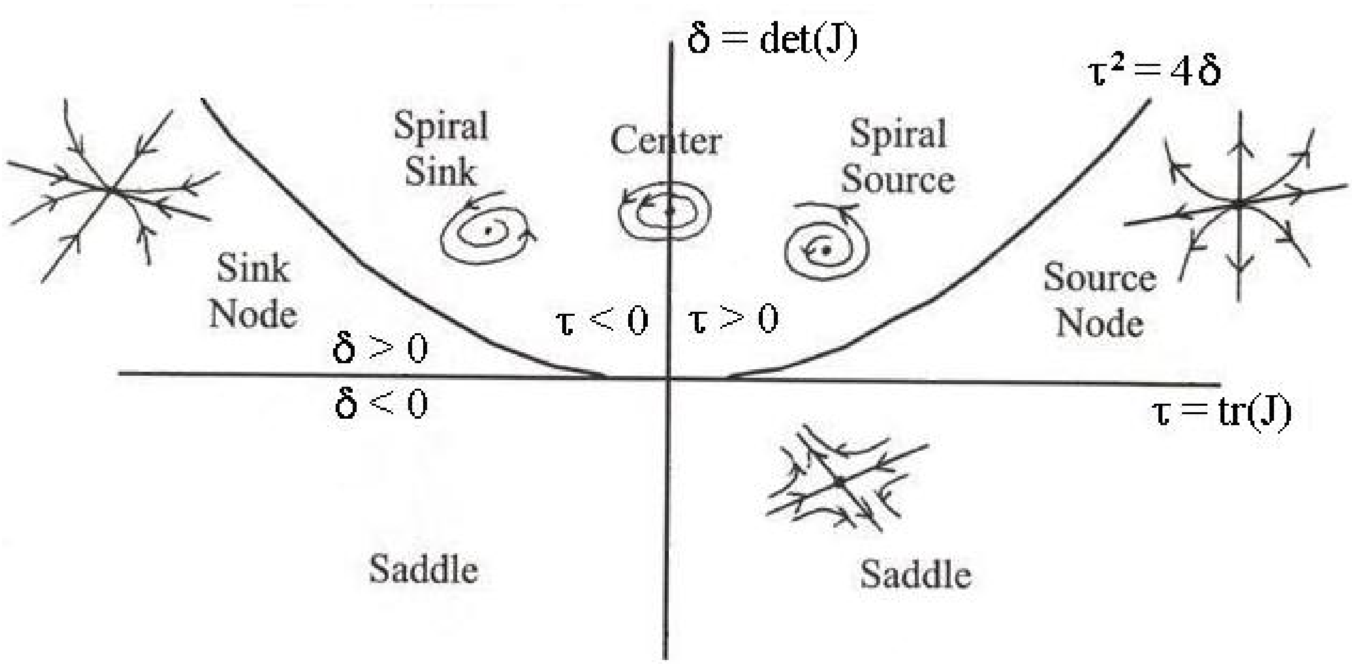
Classification of Fixed Points of Two-Dimensional Continuous Systems

This page summarizes much of the information in Section 3.4 (on 2-dimensional linear and non-linear autonomous systems of differential equations) in the textbook. The figure below (similar to figure 3.10 in the text) indicates the type (of 6 possible types) and stability of a fixed point in two dimensions, based on the values of `tr(J)` and `det(J)`, where `J` is the Jacobian matrix evaluated at the fixed point.

Note that for a 2-dimensional linear system, like those considered in Section 3.4.1 with `dX/dt=AX` where `A` is a matrix, the origin is a fixed point and its Jacobian matrix `J` is simply the matrix `A`.



The following theorem (which combines Theorems 3.3 and 3.6 in the text) verbally summarizes the information in the figure above and relates it to the eigenvalues of the Jacobian matrix associated with a fixed point of a 2-dimensional continuous dynamical system.

Theorem: Let $\tau = tr(J)$ and $\delta = det(J)$, where J is the Jacobian matrix evaluated at a fixed point of the two-dimensional dynamical system, $x' = f(x,y)$ and $y' = g(x,y)$, of two autonomous differential equations. Then the eigenvalues of J, λ_1 and λ_2, satisfy the quadratic equation $\lambda^2-\tau\lambda+\delta=0$, i.e. $\lambda_1=(\tau+\sqrt{(\tau^2-4\delta)})/2$ and $\lambda_2=(\tau-\sqrt{(\tau^2-4\delta)})/2$, and the fixed point is locally asymptotically stable if $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$ or, equivalently, if $tr(J) < 0$ and $det(J) > 0$. Furthermore,

a. If $\delta < 0$ (so the eigenvalues are real, one positive and one negative) then the fixed point is a *saddle*.

b. If $\delta > 0$, $\tau < 0$, and $\tau^2 - 4\delta >= 0$ (so the eigenvalues are real and negative) then the fixed point is a *sink node*.

c. If $\delta > 0$, $\tau > 0$, and $\tau^2 - 4\delta >= 0$ (so the eigenvalues are real and positive) then the fixed point is a *source node*.

d. If $\delta > 0$ and $\tau = 0$ (so the eigenvalues are complex with zero real parts) then the fixed point is a *center*.

e. If $\tau < 0$ and $\tau^2 - 4\delta < 0$ (so the eigenvalues are complex with negative real parts) then the fixed point is a *spiral sink*.

f. If $\tau > 0$ and $\tau^2 - 4\delta < 0$ (so the eigenvalues are complex with positive real parts) then the fixed point is a *spiral source*.