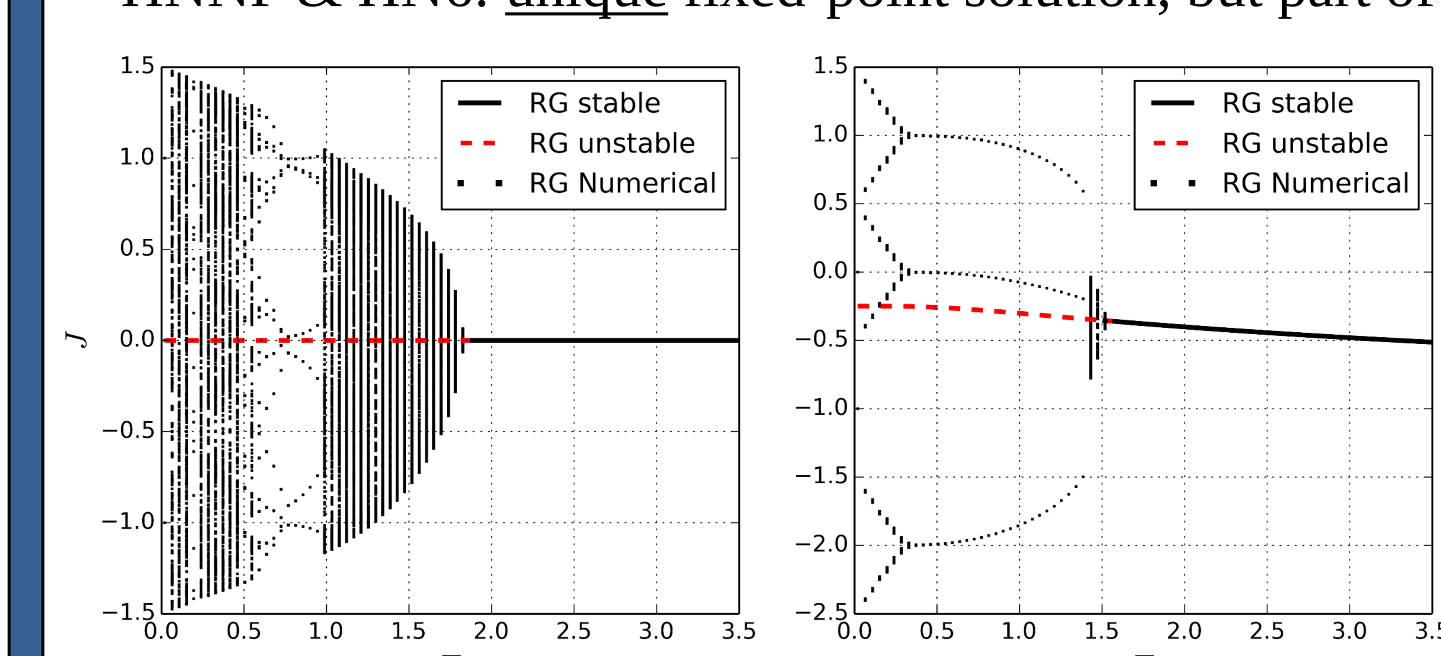


- $E(T)$: energy from SA at temperature T;
- GS: ground state energy from WL and RG;
- *HN3, HNNP, HN6*: slow relaxation at low T; HN5: may equilibrate gradually;
- *HN3, HNNP, HN6*: the energy difference $(E - GS)$ at low T follows power law with respect to annealing schedules dT. The slopes are ~ 0.27

Spin glass transition found by Renormalization Group:

The parameter of interest in RG is the *renormalized interaction strength* $\boldsymbol{J}$ whose behavior may indicate possible phase transitions. In RG flows, J has:

- HN3 & HN5: unique and stable fixed-point solutions → no phase transition.
- HNNP & HN6: unique fixed-point solution, but part of it is not stable.



- *black square-dots* numerical solutions in steps 1000 ~ 2000;
- *black solid curves* stable solution
- *red dashed curves* unstable solution

The unstable fixed-point solutions may indicate a spin glass transition.
We will calculate more parameters, such as susceptibility, to understand this better.

References
[1] C.P. Herrero, *Phys. Rev. E.* **459**, 230 (2008)
[2] Cheng and Boettcher, arXiv:1409.8313 (2014)
[3] Boettcher and Brunson, *Phys. Rev. E* **83**, 021103 (2011)
[4] Wang and Landau, *Phys. Rev. Lett.* **86**, 10 (2001)

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