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Mathematics and Physics of Disordered Systems

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Introduction by the Organisers

It was the aim of this workshop to bring together researchers from different fields, working on various aspects of the theory of disordered systems in theoretical and mathematical physics. On the one hand, this means triggering an interaction between researchers working on these phenomena using methods from theoretical physics and those working with mathematical tools on the subject, as well as scientists mainly interested in the mathematical methods themselves. On the other hand, the conference also tried to bring together researchers working on related but well distinguished physical phenomena like Anderson localization in random systems, the theory of aperiodic order, the random matrix theory its applications (e.g., in quantum chaos).

To do justice to this mix of people with different background, about half the talks were survey talks (with an extended time for the speaker), giving an introduction to the corresponding topic in general as well as to recent results. We found it important to give enough time for physicists to explain their ideas in this way. About a similar number of mathematical and physical review talks were given. It was a nice experience that there was always a very lively and intense discussion after the talks and in many cases even during the talks. Many of these discussions were continued outside the lecture hall in the free time in the afternoons and the evenings. In this way, many interactions started during the week, in particular between people who would otherwise not have met.

An important topic in the meeting was the theory of “quasiperiodic” order in various forms. Consequently, a survey talk on quasicrystals was scheduled. It triggered interesting discussions with various other disciplines. One highlight was the demonstration that and how Delone sets and Schrödinger operators defined on them form a natural bridge between the world of perfect (periodic) order and that of stochastic phenomena. Though this is still in its infancy, it became clear that a high potential for unified approaches is still to be unraveled.

One of the main topics of the conference was the theory of Anderson localization for random Schrödinger operators from different points of view. There were reviews by theoretical physicists on the subject explaining basic ideas about Anderson localization, its applications to the quantum Hall effect and the supersymmetric approach to this field. There was also special emphasis on the phenomenon of “weak localization” both from physicists and mathematicians.

Another interesting topic was the explanation of the Aizenman-Molchanov method to prove Anderson localization in the continuum. This extension of the method is rather new and is considered to be very important for future developments in the area. In the field of Anderson localization/delocalization, there were also presentations of new developments. There was also a survey talk on threshold phenomena for the random Landau Hamiltonian which is not only an interesting result by itself, but may also lead to progress toward a proof for the existence of delocalized states.

A number of talks were devoted to the theory of quantum Hall conductance. Both the physical theory and recent mathematical progress were explained. Presumably this topic encouraged the highest amount of discussion among the various disciplines. The phenomena are not yet understood from a physical point of view, even the model itself is under discussion.

The theory of random matrices is a discipline that evolved rather independently of the theory of random Schrödinger operators although there are obvious intersections between these fields. It was therefore an important task of the conference to bring together people from these fields. As a result of this attempt there are joint research papers in preparation triggered by the meeting, e.g., on Hamiltonians on random graphs which can be interpreted both as examples of random matrices and as discrete Schrödinger operators.

The use of poster presentations and the official allocation of one evening session to this proved successful in that it sparked lots of discussions that went on during the meeting. It became evident that some results can actually profit from such a presentation. In view of the fact that time for talks is limited, it might be a reasonable alternative, long used by other sciences, and some more wall space might help.

The unique atmosphere of the institute did its magic once again, and the combination with the excellent library makes Oberwolfach still one of the best possible places for meetings such as this one, also in view of the recent competition from places like BIRS in Canada.

Workshop on Mathematics and Physics of Disordered Systems

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Abstracts

Universality of Random Matrix Distributions

Thomas Kriecherbauer

In recent years interest in random matrix theory (RMT) has grown rapidly due to the appearance of random matrix distribution functions in various fields as diverse as analytic number theory, combinatorics or quantum chaos.

In this talk we review some of the basic results of RMT, focussing on spectral correlations and on the distribution of the largest eigenvalues of unitarily invariant ensembles of Hermitean matrices. (see e.g. [7], [8], [20], [22], [25] [26], see also [24] for results on Wigner ensembles which are not unitarily invariant).

We then explain the result of Baik, Deift, Johansson [4], where it was shown that the length of the longest increasing subsequence of a random permutation, appropriately rescaled, has the same distribution as the largest eigenvalue of GUE (Gaussian Unitary Ensemble). Following the work of [4], random matrix distributions have been discovered in the asymptotic description of a number of combinatorial models arising e.g. in random words, vicious walks, tilings, random growth, first and last passage percolation, see e.g. [2], [3], [5] [9], [10], [11], [12], [13], [14], [15], [16], [21], [26].

One way to show the universal occurrence of RMT distribution functions is the following. The combinatorial models mentioned above can be analyzed using a determinantal random point field (see [23]). Such a random point field is described by a family of probability measures $\mathbb{P}_n$ on n real numbers $x_1, \ldots, x_n$, where the density of $\mathbb{P}_n$ can be written in determinantal form

$$c_n \det\left[(K_n(x_i, x_j))_{1\le i,j\le n}\right],$$

for an appropriate function K_n. In RMT it has long been known (see [20] and references therein) that the joint distribution of eigenvalues of a large class of unitarily invariant ensembles of Hermitean matrices can also be written in determinantal form. It then remains to prove that the appropriately rescaled functions K_n converge to some universal kernel K. One way to see this is to show that K_n can be expressed in terms of orthogonal polynomials with respect to some weight (see [13], [12] and references therein). Using a Riemann-Hilbert formulation for orthogonal polynomials it has been established for many classes of weights, that the corresponding appropriately rescaled orthogonal polynomials of large degree display quite universal behavior (see e.g. [1], [6], [7], [8], [17], [18], [19]).

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Edge Currents for Quantum Hall Geometries

P. D. Hislop

(joint work with J. M. Combes, E. Soccorsi)

Devices exhibiting the integer quantum Hall effect can be modeled by one-electron Schrödinger operators describing the planar motion of an electron in a perpendicular, constant magnetic field, and under the influence of an electrostatic potential. The electron motion is confined to bounded or unbounded subsets of the plane by confining potential barriers. The edges of the confining potential barrier create edge currents. We study the existence of edge currents for a variety of geometries involving one-, and two-edges in the papers [3, 4, 5]. The one-edge geometries describe the electron confined to certain unbounded regions obtained by deforming half-plane regions. The two-edge geometries describe the electron confined to a strip region or to a bounded region such as a finite cylinder. We prove the existence of states carrying edge currents, and provide explicit lower bounds on the edge currents that are proportional to the square root of the magnetic field strength. The states that carry edge currents are spectrally localized in energies to intervals between the Landau energies. These states are also localized to a region of width on the order of the cyclotron radius near the edge. The confining potentials are classified into two types: Hard confining potentials, such as formed by a step function of large amplitude or Dirichlet boundary conditions, and soft confining potentials, such as given by strictly increasing, polynomially growing potentials. Both types of confining potentials lead to the formulation of edge currents. We prove that the currents are stable under various potential perturbations, provided they are suitably small relative to the magnetic field strength, including perturbations by random potentials. For these cases of one-edge geometries, the existence of, and the estimates on, the edge currents imply that the corresponding Hamiltonian has intervals of absolutely continuous spectrum between the Landau levels. The existence of edge currents, however, does not have any implication for the spectral type of the operator, as illustrated by two-edge geometries. Edge-currents exist for the two-edge, cylinder geometry even though the Hamiltonian has purely discrete spectrum.

There are several papers directly related to this work [6, 8, 9, 10, 11, 12, 14]. References to mathematical papers describing the relationship between edge currents and the integer quantum Hall effect can be found in [1, 2, 7, 13]. References to the physical literature can also be found in these papers.

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Introduction into Localisation

D.E. Khmelnitskii

An attempt to present at an intuitive level several very basic concepts related to Anderson Localisation and Integer Quantum Hall Effect.

Effects of weak disorder are discussed using a semi-classical picture of self-crossing classical random walks and associated with them quantum-mechanical interference. An estimate to the quantum correction to classical conductivity is obtained. Its dependence on space dimension is discussed.

A scaling conjecture is discussed and related to it Renormalization Group. Basic consequences from this conjecture are derived. Three Dysonian ensembles and physical condition of their implementation.

Loclaisation of a quantum particle in 2D in presence of strong perpendicular magnetic field gives rise to the Integer Quantum Hall Effect. A percolation based picture of this localisation is presented and the two-parameter scaling conjecture is discussed. It is shown that the two-parameter scaling in equivalent to the Quantum Hall phenomenology.

At the very end, the picture of levitation of extended states is presented.

Transforms of Measures

M. Krishna
(joint work with A. Jensen)

We look a the Wavelet transform of probability measures on $\mathbb{R}$ and recover the components of the measures in a Lebesgue decomposition from this transform. Our motivation for this work is that in the spectral theory of selfadjoint operators essentially only the resolvent operator and the unitary group generated by the operators are widely used and there are lots of interesting operators where the spectral theoretic questions are open where it seems hard to use these functions. Therefore there was a need to find out if there are other functions of the selfadjoint operators that could be used to study spectra and such a question translates the question we address in this work.

In the following ψ is a (complex valued) bounded continuous even function with $\psi(0) \neq 0$, that satisfies the decay condition

$$|\psi(x)| + |x\psi'(x)| \leq C(1+|x|)^{-\delta}, \ \delta > 1$$

for some $\delta > 1$. Given such a ψ, we define $\psi_a(x) = \psi(\frac{x}{a}), \ a > 0$ and let $\psi_a * \mu$ to be $\int \psi_a(\cdot - y)d\mu(y)$.

Then the theorems we prove are the following for which we normalize $\int \psi(y)dy = 1$.

Theorem 1. *Let ψ be as above and let μ be a probability measure on $\mathbb{R}$. Then*

(1) $\lim_{a\to 0} \psi_a * \mu(x) = \psi(0)\mu(\{x\})$.

(2) *For every continuous function f of compact support, the following is valid.*

$$\lim_{a\to 0} \int \left(\frac{1}{a}\psi_a * \mu\right)(x)f(x)dx = \int f(x)d\mu(x).$$

(3) *Let $d_\mu^\alpha(x) = \lim_{\epsilon\to 0} \frac{\mu((x-\epsilon,x+\epsilon))}{(2\epsilon)^\alpha}$ be finite, for some $0 < \alpha \leq 1$ and x, then*

$$\lim_{a\to 0} a^{-\alpha}\psi_a * \mu(x) = c_\alpha d_\mu^\alpha(x),$$

where $c_\alpha = \int_0^\infty \alpha 2^\alpha y^{\alpha-1}\psi(y) \ dy$.

The next theorem is the anologue of the theorems of Simon [4] for the case of Borel transforms.

Theorem 2. *Let μ be a probability measure on $\mathbb{R}$ and ψ be as above. Then for any bounded interval (c, d) the following are valid.*

(1) *Let*

$$C = \int_{\mathbb{R}} |\psi(x)|^2 \ dx.$$

then

$$\lim_{a\to 0} \frac{1}{a} \int_c^d |\psi_a * \mu|^2(x) \ dx = C \left(\sum_{x\in(c,d)} \mu(\{x\})^2 + \frac{1}{2}\left[\mu(\{c\})^2 + \mu(\{d\})^2\right] \right).$$

(2) *Suppose that for some $p > 1$,*

$$\sup_{a>0} \int_c^d \left| \frac{1}{a}(\psi_a * \mu)(x) \right|^p \, dx < \infty,$$

then μ is purely absolutely continuous in (c, d). In addition, for any compact subset S of (c, d)

$$\frac{1}{a}\psi_a * \mu \to \frac{d\mu_{ac}}{dx}, \text{ in } L^p(S), \text{ as } a \to 0.$$

The converse that if μ is purely absolutely continuous with the density $\frac{d\mu_{ac}}{dx}$ in $L^p((c, d))$, then the supremum above is finite, is also valid.

(3) *For $0 < p < 1$, we have*

$$\lim_{a\to 0} \int_c^d \left| \frac{1}{a}\psi_a * \mu \right|^p (x) \, dx = \int_c^d \left| \frac{d\mu_{ac}}{dx} \right|^p \, dx.$$

And finally for the quantities

$$C_\mu^\alpha(x) = \limsup_{a\to 0} \frac{\psi_a * \mu}{a^\alpha}(x). \; D_\mu^\alpha(x) = \limsup_{\epsilon\to 0} \frac{\mu((x-\epsilon, x+\epsilon))}{(2\epsilon)^\alpha},$$

one has the following theorem.

Theorem 3. *Let μ be a probability measure and let ψ be as above. Then $C_\mu^\alpha(x)$ is finite for any x, whenever $D_\mu^\alpha(x)$ is finite for the same x and if ψ is positive then they are both finite or both infinite.*

There are lots of functions ψ satisfying the conditions we imposed in the above theorems. Therefore, there is hope that the above theorems will enlarge the useful techniques available to the spectral theory of random and deterministic Schrödinger operators.

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Pure point diffraction and measure dynamical systems

Daniel Lenz
(joint work with Michael Baake)

Since the discovery of quasicrystals twenty years ago [10] the phenomenon of pure point diffraction has attracted a lot of attention.

In a simplified manner, diffraction can be modelled as follows: The positions of the atoms of a solid are described by a set $\Lambda \subset \mathbb{R}^d$, which fills the space in not too irregular a way. To Λ, one associates the *autocorrelation measure*

$$\gamma := \gamma_\Lambda := \lim_{n\to\infty} \frac{1}{|B_n|} \sum_{x,y\in\Lambda\cap B_n} \delta_{x-y}.$$

Here, δ_z denotes the unit point measure at $z \in \mathbb{R}^d$, $B_n \subset \mathbb{R}^d$ is the open ball around the origin with radius n and $|\cdot|$ denotes Lebesgue measure. By definition, γ contains information on the differences of the positions appearing in Λ. Thus, it is a key quantity in the study of interference. More precisely, the outcome of a diffraction experiment is given by the Fourier transform $\widehat{\gamma}$ of γ. This Fourier transform is called *diffraction measure*. It describes the intensity of the scattered beam measured on a screen.

If $\widehat{\gamma}$ is a pure point measure, we speak about *pure point diffraction*. Note that this is only possible if "a lot" of interference occurs, i.e., if the positions of Λ are correlated on long range. Put differently, pure point diffraction means a high degree of order in the solid.

Indeed, the fascinating phenomenon behind quasicrystals is exactly that nature produces substances which exhibit such long range order without actually being periodic.

There are three systematic (though not always exhaustive) methods to prove pure point diffraction. Each is based on a characterisation of pure pointedness of $\widehat{\gamma}$. These are the characterisations by

- a certain almost periodicity of γ [2],
- a certain almost periodicity of Λ [4, 7],
- pure point dynamical spectrum.

Here, we will be concerned with the last characterisation. In this case, one considers not only a single Λ but rather the set Ω of all subsets of $\mathbb{R}^d$ with the same kind of (dis)order as Λ. This set is invariant unter the translation operations α_x, $x \in \mathbb{R}^d$, and constitutes a dynamical system (Ω, α, m), where m is an α-invariant probability measure on Ω. This dynamical system gives rise to a unitary representation

$$T : \mathbb{R}^d \longrightarrow \text{Unitary operators on } L^2(\Omega, m), \;\; (T_x f)(\omega) = f(\alpha_{-x}\omega).$$

If $L^2(\Omega, m)$ possesses an orthonormal basis of eigenvectors of T (i.e., functions $f \neq 0$ with $(T_x f) = \exp(i\lambda x) f$ for all $x \in \mathbb{R}^d$), then (Ω, α, m) is said to have *pure point dynamical spectrum.*

The relationship between pure point dynamical and pure point diffraction spectrum has been investigated by various people in the last 20 years (see [1] for further references).

Starting with the work of Dworkin [3], it has been shown in increasing levels of generality that pure point dynamical spectrum implies pure point diffraction [5, 9].

In a special one-dimensional situation (and a somewhat different context), equivalence of these two notions was shown by Queffélec [8]. Lee, Moody and Solomyak could then show equivalence in arbitrary dimensions provided the elements $\omega \in \Omega$ satisfied some regularity assumptions [6]. For rather general point sets, equivalence has been shown recently by Gouéré [4]. He also provides a closed formula for γ in terms of the underlying dynamical system. His work relies on a connection to stochastic processes and Palm measures.

Now, both from the physical and the mathematical point of view, the restriction to point sets (i.e., measures of the form $\delta_\Lambda := \sum_{x\in\Lambda} \delta_x$), is rather restrictive and somewhat arbitrary. In fact, weighted Dirac combs $\sum_{x\in\Lambda} a_x\delta_x$ and density distributions of the form $\sum_{x\in\Lambda} f(\cdot - x)$ have also been considered in the past.

This calls for a unified general framework to diffraction theory. Such a framework may be found quite naturally within the context of translation bounded measures.

The aim of our work [1] can be summarised as follows:

- To provide a unified framework to diffraction theory in the context of translation bounded measures;
- To give a closed formula for the autocorrelation measure in this context;
- To prove the equivalence of pure point diffraction and pure point dynamical spectrum in this context.

Details are given in [1]. Here, we would just like to mention that the generalisation from point sets to measures requires some care as essentially all information of geometric type is lost. Roughly speaking, geometric considerations have to be replaced by functional analytic ones.

We will illustrate this by discussing the two key steps behind our proof of the equivalence of pure point dynamical and pure point diffraction spectrum:

The first step is to realize that γ is actually a spectral measure for a subrepresentation $T|_{\mathcal{U}}$ of T. This shows that pure point diffraction is equivalent to pure point spectrum of this subrepresentation.

The next step is then to realize that this subrepresentation is "large" in the sense that it having pure point spectrum forces pure point spectrum of the whole representation. This is achieved by an application of the Stone-Weierstraß Theorem.

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Spectral properties of the Laplacian on bond-percolation graphs in $\mathbb{Z}^d$

Peter Müller
(joint work with Werner Kirsch)

Spectral graph theory [14, 7, 5, 6] has attracted vivid interest in the last two decades. Here, we shall be concerned with bond-percolation graphs in $\mathbb{Z}^d$, a special type of random graphs. Such graphs are obtained by randomly removing edges from the regular hyper-cubic lattice $\mathbb{L}^d$ in d dimensions [8]. Each edge is removed independently with the same probability $0 < 1 - p < 1$. The resulting graphs are of popular use in Physics for modelling various kinds of random environments [17].

We denote by Δ the combinatorial or graph Laplacian [6] associated with a bond-percolation graph. It is a bounded, self-adjoint and non-negative random operator with off-diagonal disorder acting on the Hilbert space $\ell^2(\mathbb{Z}^d)$ of square-summable, complex-valued sequences that are indexed by $\mathbb{Z}^d$. Moreover, Δ is ergodic with respect to $\mathbb{Z}^d$-translations. Finally, we remark that given any subgraph of $\mathbb{L}^d$, its combinatorial Laplacian may be viewed as the discrete analogue of the Neumann Laplacian of a corresponding subset of d-dimensional Euclidean space $\mathbb{R}^d$. In particular, Δ is super-additive with respect to the generation of Neumann boundary surfaces.

Spectral properties of Δ have been studied in the Physics literature [2, 1, 9, 15], but have remained widely unexplored from a rigorous mathematical point of view. Closely related investigations in the physics literature concern the adjacency operator on bond-percolation graphs [10, 18]. Quite often, they go under the name "quantum percolation." In contrast, the definition of corresponding operators on site-percolation graphs in $\mathbb{Z}^d$ amounts to situations with diagonal disorder. Part of the spectral properties of the latter models have also been explored with mathematical rigour [4, 19, 20].

In what follows we state some simple spectral properties of the Laplacian Δ on bond-percolation graphs [13]. Due to ergodicity, one can follow the usual procedure

[12, 11, 3, 16] to establish the non-randomness of the spectrum of Δ, which is almost surely equal to the interval $[0, 4d]$. Moreover, the spectral subsets, which arise from the Lebesgue decomposition of the spectral measure of Δ, are almost-surely non-random, too. Clearly, the whole spectrum of Δ is almost surely pure point in the non-percolating phase, where the probability for the occurrence of an infinite percolating cluster is zero.

Definition. The *integrated density of states* $N : \mathbb{R} \to [0,1]$, $N \mapsto N(E)$ of the Laplacian Δ is given by

$$N(E) := \mathbb{E}\big[\langle \delta_0, \Theta(E-\Delta)\delta_0\rangle\big] .$$

Here, $\mathbb{E}$ denotes the probabilistic expectation over all bond-percolation graphs, Θ stands for the right-continuous Heaviside unit-step function, and the standard scalar product $\langle \cdot, \cdot \rangle$ on $\ell^2(\mathbb{Z}^d)$ is taken with respect to the unit vector $\delta_0 \in \ell^2(\mathbb{Z}^d)$ which is concentrated at the origin of $\mathbb{Z}^d$.

Remarks. (i) The integrated density of states N is the right-continuous distribution function of a probability measure on $\mathbb{R}$. Its set of growth points coincides with the almost-sure spectrum $[0, 4d]$ of Δ.

(ii) Thanks to ergodicity, the non-random quantity N may be obtained almost surely as the infinite-volume limit $\Lambda \uparrow \mathbb{Z}^d$ of $|\Lambda|^{-1}\mathrm{Trace}\,\Theta(E-\Delta_\Lambda)$, where Δ_Λ denotes a restriction of Δ to some cube $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$ vertices, and the trace extends over $\ell^2(\Lambda)$.

(iii) There is a discontinuity of N at $E = 0$ whose jump is given by the average number of clusters and isolated vertices per volume.

The main result is summarised in the following

Theorem ([13])**.** *Consider the non-percolating phase. Then the integrated density of states N of the Laplacian Δ exhibits a Lifshits tail*

$$\lim_{E\downarrow 0} \frac{\ln\big|\ln[N(E)-N(0)]\big|}{\ln E} = -\frac{1}{2}$$

at the lower spectral boundary.

Remarks. (i) Apparently, the Lifshits exponent $-1/2$ does not depend upon the spatial dimension d. This is due to the fact that the lowest eigenvalues of Δ stem from *linear* clusters.

(ii) The theorem is proven by constructing an upper and a lower bound on $N(E)-N(0)$, which coincide asymptotically as E tends to zero. The upper bound invokes Cheeger's inequality [6, Thm. 3.1(2)] in order to estimate the smallest non-zero eigenvalue of an arbitrarily shaped cluster in terms of the smallest non-zero eigenvalue of a linear cluster with the same number of vertices. The lower bound simply omits all contributions to $N(E)-N(0)$ that stem from clusters other than linear. Here the crucial observation is that the probabilistic occurrence of linear clusters with n vertices decreases exponentially with n, as is also the case for arbitrarily shaped clusters.

Extensions of the present work concern the Lifshits-tail behaviour of the integrated density of states at the upper spectral boundary $E = 4d$. We conjecture this Lifshits exponent to be given by $-d/2$, because it should be determined by the most “compact” clusters such as cubes.

Since all clusters are almost surely finite in the non-percolating phase, boundary conditions of the Laplacian certainly matter. Another extension therefore deals with Lifshits tails for Dirichlet Laplacians on percolation graphs. Work along these lines is in progress.

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Recent applications of Kotani theory

David Damanik

Kotani theory, [3], connects the absolutely continuous spectrum of an ergodic family of one-dimensional Schrödinger operators with the set of energies for which the Lyapunov exponent vanishes.

Let Ω be a compact metric space and let $T : \Omega \to \Omega$ be a homeomorphism. Let $d\mu$ be a probability measure on Ω with respect to which T is ergodic. Given a bounded measurable function $f : \Omega \to \mathbb{R}$ we associate a potential to each $\omega \in \Omega$ by

$$V_\omega(n) = f\big(T^n(\omega)\big) \quad \text{for all } n \in \mathbb{Z}.$$

The corresponding Schrödinger operator is denoted by H_ω:

$$[H_\omega \phi](n) = \phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n).$$

For each $E \in \mathbb{R}$, we associate the Schrödinger cocycle

$$A_E(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}.$$

The Lyapunov exponent, $\gamma(E)$, obeys

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \log \|A_E(T^{n-1}(\omega)) \cdots A_E(\omega)\| \text{ for } \mu - a.e.\ \omega \in \Omega.$$

The essential closure of the set $N(f) = \{E : \gamma(E) = 0\}$ is equal to $\sigma_{\mathrm{ac}}(H_\omega)$ for μ-a.e. $\omega \in \Omega$. To show that the absolutely continuous spectrum is almost surely empty, it therefore suffices to show that $M(f) = |N(f)| = 0$. Here, $|\cdot|$ denotes Lebesgue measure.

Absence of absolutely continuous spectrum is expected to be a consequence of weak regularity properties of f. It was shown by Kotani, [4], that $M(f) = 0$ whenever f takes on only finitely many values (and the potentials are not periodic). Moreover, it was conjectured by Aubry, [1], and Mandel'shtam and Zhitomirskaya, [5], that $M(f)$ for all discontinuous f. This conjecture was recently settled, up to very degenerate cases, in [2]. The key idea of the proof is to use a discontinuity point of f to show that the potentials V_ω are non-deterministic, that is, they are not determined by their values on a half-line. It follows from Kotani theory, [3, 4], that $M(f) = 0$ whenever the potentials are non-deterministic.

Work in progress with Artur Avila shows that $M(f) = 0$ is a generic property in $C(\Omega)$. That is, for a residual set of f's in $C(\Omega)$, $M(f) = 0$. The key ingredient in the proof is the observation that the map $f \mapsto M(f)$ is upper semi-continuous in some appropriate sense.

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Supersymmetry for localization problems

Konstantin Efetov

A supersymmetry method developed for a description of disordered systems is reviewed. It is demonstrated how one can represent physical quantities in terms of integrals over supervectors, which allows to average over disorder in the beginning of all calculations. A supermatrix sigma-model is derived and it is demonstrated how one can calculate with it. Some solved problems are listed.

The metal insulator transport transition for random Landau Hamiltonians

Abel Klein
(joint work with François Germinet and Jeffrey Schenker)

We prove the existence of nontrivial transport near each Landau level for random Landau Hamiltonians.

Consider the random Landau Hamiltonian

$$H_{B,\lambda,\omega} = H_B + \lambda V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x), \tag{1}$$

where

$$H_B = (-i\nabla - \mathbf{A})^2 \ \text{ with } \ \mathbf{A} = \frac{B}{2}(-x_2, x_1) \tag{2}$$

($\mathbf{A}$ is the vector potential and $B > 0$ is the strength of the magnetic field), $\lambda > 0$, and the random potential V_ω is of the form

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^2} \omega_i \, u(x-i)\,, \tag{3}$$

with $u(x) \geq 0$ a bounded measurable function with compact support, strictly positive on some open set, and $\omega = \{\omega_i;\ i \in \mathbb{Z}^d\}$ a family of independent, identically distributed random variables taking values in a bounded interval $[-M_1, M_2]$ ($0 \leq M_1, M_2 < \infty$, $M_1 + M_2 > 0$). We set $\| \sum_{i\in\mathbb{Z}^2} u(x-i)\|_\infty = 1$ without loss of generality; note $-M_1 \leq V_\omega(x) \leq M_2$.

$H_{B,\lambda,\omega}$ is a $\mathbb{Z}^2$-ergodic random self-adjoint operator on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x)$; $\Sigma_{B,\lambda}$ will denote its almost sure spectrum. Recall that the spectrum of the free Landau

Hamiltonian H_B consists of a sequence of infinitely degenerate eigenvalues, the Landau levels:

$$B_n = (2n+1)B\,, \quad n = 0,1,2,\ldots\,. \tag{4}$$

It follows that

$$\Sigma_{B,\lambda} \subset \bigcup_{n=0}^{\infty} \mathcal{B}_n\,, \quad \text{where} \quad \mathcal{B}_n = \mathcal{B}_n(B,\lambda) = [B_n - \lambda M_1, B_n + \lambda M_2]\,. \tag{5}$$

Typically the requirements for a multiscale analysis as in [GK1] are satisfied everywhere for random Landau Hamiltonians: there are finite volume operators satisfying a Wegner estimate everywhere [HLMW, CHK, CHKR]. *We assume we are in this situation.*

The (random) moment of order $n \geq 0$ at time t for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in the cube of side one around the origin, and "localized" in energy by the function $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, is given by (with $\langle x\rangle = \sqrt{1+|x|^2}$, $\|\ \|_2$ the Hilbert-Schmidt norm)

$$M_{B,\lambda,\omega}(n,\mathcal{X},t) = \left\| \langle x\rangle^{\frac{n}{2}} \mathrm{e}^{-itH_{B,\lambda,\omega}} \mathcal{X}(H_{B,\lambda,\omega})\chi_0 \right\|_2^2\,, \tag{6}$$

and its time averaged expectation by

$$\mathcal{M}_{B,\lambda}(n,\mathcal{X},T) = \frac{2}{T}\int_0^\infty \mathrm{e}^{-\frac{2t}{T}} \mathbb{E}\left\{M_{B,\lambda,\omega}(n,\mathcal{X},t)\right\}\,\mathrm{d}t. \tag{7}$$

The random Landau Hamiltonian $H_{B,\lambda,\omega}$ exhibits strong HS-dynamical localization in the open interval I if for all $\mathcal{X} \in C^\infty_{c,+}(I)$ we have

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}} M_{B,\lambda,\omega}(n,\mathcal{X},t)\right\} < \infty \quad \text{for all } n \geq 0\,, \tag{8}$$

We recall some results in [GK3]: The strong insulator region Σ_{SI}, defined as the set of energies $E \in \Sigma_{B,\lambda}$ such that $H_{B,\lambda,\omega}$ exhibits strong HS-dynamical localization in some open interval $I \ni E$, is shown to be equal to the set of energies $E \in \Sigma_{B,\lambda}$ with an open interval $I \ni E$ such that

$$\sup_{T\geq 0} \mathcal{M}_{B,\lambda}(n,\mathcal{X},T) < \infty \quad \text{for all } n \geq 0 \text{ and } \mathcal{X} \in C^\infty_{c,+}(I). \tag{9}$$

The weak metallic transport region is defined as $\Sigma_{\mathrm{WMT}} = \Sigma_{B,\lambda}\backslash\Sigma_{\mathrm{SI}}$. It is proven that if $E \in \Sigma_{\mathrm{WMT}}$ then there is nontrivial transport for wave packets with energies around E, more precisely,

$$\lim_{T\to\infty} \frac{1}{T^\alpha}\mathcal{M}_{B,\lambda}(n,\mathcal{X},T) = \infty \tag{10}$$

for all $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \ni E$, $\alpha \geq 0$, and $n > 4\alpha + 22$. (See also [GK4].)

If $\lambda < \frac{2B}{M_1+M_2}$ the bands $\mathcal{B}_n$ are disjoint, and we typically have localization at the edge of the bands [CH, W, GK2]. More precisely, a multiscale analysis can be performed at the edge of the bands, which yields pure point spectrum with exponentially decaying eigenfunctions, strong HS-dynamical localization, and more (see [GK1]). In particular, the band edges are in Σ_{SI}. (It is proved in [GK3] that Σ_{SI} is exactly the set of energies in the spectrum where we can perform a multiscale analysis.) By showing nontrivial transport near each Landau level we prove that there is a metal insulator transport transition (see [GK3]) in each band $\mathcal{B}_n$.

Theorem. *Let $H_{B,\lambda,\omega}$ be a random Landau Hamiltonian as in* (1) *satisfying a satisfying a Wegner estimate everywhere. If $\lambda < \frac{2B}{M_1+M_2}$ we have*

$$\Sigma_{\mathrm{WMT}} \cap \mathcal{B}_n \neq \emptyset \quad \textit{for all} \quad n = 0, 1, 2, \ldots \tag{11}$$

The fact that some sort of delocalization occurs in quantum Hall models at an energy near each the Landau goes back to Halperin [Ha]. Kunz [Ku] argued that the “localization length” diverges near each Landau level, which was shown by Bellissard, van Elst and Schulz-Baldes [BES] in a discrete setting.

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KdV-flow and Floquet exponent

Shinichi Kotani

The KdV equation is

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x},$$

and this describes the dynamics of shallow waters. As is well known, n-soliton solutions for the KdV equation are given by

$$\begin{cases} u(t,x) = -2\dfrac{\partial^2}{\partial x^2}\log\det(I+A(t,x)), \\ \text{where } A(t,x) = \left(\dfrac{\sqrt{m_i m_j}}{\eta_i+\eta_j}e^{-(\eta_i+\eta_j)x+4\left(\eta_i^3+\eta_j^3\right)t}\right)_{1\le i,j\le n} \text{ with } m_i,\eta_i>0. \end{cases}$$

For each fixed $t\in\mathbf{R}$, $u(t,\cdot)$ is a reflectionless potential which appears in one–dimensional scattering theory. Marchenko[1] considered the compact uniform closure of reflectionless potentials, which we denote by $\Omega\left([-\lambda_0,0]\right)$. We assume here that the Schrödinger operator $L(u)$ with potential $u\in\Omega\left([-\lambda_0,0]\right)$ has its spectrum in $[-\lambda_0,\infty)$. He attempted to solve the KdV equation starting from an element of this closure. However, he had to impose a solvability condition on an integral equation, which made it impossible to solve the KdV equation in its full generality. On the other hand, M.Sato and Y.Sato(an explanation of their theory is given in Date, Jimbo, Kashiwara and Miwa[2]) established a unified approach for a large class of completely integrable systems. They constructed solutions based on dynamics(flows) on infinite dimensional Grassmann manifold, and it was rewritten from an analytic point of view by Segal and Wilson[3]. Johnson[4] mentioned the applicability of their approach to this space. However, to apply this method we have to prove the transversality and our first task is to show the transversality. Let

$$\Gamma = \left\{\begin{array}{c} g;\ g(z) \text{ is holomorphic on } \mathbf{D},\ g(0)=1,\ g(z)\neq 0 \text{ for } \forall z\in\mathbf{D}, \\ \text{takes real values on } \mathbf{R} \text{ and } g(-z)=g(z)^{-1} \text{ for } \forall z\in\mathbf{D} \end{array}\right\},$$

where $\mathbf{D}$ is the closed unit disc. We construct a homomorphism K between the group Γ and the group of all homeomorphism on $\Omega\left([-\lambda_0,0]\right)$ by applying Sato's theory. This K induces the shift operation if we choose $g_x(z)=e^{-xz}\in\Gamma$ and solutions for the KdV equation if $g_{x,t}(z)=e^{-xz+4tz^3}\in\Gamma$. Any other higher order KdV equation can be solved in this way on $\Omega\left([-\lambda_0,0]\right)$. It is also known that for $u\in\Omega\left([-\lambda_0,0]\right)$, $L(u)$ and $L(K(g)u)$ are unitarily equivalent.

The motivation is to construct a nice solution for the KdV equation starting from a certain random initial data. This problem was raised by V.E.Zakharov and the author learned it from S.A.Molchanov. We would like to construct a solution as a typical random field $\{u(t,x)\}_{t,x\in\mathbf{R}}$ which is shift invariant with respect to t and x. In this respect, there are already solutions which are quasi-periodic in time and space, which is a special case of shift invariant random fields. However our aim is to give a very random solution. The construction of a KdV flow is the starting point to solve this problem. Since $\Omega\left([-\lambda_0,0]\right)$ is compact and the KdV flow $\{K(g)\}_{g\in\Gamma}$ is commutative, the space of all probability measures on $\Omega\left([-\lambda_0,0]\right)$ invariant with respect to $\{K(g)\}_{g\in\Gamma}$ is a non-empty compact convex set. Therefore we have many ergodic $\{K(g)\}$ −invariant probability measures on $\Omega\left([-\lambda_0,0]\right)$. Since a KdV-flow invariant probability measure μ is automatically shift invariant, we can define the Floquet exponent

$$w_\mu(\lambda) = -\frac{1}{2}\int_{\Omega([-\lambda_0,0])} g_\lambda(0,0,u)^{-1}\mu(du),$$

where $g_\lambda(x,y,u)$ is the Green function of the Schrödinger operator with potential u. It is known that Floquet exponents arising from finite band spectrum determine uniquely a $\{K(g)\}$ −invariant probability measures.

Open problem: To what extent does the Floquet exponent w_μ characterize the measure μ?

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Lifshitz tails in constant magnetic fields

Frédéric Klopp

(joint work with Georgi Raikov)

Consider the standard Landau Hamiltonian

$$H_0 := (-i\nabla - A)^2, \quad A := \left(-\frac{by}{2}, \frac{bx}{2}\right)$$

i.e. the Schrödinger operator with constant (scalar) magnetic field $b > 0$, self-adjoint in $L^2(\mathbb{R}^2)$. Its spectrum is the set $\{(2n+1)b;\ n \in \mathbb{N}\}$ and all the eigenvalues are of infinite multiplicity.

Consider the random potential

$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^2} \omega_\gamma V(x-\gamma)$$

where V is any non-negative single site potential satisfying $|V(x)| \leq C(1+|x|)^{-\beta}$ for some $\beta > 2$, and the $(\omega_\gamma)_\gamma$ are non trivial, i.i.d. random variables.

The operator $H_\omega = H_0 + V_\omega$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. Its almost sure spectrum consists of a union of intervals that may be separated by open gaps.

One defines the density of states of H_ω in the usual way. Namely, for any energy E, one shows that the limit

$$N(E) := \lim_{l\to+\infty} \frac{2\pi}{bl^2} \sharp\{\text{eigenvalues of } H^l_\omega \leq E\}$$

exists; here, the operator H_ω^l is H_ω restricted to the square centered at 0 of side length l. It defines a non decreasing, non random function of energy E. The almost sure spectrum of H_ω is the set of growth points of N ([6]).

The main purpose of these notes is to present some results on the behavior of N near the edges of $\sigma(H_\omega)$. It is well known that, for many random models, this behavior consists in a very fast decay which goes under the name of "Lifshitz tails". It was studied extensively in the absence of a magnetic field (see [6, 7]) and also in the presence of magnetic field for other types of disorder (see [2, 3, 1]).

To fix the picture of the almost sure spectrum, let us assume that the support of the random variables $(\omega_\gamma)_\gamma$ consists of an interval containing 0; for the sake of simplicity, assume, moreover, that

$$\operatorname*{ess-sup}_{\omega} \|V_\omega\|_\infty < b.$$

Then, the spectrum of H_ω is a countable disjoint union of intervals, each of them containing exactly one Landau level. Let us denote the n-th band i.e. the interval containing the n-th Landau level $(2n+1)b$ by $[E_n^-, E_n^+]$. We describe the behavior of the integrated density of states near E_n^- (the behavior near E_n^+ can be described in a similar way), and, therefore, introduce $\rho_n(E) = N(E) - N(E_n^-)$.
One has to distinguish between two cases :

- if $E_n^- = (2n+1)b$. This is the case if and only if the random variables $(\omega_\gamma)_\gamma$ are non negative (as 0 is assumed to be in their support). In this case, we prove

  Theorem 1 ([4]). *Assume that $\mathbb{P}(0 \le \omega_0 \le E) \sim CE^\kappa$ for some $C > 0$ and $\kappa > 0$.*
  - *Assume $V(x) \ge \varepsilon$ for $|x| \le \varepsilon$. Then, for some $\beta > 0$ and for $E > E_n^-$ close to E_n^-, one has*

$$\log \rho_n(E) \le -C|\log(E - E_n^-)|^{1+\beta} \tag{1}$$

  - *Assume $V(x) \ge \dfrac{e^{-Cx^2}}{C}$ for some $C > 0$. Let $\mu : \mathbb{R} \to \mathbb{R}^+$ be a positive function such that $\mu(E) \to 0$ and $\mu(E)\log E \to +\infty$ when $E \to 0$. There exists $C > 0$ such that, for $E > E_n^-$ close to E_n^-, one has*

$$\log \rho_n(E) \le -\frac{1}{C} \cdot \mu(E - E_n^-) \cdot \log^2(E - E_n^-). \tag{2}$$

  If V decays as least at a Gaussian speed at infinity, one expects that

$$\log \rho_n(E) \underset{\substack{E \to E_n^- \\ E > E_n^-}}{\asymp} -\log^2(E - E_n^-). \tag{3}$$

  The lower bound corresponding to (3) is easily obtained using trial functions; (1) and (2) do not quite give the corresponding upper bound.
- if $E_n^- < (2n+1)b$. In this case, the behavior of ρ_n is radically different. Let $\omega_- = \text{ess-inf}\,\omega_0$; as said above, $\omega_- < 0$ as $E_n^- < (2n+1)b$. One proves

Theorem 2 ([4]). *Assume $b \in 2\pi\mathbb{Q}$ and $\mathbb{P}(0 \leq \omega_0 - \omega_- \leq E) \sim CE^\kappa$ for some $C > 0$ and $\kappa > 0$. Then, there exists $\alpha > 0$ such that*

$$\lim_{\substack{E \to E_n^- \\ E > E_n^-}} \frac{\log|\log \rho_n(E)|}{\log|E - E_n^-|} = -\alpha \tag{4}$$

So we obtain a behavior similar to that obtained for random perturbations of periodic Schrödinger operators in dimension 2 (see [5]).

In the case $b \notin 2\pi\mathbb{Q}$, we expect a behavior that is intermediate between (4) and (3).

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Limit distributions for models of exactly solvable walks

Christoph Richard

(joint work with P. Duchon and M. Nguên Thê)

Dyck paths are classical objects in enumerative combinatorics [10]. They have been analysed according to the distribution of area under the path, in the limit of large path length [2]. The limit distribution is known as the Airy distribution [3]. This distribution appears in different contexts such as the distribution of path lengths in varieties of trees [11] in the limit of large tree size, and as the distribution of area of exactly solvable subclasses of self-avoiding polygons [2, 9], in the limit of large polygon perimeter. The relation between these results, which were obtained independently within probability theory [11], combinatorics [2] and statistical physics [9], can be understood by bijections between corresponding combinatorial models [10]. It was recently found that the Airy distribution also describes aspects of one-dimensional fluctuating interfaces in physics [5], and there is evidence based on extensive numerical investigations that it describes the limit distribution of area in the unsolved model of planar self-avoiding polygons [8].

In the limit of large path lengths, Dyck paths can be described by Brownian excursions [1]. This hints at a connection between discrete models, analysed by methods of asymptotic analysis of generating functions, and their continuous variants, analysed by stochastic methods [6]. Within the stochastic description, general results have been obtained concerning excursions, e.g., Louchard's formula [4] for the double Laplace transform of the distribution of certain functionals of the excursion. Particular examples, where explicit expressions for distributions have been derived, are the absolute value functional, which describes the excursion area, and the quadratic functional. For more complicated functionals, it is however difficult to extract detailed information about the distributions, such as values of their moments.

Within the discrete description using Dyck paths, it has recently been realised [7] that the so-called higher rank parameters, which were introduced earlier [2], correspond to polynomial functionals in the stochastic description. This insight led to a new proof of Louchard's formula in the case of polynomial functionals, and to a new recurrence relation for the moments of the joint distribution of the absolute value and the quadratic functional.

Methods originating from statistical physics [9] prove useful for a refined analysis of the discrete problem. In fact, it is possible to obtain a recursion for the moments of the joint distribution of polynomial functionals $V(x) = s_1|x| + s_2|x|^2 + \ldots + s_M|x|^M$. This generalises previous results [7], obtained for $M = 2$, to arbitrary values of M.

More precisely, consider the generating function $D(u_0, u_1, \ldots, u_M)$ of Dyck paths with rank M parameters. The arch decomposition of Dyck paths can be used to derive the functional equation

$$D(u_0, \ldots, u_M) = \frac{1}{1 - u_0^2 u_1 \cdots u_M D(v_0, \ldots, v_M)}, \tag{1}$$

where the arguments $v_0, \ldots, v_M$ are given by $v_n = \prod_{k=n}^{M} u_k^{\binom{k}{n}}$. Consider the factorial moment generating functions

$$g_{k_1,\ldots,k_M}(u_0) = \frac{1}{k_1! \cdots k_M!}\, \partial_{u_1}^{k_1} \cdots \partial_{u_M}^{k_M} D(u_0, u_1, \ldots, u_M)\big|_{u_1=\ldots=u_M=1}. \tag{2}$$

Using the framework of [7], it can be shown that these functions are algebraic and have asymptotically the singular behaviour

$$g^{(sing)}_{k_1,\ldots,k_M}(u_0) \sim \frac{f_{k_1,\ldots,k_M}}{(1-4u_0^2)^{\gamma_{k_1,\ldots,k_M}}} \qquad \left(u_0 \to \frac{1}{2}\right), \tag{3}$$

with exponents $\gamma_{k_1,\ldots,k_M} = -\frac{1}{2} + \sum_{i=1}^{M} \left(1 + \frac{i}{2}\right) k_i$ and amplitudes $f_{k_1,\ldots,k_M}$. Consider the generating function $F(\mathbf{s})$ of the amplitudes $f_{k_1,\ldots,k_M}$, given by

$$F(\mathbf{s}) = \sum_{k_1,\ldots,k_M} f_{k_1,\ldots,k_M} s_1^{k_1} \cdots s_M^{k_M}. \tag{4}$$

The method of dominant balance [9] can be applied to the functional equation (1) in order to obtain a quasilinear partial differential equation for the amplitude generating function $F(\mathbf{s})$. It is

$$(5)\quad 1-\frac{1}{4}F(\mathbf{s})^2+\frac{1}{2}s_1F(\mathbf{s})-\sum_{k=1}^{M}\frac{k+2}{2}s_1s_k\frac{\partial F}{\partial s_k}(\mathbf{s})-\sum_{k=1}^{M-1}\frac{k+1}{2}s_{k+1}\frac{\partial F}{\partial s_k}(\mathbf{s})=0.$$

The recurrence for the amplitudes $f_{k_1,\dots,k_M}$, as implied by (5), translates into a recurrence for the moments of the joint distribution, via transfer theorems for coefficients of algebraic generating functions. For $M=2$, this is described in [7]. Moreover, we have convergence of (normalised) moments of all orders to their continuous counterparts [1]. We obtain the following result for excursions. Let $X_k=\int_0^1 e^k(s)\,\mathrm{d}s$, where e is a normalised excursion, and let $\eta_{k_1,\dots,k_M}=\mathbb{E}[X_1^{k_1}\cdots X_M^{k_M}]$. Then

$$(6)\qquad \eta_{k_1,\dots,k_M}=\frac{k_1!\cdots k_M!}{2^{\gamma_{k_1,\dots,k_M}}}\frac{\sqrt{2\pi}}{\Gamma(\gamma_{k_1,\dots,k_M})}f_{k_1,\dots,k_M},$$

where $\Gamma(z)$ denotes the Gamma function of argument z. For $M=2$, this reduces to previous results [7, Table 2].

The method described above can also be applied to discrete counterparts of Brownian motion, Brownian bridges, and meanders. More generally, it can be applied to models of trees or polygons with higher rank parameters, where similar functional equations appear.

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On spectral norm of large band random matrices

A. Khorunzhy

1. Band random matrices and semicircle law

We consider the ensemble of $N \times N$ hermitian random matrices $H^{(N,b)}$ whose elements are equal to zero inside the band of width b along the principal diagonal. Inside of this band, the elements $\{H_{xy}^{(N,b)},\ x \le y\}$ are given by Gaussian jointly independent random variables with zero mean value and variance v^2/b. In other words

$$[H^{(N,b)}]_{xy} = \frac{1}{\sqrt{b}} \begin{cases} a_{xy}, & \text{if } |x-y| \le b, \\ 0 & \text{otherwise}, \end{cases} \qquad 1 \le x \le y \le N, \tag{1}$$

where $\{a_{xy} = \alpha_{xy} + i\beta_{xy},\ 1 \le x \le y \le N\}$ are complex gaussian random variables with the density of probability distribution proportional to

$$\exp\left\{ -\frac{1}{2v^2} \sum_{x,y=1}^{N} |a_{xy}|^2 \right\}, \quad a_{xy} = \overline{a_{yx}}.$$

Regarding $b = 1$, one obtains three-diagonal matrices that correspond to the discrete version of random Schrödinger operator; in the case of $b = N$, matrices (1) determines the Gaussian Unitary [Invariant] Ensemble of random matrices (GUE) [4].

The spectral properties of GUE $\{H^{(N,N)}\}$ are well understood. Since the pioneering works by E. Wigner [6], it is known that the normalized eigenvalue counting function

$$\sigma(\lambda; H^{(N,N)}) = \#\{\lambda_j^{(N)} \le \lambda\} N^{-1}$$

converges to a nonrandom function $\sigma_W(\lambda)$ with the density of the semicircle form over $(-2v, 2v)$. This proposition is referred to as the semicircle (or Wigner) law. Later it was proved by S. Geman [2] that

$$\lim_{N\to\infty} \|H^{(N,N)}\| = 2v \quad \text{with probability 1.}$$

These two results are cornerstones for the spectral theory of random matrices.

2. Main result and the scheme of the proof

We have studied the spectral properties of band random matrices $H^{(N,b)}$ in the limit when $1 \ll b \ll N$. It is known (see e.g. [5]) that in this case the normalized eigenvalue counting function $\sigma_{N,b}(\lambda) = \sigma(\lambda, H^{(N,b)})$ converges to $\sigma_W(\lambda)$; the semicircle law is valid. Much less is known about the limiting behaviour of the spectral norm of band random matrices. An elementary arguments show that if $1 \ll b \ll \log N$, then $\lambda_{\max}^{(N,b)} \to \infty$ with probability 1.

Our main result is that in the limit when $b \gg (\log N)^3$, the following estimate holds with probability 1

$$\limsup_{N,b\to\infty} \lambda_{\max}^{(N,b)} \le 2v. \tag{2}$$

Also one can show that if $b = N^\alpha$ with $\alpha \in (0,1)$, then $\|H^{(N,b)}\|$ converges to $2v$ with probability 1.

Following the arguments of [2], we consider the moments

$$L_{2k}^{(N,b)} = \frac{1}{N} \text{ Tr } [H^{(N,b)}]^{2k}$$

and their averages $M_{2k}^{(N,b)} = \mathbf{E}\{L_{2k}^{(N,b)}\}$. Applying the method developed in [1], we derive recurrent relations for $M_{2k}^{(N,b)}$ that involve the variances of L_{2j}, $j \le k$. We show that the variance terms also verify a system of closed recurrent relations.

Basing on these relations and modifying computations of [1], we show that if $k^3 \le b$, then the following estimate holds

$$M_{2k}^{(N,b)} \le \left(1 + \frac{k^3}{b^2}\right)^k m_{2k} \tag{3}$$

where

$$m_j = \int_{-2v}^{2v} \lambda^j \, \mathrm{d}\sigma_W(\lambda).$$

Taking into account the bound $m_{2k} \le (2v)^{2k}$, one deduce from (3) that for any given $\epsilon > 0$

$$M_{2k}^{(N,b)} \le [2v(1+\epsilon)]^{2k}$$

for all $k^3 \le b \le N$. To obtain (2), it is sufficient to consider the limit $N \to \infty$ with $k = O(\log N)$. The complete proof is given in [3].

The estimate (3) is not optimal. The main open problem is how to modify our method to get the most precise inequalities. This is important to determine the local scale of the eigenvalue spacing at the spectral edge. One could expect to get the estimate $M_{2k} \le (1 + k^3/b^2)m_{2k}$.

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Weak disorder expansion for localization lengths of quasi-1D systems

Rudolf A. Römer
(joint work with Hermann Schulz-Baldes)

The Anderson model describes the generic behavior of the motion of an electron in a disordered solid. In the one-dimensional (1D) situation, rigorous proofs of strong localization have been given [1]. Moreover, the localization length has been calculated [2, 3] for weak disorder and its inverse, the Lyapunov exponent, is given by $\gamma \approx w^2/(96\sin^2 k) = w^2/(96 - 24\,E^2)$, where $E = 2\cos k$ is an energy in the Bloch band of the unperturbed operator and w is the disorder strength. For the 2D case, scaling theory [4] predicts also strong localization with an disorder dependence of the localization length of the form $\exp(1/w^2)$. This was confirmed by high-precision numerical studies based on the transfer-matrix method (TMM) [5, 6]. A rigorous proof of strong localization in 2D and 3D exits only for the band edges and at high disorder [7, 8] and, in particular, not for energies in the band and small disorder in the 2D situation.

In order to approach the higher dimensional cases, a detailed understanding of the quasi-1D situation, *i.e.* an infinite wire with many channels, is of crucial importance. The TMM [5, 6] is here a very reliable tool for a numerical study of the inverse localization length, namely the smallest Lyapunov exponent γ_L. The rigorous perturbative formula for the smallest Lyapunov exponent within the band is then, under a hypothesis on the incommensurability of the rotation phases η_l which excludes energies with Kappus-Wegner-type anomalies [10] and interior band edges, given by [9]

$$\gamma_L \;=\; \frac{w^2}{96\,L}\sum_{l,k}\frac{2-\delta_{l,k}}{\sin\eta_l\sin\eta_k}\;\langle\rho_{L,l}\;\rho_{L,k}\rangle \;+\; O(w^4)\;, \tag{1}$$

where the sum runs over elliptic channels only. Also, $\rho_{p,k}(n) = \langle u_p(n)|\pi_k|u_p(n)\rangle$ is the weight of the pth frame vector in the kth channel at iteration n, $\langle\cdot\rangle$ denotes the Birkhoff mean along the strip.

In the present work [11], we compare (1) numerically with the TMM. Our main result is that the perturbative formula works remarkably well for all but a discrete set of energies and, quite surprisingly, relatively large values of the disorder strength. This allows to understand the rich structure of the energy dependence of the smallest Lyapunov exponent. We also study various quantities associated to the random dynamical system underlying the TMM.

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The mathematics of aperiodic order

Robert V. Moody

There is by now a diverse and active area of mathematics concerned with long-range aperiodic order. The term itself, long-range aperiodic order, is not particularly well-defined, but generally it refers to discrete structures in real space $\mathbb{R}^d$ that are highly ordered, not periodically, but in ways which are clearly generalizations of periodicity.

The epitome of a discrete periodic structure is a lattice and it is useful to start with a list of some of the basic properties of lattices and their translates. Let $\Lambda \subset \mathbb{R}^d$ be a lattice, that is, Λ is the $\mathbb{Z}$-span some basis of $\mathbb{R}^d$.

The Delone property: Λ is uniformly discrete (there is a minimal distance of separation between its points) and relatively dense (there is a covering radius R for which balls of radius R centred on the points of Λ cover all of $\mathbb{R}^d$).

Algebraic Structure: $\Lambda - \Lambda \subset \Lambda$, or for translates, $(a+\Lambda) - (a+\Lambda) \subset (a+\Lambda) - a$

Repetition: everything repeats with perfect translational symmetry.

Dynamical System: $\mathbb{R}/\Lambda$ is a compact space (in fact an Abelian group) on which $\mathbb{R}^d$ acts continuously via translation.

Coherence: there is a dual object, the reciprocal lattice

$$\Lambda^\circ = \{k \in \mathbb{R}^d : \exp 2\pi i k \cdot u = 1 \text{ for all } u \in \Lambda\}$$

Diffraction: Λ has pure point diffraction.

One of the most useful constructions of the theory of aperiodic sets is that of the cut-and-project sets, or model sets. A *cut and project scheme* is a set-up of the following kind:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \bigcup & & \\ & & \widetilde{L} & & \end{array}$$

where H is a locally compact Abelian group, $\widetilde{L} \subset \mathbb{R}^d \times H$ is a lattice, i.e. a discrete subgroup for which the quotient group $(\mathbb{R}^d \times H)/\widetilde{L}$ is compact, $\pi_1|_{\widetilde{L}}$ is injective, and $\pi_2(\widetilde{L})$ is dense in H.

For a subset $V \subset H$, we denote $\Lambda(V) := \{\pi_1(x) \in \mathbb{R}^d : x \in \widetilde{L}, \pi_2(x) \in V\}$.

A *model set* in $\mathbb{R}^d$ is a subset $\varLambda$ of $\mathbb{R}^d$ for which, up to translation, $\Lambda(W^\circ) \subset \varLambda \subset \Lambda(W)$, where W is compact in H, $W = \overline{W^\circ} \neq \emptyset$. The model set $\varLambda$ is *regular* if the boundary $\partial W = W \backslash W^\circ$ of W is of (Haar) measure 0.

Model sets are based on partial projection of a lattice in some 'higher' space. They are generally aperiodic. If one wants periods one has to do something special – for example, making H a finite group gives model sets that are finite unions of cosets of a lattice. Nonetheless, model sets $\varLambda$ have exemplary properties which beautifully generalize those of lattices that we saw above. [1]

The Delone property: $\varLambda$ is always a Delone set.

Algebraic Structure: there is a *finite* set F so that $\varLambda - \varLambda \subset \varLambda + F$.

Repetitivity: Suppose that $\partial W \cap \pi_2(\widehat{L}) = \emptyset$. Then for each $r > 0$ there is an $R > 0$ so that a translated copy of every configuration of points of $\varLambda$ lying in any ball of radius r in $\mathbb{R}^d$ appears in every ball of radius R in $\mathbb{R}^d$. In short, everything repeats, and things repeat in a very strict way. Of course these repetitions are not periodic!

Dynamical System: There is an obvious dynamical system in sight, namely the compact group $(\mathbb{R}^d \times H)/\widetilde{L}$ with $\mathbb{R}^d$ acting on it in the obvious way. Below we will see that there are two other possibilities for a dynamical system here.

Coherence: for each $\epsilon > 0$ there is a dual object, the ϵ-dual of $\varLambda$

$$\varLambda^\epsilon = \{k \in \mathbb{R}^d : |\exp 2\pi i k \centerdot u - 1| < \epsilon \text{ for all } u \in \varLambda\}$$

and these also are Delone sets.

Diffraction: If $\varLambda$ is a regular model set then it has pure point diffraction.

This last result is due to A. Hof and M. Schlottmann, and is difficult to show.

Model sets are very nice objects, and they have played a significant role in the theoretical underpinning of physical quasicrystals. But they seem to involve a number of exotic features: hidden spaces, lattices in higher dimensions, and windows which can almost be anything. How can one characterize them in some reasonable way?

For Delone sets the two properties that we have listed under algebraic structure and coherence are actually equivalent (another difficult result, due to Y. Meyer). We call Delone sets with either, hence both, of these properties *Meyer* sets. Meyer sets have other simple (to state) characterizations, including the Lagarias condition:

[1] Rather than put in a lot of references, I will refer to just one recent book, *Directions in Mathematical Quasicrystals*, eds. M. Baake and R. V Moody, CRM Monograph Series 13, AMS, 2000, which is a collection of papers on a number of areas of interest in aperiodic order at the moment. It also has a useful guide to the literature. This perhaps will compensate for the narrow view of the subject that I am presenting here.

$$\text{Meyer set} \Longleftrightarrow \text{Delone} + \ \Lambda - \Lambda \text{ is uniformly discrete.}$$

To characterize model sets we introduce two dynamical systems. Let $\Lambda \subset \mathbb{R}^d$ be a repetitive Delone set. We let $\mathbb{X}(\Lambda)$ be all the subsets Λ' of $\mathbb{R}^d$ that are locally indistinguishable from Λ – that is, whatever pattern of points shows up locally in Λ shows up (possibly after translation) in Λ', and vice-versa. This is topologized in the following way: two subsets Λ', Λ'' in $\mathbb{X}(\Lambda)$ are close if after a small translational shift they agree on a large ball around the origin. This can be made precise of course, and the topology is actually a metric topology. Furthermore $\mathbb{X}(\Lambda)$ is complete, compact, and a dynamical system under the obvious translation action of $\mathbb{R}^d$ on it. We call this the *local hull* of Λ, since its topology is based on local considerations.

Now we start again. This time two uniformly discrete sets Λ', Λ'' are close if after a small translational shift v we have

$$d(\Lambda', \Lambda'') := \mathrm{dens}((v + \Lambda') \triangle \Lambda'')$$

is small ($\triangle$ being the symmetric difference operator).

This assumes that densities exist. In this notion of closeness it is the long-range behaviour that counts. Two sets can differ in zero density without being equal. This introduces an equivalence relation $\equiv$ through which we may identify such sets.

Starting with a *Meyer* set Λ we now form the closure, in this new topology, of its translation orbit

$$\mathbb{A}(\Lambda) := \overline{\mathbb{R}^d + \Lambda}$$

which we call the *autocorrelation hull* of Λ. It is again carries a natural $\mathbb{R}^d$ action, it is an Abelian group, and it is compact if and only if Λ is pure point diffractive. In the case of regular model sets the cut and project scheme can be arranged so that $\mathbb{A}(\Lambda) \simeq (\mathbb{R}^d \times H)/\widetilde{L}$, giving the latter a nice geometrical interpretation.

It is natural to ask for an $\mathbb{R}^d$-map $\mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ (but not the other way around due to collapsing under $\equiv$) with $\Lambda \mapsto \Lambda \pmod{\equiv}$. If it exists then it is unique, and we call it the *torus parametrization*. But why should there be any relationship between the two topologies? In general one cannot expect such a thing. Nonetheless …

Proposition 1. *Let Λ be a repetitive regular model set. Then the torus parametrization exists and it is one-to-one almost everywhere with respect to the Haar measure on $\mathbb{A}(\Lambda)$. In the opposite direction, suppose that Λ is a repetitive Meyer set and the torus parametrization exists, and it is one-to-one almost everywhere with respect to the Haar measure on $\mathbb{A}(\Lambda)$. Then there is a cut and project scheme and a window W in the corresponding internal space with respect to which Λ is a regular model set and $\mathbb{X}(\Lambda)$ is the local indistinguishability class of Λ.*

All harmony is restored for crystals

Proposition 2. *A Meyer set $\varLambda$ is a crystal (a finite union of cosets of a lattice) if and only if the torus parametrization exists and is a bijection.*

These results are due to M. Baake, D. Lenz, and myself in the setting here. Jeong-Yup Lee and I have generalized it to point sets with multi-colours. Details are quite tricky in both cases. For exact statements and proofs one should go to the preprints which will soon be on the website:

`http://www.math.ualberta.ca/~rvmoody/rvm/`

The nature of the failure to be one-to-one is amazingly intricate. The set of singular points of $\mathbb{A}(\varLambda)$, those at which the map is not one-to-one, is typically dense in $\mathbb{A}(\varLambda)$.

The main point though is that, at least in the context of Meyer sets, the model sets are those in which the two topologies are in agreement with each other. The first topology is about local structure, the second about the long-range order. That these should be linked in quasicrystals seems natural. And so the cut and project formalism, strange as it may seem, somehow enters quasicrystal theory in a fundamental way .

This brief description of long-range aperiodic order from the point of view of discrete point sets is already convincing evidence of the beautiful way in which periodicity can be lifted into the aperiodic domain, retaining along the way almost all the properties of lattices, albeit suitably generalized. It is the gentlest form of disorder.

Quantum diffusion

László Erdős

(joint work with M. Salmhofer and H.T.Yau)

The mystery of the erratic motion of pollen grains suspended in water, named after its explorer, Robert Brown, was solved by Einstein in his miraculous year of 1905. His kinetic theory, based upon light water molecules continuously bombarding the heavy pollen, provide not just an explanation of diffusion from the Newtonian mechanics, but also the most direct evidence yet for the existence of atoms and molecules. Since the discovery of quantum mechanics it has been a major challenge to verify the emergence of diffusion from the Schrödinger equation. In this talk I reported on a mathematically rigorous derivation of a diffusion equation as a long time scaling limit of a random Schrödinger equation in a weak, uncorrelated disorder potential.

We considered the usual Anderson model

$$H = -\Delta + \lambda V_\omega$$

in $d \geq 3$ dimensions, with small coupling, $\lambda \ll 1$, and with an uncorrelated random potential, $\mathbf{E}V(x) = 0$, $\mathbf{E}V(x)^2 = 1$, $\mathbf{E}V(x)V(y) = 0$, $x \neq y$. For simplicity we work on the lattice $\mathbf{Z}^d$, although our method easily extends to the continuous model. The dispersion relation is given by

$$e(p) := \sum_{i=1}^{d}(1-\cos(p^{(i)}))\ , \qquad p = (p^{(1)}, p^{(2)}, \ldots, p^{(d)})$$

We consider the long time evolution of the Schrödinger equation

$$i\partial_t \psi_t = H\psi_t$$

with a regular initial data. The time scale be a negative power of the coupling constant λ.

In our earlier work with H.T. Yau we showed that under the kinetic time scaling, $t = T\lambda^{-2}$, $x = X\lambda^{-2}$, $T = O(1), X = O(1)$, the evolution of the rescaled Wigner transform, $W^\varepsilon(x,v) := \varepsilon^{-d}W_\psi(x/\varepsilon, v)$, $\varepsilon = \lambda^2$, is given by the Boltzmann equation for arbitrary T, improving an earlier result by Spohn, who proved the same statement for sufficiently small T.

In the current work we extend this result to a time scale $t = T\lambda^{-2-\kappa}$ for some fixed positive κ. The space is rescaled accordingly: $x = X\lambda^{-2-\kappa/2}$. The limiting equation on this time scale is a heat equation in position space. More precisely, we prove the following:

Theorem Let $\psi_0 \in \ell^2(\mathbf{Z}^d)$ be an arbitrary initial wave function with $\widehat{\psi}_0 \in L^\infty$ and let

$$g(e) := \Big\langle |\widehat{\psi}_0(v)|^2 \Big\rangle_e := \int |\widehat{\psi}_0(v)|^2 \delta(e - e(v))dv$$

be its projection in the energy space of the free Laplacian. Let $\psi(t) = \psi^\lambda_{t,\omega}$ solve the Schrödinger equation. Then for $d \geq 3$ and $\kappa < 1/50$, the energy space projection of the solution satisfies

$$\mathbf{E}\Big\langle\ W^\varepsilon_{\psi(\varepsilon^{-1}\lambda^{-\kappa/2}T)}(X,v)\ \Big\rangle_e \rightharpoonup f(T,X,e)$$

weakly in $L^1(\mathbf{R}^3_X \times \mathbf{R}_e)$ as $\varepsilon \to 0$, where $\varepsilon = \lambda^{2+\kappa/2}$. The limit density f satisfies the heat equation

$$\partial_T f(T,X,e) = \nabla_X \cdot D(e) \nabla_X f(T,X,e)$$

with the initial condition

$$f(0,X,e) := \delta(X)g(e)$$

and with diffusion matrix D given by

$$D_{ij}(e) := \frac{1}{\langle 1\rangle_e^2}\ \Big\langle\ (\sin v_i)(\sin v_j)\ \Big\rangle_e$$

The limiting diffusive equation is the same as the one that is obtained from the long time rescaling of the Boltzmann equation. Therefore we show that quantum interferences, that are responsible for possible quantum corrections, remain negligible on our time scale in $d \geq 3$. Note that the Boltzmann picture on the kinetic time scale is correct also in $d = 2$, but it is not expected that the long time rescaling of the Boltzmann equation describes the true quantum dynamics in this case.

Signatures of order, chaos and time-reversal symmetry over finite fields

John A G Roberts

In this talk, we describe the dynamics of rational maps of the affine plane over a finite field. Many such maps have previously been studied over the real or complex plane. For example, investigations of area-preserving maps like the Henon map: $x' = y$, $y' = -x + y^2 + C$ (where C is a parameter and primes denote the image point), helped to understand the dynamics of low dimensional Hamiltonian systems. If such polynomial or rational maps have coefficients that are rational, then for all but finitely many primes p, the map reduces to one over $\mathbb{F}_p^2$ (where $\mathbb{F}_p$ is a finite field with p elements) or its projective version. Some important properties of area-preserving maps can be expressed in algebraic terms. If we denote the map by L, two such properties are: (i) integrability, or the possession of an integral $I(x, y)$, defined by $I \circ L = I$; and (ii) reversibility or time-reversal symmetry, which means the existence of a map G that conjugates L to its inverse L^{-1}, i.e. $L^{-1} = G \circ L \circ G^{-1}$. We ask: can one see the *signature* of such properties of an area-preserving map when studying its reduction over $\mathbb{F}_p^2$? This relates to the question of identifying whether a given map has the property or not. Over a phase space that is a continuum, this question is inherently difficult to answer because it is hard to separate systems that possess a property from those perturbations that do not possess the property.

Over a finite field, a rational map will in general possess periodic orbits and non-periodic orbits. Suppose the point $z \in \mathbb{F}_p^2$ is periodic under the map L and let $T(z)$ be its period. We define the distribution $\mathcal{D}_p(x) := \#\{z : T(z) \le rx\}/\#C_p$, where C_p is the set of periodic points in the phase space and r is a chosen scaling factor. Hence $\mathcal{D}_p(x)$ represents the probability that a periodic point chosen at random belongs to a cycle whose scaled period $T(z)/r$ does not exceed x. We find that the periodic orbit statistics (their number, their lengths) are markedly different depending on whether the map is integrable or non-integrable, reversible or irreversible. Numerical experiments suggest that there are universal distribution functions for the scaled periodic orbits of the map in each case (in the limit of large prime p). For integrable maps, the cumulative distribution function has plateaus, corresponding to the fact that only certain values of scaled period are possible [1, 2]. This derives from the theory of algebraic curves over finite fields and the so-called Hasse-Weil bound that restricts the number of points on such curves. The curves here are precisely the level sets of the integral of the map. In the non-integrable case, we study the effect of a particular type of reversibility, called *R-reversibility*, on polynomial automorphisms which induce permutations of $\mathbb{F}_p^2$. The resulting distribution is smooth and we have conjectured its form based on numerical evidence that suggests there is an underlying Poisson statistics present [3]. Finally, in the non-integrable case in which R-reversibility is absent, the appropriately scaled period distribution is consistent with that of a random permutation of the p^2 elements of $\mathbb{F}_p^2$ [3]. Taken together, our results furnish, upon reduction, various necessary conditions for integrability and time-reversal symmetry to exist in rational maps of the real or complex plane.

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The random magnetic impurity model

Stéphane Ouvry

(joint work with Jean Desbois, Cyril Furtlehner)

We review the present status of the random magnetic impurity model: its perturbative expansion, its path integral formulation in terms of Brownian winding. Some open questions are adressed.

The Random Magnetic Impurity model was introduced[1] ten years ago in relation with the Integer Quantum Hall Effect. It consists, on the plane, of a charged particle coupled to a Poissonian (with mean impurity density ρ) spatial distribution of point Aharonov-Bohm vortices (magnetic impurities) carrying a fraction α of the flux quantum Φ_o of the test particle. Periodicity and symmetry considerations allow to take the parameter $\alpha \in [0, 1/2]$. It has been shown via path integral random walk simulations that, when α is small ($\alpha \to 0$), the average density of states of the test particle narrows down to the Landau density of states for the average magnetic field $< B >= \rho\alpha\Phi_o$ with broadened Landau levels (weak disorder). On the contrary, when α is big ($\alpha \to 1/2$), the density of states has no Landau level oscillations and rather exhibits a Lifschitz tail at the bottom of the spectrum (strong disorder). The transition between the two regimes takes place at $\alpha_c \simeq 0.3$.

Only the one-impurity case is solvable (standard Aharonov-Bohm problem), whereas $(\rho\alpha)^n$ n-impurity mean field contributions are also easily obtained. Corrections to the mean field are due to disorder and can be obtained by a perturbative expansion of the average density of states or, equivalently, of the average partition function, in terms of Feynman diagrams. The first such non trivial corrections appear at order $\rho^2\alpha^4$ (i.e. 2 impurities interacting 4 times).

On the other hand, specific heat thermal expansion indicate that, in order to support the numerical estimation for α_c, the 2 impurity case is needed.

At order $\rho^2\alpha^4$, the diagram with maximal impurity lines crossing was shown to be the only one to contribute. Evaluating this diagram, one came across with the multiple integral

$$I_{1100} = \int_{a,b,c,d=0}^{\infty} da\, db\, dc\, dd\, \frac{ab}{abc+bcd+cda+dab} e^{-(a+b+c+d)}$$

which can be rewritten as

$$I_{1100} = 2^4 \int_0^{\infty} \frac{udu}{2} (\frac{u}{2})^2 K_1(u)^2 K_0(u)^2$$

where the $K_\nu(u)$'s are modified Bessel functions also called Macdonald functions. One obtained by direct integration $I_{1100} = (1 + 7\zeta(3)/2)/8$. It was recently realized[2] that one can generalize this result to the family of integrals

$$I_{n_a n_b n_c n_d} = \int_{a,b,c,d=0}^{\infty} da\, db\, dc\, dd\, \frac{a^{n_a} b^{n_b} c^{n_c} d^{n_d}}{abc+bcd+cda+dab} e^{-(a+b+c+d)}$$

or

$$I_{n_a n_b n_c n_d} = 2^4 \int_0^{\infty} \frac{udu}{2} (\frac{u}{2})^{n_a+n_b+n_c+n_d} K_{n_a}(u) K_{n_b}(u) K_{n_c}(u) K_{n_d}(u)$$

so that

$$I_{n_a n_b n_c n_d} = u_{n_a n_b n_c n_d} + v_{n_a n_b n_c n_d} 7\zeta(3)/2$$

where $u_{n_a n_b n_c n_d}$ and $v_{n_a n_b n_c n_d}$ are positive or negative rational numbers. This result follows from integrations by parts manipulations which allow for the u's and v's to be determined by a recurrence relation, with initial conditions obtained by direct integration. The situation here is quite reminiscent of the one encountered to prove[3] the irrationality of $\zeta(3)$.

The appearance of the Riemann zeta function at odd integer (here $\zeta(3)$) is interesting per se even though not unique[4]. Pushing forward the perturbative expansion up to order $\rho^2\alpha^6$ (2 impurities interacting 6 times), one might expect a possible pattern in terms of zeta at odd integers, or more complicated objects. Again, the diagram with maximal impurity lines crossing is the only one to contribute at this order. Evaluating this diagram one came across yet another family of integrals still involving the product of four modified Bessel functions but now related to $\zeta(2)$. One finally arrives[5] at double nested integrals which have a structure reminiscent of the polyzeta function $\zeta(2,3)$. It still remains to be seen, by pushing the numerics further, what exactly are these double nested integrals in terms of Riemann polyzeta and Euler sums, and at which level.

Coming back to the path integral formulation[1] of the model, the average partition function rewrites as an average over the set C of closed Brownian curves of a given length t, the inverse temperature (Z_o is the free partition function)

$$< Z >= Z_o < e^{\rho \sum_n S_n (e^{i2\pi\alpha n} - 1)} >_{\{C\}}$$

where S_n is the arithmetic area of the n-winding sector of a given path in $\{C\}$. Roughly speaking, it amounts to say that random Aharonov-Bohm vortices couple to the S_n, a different (intermediate) situation from the well-known cases of a single vortex, which couples to the angle spanned by the path around the vortex, or of a homogeneous magnetic field, which couples to the total algebraic area $\sum_n nS_n$ enclosed by the path.

$< Z >$ rewrites as

$$< Z >= Z_o \int e^{-\rho t(S+iA)} P(S,A) dS dA$$

where $S = \frac{2}{t}\sum_n S_n \sin^2(\pi\alpha n)$, $A = \frac{1}{t}\sum_n S_n \sin(2\pi\alpha n)$ are random Brownian loop variables and $P(S,A)$ is their joint probability distribution. If one notices[6] that S_n scales like t -in fact $< S_n >= t/(2\pi n^2)$, even more, for n sufficiently large, $n^2 S_n \to < n^2 S_n >= t/(2\pi)$-, the variables S and A are indeed t independent.

Let us examine the small and big α limits:

i) when $\alpha \to 0$, one obtains $< Z > \to Z_{<B>} = Z_o < e^{i<B>\sum_n nS_n} >_{\{C\}}$, i.e. the partition function for the homogeneous mean magnetic field $< B >$, since $\sum nS_n$ is nothing but, as already said, the algebraic area enclosed by the path.

ii) when $\alpha \to 1/2$, now $< Z >= Z_o < e^{-\rho t \sum_{n\, odd} S_n} >_{\{C\}}$ and the average density of states simply reads

$$< \rho(E) >= \rho_o(E) \int_0^{\frac{E}{\rho}} P(S')dS'$$

with $P(S')$ the probability distribution for the random variable $S' = \sum_n S_n$, n odd and $\rho_o(E)$ the free density of states. In this situation of maximal disorder, it would certainly be interesting to have an exact analytical expression of $< \rho(E) >$, that is to say, of $P(S')$.

iii) more generally, when $0 < \alpha < 1/2$, i.e. when the variables S and A are at work, a more precise analysis of the joint distribution function $P(S,A)$ is needed.

In this context, one should stress that important progresses[7] have been made recently in the determination of critical exponents related to 2d random walk based on a family of conformally invariant stochastic processes, the stochastic Loewner evolution (SLE). As an example, and to focus on the specific problem at hand when $\alpha = 1/2$, some numerical simulations have been made by Richard[8] on the arithmetic area $\sum_n S_n$ statistics of the outer boundary of planar random loops of a given outer perimeter, in connection with the scaling dimension of self avoiding loops obtained from a particular class of SLE. However, the situation is here different since one has to deal with the variable $S' = \sum_{n\, odd} S_n$, and the statistics has to be done on Brownian loops with a given length.

It certainly would be quite rewarding to see at possible, if any, interpretations of the random variable S', and more generally of the random variables S and A, in terms of conformally invariant stochastic processes.

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A Kubo formula in disordered media and some properties of the edge conductance

François Germinet

(joint work with J.-M. Bouclet, J.-M. Combes, A. Klein, J. Schenker)

We report here two works: the first one is a joint work with J.M. Bouclet, A. Klein and J. Schenker; the second one is a joint work with J.M. Combes.

In the work [BGKS] we consider a magnetic Schrödinger operator in a disordered medium,

$$H_\omega = (-i\nabla - A)^2 + V_\omega \quad \text{on} \quad \mathcal{H} \equiv \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{1}$$

(where the parameter ω runs in a probability space $(\Omega, \mathbb{P})$; we assume the usual covariance property), and give a mathematically rigorous derivation of a Kubo formula for the electric conductivity tensor, validating the linear response theory. The electric field is switched on adiabatically. We recover the expected expression for the quantum Hall conductivity whenever the Fermi energy lies in either a gap of the spectrum of H_ω or in a region of localized states.

To perform our analysis we develop an appropriate mathematical apparatus for the linear response theory. We describe quite precisely some normed spaces of operators, closely related to the trace per unit volume, which are crucial for our analysis. (Similar spaces and their relevance were already discussed in [BESB].) We require detailed knowledge of the operators in these spaces. We then use the mathematical tools we developed to compute rigorously the linear response of the system. This is achieved in two steps. First we solve the Liouville equation

$$\begin{cases} i\partial_t \varrho_\omega(t) = [H_\omega(t), \varrho_\omega(t)] \\ \lim_{t\to-\infty} \varrho_\omega(t) = P_\omega^{(E_F)} \end{cases} . \tag{2}$$

which describes the time evolution of a density matrix $\varrho_\omega(t)$ under the action of a time-dependent electric field $\mathbf{E}(t) = \mathrm{e}^{\eta t}\mathbf{E}$, for $t \le 0$, and any $\eta > 0$, $\mathbf{E} \in \mathbb{R}^d$. Here $P_\omega^{(E_F)}$ is the Fermi projector at Fermi energy E_F. We work in a suitable gauge in which

$$H_\omega(t) = (-i\nabla - A - \mathbf{F}(t))^2 + V_\omega(x) = G(t) H_\omega G(t)^*\,, \tag{3}$$

where $\mathbf{F}(t)=\int_{-\infty}^{t}\mathbf{E}(s)\mathrm{d}s$, and $G(t)=\mathrm{e}^{i\mathbf{F}(t)\cdot x}$. The assumption under which the Liouville equation can be solved reads

$$\mathbb{E}\left\{\left\|[x_k,\,P_\omega^{(E_F)}]\chi_0\right\|_2^2\right\}<\infty\,, \tag{4}$$

for $k=1,\cdots,d$. (This is essentially the condition identified in [BESB].) Of course, if E_F falls inside a gap of the spectrum of H_ω, then (4) is readily fulfilled by general arguments. The main challenge is to allow for the Fermi energy E_F to be inside the region of localization (the operator $[x_k,\,P_\omega^{(E_F)}]$ is then unbounded).

Second, we compute the net current per unit volume in the j-th direction $J^{(E_F)}(\eta,\mathbf{E})=\mathcal{T}(v_{j,\omega}(0)\varrho_\omega(0))-\mathcal{T}(v_{j,\omega}P_\omega^{(E_F)})$ induced by the electric field and show it is differentiable with respect to $\mathbf{E}$ at $\mathbf{E}=0$ ($\mathcal{T}$ is the trace per unit volume; $v_{j,\omega}(0)=\overline{i[H_\omega(0),x_j]}$ is the velocity). This yields the desired Kubo formula for the electric conductivity tensor.

Theorem. *Assume* (4)*, and let* $\eta>0$*. The map* $\mathbf{E}\to J^{(E_F)}(\eta,\mathbf{E})$ *is differentiable with respect to* $\mathbf{E}$ *at* $\mathbf{E}=0$ *and the derivative* $\sigma^{(E_F)}(\eta)$ *is given by*

$$\begin{aligned}\sigma_{jk}^{(E_F)}(\eta) &= \tfrac{\partial}{\partial\mathbf{E}_k}J_j^{(E_F)}(\eta,0)\\ &= -\mathcal{T}\left\{\int_{-\infty}^{0}\mathrm{d}r\,\mathrm{e}^{\eta r}v_{j,\omega}\,\mathcal{U}_\omega^{(0)}(-r)\left(i[x_k,P_\omega^{(E_F)}]\right)\right\},\end{aligned}$$

where $\mathcal{U}_\omega^{(0)}(r)(Y_\omega)=\mathrm{e}^{-irH_\omega}\odot_L Y_\omega\odot_R\mathrm{e}^{irH_\omega}$ *(here* $\odot_L$ *and* $\odot_R$ *refer to left and right multiplication in suitable normed spaces of operators).*

We then push the analysis further to recover the expected expression for the quantum Hall conductivity [NB, BESB, AG].

Theorem. *Assume* (4)*. For all* $j,k=1,2,\ldots,d$*, we have*

$$\sigma_{jk}^{(E_F)}=\lim_{\eta\downarrow0}\sigma_{jk}^{(E_F)}(\eta)=-i\mathcal{T}\left\{P_\omega^{(E_F)}\odot_L\left[[x_j,P_\omega^{(E_F)}],[x_k,P_\omega^{(E_F)}]\right]_\ddagger\right\}.$$

(Here $[Y_\omega,Z_\omega]_\ddagger$ *refers to a commutator of unbounded operators in suitable normed spaces of operators.)*

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In the work [CG], we prove an invariance principle for the edge conductance and discuss some consequences of it.

Consider the two-dimensional magnetic operator

$$H=H_B+V_L+V_R,\quad H_B=(-i\nabla-A)^2\ , \tag{5}$$

where A is the vector potential of a constant magnetic field of strength B; here V_L lives in the region $x_1\le0$, and it is either 0 or a wall V_0; next, V_R is located in the region $x_1\ge0$, and is typically an impurity potential. We further denote by $g(\lambda)$ a smooth decreasing function whose derivative lives in an interval $I=[E_1,E_2]$, that equals 1 if $\lambda\le E_1$ and 0 if $\lambda\ge E_2$, and by $\Lambda_2(x_1,x_2)=\Lambda_2(x_2)$ a similar smooth decreasing function, but two-dimensional, independent of x_1, and whose derivative lives in the region $\mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}]$. Then the edge conductance (in the x_2 direction) associated to this system (V_L, V_R, g) is defined as

$$\sigma_e(V_L, V_R, g) = -\mathrm{tr}(g'(H) i [H_B, \Lambda_2]), \tag{6}$$

whenever the trace is well-defined.

Assume that I lies in between two successive Landau levels, say the N^{th} and the $(N+1)^{\text{th}}$. While clearly $\sigma_e(0,0,g) = 0$, for any g as above, a straightforward computation shows that $\sigma_e(V_0, 0, g) = N$, provided $V_0(x_1, x_2) = V_0(x_1)$ is such that $\lim_{x_1 \to -\infty} V_0(x_1) > E_2$. As a first result, we show that the edge conductance is stable under a perturbation by a potential W located in a strip $[L_1, L_2] \times \mathbb{R}$: whenever the traces are finite,

$$\sigma_e(V_L + W, V_R, g) = \sigma_e(V_L, V_R + W, g) = \sigma_e(V_L, V_R, g). \tag{7}$$

That (7) does not hold anymore for any perturbation in the half-space $x_1 \geq 0$ is easy to see. However, we prove the following invariance principle.

Theorem. *Let V_0 and I be as above. Let V_R be a bounded potential located in the region $x_1 \geq 0$. Then the operator $(-g'(H_B + V_0 + V_R) + g'(H_B + V_R)) i [H_B, \Lambda_2]$ is trace class, and has trace N. In particular if one trace is finite then so is the second one, one has*

$$\sigma_e(V_0, V_R, g) - \sigma_e(0, V_R, g) = N\,. \tag{8}$$

As an immediate, but noticeable, consequence, if $\|V_R\|_\infty < B$, then $\sigma_e(V_0, V_R, g) = N$, whenever $I \subset](2N-1)B + \|V_R\|_\infty, (2N+1)B - \|V_R\|_\infty[$, recovering a recent result of Kellendonk Schulz-Baldes [KSB].

Notice that $\sigma_e(0, V_R, g) \neq 0$ means that there exists a current carrying states due to the impurity potential V_R. Existence of such "Edge current without edges" is thus responsible for the deviation of Hall conductance from its ideal value N in I. Typically this is expected to happen in a regime of strong disorder (with respect to the magnetic strength B). As an example of this phenomenon we revisit a model studied by S. Nakamura and J. Bellissard in [NB] and show that in this case $\sigma_e(V_0, V_R, g) = 0$ and thus $\sigma_e(0, V_R, g) = -N$.

As a counterpart, in the weak disorder regime (i.e. weak impurities in the region $x_1 \geq 0$ and no electric potential in the left half-plane), one expects that no current will flow near the region $x_1 = 0$. After a regularizing procedure we argue that this is exactly what happens with the model studied in [CH, GK, W], if I lies in a region of the spectrum where localization has been shown.

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Towards a constructive approach to Lyapunov exponents

Ilya Goldsheid

In his famous paper [1] H. Furstenberg proved that the top Lyapunov exponent of a typical product of unimodular independent identically distributed random matrices is *strictly* positive (see below). Already in this work the estimate of the spectral radius of an average of a unitary representation of a certain group plaid a major role. In 1980 A. Virtser [3] extended the above result to the case of Markov chains; in his work the idea of averaging of unitary operators was explained explicitly (and separately for the case of independent random matrices). The purpose of the present work is to establish the power k such that the L_p-norm of the relevant averaged operator raised to this power is strictly less than 1. The method we use is an extension of the one used by Virtser. It also turns out that G. Margulis used similar considerations in a more special context as early as in 1974.

Two beautiful papers by T. W. Wolff should be mentioned here. In [4] the case of 2×2 matrices was completely resolved (we note that Wolff's terminology is purely algebraic). In [5] he studied averages of general unitary L_2 representations and the relevant spectral gap property. It is likely that he was not familiar with the work of Virtser.

Let μ be a probability distribution on $SL(m,\mathbb{R})$ - the group of real matrices with determinant one. By $\operatorname{supp}\mu$ we denote the support of μ and by G_μ the group generated by this support. G_k denotes a group generated by the following set:

$$\{g_1^{-1}g_2^{-1}...g_k^{-1}g_{k+1}g_{k+2}...g_{2k} : \\ \text{where } g_j,\ 1 \le j \le 2k, \text{ are any matrices from the } \operatorname{supp}\mu\}$$

It is obvious that $G_1 \subset G_2 \subset ... \subset G_\mu$.

If g is a matrix and x is a unit vector then we put $g.x = gx/||gx||$. By $\mathcal{S}$ we denote the unit sphere in $\mathbb{R}^m$. A probability measure κ on $\mathcal{S}$ is said to be preserved by g if $\kappa(B) = \kappa(g.^{-1}B)$ for any Borel subset $B \subset \mathcal{S}$. A group G preserves a measure κ on $\mathcal{S}$ if every $g \in G$ preserves κ.

For a fixed $p > 1$ we consider the usual $L_p(\mathcal{S}$ space of functions with the uniform measure on $\mathcal{S}$. The 'p - unitary' representation of $SL(m,\mathbb{R})$ is then defined as follows: given a $\mathbb{C}$ valued function $f \in L_p(\mathcal{S})$, we put

$$V_g f(x) = f(g.x)||gx||^{-\frac{m}{p}}.$$

It is easy to verify that

$$||f||_p = ||V_g f||_p \quad \text{and that} \quad V_{g_1 g_2} = V_{g_2} V_{g_1}.$$

Finally we put

$$V = \int_{SL(m,\mathbb{R})} V_g d\mu(g).$$

Theorem 1. *If no probability measure on $\mathcal{S}$ is preserved by G_k then $||V^k||_p < 1$.*

Note that k does not depend on p.

Theorem 1 has a number of applications. Here I shall discuss only the case of products of identically distributed matrices.

From now on we suppose that the following condition is satisfied:

$$\int_{SL(m,\mathbb{R})} \ln ||g|| d\mu(g) < \infty.$$

Remember that the Lyapunov exponent of a product of matrices $S_n = g_n g_{n-1} ... g_1$ can defined by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln ||S_n||.$$

The existence of this limit (with probability 1) follows from the Kingman's subadditive ergodic theorem. Moreover, this limit is not random and its value is defined by the distribution μ of our random matrices. The two results below follow easily from Theorem 1 (and I even prove one of them).

Theorem 2. *If no probability measure on $\mathcal{S}$ is preserved by G_k, then*

$$\lambda \geq \frac{-p \ln ||V^k||_p}{km}.$$

Remarks. 1. Obviously $\lambda > 0$.

2. It would be important to estimate the $||V^k||_p$ in terms of the distribution μ. Such an estimate would allow one to drop the word 'Towards' in the title of this work.

Theorem 3. *(H. Furstenberg* [1]*) If no probability measure on $\mathcal{S}$ is preserved by G_μ, then*

$$\lambda > 0.$$

Proof: why multiplying by identity may be useful. Let us first consider the case when the unit matrix I belongs to the support of the measure μ. Then $G_1 = G_\mu$ and hence $||V||_p < 1$ (Theorem 1). According to Theorem 2 this implies $\lambda > 0$.

In the case when $I \notin \operatorname{supp} \mu$, we consider a modified distribution $\tilde{\mu} = (1 - q)\delta_I + q\mu$ with any q which lies strictly between 0 and 1. Here δ_I is a unit mass distribution concentrated at I. Obviously, $G_{\tilde{\mu}} = G_\mu$ but $\tilde{\mu}$ has the additional property that $I \in \operatorname{supp} \tilde{\mu}$. Hence the corresponding $\tilde{\lambda} > 0$. On the other hand it is clear that

$$\tilde{\lambda} = \lim_{n\to\infty} \frac{1}{n} \ln ||\tilde{g}_n \tilde{g}_{n-1} ... \tilde{g}_1|| = q \lim_{n\to\infty} \frac{1}{n} \ln ||g_n g_{n-1} ... g_1|| = q\lambda.$$

This implies that λ is strictly positive too. □

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Fractional moment methods for Anderson models in the continuum

Günter Stolz

(joint work with M. Aizenman, A. Elgart, S. Naboko, J. Schenker)

There are two methods available which provide rigorous proofs of localization properties in multi-dimensional Anderson models. The method of multiscale analysis (MSA) was introduced in 1983 by Fröhlich and Spencer [5]. By now, this method has been built into a powerful tool which applies in a broad range of situations for lattice as well as continuum models. A self contained introduction in book form can be found in [9]. For a state-of-the-art account see [6].

Ten years later, in 1993, Aizenman and Molchanov pioneered the fractional moment method (FMM) in [3]. It provided an elegant and rather short (e.g. [7]) new proof of Anderson localization for the case of lattice hamiltonians. The FMM lead to a proof of exponential decay of spatial correlations in the Schrödinger time evolution (e.g. [4]), a dynamical localization property which was expected for the Anderson model by physicists.

It took an additional ten years after the introduction of the FMM to extend it to continuum Anderson models. This was finally achieved in [1], see also [2] for a brief announcement of the results. A major technical obstacle was to find a continuum analogue of the finiteness of expectations of fractional powers of the Green function. Below we will describe briefly how a result of Naboko [8] on the value distribution of boundary values of resolvents of dissipative operators was used to overcome this difficulty.

In the MSA and the FMM mathematical physicists now have two general methods at hand to study localization properties. The two methods do in several ways complement each other. While the FMM gives stronger results on dynamics and (so far at least in the lattice case) allows for shorter proofs, MSA seems to allow for weaker assumptions on the regularity of the random parameters in the Anderson model.

Let us now briefly discuss the main ideas behind the proof of finiteness of fractional moments of the Green function in the continuum. This serves as the starting point of the detailed analysis in [1]. We work with a simplified model here, see [1] for generalizations.

Consider the continuum Anderson model

$$H_\lambda = H_\lambda(\omega) = -\Delta + \lambda \sum_{n\in\mathbb{Z}^d} \omega_n u_n(x)$$

in $L^2(\mathbb{R}^d)$. Here $u_n(x) = u(x-n)$ with u a non-negative, compactly supported and bounded bump function in $\mathbb{R}^d$. For technical reasons (applicability of the Birman-Schwinger method) we also need that the boundary of $\operatorname{supp} u = \{x : u(x) > 0\}$ has Lebesgue-measure zero. The ω_n are a sequence of i.i.d. random variables, such that their common distribution has a bounded density ρ supported in $[0, 1]$. We also include a disorder parameter $\lambda > 0$.

Finiteness of fractional moments of the Green function is established in the following form:

Theorem 1: *Let $0 < s < 1$. Then there exists $C_s < \infty$ such that*

$$\sup_{\varepsilon>0} \mathbb{E}\left(\|u_n(H_\lambda - E - i\varepsilon)^{-1}u_m\|^s\right) \le C_s(1+|E|)^{s(d+2)}(1+\frac{1}{\lambda})^s \tag{1}$$

for all $n, m \in \mathbb{Z}^d$, $E \ge 0$ and $\lambda > 0$.

To sketch the proof of this, consider the simplest case $n = m$ and set $\hat{H}_\lambda = H_\lambda - \lambda\omega_n u_n$. The resolvent equation leads to the Birman-Schwinger identity

$$u_n(H_\lambda - z)^{-1}u_n = u_n^{1/2}(K_n^{-1} + \lambda\omega_n)^{-1}u_n^{1/2}, \tag{2}$$

where

$$K_n = u_n^{1/2}(\hat{H}_\lambda - z)^{-1}u_n^{1/2},$$

as an operator in $L^2(\operatorname{supp} u_n)$, does not depend on ω_n. In taking the expectation of fractional powers of the norm of (2), we first integrate over ω_n. Due to the assumptions on the distribution on ω_n this leads us into needing a bound for

$$\int_0^1 d\omega_n \, \|u_n^{1/2}(A - \omega_n)^{-1}u_n^{1/2}\|^s, \tag{3}$$

where $A = -K_n^{-1}/\lambda$. A is maximally dissipative, which suggests to use the following fact, an immediate consequence of results due to Naboko [8]:

Theorem 2: *Let $\mathcal{H}, \mathcal{H}_0$ be separable Hilbert spaces and $0 < s < 1$. Then there exists $C(s) < \infty$, such that for all Hilbert-Schmidt operators $K : \mathcal{H} \to \mathcal{H}_0$, all maximally dissipative operators $A : \mathcal{H}_0 \to \mathcal{H}_0$ and all bounded probability densities*

ρ, it holds that

$$\int_{\mathbb{R}} \|K^*(A-q+i0)^{-1}K\|_{HS}^s \rho(q)\,dq \leq C(s)\|\rho\|_\infty^s \|K\|_{HS}^{2s}.$$

Of course, this result is not directly applicable to study (3), as multiplication by $u_n^{1/2}$ is not a Hilbert-Schmidt operator in $L^2(\mathbb{R}^d)$. But it nevertheless can be combined with perturbative arguments (using the relative trace ideal properties of the multipliers) to lead to a proof of Theorem 1 based on bounds of (3). See the details in [1].

Theorem 1 is somewhat weaker than the corresponding result in the lattice case, as the right hand side of (1) does not decay for $\lambda \to \infty$. It would be interesting to know if the bound can be improved to yield this decay. This would lead to simplifications in the localization proof provided in [1] for the large disorder regime.

We mention also that the localization proofs in [1] require that (1) is known with u_n, u_m replaced by χ_n, χ_m, characteristic functions of unit cubes centered at n and m. This leads to the additional "covering" assumption on the single site potential that $u \geq c\chi_0$ for some $c > 0$. Allowing for single site potentials with smaller support is another open question in the approach to localization via FMM (but is possible with MSA).

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Wegner estimates and localization for Gaussian random potentials and random vector potentials

Naomasa Ueki

On the spectral localization of a random Schrödinger operator

$$H^\omega = (i\nabla + A^\omega(x))^2 + V^\omega(x)$$

on a general dimensional Euclidean space, we have many results for the case that the scalar potential $V^\omega(x)$ is the alloy type random field and the vector potential $A^\omega(x)$ is deterministic. The subject of this talk is the extension to the case that

$V^{\omega}(x)$ is given by a Gaussian random field and $A^{\omega}(x)$ is also random. To obtain the strong dynamical localization and the semi-uniformly localized eigenfunctions, Germinet and Klein's theory [2] on the bootstrap multiscale analysis is extended to operators of unbounded below so that the above cases are included. Moreover, to obtain the Wegner estimate for the case that the vector potential is random, Klopp's method [3], [4] is extended to a short correlated Gaussian random field by the random Fourier series. Consequently we obtain the results on the localization for a Schrödinger operator with a bounded random vector potential and a Gaussian random scalar potential at sufficiently low energies. These are results in [6] and a refinement and a generalization of a special case of the result in Fischer, Leschke and Müller [1].

In the second part, for a Pauli Hamiltonian with a constant magnetic field perturbed by a centered Gaussian random magnetic field, the similar results are obtained on middle parts of the gap of the Landau level of the unperturbed operator. For this, the problem is reduced to the same problems for the random Dirac operator and a Birman-Schwinger type operator by referring a Raikov's work [5].

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Distribution of the maximal height of a fluctuating interface

Alain Comtet

(joint work with Satya Majumdar)

We present an exact solution for the probability distribution of the maximal height, measured with respect to the average height, in the steady state of a fluctuating one dimensional Edwards-Wilkinson interface of length L.

The height $H(x,t)$ satisfies the linear stochastic differential equation

$$\frac{\partial H(x,t)}{\partial t} = \frac{\partial^2 H(x,t)}{\partial x^2} + \eta(x,t), \tag{1}$$

where $\eta(x,t)$ is a Gaussian white noise with zero mean and correlation $\langle \eta(x,t)\eta(x't')\rangle = 2\delta(x-x')\delta(t-t')$. The relative height $h(x,t) = H(x,t) - 1/L\int H(y,t)dy$ converges at large time to a stationary Gaussian process $h(x)$. Our aim is to compute the probability distribution of the maximal height $F(h_m, L) = \text{Prob}\,[\max\{h\} < h_m, L]$.

The density $f(h_m, L) = \frac{\partial F(h_m,L)}{\partial h_m}$, which only depends of the scaling variable $x = h_m/\sqrt{L}$ is obtained in the following two cases:

1) Periodic boundary conditions $h(0) = h(L)$.
The density is given by the Airy distribution function

$$f_1(x) = \frac{2\sqrt{6}}{x^{10/3}} \sum_{k=1}^{\infty} e^{-b_k/x^2} b_k^{2/3} U(-5/6, 4/3, b_k/x^2), \tag{2}$$

α_k are the magnitude of the zeros of the Airy function, $b_k = 2\alpha_k^3/27$ and $U(a, b, c)$ is the confluent hypergeometric function.

2) Free boundary conditions.
We obtain a new distribution

$$\begin{aligned} f_2(x) &= \frac{1}{2^{1/3} x^{7/3} \sqrt{3\pi}} \\ &\quad \times \sum_{k=1}^{\infty} \alpha_k C(\alpha_k) e^{-b_k/x^2} \left(U(\tfrac{1}{6}, \tfrac{4}{3}, \tfrac{b_k}{x^2}) + 2U(-\tfrac{5}{6}, \tfrac{4}{3}, \tfrac{b_k}{x^2}) \right), \end{aligned} \tag{3}$$

where $C(\alpha) = [\int_{-\alpha}^{\infty} \text{Ai}(z)dz]^2 / [\text{Ai}'(-\alpha)]^2$.

These results have been derived by a path integral method by a mapping onto the quantum mechanical problem of a particle in a one dimensional electric field with an infinite wall at the origin $y = 0$. The corresponding hamiltonian is given by $\hat{H} \equiv -\frac{1}{2}\frac{\partial^2}{\partial y^2} + \lambda y$. For periodic boundary conditions, the characteristic function is given in terms of the partition function

$$\text{E}(e^{-\lambda L h_m}) = \sqrt{2\pi} L^{3/2} \text{Tr}(e^{-\hat{H} L}). \tag{4}$$

For free boundary conditions it involves the Hilbert-Schmidt norm of the resolvent.

These results can also be derived by using the Denisov and Verwaat decomposition of the Brownian motion around its maximum. In the periodic case, $\max h(x)$ is distributed as the area under a Brownian excursion (Takacs 1991). For free boundary conditions it is distributed as the sum of two independent Brownian meanders. The fact that the Airy distribution arises in a variety of contexts has been recently noticed by many authors. Our model adds a new example and raises some interesting questions in the statistical physics of interfaces. Possible relations with the recent work of C. Richard are certainly worth investigating.

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Bounds on the spectral shift function and the (integrated) density of states

Ivan Veselić

(joint work with D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann)

Summary

We study spectra of Schrödinger operators on $\mathbb{R}^d$. First we consider a pair of operators which differ by a compactly supported potential, as well as the corresponding semigroups. We prove almost exponential decay (as $n \to \infty$) of the singular values μ_n of the difference of the semigroups and a bound on the spectral shift function ξ of the operator pair.

Thereafter we consider alloy type random Schrödinger operators. The single site potential u is assumed to be non-negative and of compact support. Therefore we can apply the estimates on ξ mentioned above. It allows us to improve on earlier results on the Hölder continuity of the integrated density of states [2].

We are able to treat alloy type models with singular random coupling constants. More precisely, we can deal with Hölder continuous single site distributions. This improves earlier results [9] where a Wegner estimate with quadratic volume dependence was derived for such models.

Results

Let A be a vector-potential with components in L^2_{loc} and $H_A := (-i\nabla - A)^2$. Let V, u be potentials such that for both $W \in \{V, V+u\}$ the positive and negative part satisfy: $W_+ \in L^1_{\text{loc}}$ and W_- is Δ bounded with relative bound $1-\delta < 1$. Set $H_1 = -\Delta + V$ and $H_2 = H_1 + u$ and assume that u is compactly supported. Denote by H_1^l, H_2^l the corresponding Dirichlet restrictions to the cube $\Lambda_l = [-l/2, l/2]^d, l \geq 1$. Set $V_{\text{eff}} := e^{-H_1} - e^{-H_2}$ and $V_{\text{eff}}^l := e^{-H_1^l} - e^{-H_2^l}$.

Theorem 1. There are finite positive constants c_1, c_2 such that the singular values μ_n of the operator V_{eff}^l obey

$$\mu_n \leq c_1\, e^{-c_2\, n^{1/d}}.$$

The constants depend only on d, δ and the diameter of the support of u. The same estimate holds for the singular values of V_{eff}.

The proof uses Wely's asymptotic law for eigenvalues on balls, the Feynman-Kac formula for Schrödinger semigroups and exit time estimates for the Brownian motion.

Denote by $\xi(\cdot, H_2, H_1)$ the spectral shift function of the pair of operators H_1, H_2.

Theorem 2. Let $f \in C_c$ and set $b = \sup \operatorname{supp} f$. There exist constants K_1, K_2 depending only on d, δ and $\operatorname{diam} \operatorname{supp} u$ such that

$$\int f(\lambda)\, \xi(\lambda, H_2^l, H_1^l)\, d\lambda \leq K_1 e^b + K_2 \{\log(1 + \|f\|_\infty)\}^d \|f\|_1 \tag{1}$$

The same estimate holds for $\xi(\cdot, H_1, H_2)$.

Theorem 2 is derived from Theorem 1 using Young's inequality and [4]. It improves upon estimates established in [3].

Bounds like (1) are related to the question which operator pairs have a locally bounded spectral shift function. Negative results concerning this question can be found in [5, 7] and positive in [8].

An *alloy type model* is a random Schrödinger operator $H_\omega = H_0 + V_\omega$, where $H_0 = -\Delta + V_{\text{per}}$ is periodic. The random part of the potential has the following form $V_\omega(x) = \sum_{k \in \mathbb{Z}^d} \omega_k\, u(x-k)$. The *coupling constants* $\omega_k, k \in \mathbb{Z}^d$ are a sequence of bounded random variables, which are independent and identically distributed. Assume that the *single site potential* $u \in L^1(\mathbb{R}^d)$ is non-negative and of compact support. Using Theorem 2 we derive a new Wegner estimate [11] for certain alloy type Schrödinger operators.

Theorem 3. Let H_ω be an alloy type model. Assume that ω_0 has a Hölder continuous distribution with exponent α and that $u \geq \beta \chi_{[0,1]^d}$ for some $\beta > 0$. Then for each $E_0 \in \mathbb{R}$ there exists a constant C_W such that, for all $E \leq E_0$ and $\epsilon \leq 1/2$

$$\mathbb{E}\{\operatorname{Tr}[\chi_{[E-\epsilon, E+\epsilon]}(H_\omega^l)]\} \leq C_W\ \epsilon^\alpha\, (\log \tfrac{1}{\epsilon})^d\ l^d$$

The proof of Theorem 3 uses the following elementary

Lemma Let μ be a probability measure with support in $]a, \infty[$. Denote for $\epsilon > 0$

$$s(\mu, \epsilon) = \sup\{\mu([E-\epsilon, E+\epsilon]) \mid E \in \mathbb{R}\}$$

Let $\phi \in C^1(\mathbb{R})$ be non-decreasing with $\lim_{\lambda \to \infty} \phi(\lambda) \leq 0$. Then for any $0 < \epsilon < 1$,

$$\int_{\mathbb{R}} (\phi(\lambda + \epsilon) - \phi(\lambda))\, d\mu(\lambda) \leq s(\mu, \epsilon) \cdot (\phi(\infty) - \phi(a))$$

Combining our techniques with those of [2], we obtain

Theorem 4. Let H_ω be an alloy type model. Assume that ω_0 is distributed according to a bounded density, H_0 has the unique continuation property, and u is positive on an open set. Then for each $E_0 \in \mathbb{R}$ there exists a constant C_W such that, for all $E \leq E_0$ and $\epsilon \leq 1/2$

$$\mathbb{E}\{\operatorname{Tr}[\chi_{[E-\epsilon, E+\epsilon]}(H_\omega^l)]\} \leq C_W\ \epsilon\, (\log \tfrac{1}{\epsilon})^d\ l^d \tag{2}$$

Under more restrictive conditions, bounds like (2), which are linear in ϵ, have been established [6, 1, 10].

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Ergodicity and continuity properties of the integrated density of states

P. Stollmann

(joint work with S. Klassert, D. Lenz)

We study strictly ergodic Delone dynamical systems and the associated tight binding Hamiltonians [2, 4].

We prove an ergodic theorem for Banach space valued functions on the associated set of pattern classes. As an application, we prove existence of the integrated density of states in the sense of uniform convergence in distribution for the associated random operators.

This very strong convergence gives the following result (see [2]), saying that jumps in the integrated density of states are a consequence of the existence of eigenfunctions with compact support, i.e., a consequence of the lack of unique continuation.

Theorem 1. *Let (Ω, T) be a strictly ergodic DDSF and A an operator of finite range on (Ω, T). Then, E is a point of discontinuity of ρ^A if and only if there exists a locally supported eigenfunction of $A_\omega - E$ for one (every) $\omega \in \Omega$.*

The phenomenon that such localized eigenfunctions can appear has been known for quite some time, as seen from the list references.

We finish this section by giving the precise version of the ergodic theorem proven in [4].

Theorem 2. *For a minimal, aperiodic DDSF* (Ω, T) *the following are equivalent:*
(i) (Ω, T) *is uniquely ergodic.*
(ii) The limit $lim_{k\to\infty}|P_k|^{-1}F(P_k)$ *exists for every van Hove sequence* (P_k) *and every almost additive* F *on* (Ω, T) *with values in a Banach space.*

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Kubo's formula for quantum transport

Jean Bellissard

This lecture gives a review of various recent works made with collaborators on quantum transport in aperiodic media. In the first part of the talk, several mechanisms observed in experiments and investigated by theoreticians in the past are summarized. A special emphasize concerns the notion of Mott's variable range hopping conductivity that seems to dominate the low temperature behavior of lightly doped compensate semiconductors, in particular the ones used in experiments on the quantum Hall effect (QHE). This is because the relaxation time approximation does not explain the experiments, leading to a need for a more accurate model for dissipation.

In the second part of the talk, coherent transport is described within the the *relaxation time approximation* (RTA). Starting from the Drude model, the corresponding model of dissipation for the electron gas in an aperiodic medium is constructed. Namely, without dissipation, the electrons can be described by ignoring interactions: this is the independent particle approximation. Then their quantum motion (coherent transport) is described by a one-particle covariant Hamiltonian H depending upon the medium investigated. At random Poissonian times (collisions times), the actual state of the Fermi gas is updated to the equilibrium state. The average collision time τ_{rel} is a parameter of this model that must be fit by experiments. Then, if at time zero the electron gas is at equilibrium and if an external electric field is switched on, the (quantum, thermal and collision) averaged current satisfies the linear response hypothesis and the conductivity tensor can be proved to be given by Kubo's formula

$$\sigma_{i,j} \;=\; \frac{q^2}{\hbar}\,\mathcal{T}_{\mathbb{P}}\left(\partial_j\left(\frac{1}{1+e^{\beta(H-\mu)}}\right)\,\frac{1}{1/\tau_{rel}-\mathcal{L}_H}\,\partial_i H\right)$$

where $\mathcal{L}_H = \imath[H,\cdot]$ and $\partial_i A = \imath[R_i, A]$ whenever $\vec{R} = (R_1, \cdots, R_d)$ is the position operator. A short discussion follows to discuss the notion of transport exponent, β_F being the transport exponent at the Fermi energy, describing the coherent part of the transport. In particular the *anomalous Drude formula* predicts that the direct DC conductivity behaves like $\sigma \stackrel{\tau_{rel}\uparrow\infty}{\sim} \tau_{rel}^{2\beta_F-1}$, valid at at small temperature.

In the third part, a model for the variable range hopping is considered. Due to correlation introduced by the fermion nature of the charge carriers, it requires to work with the second quantized quasilocal algebra of observables. Moreover, since there is an infinite family of relaxation times, the dissipation must be described through a *Lindbladian*. To built this operator, acting on the quasilocal algebra, the probability amplitude of quantum jumps from one eigenstate of the one-particle Hamiltonian to another, including thermal baths, must be given. Then a series of rigorous results insures that such an operator is well defined in the infinite volume limit. Moreover it induces a norm pointwise continuous Markov semigroup on the quasilocal algebra which defines the dissipative evolution of the electron gas.

The last part of the talk is devoted to the general derivation of the Green-Kubo formula for generalized thermodynamical forces and currents (such as electric, thermol, spin, currents) in the linear response theory, within the previous framework. The validity of this approach requires the existence of at least three scales of times, space and energies, a microscopic one (where perturbation theory describes each collision), the microscopic one (given by the scales at which the experiments are performed) and a mesoscopic one, large enough to accomodate a large number of collisions, but small enough to be considered as infinitesimal compared to macroscopic scales. Moreover the Green-Kubo formula ia valid if and only if the Linbladian describing the dissipative evolution is invertible in some suitable space. Criterions insuring such an invertibility are not discussed in this lecture.

A short series of comments is given as a conclusion.

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Equality of the bulk and edge Hall conductances

J. H. Schenker
(joint work with A. Elgart and G. M. Graf)

Two conductances, σ_B and σ_E, are associated to the Quantum Hall Effect (QHE), depending on whether the currents are ascribed to the bulk or to the edge. The equality $\sigma_B = \sigma_E$ has been derived in a microscopic treatment of the integral QHE [SBKR00, EG02] in the case that the Fermi energy lies in a spectral gap Δ of the single-particle Hamiltonian H_B. We prove this equality, by quite different means, in the more general setting that H_B exhibits Anderson localization in Δ – precisely, H_B admits a *semi-uniformly localized eigenfunction* (SULE) [dRJLS96] basis (see (2) below.)

The bulk is represented by the lattice $\mathbb{Z}^2 \ni x = (x_1, x_2)$ with Hamiltonian $H_B = H_B^*$ whose matrix elements $H_B(x, x')$ are short range,

$$\sup_{x\in\mathbb{Z}^2} \sum_{x'\in\mathbb{Z}^2} |H_B(x,x')| \, (\mathrm{e}^{\mu|x-x'|} - 1) \ =: \ C_1 \ < \ \infty \tag{1}$$

for some $\mu > 0$, where $|x| = |x_1| + |x_2|$. The spectrum of H_B is assumed to be pure point in $\Delta \subset \mathbb{R}$ with a dense set of eigenvalues $\mathcal{E}(\Delta)$ and a SULE eigenfunction basis, i.e., each eigenfunction ψ_λ associated to an eigenvalue $\lambda \in \mathcal{E}(\Delta)$ has a center of localization $x(\lambda) \in \mathbb{Z}^2$ such that for any $\varepsilon > 0$

$$|\psi_\lambda(x)| \ \le \ C_\varepsilon \mathrm{e}^{\varepsilon|x(\lambda)|}\mathrm{e}^{-\mu|x-x(\lambda)|} \quad \text{for all } \lambda \in \mathcal{E}(\Delta) \ . \tag{2}$$

We consider simple spectrum for ease of notation (finite degeneracy + SULE would suffice).

The above hypotheses hold almost surely for an ergodic Schrödinger operator whose Green's function satisfies a moment condition such as derived in ref. [AM93, ASFH01]. The existence of a SULE basis [dRJLS96] follows from a dynamical localization bound [Aiz94] and simplicity of the spectrum [Sim94].

The bulk Hall conductance at Fermi energy λ is defined to be

$$\sigma_B(\lambda) \ = \ -\mathrm{i}\,\mathrm{tr}\, P_\lambda \, [\, [P_\lambda, \Lambda_1] \, , [P_\lambda, \Lambda_2] \,] \ , \tag{3}$$

where $P_\lambda = E_{(-\infty,\lambda)}(H_B)$ and $\Lambda_i(x)$ is the characteristic function of $\{x = (x_1, x_2) \in \mathbb{Z}^2 \mid x_i < 0\}$. Because of localization, $\sigma_B(\lambda)$ is well-defined for $\lambda \in \Delta$ and independent thereof, i.e., it shows a plateau.

The edge sample is modelled as a half-plane $\mathbb{Z}\times\mathbb{Z}_a$, where $\mathbb{Z}_a = \{n \in \mathbb{Z} \mid n \ge -a\}$ with the height $-a$ of the edge eventually tending to $-\infty$. The Hamiltonian $H_a = H_a^*$ on $\ell^2(\mathbb{Z}\times\mathbb{Z}_a)$ is obtained by restriction of H_B under Dirichlet, $H_a = \mathcal{J}_a^* H_B \mathcal{J}_a$, or more general boundary conditions. We remark that eq. (1) is inherited by H_a with a constant C_1 that is uniform in a, but not the existence of a SULE basis as a rule.

The definition of the edge conductance is simplest if we temporarily assume that Δ is a gap for H_B, i.e., if $\sigma(H_B) \cap \Delta = \emptyset$, in which case one may set [SBKR00]

$$\sigma_E \ := \ -\mathrm{i}\,\mathrm{tr}\, \rho'(H_a) \, [H_a, \Lambda_1] \ , \tag{4}$$

where $-\mathrm{i}\,[H_a, \Lambda_1]$ is the current operator across the line $x_1 = 0$ and $\rho \in C^\infty(\mathbb{R})$ satisfies $\rho(\lambda) = 1$ if $\lambda < \Delta$ and $= 0$ if $\lambda > \Delta$.

To motivate (4), we interpret $\rho(H_a)$ as the 1-particle density matrix of a stationary state at the edge, imagining that any steady state current in this state is cancelled by an equal but opposite current at another edge located at $x_2 = +\infty$. Lowering the chemical potential by δ at the first edge, but not the second, induces a net current

$$I \;=\; -\mathrm{i}\,\mathrm{tr}\,((\rho(H_a+\delta)-\rho(H_a))\,[H_a, \Lambda_1])\ , \tag{5}$$

so eq. (4) gives the conductance I/δ in the limit of small δ.

When H_B has spectrum in Δ, a regularization is required to define σ_B because the operator in (4) is no longer trace class. A possible regularization is to consider only the current flowing across the line $x_1 = 0$ within a finite window $-a < x_2 < 0$ near the edge, which amounts to taking the quantity

$$\lim_{a\to\infty} -\frac{\mathrm{i}}{2}\,\mathrm{tr}\,\rho'(H_a)\,\{\,[H_a, \Lambda_1]\,, \Lambda_2\} \tag{6}$$

as a candidate for σ_E. Although this limit exists, it is not the physically correct choice.

Theorem 1. *We have*

$$\lim_{a\to\infty} -\frac{\mathrm{i}}{2}\,\mathrm{tr}\,\rho'(H_a)\,\{\,[H_a, \Lambda_1]\,, \Lambda_2\} \;=\; \sigma_B \;-\; \sum_{\lambda\in\mathcal{E}(\Delta)} \rho'(\lambda)\,\mathrm{Im}\,(\psi_\lambda, \Lambda_1 H_B \Lambda_2 \psi_\lambda) \tag{7}$$

where the sum is absolutely convergent.

The limit has two contributions: i.) the bulk conductance σ_B and ii.) a term arising from persistent currents in bound states ψ_λ supported near the point $(0, 0)$. These bound states have no <u>net</u> current across the line $x_2 = 0$,

$$-\mathrm{i}\,(\psi_\lambda, [H_B, \Lambda_1]\,\psi_\lambda) \;=\; 0\ , \tag{8}$$

a fact which can (and should) be preserved by the regularization provided the spatial cutoff Λ_2 is time averaged.

Theorem 2. *Let*

$$\sigma_E \;:=\; \lim_{T\to\infty}\lim_{a\to\infty} -\frac{\mathrm{i}}{2T}\int_0^T \mathrm{d}t\,\mathrm{tr}\,\rho'(H_a)\,\big\{\,[H_a, \Lambda_1]\,, \mathrm{e}^{\mathrm{i}H_a t}\Lambda_2\mathrm{e}^{-\mathrm{i}H_a t}\big\}\ . \tag{9}$$

The iterated limit (9) *exists and* $\sigma_E \;=\; \sigma_B$.

The use of a regularization preserving (8) is analogous to the procedure [Rob73] used in the theory of electro-magnetism of material media to show that the macroscopic current in a macroscopically homogeneous medium vanishes, although microscopic currents <u>need not</u> vanish there. It has been emphasized [FS93] that σ_B and σ_E should be understood as macroscopic properties. The sum in (7) depends on the choice of origin and is best understood as a microscopic quantity, which need not vanish – the asymptotic expansion for this quantity at large energy with

H_B the Harper hamiltonian plus an i.i.d. Cauchy random potential has a non-zero term. However the time averaged current vanishes:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^\infty \sum_{\lambda\in\mathcal{E}(\Delta)} \rho'(\lambda)\,\mathrm{Im}\left(\psi_\lambda, \Lambda_1 H_B \mathrm{e}^{\mathrm{i}H_B t}\Lambda_2 \mathrm{e}^{-\mathrm{i}H_B t}\psi_\lambda\right) \;=\; 0\,. \tag{10}$$

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Thouless formula for non-Hermitian random Jacobi matrices

Boris Khoruzhenko
(joint work with Ilya Goldsheid)

Let (a_j, b_j, c_j) be a sequence of independent and identically distributed random three-component vectors. In this work we are concerned with the limiting distribution of eigenvalues for the Jacobi matrices

$$J_n = \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{pmatrix}$$

in the limit when the matrix dimention n tends to infinity. If all b_j are real and $c_j = a_{j+1}^*$ for all j the matrices J_n are Hermitian. The eigenvalue distribution of such matrices was studied extensively in the past in the context of the Anderson model, see e.g. [6, 2]. In this case the eigenvalues are always real and there are several ways to prove that the normalized eigenvalue counting measure of J_n converges to a limiting measure as $n \to \infty$. None of these proofs works in the non-Hermitian case and little is known about the limiting eigenvalue distribution of random non-Hermitian Jacobi matrices.

In this work we consider the general case when there are no restrictions on the coefficients a_j, b_j, c_j (and hence J_n may have complex eigenvalues) and prove that with probability one the normalized eigenvalue counting measure of J_n converges weakly to a non-random measure μ as $n \to \infty$. We also extend the Thouless formula to the non-Hermitian case. This formula relates μ and the Lyapunov exponent

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{2n} \log[|f_{n+1}(z)|^2 + |f_n(z)|^2] \tag{1}$$

of the second order difference equation

$$a_j f_{j-1} + b_j f_j + c_j f_{j+1} = z f_j, \qquad j = 1, 2, \ldots.$$

In (1) $f_n(z)$ is the solution of the above equation satisfying the initial condition $f_0 = 0$, $f_1 = 1$.

It is well known that $\gamma(z)$ is a (non-random) subharmonic function in the entire complex plane. The subharmonicity of γ implies that $\frac{1}{2\pi}\Delta\gamma$, where Δ is the distributional Laplacian in variables $\mathrm{Re}\, z$ and $\mathrm{i}z$, defines a measure on $\mathbb{C}$. Our main result is that this measure describes the limiting eigenvalue distribution for J_n:

Theorem 1. *Assume that $\{(a_j, b_j, c_j)\}_{j=1}^{\infty}$ is a sequence i.i.d. random vectors with a non-degenerate probability distribution and such that for some $\delta > 0$, $E[|a_j|^\delta + |a_j|^{-\delta} + |b_j|^\delta + |c_j|^\delta + |c_j|^{-\delta}] < \infty$. Then:*

(a) *With probability one, the normalized eigenvalue counting measure μ_n of J_n converges weakly to $\mu = \frac{1}{2\pi}\Delta\gamma$ as $n \to \infty$.*
(b) *(Thouless formula) $\gamma(z) = \int_{\mathbb{C}} \log|w - z|\, d\mu(w) - E\log|c_1|$ for every $z \in \mathbb{C}$.*
(c) *The limiting eigenvalue counting measure μ is log-Hölder continuous. More precisely, for any $B_{z_0,\delta} = \{z : |z - z_0| \le \delta\}$, $0 < \delta < 1$, $\mu(B_{z_0,\delta}) \le C(z_0, \delta)/\log\frac{1}{\delta}$, where $C(z_0, \delta) \to 0$ as $\delta \to 0$.*

We deduce Theorem 1 from Theorem 2 which is of independent interest in the context of second order difference equations.

Theorem 2. *With probability one,*

$$\lim_{n\to\infty} \frac{1}{n} \log|f_{n+1}(z)| = \gamma(z) \tag{2}$$

for almost all z with respect to the Lebesgue measure on $\mathbb{C}$.

To prove Theorem 2, we use the theory of products of random matrices. Of course this is unnecessary in the Hermitian case. In this case (2) and the Thouless formula follow directly from the fact that μ_n converges weakly to a limiting measure μ [1, 4] and the latter can be established independently and by more elementary means. We would like to emphasize that in the non-Hermitian case we follow the opposite direction route: the weak convergence of μ_n and the Thouless formula are deduced from (2). To this end we make use of the relation

$$\mu_n = \frac{1}{2\pi n}\Delta \log|f_{n+1}|. \tag{3}$$

[Note that $f_n(z)$ is a polynomial in z of degree n, its roots coincide with the eigenvalues of J_n, hence the above formula.]

Theorem 2 asserts that $\frac{1}{2\pi n}\log|f_{n+1}|$ converges in the limit $n\to\infty$ for almost all z. This convergence implies convergence in the sense of distribution theory, see e.g. [7], and as the distributional Laplacian is continuous on distributions, it also implies the weak convergence of μ_n to $\frac{1}{2\pi}\Delta\gamma$ and the Thouless formula.

Details of our proofs can be found in [5].

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A variational principle for spin glass models

Robert Sims

(joint work with M. Aizenman and S. Starr)

Spin glass systems are mean-field models of disordered magnetic structures where the interactions between spins are chosen at random. Perhaps the most scrutinized example of such a system is the original Sherrington-Kirkpatrick (SK) model, introduced in [12],

$$H_N(\sigma,h) := -\frac{1}{\sqrt{N}}\sum_{1\le i<j\le N} J_{ij}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \tag{1}$$

where the spin configurations $\sigma=\{\sigma_i\}_{i=1}^N\in\{\pm1\}^N$, the couplings J_{ij} are independent, identically distributed $N(0,1)$ gaussian random variables, and $h\in\mathbb{R}$ is an external magnetic field. As the interactions are gaussians, almost every representative $J(\omega)=\{J_{ij}(\omega)\}$ will possess some pairs (i,j), corresponding to spins (σ_i,σ_j), for which the couplings J_{ij} are positive, as well as others for which $J_{k,\ell}$ is negative. In this case, the system is said to be *frustrated*, or in other words, the selection of a particular configuration σ which minimizes the energy (i.e., the

Hamiltonian $H_N(\sigma, h)$) is highly non-trivial. For this reason, it is expected that the *overlaps*, i.e.,

$$q_{\sigma,\sigma'} := \frac{1}{N}\sum_{i=1}^{N} \sigma_i \sigma_i', \tag{2}$$

corresponding to those low energy configurations will have a rather intricate structure, and important insights on these heuristics have been obtained with numerics, indepth theory, and, in the past few years, rigorous proof.

For mathematical results concerning (1), one is often interested in the finite volume *partition function*

$$Z_N(\beta, h) := \sum_{\sigma} e^{-\beta H_N(\sigma,h)} \tag{3}$$

and *pressure*

$$P_N(\beta, h) := \ln\left[Z_N(\beta, h)\right], \tag{4}$$

which is a constant multiple of the *free energy.* It was not until quite recently that Guerra and Toninelli, see [6, 7], were able to verify the existence of the thermodynamic limit:

$$P(\beta, h) = \lim_{N\to\infty} P_N(\beta, h). \tag{5}$$

Their method of proof introduces an ingenious interpolation scheme which readily demonstrates that certain quantities are superadditive; from this observation, the existence of the infinite volume limit follows easily.

Much earlier, G. Parisi, in [8] (see also [9]), made an intriguing conjecture that the observed spin configurations, with respect to a Gibb's state, will have overlaps which are governed by a hierarchical structure, referred to as "ultrametricity" in the physics literature. Using this marvelous, calculation-facilitating ansatz, Parisi was able to determine an explicit "solution" describing the pressure of the SK system, i.e., (5). Quickly following their original result, Guerra, in [5], was able to push these interpolation techniques further and prove that "Parisi's solution" does indeed provide a rigorous upper bound on (5). With this proof, "one-half" of Parisi's conjecture was verified; the lower bound was still lacking.

In [2], we characterize the pressure of the SK system in terms of a variational principle. To do so, we consider larger systems composed of a ROSt reservoir, or a set of configurations which interact via some random structure, and a block of SK spins. Motivated by the cavity perspective introduced in [9], we calculate the change in free energy of such a system as a result of allowing the reservoir configurations to interact with the SK spins. We obtain the SK pressure by infimizing a particular functional over all possible ROSt reservoirs. By restricting the class of ROSts to those generated by random probability cascades, see [4] and [11], we reproduce Guerra's upper bound.

More recently, M. Talagrand has proven a result which shows that Parisi's solution is indeed correct, see [13, 14, 15]! This confirms a long standing conjecture in the physics community and points the way towards the resolution of many interesting related problems.

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Multiparticle random Schrödinger operators

Heribert Zenk

We consider a fixed number N of interacting electrons moving under the influence of an additional random perturbation in d-dimensional configuration space $\mathbb{R}^d$. The Hamiltonian acting in $L^2(\mathbb{R}^{Nd})$ is of the form

$$H(\omega) := - \triangle + V_I - V_N(\omega)$$

The interaction given by

$$V_I(x_1, ..., x_N) = \sum_{1 \leq k < l \leq N} v(x_k - x_l),$$

is the sum of repulsive two body interaction potentials $v : \mathbb{R}^d \to [0, \infty[$. We assume that V_I is infinitesimally $-\triangle$-bounded in $L^2(\mathbb{R}^{Nd})$, so $-\triangle + V_I$ is selfadjoint,

$$V_N(\omega, x_1, ..., x_N) = \sum_{k=1}^{N} V_1(\omega, x_k)$$

is the sum of one particle random potentials. The realizations $\omega = (\omega_j)_{j\in\mathbb{Z}^d}$ are supposed to be in the probability space $\Omega = ([0,1]^{\otimes\mathbb{Z}^d}, \mathcal{B}, \mathbb{P} = \mathbb{P}_0^{\otimes\mathbb{Z}^d})$, where $\mathbb{P}_0$ is a probability measure on $[0, 1]$ and the one-particle potential

$$V_1(\omega, x) = \sum_{j\in\mathbb{Z}^d} \omega_j \wp(x - j)$$

is of Anderson type with single site potential $\wp$ with compact support. We prove some basic properties of these family of random Schrödinger operators.

Theorem. *There are closed sets* $\Sigma = \Sigma_{\mathrm{ess}}, \Sigma_{\mathrm{ac}}, \Sigma_{\mathrm{sc}}, \Sigma_{\mathrm{pp}}, \Sigma_{\mathrm{c}}, \Sigma_{\mathrm{s}} \subseteq \mathbb{R}$, *such that* $\Sigma = \sigma(H(\omega)) = \sigma_{\mathrm{ess}}(H(\omega))$, $\Sigma_{\mathrm{ac}} = \sigma_{\mathrm{ac}}(H(\omega))$, $\Sigma_{\mathrm{sc}} = \sigma_{\mathrm{sc}}(H(\omega))$, $\Sigma_{\mathrm{pp}} = \sigma_{\mathrm{pp}}(H(\omega))$, $\Sigma_{\mathrm{c}} = \sigma_{\mathrm{c}}(H(\omega))$ *and* $\Sigma_{\mathrm{s}} = \sigma_{\mathrm{s}}(H(\omega))$ *for* $\mathbb{P}$*-almost every* $\omega \in \Omega$.

The proof uses the theory of ergodic operator families, as $H(\omega)$ is ergodic with respect to translations shifting with $(x, x, ..., x) \in \mathbb{Z}^{Nd}$. This implies, that the sub-/superadditive ergodic theorem is not appliciable for the existence proof of the integrated density of states, but there is an explicit formula:

Theorem. *(joint work with Frédéric Klopp) The integrated density of states for the interacting* N*-electron Anderson model given by*

$$N_N(E, \omega) := \lim_{L\to\infty} \frac{1}{L^{Nd}} \mathrm{Trace}(\mathbf{1}_{]-\infty,E]} H_L(\omega)) \tag{1}$$

($H_L(\omega)$ *is the restriction of* $H(\omega)$ *to* $\{x = (x_1, ..., x_{Nd}) \in \mathbb{R}^{Nd} : |x_j| < \frac{L}{2}\}$ *with Dirichlet boundary conditions) is for any fixed* $E \in \mathbb{R}$ *a* $\mathbb{P}$*-almost everywhere constant random variable. The expectation value is given by*

$$N_N(E) = (N_1 * \nu * \cdots * \nu)(E), \tag{2}$$

where N_1 *is the integrated density of states for the one electron model and* ν *is the density of states (i.e. the measure on* $\mathbb{R}$ *given by the distribution function* N_1*).*

Regularity properties of N_N are obtained from the Wegner estimate, whose proof is very similar to the one particle case:

Theorem. *Let* $\mathbb{P}_0$ *have a density* $g \in L^\infty(\mathbb{R})$ *of compact support and* $\wp(x) \geq 1$ *for* $\|x\| \leq 1$ *and let* π_L *be the spectral projection of* H_L*. For each sidelength* $L \geq 1$ *and each* $E_0 \in \mathbb{R}$*, there is a* $C_W = C_W(E_0, \|g\|_{L^\infty}) < \infty$*, such that*

$$\mathbb{E}(\mathrm{Trace}(\pi_L(]E_0 - \eta, E_0 + \eta[))) \leq C_W \eta L^{Nd}$$

for all $\eta > 0$.

Many interesting problems like a localization proof remain open for this model. For a localization proof the problem is the lack of independence of the restrictions H_Λ and $H_{\Lambda'}$ of H to disjoint regions Λ and $\Lambda' \subseteq \mathbb{R}^{Nd}$.

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