Introduction to Disordered Systems and Spin Glasses

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Abstract This chapter reviews some basic material required for the study of disordered systems, establishing some of the terminology used in later chapters. It contains a summary of the necessary background in statistical mechanics, including mean-field theory, scaling at continuous phase transitions, and Landau theory. It also provides a description of the percolation problem as well as informal derivations of some useful general results in disordered systems, such as the Harris criterion for the relevance of disorder and the Imry-Ma argument for the stability of ordered states to random-field perturbations. Spin glasses are introduced as a prototypical example of disordered systems. The replica approach to the Sherrington-Kirkpartrick model of infinite range Ising spin glasses is briefly described, together with a summary of more recent developments in spin glass physics. The connection between theories of spin glasses and satisfiability problems in theoretical computer science is outlined briefly.

1 Thermodynamics and Statistical Mechanics

Thermodynamics is grounded in the following assertion: Sufficiently large systems can attain macroscopic states in which measured properties are independent of preparation history. Provided no currents flow through such a system, it is said to be in **thermal equilibrium**.

The physical properties, at a coarse-grained level, of systems in thermal equilibrium are determined by a small number of macroscopic variables. Two distinct systems in thermal equilibrium which possess identical values of these variables will behave similarly. Such systems can be said to be in the same **macrostate**. A complete set of macroscopic variables describing a monoatomic gas are the num-

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ber of particles N in the gas, the volume V of the gas and the internal energy E of the gas. These are **extensive variables** *i.e.* for a composite system made up of two subsystems, each having its own value of E, V and N, the values of E, V and N characterizing the composite system are obtained by adding their values for the subsystems.

The internal energy E of a thermodynamic system can be changed in several ways. One could modify, for example, a **mechanical variable**, such as the volume V of the system. Alternatively, one could alter a **chemical variable**, such as the number of molecules N of a specific type. Thermodynamics recognizes that there is yet another way of changing E which does not involve changes in either mechanical or chemical variables. This involves the transfer of heat to or from the system. Operationally, heat may be transferred by placing the system in contact with another macroscopic system at a different temperature. (The existence of a temperature is a fundamental postulate of thermodynamics, the zeroth law of thermodynamics.) Heat transfer alters the internal energy of the system without affecting other macroscopic variables. Understanding the physical consequences of heat transfer requires the introduction of another extensive variable, the **entropy**.

Thermodynamics postulates that a single function of the extensive variables suffices to fully characterize the thermodynamic behaviour of the system. Such a function, termed a **fundamental relation**, can be written as $S = S(E,N,V)$, defining a quantity called the entropy S as a function of E, V and N.

Thermodynamics, by definition, is concerned with macroscopic quantities. However, macroscopic behaviour is determined by microscopic detail *i.e.* by the probability of finding the system in each of its *microstates*. The entropy S is the single central quantity connecting the system description at the level of its microstates to the macrostate specification. Adding heat to the system, keeping other macroscopic variables of the system such as N and V constant, changes the distribution of the system across its microstates, thereby altering the entropy.

Postulating the existence of S, coupled together with a few additional, reasonable assumptions concerning its monotonicity and analyticity as a function of E, V and N, reproduces all the hard-won results of classical thermodynamics. In textbooks, this discussion is always presented in terms of engines and processes, but the approach of postulating the existence of an entropy *a priori* is more elegant, as well as easier to understand.

It must be assumed that that S is a monotonically increasing function of the internal energy E. For a composite system, made of two independent systems with extensive variables N_1, V_1 and E_1 and N_2, V_2 and E_2, and thus the fundamental relations $S_1(E_1, N_1, V_1)$ and $S_2(E_2, N_2, V_2)$, the entropy S is additive *i.e.* $S = S_1 + S_2$. Given that S is a monotonic function of E, the relationship can be reversed to provide $E = E(S,N,V)$, a form which is often more convenient. Provided the subsystems composing the composite system do not exchange energy, particles or volume, these quantities are separately conserved for the subsystems.

Having defined the entropy as an additional macroscopic variable required in the description of the system, the central idea is then the following: If the two subsystems are allowed to exchange energy, ensuring that only $E = E_1 + E_2$ is

conserved and not E_1 and E_2 separately then the system seeks a final state such that $S = S(E,V,N)$ is maximized over all ways of partitioning E over the subsystems. (If the subsystems exchange particles or volume, a similar statement applies to $N = N_1 + N_2$ and to $V = V_1 + V_2$.

We can think of relaxing the constraint that the two subsystems do not exchange energy, volume or particle number as relaxing an internal constraint of the composite system made up of the two subsystems. Then, the second law of thermodynamics can be phrased in the following way:

> Removing a constraint internal to the system results in a final entropy which cannot be lower than the initial entropy.

An **extremum principle**, the maximization of entropy, thus controls the equilibrium behaviour of a thermodynamic system. Allowing the subsystems to exchange energy and thus achieve equilibrium with each other requires that the entropy be maximized at fixed total energy E over the its distribution over the subsystems. Alternatively, the energy E should be minimized at fixed overall entropy S. This is just the law of entropy increase, summarized in textbook formulations, of the **second law of thermodynamics**.

From this, all of thermodynamics can be shown to follow, including the definitions of temperature T, chemical potential μ and pressure P. These definitions are

$$T = \left(\frac{\partial E}{\partial S}\right)_{N,V}, \quad P = -\left(\frac{\partial E}{\partial V}\right)_{S,N}, \quad \mu = \left(\frac{\partial E}{\partial N}\right)_{S,V}. \tag{1}$$

Such a definition ensures that two systems which can exchange energy are in thermal equilibrium only if their temperatures are equal. Similarly, mechanical equilibrium between two systems which can exchange matter (volume) requires that the chemical potentials (pressures) of both subsystems be equal.

In calculations or in experiments, it is often easier to constrain the temperature T defined above, rather than E. This can be done by the simple expedient of allowing the system to exchange energy with a **thermal bath** at temperature T. Such a bath is realizable in terms of a system with a large number of degrees of freedom whose temperature is unchanged if an arbitrary finite amount of energy is added to or removed from it. Defining an appropriate extremum principle if T, as opposed to E, is held fixed, is done using the idea of a Legendre transform. Since T and S are conjugate variables, the Legendre transform $F(T,V,N)$ is defined through

$$F(T,V,N) = E(S,V,N) - TS. \tag{2}$$

The Legendre transform F of E, termed the Helmholtz free energy, provides a new fundamental relation in terms of the quantities T,V and N. Like the entropy, it provides a complete thermodynamic description of the system when T, V and N are fixed. It also obeys a similar extremum principle – the Helmholtz free energy is minimized in equilibrium. Other potentials follow by eliminating other conjugated quantities *e.g.* the number of particles N in favour of the chemical potential or the volume V in favour of the pressure P.

Entropy is a concept which is central to thermodynamics. Changes in the entropy represent the effects of the transfer of energy (heat) to microscopic degrees of freedom. In a theory which begins with these microscopic degrees of freedom *i.e.* statistical mechanics, the entropy must have a suitable definition.

The crucial connection between microscopic and macroscopic behaviour was made by Boltzmann, who derived the fundamental formula

$$S = k_B \ln \Omega(E, V, N),\tag{3}$$

with $\Omega(E, V, N)$ defined as the number of microstates accessible to a system once a fixed E, V and N are specified. The quantity k_B is Boltzmann's constant, required for consistency with the units in which thermodynamic measurements are conventionally made.

This defines, for statistical mechanics, the fundamental relation of the system, in the **microcanocanical ensemble**. Given this fundamental relation, other thermodynamic potentials can be easily derived. The results for thermodynamic quantities must be the same in these diferent ensembles in the thermodynamic limit. This requires that that the most probable value and the mean value of thermodynamically measurable quantities should coincide in the limit of large systems.

While the broad formalism of classical thermodynamics carries over to the study of disordered systems, such systems do have some unique features. Appropriately defined coarse-grained free energies in disordered systems have more complex structure than their pure system counterparts. Such free energies typically possess multiple low-lying minima, separated by large barriers. The degeneracy of the ground state can, in some cases, be extensive in the size of the system. Relaxation out of such metastable states can dominate the dynamics, often ensuring that the system does not equilibrate over experimental time scales and thus obscuring the relationship to the underlying thermodynamic system at equilibrium.

2 Statistical Mechanics: Once over lightly

A physical system is characterized by listing the states available to the system, as well as a dynamics which takes the system between these states. For systems described by conventional thermodynamics, one assumes ergodicity. This corresponds to the statement that time averages, a property of the dynamics of the system, can be replaced by statistical averages over the probability of finding the system in its different states.

For concreteness, consider Ising spin models. These are models for spins placed on a lattice in which the spins can take values $\sigma_i = \pm 1$. A model in which N Ising variables σ_i interact pairwise with each other as well as separately with an external field H_{ext} is

$$\mathcal{H} = -J \sum_{\langle ij \rangle}^{N} \sigma_i \sigma_j - H_{ext} \sum_{i=1}^{N} \sigma_i. \tag{4}$$

Interactions between spins are modeled here via an exchange coupling J. The notation $\langle ij \rangle$ normally implies a sum over distinct pairs of spins. This sum is often restricted to the case in which the pairs considered are neighbouring sites. The lowest energy state of this model, or its **ground state**, has all Ising spins aligned with the external field: $\{\sigma_i = 1\}$.

This model system is the simplest example of a model which exhibits non-trivial behaviour as a consequence of interactions, including a finite temperature phase transition at zero field in dimensions $d = 2$ and above.

A microstate of this model is a particular spin configuration, for example the configuration:

$$\{01010101100000101010\ldots\}. \tag{5}$$

There are 2^N microstates in this simple model.

Statistical mechanics prescribes rules for connecting averages over microstates to macrostates. The properties we would like to calculate are thermodynamic properties such as the specific heat, the average energy or susceptibility. Ensembles in statistical mechanics provide specific rules for calculating these properties, corresponding to the constraints placed on the system.

In the canonical ensemble, intended to describe systems in contact with a large bath at fixed temperature, but with fixed volume and number of particles, the rule is that the contribution of each microstate of energy E is weighted by a factor $\exp(-\beta E)$, with $\beta = 1/k_B T$ and k_B the Boltzmann constant. This associates a probability p_k with each microstate, of the form

$$p_k = \exp(-\beta E_k)/\mathcal{Z}, \tag{6}$$

where $\mathcal{Z}$ is a normalizing factor. Any macroscopic quantity may be obtained as a sum of its value in each microstate, weighted by the probability of obtaining the microstate. Thus

$$\langle O \rangle = \sum_k O(k) p_k, \tag{7}$$

where the sum ranges over all microstates and $O(k)$ is the value of the observable O in microstate k. We assume that the probabilities are appropriately normalized, so that $\sum_k p_k = 1$.

The calculation of thermodynamic quantities is made easier through the calculation of the **partition function** $\mathcal{Z}$, defined by

$$\mathcal{Z} = \sum_{\{\sigma\}} e^{-\beta \mathcal{H}}. \tag{8}$$

where we sum over all values the spins can take, *i.e* over all the microstates of the system. (Sometimes, for notational reasons, we replace $\sum_{\{\sigma\}}$ by the trace notation Tr_{σ}, but the calculational operation is the same in both cases.)

The link between the partition function and the thermodynamic potential of relevance here, the Helmholtz free energy, since the system is in contact with a heat bath at temperature T, is

$$F = -k_B T \ln \mathscr{Z}. \tag{9}$$

A consistency check is

$$-\beta F = \ln \mathscr{Z} = \ln \sum_{\sigma_i} \exp(-\beta H), \tag{10}$$

or

$$\exp(-\beta F) = \sum_E \Omega(E) \exp(-\beta E), \tag{11}$$

where we used

$$S = k_B \ln \Omega(E), \tag{12}$$

and transformed this expression into one involving a density of states $\Omega(E)$. Thus

$$\exp(-\beta F) = \sum_E \exp(S/k_B) \exp(-\beta E) = \sum_E \exp \frac{1}{k_B T}(-E + TS) = \sum_E \exp -\beta(E - TS) \tag{13}$$

The standard definition of the Helmholtz free energy follows, if one assumes, via the saddle point approximation, that the most probable value of the energy dominates the sum. Then

$$F = \langle E \rangle - TS(\langle E \rangle). \tag{14}$$

The free energy can be used to calculate all thermodynamic properties of interest.

The calculational bottleneck is almost always the sum over microstates. The difficulty arises from the large number of microstates to be considered. Even for 100 spins, the number of states is $2^{100} \sim 10^{30}$, which is prohibitively large. In some special cases, this can be circumvented, as for example in the one-dimensional Ising model which can be solved exactly.

In two dimensions, this is a technically formidable calculation even in the simplifying limit where H_{ext} is zero. In the presence of a non-zero applied magnetic field and in dimensions greater than 2 for the zero-field case, it has so far proved impossible to solve the model exactly.

The formulation of the statistical mechanics of disordered systems requires an additional ingredient, the distinction between **quenched** and **annealed** variables. Annealed variables are those which are summed over in the standard way when taking the trace in the partition function. The σ's, for example, are annealed variables.

Quenched variables, on the other hand, are variables which appear in the definition of the partition function but which are *not dynamical* and thus are not summed over.

It is often convenient to specify such quenched variables at the level of a probability distribution, since they are also microscopic. The question of which variables are to be averaged over such a probability distribution is, however, non-trivial.

Conventional statistical mechanics approaches work in the case of annealed variables. They just introduce more variables into the partition sum. At a basic level, however, the statistical mechanical problem is identical. In contrast to this, the appearance of quenched variables in the theory introduces qualitatively new physics.

3 Mean Field Theory and Phase Transitions in Pure systems

At the simplest level, phase transitions separate regimes of behaviour in which a suitably defined macroscopic variable of thermodynamic significance, called an **order parameter**, changes from being zero in the disordered phase to a non-zero value in the ordered phase. More generally, phase transitions represent singularities in the free energy which appear in the thermodynamic limit.

Order parameters can, however, be subtle. For example, a significant achievement of the Landau-Ginzburg theory of phase transitions in liquid Helium and superconductors is the realization that an order parameter need not be an experimentally measurable quantity. Sometimes no appropriate *local* definition of an order parameter exists and an appropriate non-local, topological definition must be sought.

The issue of a suitable order parameter for the sorts of disordered systems we will encounter is a delicate one. At least in one celebrated case, that of replica symmetry breaking in the Edwards-Anderson spin glass, the order parameter is a function in the interval $[0,1]$.

A simple and intuitive approach to understanding several important properties of the model, including the appearance of a phase transition, can be obtained from **mean-field theory**. The rationale for this approach is the following: One can always solve the problem of a *single spin* in an external field, since there is only one variable here as opposed to 10^{30}. So one strategy would be to attempt to reinterpret the original interacting problem in terms of the problem of a single spin in an effective magnetic field which is self-consistently determined.

3.1 Single site mean-field theory

Start with the Ising Hamiltonian in d-dimensions, with the energy

$$\mathcal{H} = -J \sum_{\langle ij \rangle}^{N} \sigma_i \sigma_j - H_{ext} \sum_{i=1}^{N} \sigma_i. \tag{15}$$

Consider a given spin. It has z neighbours which it interacts with via the exchange term J. For simplicity we set the external field to zero. We start with the obvious result

$$\sigma_i = \sigma_i + m - m. \tag{16}$$

We assume that

$$\langle \sigma_i \rangle = \langle \sigma_j \rangle = m, \tag{17}$$

and that we can neglect fluctuations. Thus

$$\sigma_i \sigma_j = (\sigma_i - m + m)(\sigma_j - m + m) \simeq m^2 + m(\sigma_i - m) + m(\sigma_j - m) = m(\sigma_i + \sigma_j) - m^2 \tag{18}$$

We drop the m^2, since it will drop out when we calculate averages. Now

$$\mathscr{H} = -J\sum_{\langle ij \rangle}^{N} \sigma_i \sigma_j = -\frac{Jzm}{2}\sum_i^N (2\sigma_i) = -Jzm\sum_i^N \sigma_i = -Jzm\sum_i^N \sigma_i. \tag{19}$$

The magnetization, m is defined by

$$m = \sum_i \langle \sigma_i \rangle / N. \tag{20}$$

This then yields

$$Z = 2\cosh(\beta Jzm). \tag{21}$$

The average value of the spin σ_i must then self-consistently be the magnetization, yielding

$$m = \tanh(\beta Jzm), \tag{22}$$

a self-consistent equation which must be solved for m.

The solution to this equation can be obtained graphically at any value of β. There is no solution except at $m = 0$ for small β (high temperatures). Two solutions $\pm m_0$ appear at sufficiently low temperatures, with $m_0 \to 1$ as $T \to 0$, as shown in Fig. 1.

Close to the transition between these two different types of behaviour, which occurs at a non-zero and finite value of β, the value of m is small and one can expand.

$$\tanh(\beta Jmz) = \beta Jmz + \frac{(\beta Jmz)^3}{3} + \dots \tag{23}$$

which yield the equation determining m

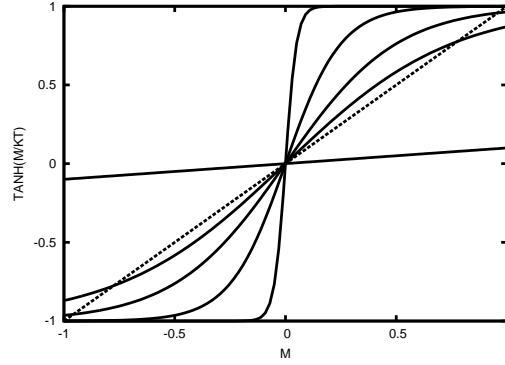$$m(1 - \beta Jz) = \frac{(\beta Jmz)^3}{3}. \tag{24}$$

Fig. 1 Plot of the self-consistent solution to $tanh(\beta M) = M$, obtained by plotting the two curves with different values of β. For small β, *i.e.* large temperatures, the curves intersect only at $M = 0$, indicating that only one (stable) solution to the self-consistent equation exists. Below a critical temperature, while the solution at $M = 0$ is always present, additional non-trivial solutions at nonzero $\pm M_0$ are obtained, which evolve smoothly out of the $M_0 = 0$ state as β is increased. The states with non-zero magnetization are lower free energy states below the critical point.

Note that $m = 0$ always trivially solves this equation. However, a non-trivial solution is obtained at values of β larger than

$$\beta_c = 1/Jz, \tag{25}$$

implying a critical temperature T_c, given by

$$k_B T_c = Jz. \tag{26}$$

Below this critical temperature, the magnetization evolves smoothly from zero, saturating to unity at $T = 0$. Such a smooth, continous evolution of the order parameter from zero to non-zero across the transition, characterizes a **continuous phase transition**.

Note that mean-field theory predicts a transition in all dimensions, since dimensionality enters only indirectly in this argument, via the relation between z and d.

3.2 Mean-field theory via the Bragg-Williams approximation

We now discuss the infinite range version of the model above, with the Hamiltonian

$$\mathcal{H} = -\frac{J}{2N} \sum_{ij} \sigma_i \sigma_j \tag{27}$$

Here the sum is over all pairs. The factor of N ensures proper extensivity, since the all aligned $T = 0$ state must have an energy which is properly extensive in N.

We have

$$m^2 = \left(\frac{1}{N}\sum_i \sigma_i\right)\left(\frac{1}{N}\sum_j \sigma_j\right) = \left(\frac{1}{N^2}\sum_{ij}\sigma_i\sigma_j\right), \tag{28}$$

from which $\sum_{ij}\sigma_i\sigma_j = N^2 m^2$. Then we may rewrite the Hamiltonian as

$$\mathcal{H} = \frac{NJm^2}{2}. \tag{29}$$

We can now integrate over all allowed m with the appropriate weight. To do this, we calculate the number of configurations with a fixed number of up and down spins consistent with a given magnetization. Given $m = (N_{up} - N_{down})/N$ and $N = N_{up} + N_{down}$, we can write the entropy as

$$S = \ln\frac{N!}{N_{up}!N_{down}!}, \tag{30}$$

the standard Bragg-Williams form, which is then

$$S = \ln\left(\frac{N!}{(N(1+m)/2)!(N(1-m)/2!)}\right)$$
$$= N \times \left[\ln 2 - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m)\right]. \tag{31}$$

Thus we may write the free energy per spin as

$$\frac{F}{N} = -\frac{1}{2}Jm^2 + \frac{T}{2}\left[(1+m)\ln(1+m) + (1-m)\ln(1-m)\right] - T\ln 2. \tag{32}$$

The equation of state follows from $\partial f/\partial m = 0$ after which,

$$m = \tanh(\beta Jm), \tag{33}$$

reproducing our earlier answer. This indicates that the mean-field approximation we made when we attempted to solve the short-range version of the model is exact in the case where interactions are long-ranged. (The missing factor of z arises because $z = N$ in a limit where every spin is connected to every other. This factor thus cancels with the $1/N$ required for the right thermodynamic limit.)

3.3 Mean-field theory from the saddle point approximation

Finally, it is also instructive to describe another method to solve this problem. Consider

$$\mathscr{H} = -\frac{J}{2N}(\sum_i \sigma_i)^2. \tag{34}$$

Now

$$\mathscr{Z} = Tr_\sigma \exp\left[\frac{\beta J}{2N}(\sum_i \sigma_i)^2.\right] \tag{35}$$

We can then do

$$\mathscr{Z} = \left(\frac{\beta NJ}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dm\, Tr_\sigma \exp\left[-\frac{N\beta J}{2}m^2 + \beta Jm(\sum_i^N \sigma_i)\right]. \tag{36}$$

Taking the trace inside, we have

$$\mathscr{Z} = \left(\frac{\beta NJ}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dm \exp\left[-\frac{N\beta J}{2}m^2\right] Tr_\sigma \exp\left[\beta Jm\sum_i^N \sigma_i\right]. \tag{37}$$

We can now perform the trace

$$\mathscr{Z} = \left(\frac{\beta NJ}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dm \exp\left[-\frac{N\beta J}{2}m^2\right] \exp\left[N\ln(2\cosh\beta(Jm))\right], \tag{38}$$

which gives us

$$\mathscr{Z} = \left(\frac{\beta NJ}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dm \exp\left[-\frac{N\beta J}{2}m^2 + N\ln 2\cosh(\beta(Jm)\right]. \tag{39}$$

which can be finally written as

$$\mathscr{Z} = \left(\frac{\beta NJ}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dm \exp\left[-\beta JNf(m)\right], \tag{40}$$

where

$$f(m) = \frac{1}{2}m^2 + \frac{1}{\beta J}\log[2\cosh\beta(Jm)]. \tag{41}$$

Notice the way the system size N appears in the argument of the exponential. As $N \to \infty$, a saddle point evaluation of the integral becomes possible. The integral is dominated by its value at the minimum, located via

$$\frac{\partial f}{\partial m} = 0, \tag{42}$$

which then gives us

$$m = \tanh\beta(Jm). \tag{43}$$

This is identical to the simple expression evaluated earlier.

These ideas can be used to elaborate a fundamental and intuitive approach to phase transitions, called Landau-Ginzburg theory after its founders. We start with the free energy derived earlier

$$\frac{F}{N} = -\frac{1}{2}Jm^2 + \frac{T}{2}[(1+m)\ln(1+m) + (1-m)\ln(1-m)] - T\ln 2, \tag{44}$$

which can now be expanded, using

$$(1+m)\ln(1+m) + (1-m)\ln(1-m) = m^2/2 + m^4/12, \tag{45}$$

to yield

$$\frac{F}{N} = -\frac{1}{2}(\beta J - 1)m^2 + \frac{T}{2}\left[m^4/12 + O(m^6)\dots\right]. \tag{46}$$

Note that the free energy is minimized with $m = 0$ for temperatures greater than the critical temperature. For temperatures less than this critical temperature, the free energy is minimized with a non-zero value of m. A plot of this free energy for different temperature values both above and below the critical temperature, would look as illustrated in Fig. 3.3

We can use this example as a more general way of thinking about phase transitions. As discussed, the order parameter is a new thermodynamic variable which must enter the thermodynamic description of the model. For magnetic systems which order ferromagnetically, the order parameter is the magnetization. For anti-ferromagnets, the order parameter is the staggered magnetization. Landau proposed that the dependence of the free energy on the order parameter could be modelled in a very general manner, with a form dictated by reasoning from symmetry arguments.

Consider a generic phase transition in which a quantity m is the order parameter and contemplate the general ingredients of an expansion of the free energy f in terms of m. Landau argued that an appropriate free energy density for the Ising problem could be obtained in a systematic expansion in the order parameter. This form, as generalized by Landau and Ginzburg is:

$$f = f_0 + am^2 + dm^4 + c(\nabla m)^2 \dots \tag{47}$$

The assumptions which enter this are the following: There is always a non-singular part to the free energy density which we will lump into f_0. We neglect all powers higher than 4, since we are interested in the vicinity of the critical point, where m is small. We assume that at large M the free energy should be large, so $d > 0$. We will
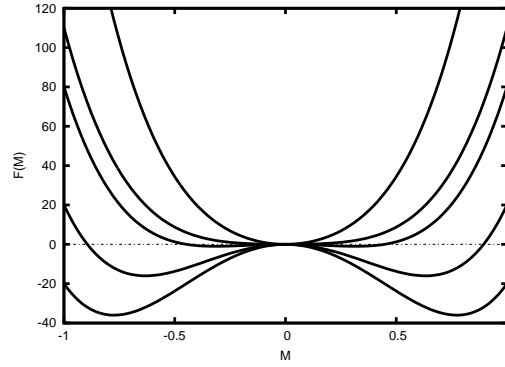
Fig. 2 Landau Free energy as a function of the order parameter for a range of temperatures above and below the transition point. Above the transition temperature, this free energy has a minimum only at $M = 0$. Below the transition, there are minima, at $\pm M_0$; the $M = 0$ state is a local maximum of the free energy.

see below that at $a > 0$ there is one phase, whereas for $a < 0$, there are two phases. Thus, a should change sign at $T = T_c$. At the simplest level, we can expand it as

$$a = \alpha(T - T_c). \tag{48}$$

If $c > 0$, the free energy can only increase if the field is non-uniform. Thus, the full free energy is

$$F = F_0 + V \left(\alpha(T - T_c)m^2 + dm^4 \right) \tag{49}$$

The minimum of the free energy gives,

$$2\alpha(T - T_c)m + 4dm^3 = 0, \tag{50}$$

with solutions

$$m = 0 \text{ or } m^2 = -\frac{\alpha(T - T_c)}{2d}. \tag{51}$$

This free energy can be used to calculate all the thermodynamic properties of the model.

If we define a **critical exponent** β which captures the increase of the magnetization m below T_c, as

$$m \sim (T - T_c)^\beta, \qquad T < T_c \tag{52}$$

then $\beta = 1/2$ within the Landau-Ginzburg theory. Our interest will be in the calculation of how similar thermodynamic functions become singular at the transition. Another example is the calculation of the magnetic susceptibility χ, which defines

a related exponent γ *i.e.*

$$\chi \sim |T - T_c|^{-\gamma}, \tag{53}$$

where we anticipate that the value of γ is the same as the transition is approached from below or from above.

To obtain this exponent, we add a small external field H. This enters the Landau-Ginzburg expansion as

$$F = F_0 + V(\alpha(T - T_c)m^2 + dM^4 - Hm) \tag{54}$$

We want to calculate

$$\chi = \frac{\partial m}{\partial H}. \tag{55}$$

To obtain this, we use

$$2\alpha(T - T_c)m + 4dm^3 = H, \tag{56}$$

and distinguish two cases:

1. If $T > T_c$, then $m(H = 0) = 0$. For small H we neglect m^3, then

$$m = \frac{H}{2\alpha(T - T_c)} \tag{57}$$

   and the susceptibility

$$\chi = \frac{1}{2\alpha(T - T_c)}. \tag{58}$$

   This diverges at $T \to T_c$.
2. If $T < T_c$, $m(H = 0) = \pm m_0$ with $m_0 = \pm\sqrt{-\alpha(T - T_c)/2d}$. Let

$$m = m_0 + \delta m \tag{59}$$

$$\delta m = -\frac{H}{4\alpha(T - T_c)} \tag{60}$$

   yielding

$$\chi = |\frac{1}{4\alpha(T - T_c)}| \tag{61}$$

Note that the susceptibility exponent is $\gamma = 1$, within Landau-Ginzburg theory.

The Ginzburg-Landau free energy can be used to derive predictions for the singular behaviour of a variety of observables, including the specific heat, the magnetization and others. The predictions here are identical to those obtained from the

sorts of mean-field theories discussed here. Some of these observables diverge at the transition, with well-defined critical exponents, while quantities such as the specific heat are discontinous. This non-analytic behaviour at the transition, with associated algebraic dependence on the distance from the critical point, defines a number of **critical exponents**.

This derivation of mean-field theory illustrates several important features:

- Mean field theory is exact as the number of neighbours goes to infinity. This happens because fluctuations are averaged over and correlations can be ignored.
- With a thermodynamic phase transition is associated an order parameter. The order parameter is usually defined to be zero on the high temperature side of the transition and non-zero below it.
- A common property of most (but not all) phase transitions is that a symmetry is broken in the low temperature phase. What this means is that although the Hamiltonian may have a certain symmetry, the low-temperature symmetry-broken state need not have that symmetry. The ferromagnetic Ising model has two ground states and the system chooses one or the other at the transition.
- The barriers between possible symmetry broken states must become infinitely large, as the system size goes to infinity, if the symmetry broken state is to be stable.
- Several thermodynamic quantities behave in a singular manner at the phase transition point. Mean-field theory provides precise predictions, typically involving simple rational numbers, for this singular behaviour.

These general results will be useful in developing intuition for the statistical mechanics of disordered systems.

3.4 Critical Point Exponents

Scaling descriptions are ubiquitous in statistical physics. Such descriptions are appropriate whenever the system behaves as though it lacks a "typical" or dominant scale set by microscopic physics. In the context of equilibrium systems, behavior close to the critical point, in particular the study of the non-analytic behaviour of thermodynamic functions, is of special interest to the statistical mechanic. Early work on understanding such behaviour led to the development of the concepts of scale-invariance and of universality. Many classes of apparently very disparate systems appear to have the same critical exponents, defining what are called **universality classes** of physical behavior.

The reason for this is the fact that correlation lengths diverge at continous phase transitions, thus coupling degrees of freedom from microscopic scales to macroscopic scales. The divergence of the correlation length implies the absence of a characteristic length scale, as required for a scaling description. Universality is obtained because microscopically different systems have the same large-scale description.

This is again an emergent property, inaccessible in discussions which focus solely on microscopics.

Mean-field theory provides one way in which the behaviour in the vicinity of the critical point can be calculated. Unfortunately, the predictions of mean-field theory are quantitatively wrong, although such theories are powerful ways of understanding statistical mechanical systems. Mean-field theory fails, at least for dimensions $d < 4$ in the short-ranged Ising case, because it ignores fluctuations, assuming that a single configuration dominates the partition sum. Mean-field behaviour is restored in models in high enough dimensions or if the interactions are sufficiently long-ranged.

Begin by defining a dimensionless scaled temperature which measures the distance from the critical point, $t = (T - T_c)/T_c$. In terms of this, if a quantity behaves as $F(t) \sim |t|^\lambda$ close to the critical point, then the critical exponent λ can be obtained from $\lambda = lim_{t \to 0} \frac{\ln |F(t)|}{\ln t}$. Two-point correlation functions are defined through

$$G(r) \equiv \langle m(0)m(\mathbf{r}) \rangle. \tag{62}$$

Such a correlation function will generically decay as

$$G(r) \sim e^{-r/\xi}/r^{d-2}. \tag{63}$$

As one approaches the critical point, ξ the correlation length diverges as

$$\xi \sim |t|^{-\nu}. \tag{64}$$

Precisely at the critical point, the correlation function decays as

$$G(r) \sim 1/r^{d-2+\eta}. \tag{65}$$

A full list of the conventionally defined critical exponents is the following:

$$
\begin{aligned}
\text{Specific heat } & C \sim |t|^{-\alpha} \\
\text{Magnetization } & M \sim |t|^{-\beta} \\
\text{Isothermal susceptibility } & \chi \sim |t|^{-\gamma} \\
\text{Critical isotherm } & M \sim |H|^{-\delta} \\
\text{Correlation length } & \xi \sim |t|^{-\nu} \\
\text{Correlation Function at T}_c \ & G(r) \sim 1/|r|^{d-2+\eta}
\end{aligned}
\tag{66}
$$

We do not define different exponents for $T \to T_c^-$ and $T \to T_c^+$; the current understanding is that, where both exponents exist and are well defined, they are equal.

From thermodynamics, we can obtain various inequalities connecting the critical exponents.

$$
\begin{aligned}
\alpha + 2\beta + \gamma \geq 2 \\
\alpha + \beta(1 + \delta) \geq 2
\end{aligned}
$$

$$\gamma \leq (2 - \eta)\nu$$
$$d\nu \geq 2 - \alpha \qquad\qquad (67)$$

These relations are believed to hold as equalities, a result which follows from the structure of the renormalization group approach. The last of these relations, due to Josephson, explicitly involves the dimension d and is called a hyperscaling relation. Such a relation is only valid at and below a special value for the dimension called the **upper critical dimension**. This critical dimension is 4 for the Ising case.

In the table below, we tabulate the exponents as defined above. These exponents are (i) calculated within mean-field (Landau-Ginzburg) theory, (ii) obtained from an exact calculation for the two-dimensional Ising model due to Onsager, (iii) extracted from series expansions for the three dimensional Ising model, and finally, (iv), obtained from experiments on fluids near the liquid gas critical point.

Exponent	Value (Mean Field)	2d Ising Model	3d Ising Model	Experiments (Fluids)
α	0	0 (log)	0.109	0.15
β	0.5	1/8	0.327	0.33
γ	1.0	7/4	1.237	1.3
δ	3.0	15	4.78	4.5
ν	0.5	1	0.63	0.6
η	0.0	1/4	0.03	0.07

Note the close similarity of exponents for the 3-d Ising model and those obtained from a variety of experiments on fluid systems near the critical point. This is no accident. Universality arguments suggest that these nominally very dissimilar systems belong to the same universality class and thus exhibit continous transitions characterized by the same set of critical exponents.

4 The Percolation Problem and Geometric Phase Transitions

The **percolation** problem is a classic example of a geometrical phase transition. It is is exceedingly simple to pose. Consider a regular lattice, say square or triangular. Dilute this lattice by removing either sites (**site percolation**) or bonds (**bond percolation**). This is done with probability $(1 - p)$ independently at each site. Thus, for large enough systems, Np sites or bonds are obtained on average. Obviously, once one has specified the lattice type, the only parameter of relevance is p.

Consider the behaviour of the bond percolation problem at both large and small p. For large p, very few bonds are removed. It is intuitively reasonable that, given any two sites on the lattice, a connected path exists from one to the other. In the opposite limit, where p is small, very few bonds survive and it is overwhelmingly likely that no such path should exist. It is reasonably to expect that these two asymptotic behaviours should be separated by a well-defined transition point. The transition can be described as a change in connectivity, where connecting two points whose

asymptotic separation can be taken to be very large is either possible or impossible, on either sides of the transition.

These arguments make it intuitively reasonable that a transition must exist. More quantitative arguments are required, however, to sharpen and quantify the argument. We do this by defining several quantities characterizing the connectivity in the different phases.

Let $n_s(p)$ be the number of clusters per lattice site of size s. The probability that a given site is occupied and belongs to a cluster of size s is thus $sn_s(p)$. Let $P(p)$ be the fraction of occupied sites belonging to the infinite cluster. Clearly, $P(p) = 1$ for $p = 1$ and $P(p) = 0$ for $p \ll p_c$. Thus $P(p)$ is analogous to an order parameter for a system which undergoes a thermal phase transition. We also have the relation

$$\sum_s sn_s(p) + pP(p) = p, \tag{68}$$

where the summation extends over all finite clusters.

One can also define the mean size of clusters, denoted by $S(p)$. This is related to $n(p)$ through the following:

$$S(p) = \frac{\sum_s s^2 n_s(p)}{\sum_s sn_s(p)}, \tag{69}$$

where the sum is again over finite clusters.

The pair connectedness $C(p, r)$ is the probability that occupied clusters a distance r apart belong to the same cluster. We can also define $G(p)$, the total number of finite clusters per lattice site, via

$$G(p) = \sum_s n_s(p). \tag{70}$$

This quantity is analogous to the free energy per site for magnetic phase transitions.

We now describe a scaling theory for the percolation problem. The percolation probability, or the probability that a given site belongs to the infinite cluster, behaves as

$$P(p) \sim (p - p_c)^\beta. \tag{71}$$

Similarly, we have

$$S(p) \sim (p_c - p)^{-\gamma} \tag{72}$$

$$G(p) \sim |p - p_c|^{2-\alpha}. \tag{73}$$

In percolation, the linear size of the finite clusters, below and above p_c, is characterized by the connectivity length ξ. This is similar to the correlation length in thermodynamic phase transition. Here ξ is defined as the root mean square distance between two sites on the same finite cluster, averaged over all finite clusters. It can

be calculated by measuring the radius of gyration of a cluster

$$R_s^2 = \frac{1}{s^2} \sum_{i=1}^{s} |\mathbf{r_i} - \mathbf{r_0}|^2$$

where $\mathbf{r_0} = \sum_{i=1}^{s} \mathbf{r_i}/s$ is the position of the center of mass of the cluster and $\mathbf{r}_i$ is the position of the ith site of the cluster. The connectivity length is the average radius of gyration over all finite clusters and given by

$$\xi^2 = \frac{2\sum_s R_s^2 s^2 n_s(p)}{\sum_s s^2 n_s(p)}. \tag{74}$$

At p_c, clusters of all possible sizes, starting from a single site up to clusters spanning the system, appear. There are, correspondingly, very large fluctuations in cluster sizes. Due to the appearance of large finite clusters at the critical point, the connectivity length diverges as $p \to p_c$ with an exponent ν given by

$$\xi \sim |p - p_c|^{-\nu}.$$

The connectivity length exponent ν is also found to be related to the moment exponents of $n_s(p)$ via a hyper scaling relation. The infinite cluster at $p = p_c$ contains holes of all possible sizes; the percolation cluster is self similar and fractal. A hyperscaling relation $\gamma + 2\beta = d\nu$ is satisfied between the scaling exponents. The pair connectedness function behaves as:

$$C(p,r) \sim \frac{\exp -r/\xi(p)}{r^{d-2+\eta}}. \tag{75}$$

The scaling theory assumes that for p near p_c, there is a typical cluster size s_ξ which leads to the dominant contribution to these divergences. Since this diverges as $p \to p_c$, we can assume that

$$s_\xi \sim |p - p_c|^{-1/\sigma} \tag{76}$$

We will also assume that

$$n_s(p) = n_s(p_c) f(\frac{s}{s_\xi}) \tag{77}$$

We require that the function $f(x) \to 0$ as $x \to \infty$ and $f(x) \to 1$ as $x \to 0$. We know that, for large s, the scaling form $n_s(p_c) \sim s^{-\tau}$ is obtained. Thus we get

$$n_s(p) = s^{-\tau} \phi(s|p - p_c|^{1/\sigma}) \tag{78}$$

The percolation exponents can therefore be obtained in terms of two independent exponents σ and τ.

The percolation problem can be solved exactly in one dimension. It can be easily seen that, for long range connectivity, all the sites must be present, which implies

that $p_c = 1$. For clusters containing s-sites: $n_s = p^s(1-p)^2$. The $(1-p)^2$ term comes from the probability of two empty sites that bind the cluster from two ends. Once this distribution is known it is easy to obtain the behavior of various observables near $p_c = 1$. The values of the critical exponents can be obtained by taking suitable moments of the cluster size distribution as stated before.

In two dimensions, the exponents of the percolation problem are given in terms of rational numbers. The exponents are listed below in table 1.

Dimension	d=2	d=3	mean-field
β	5/36	0.41	1
γ	43/18	1.80	1
ν	4/3	0.88	1/2
α	$-2/3$	-0.62	-1
σ	36/91	0.45	1/2
τ	187/91	2.18	5/2

Table 1 The values of the critical exponents of the percolation phase transition.

The relationship between thermal phase transitions and percolation was first discovered by Fortuin and Kastelyn who showed that the bond percolation problem could be mapped to the $q-$state Potts model in the limit that $q \rightarrow 1$. This mapping can be used to obtain the critical exponents in the equations above.

The values of the critical exponents are found to be same on different lattices and for site or bond percolation. The exponents depend only on the space dimension, another illustration of universality.

5 Disordered Spin Systems

There are many ways in which disorder can enter the specification of a simple model such as the one we have been studying. For one, some of the spins could be absent from their regular sites. For another, the interaction between two spins could depend on the labels of the spins with $J \rightarrow J_{ij}$. A third possibility is the presence of random fields acting on the spins independently.

Consider a model Hamiltonian, for Ising variables, of the form

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j - \sum_i h_i\tau_i \tag{79}$$

where the disorder is contained in the variables J_{ij} and the random fields h_i. The problem in which all the h_i's are set to zero and the J_{ij}'s are random is the **random exchange model**. The problem with $J_{ij} = J$ i.e. *uniform* exchange couplings but with h_i random is the **random field model**. In general, in experimental systems, both types of disorder could be present. One could also have situations where spins are missing from sites, the case of **site diluted magnets** or where the magnitude of

the spin could differ from site to site. Further, instead of considering Ising spins, one could consider spin variables which take values on a circle (XY spins) or on the surface of a sphere (Heisenberg spins).

In an Ising spin system, if $J \to J_{ij}$ with the J_{ij} fixed and non-dynamical, then the J_{ij}'s would be quenched variables. However, if the J_{ij}'s themselves were derived, for example, from $J_{ij} = \tau_i \tau_j$ with the τ's allowed to fluctuate, the τ variables would then be annealed and so would the J_{ij}'s in this case.

How does one perform the average over the quenched variables? Imagine subdividing the macroscopic system into many sub-parts, each large enough to be treated macroscopically. The interaction between each subsystem is only a surface effect, one which is subleading with respect to the bulk in the limit of large subsystem size. Note that the free energy in each subsystem is a well-defined quantity but fluctuates from subsystem to subsystem since the microscopic configuration of disorder is different in each subsystem.

We may thus obtain the free energy and thermodynamic functions within each subsystem, in principle by doing the partition function sum keeping the quenched variables fixed. We must then decide how to average over disorder, in order to elimi-
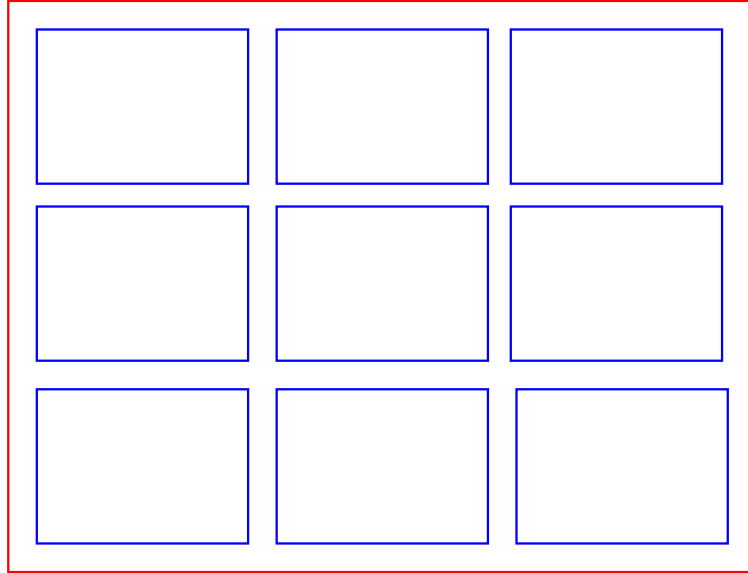

Fig. 3 The decomposition of a thermodynamic system into subsystems

nate the dependence on the microscopic specification of the disorder variables. This is done using the following argument. Each subsystem provides an independent realization of disorder, derived from the probability distribution for the quenched variables. In the limit of an infinite number of subsystems whose interactions can be ignored to a first approximation, the disorder average is automatically the aver-

age over the subsystems. Averaging over disorder is equivalent to averaging over subsystems for variables such as the free energy whose values can be appropriately defined by summing up over subsystems.

A quantity is called **self-averaging** if its distribution over the subsystems as described above is not anomalously broad. This is equivalent to demanding that the mean-value and the most probable value coincide in the thermodynamic limit. Note that the partition function is *not* self-averaging and cannot be averaged in the same way as the free energy.

Checking whether a given thermodynamic quantity is properly self-averaging or not is an important task: otherwise the statistical averaging may yield mean values that are not the values which will be measured in a typical experiment. The argument above, due initially to Brout, indicates that the free-energy and similar observables are self-averaging, for interactions which are short-ranged and for probability distributions over disorder which are "well-behaved".

A useful example of the distinction between typical and average is the following: Consider a random variable x that takes two values

$$X_1 = e^{\alpha\sqrt{N}} \text{ and } X_2 = e^{\beta N}, \beta > 1, \tag{80}$$

with probabilities

$$p_1 = 1 - e^{-N}, \text{ and } p_2 = e^{-N}. \tag{81}$$

In the limit $N \to \infty$, the average value $\to e^{(\beta-1)N}$ while the typical or most probable value is $x = X_1$ with probability 1. On the other hand the average value of $\ln x \to \alpha\sqrt{N}$ in the same limit, showing that the most probable value is determined by the typical value of the variable while the moments are controlled by the rare events. Provided x has a probability distribution which does not have special or rare events which occur with low probability but contribute anomalously to mean values and higher moments, this problem of matching the average value with the most probable value does not exist.

We have indicated that disorder is crucial and must be averaged over in a non-trivial sense in the statistical mechanics. Disorder however is not the only ingredient which leads to the non-trivial statistical mechanics of disordered systems. The other ingredient is **frustration**. To understand frustration, let us return to the other model we described briefly earlier, the model in which

$$\mathcal{H} = \sum_{\langle ij \rangle} \tau_i \tau_j \sigma_i \sigma_j, \tag{82}$$

where we have taken the special case in which $J_{ij} = \tau_i \tau_j$. and $\tau_i = \pm 1$.

We assume that the τ's are genuine quenched variables. Nevertheless all of the statistical mechanical content of this model can be shown to be trivially related to the pure version of the Ising model. To do this, simply redefine

$$\sigma_i' = \tau_i \sigma_i. \tag{83}$$

The partition function now reduces to

$$\mathcal{Z} = Tr_{\sigma_i'} \exp + \beta J \sum_{\langle ij \rangle} \sigma_i' \sigma_j'. \tag{84}$$

This is, however, just the Ising model in another guise, since disorder now does not appear explicitly in the specification of the partition function. This model, the Mattis model, provides an example of the way in which, in some cases, disorder can be transformed away. What remains is an effective pure problem in a redefined set of variables.

There is an interesting peculiarity of the transformation of spin degrees of freedom discussed in the previous section. If we multiply the spin variables τ_i across any closed circuit, the answer is always 1. This is easily traced to the fact that each τ_i appears twice when multiplied along a closed loop. What this means in practice is that the ground state of the model is non-degenerate, since one can minimize the energy of each spin separately to obtain the overall ground state. Since all the variables $\tau_i \sigma_1$ must be equal in the ground state, the sigma values which minimize the energy can be trivially extracted.
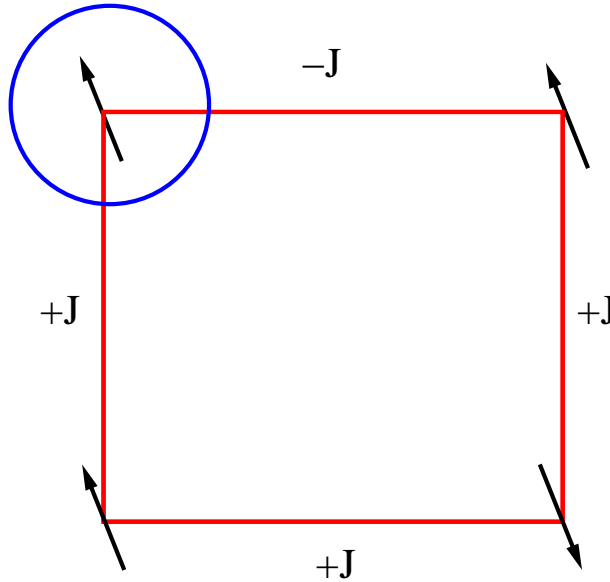

Fig. 4 Illustration of Frustration: The circled spin receives conflicting information from its neighbours and can point either up or down with the same energy. There is no way of minimizing its energy.

Models where this does not happen are frustrated models. The Mattis model exhibits disorder but not frustration. The idea of frustration is easily illustrated by considering spins on a square block with nearest neighbor interaction as shown in Fig. 4. Take the spins on the square. While the energy of any pair of spins can be appropriately minimized by minimizing the bond energy, one spin indicated by

circle in the figure receives contradictory instructions from each of the spins it is bonded to. Thus not all the interactions can be satisfied automatically.

Frustration refers to such an impossibility of satisfying all interactions at the same time to achieve an absolute minimum of the free energy. Non-trivially disordered systems display both quenched disorder and frustration. It is easy to come up with examples of systems which exhibit frustration in the absence of quenched disorder and systems which exhibit quenched disorder in the absence of frustration. Frustration and disorder are essential ingredients of all models of spin glasses, since the combination of these two yield the most important property of spin-glass materials, the larger number of low-lying (or degenerate) free energy minima and extremely complex patterns of ergodicity and symmetry breaking.

There are three basic scenarios for spin glass-like behaviour. In the first one, barriers separating low free energy minima remain finite at all nonzero temperatures. Thus the relaxation of the system may be anomalous but the system remains paramagnetic. In the second, it could happen that there is a certain spin state, or a finite number of such states related by symmetry, with a lower free energy than any other. In such a frozen state, an appropriate order parameter describing the freezing of the spins along random directions, the Edwards-Anderson order parameter to be defined later, would be non-zero. This case can still be studied along traditional lines. In the last scenario, we could have an extensive number of ground states with the system frozen in any one of these states. The infinite-ranged spin glass model to be discussed in the following sections corresponds to the last case.

5.1 The Harris Criterion

When does the presence of disorder change the universality class of a transition? The question of when disorder is relevant and when it is not was answered some decades ago by Harris. Harris argued the following: Consider a system where the disorder couples to the temperature, say through a variable p. Thus, different regions of the sample have different critical temperatures $T_c(p)$. We may thus ask: Does allowing the local T_c to vary in this manner alter the critical exponents of the pure transition or leave them unchanged?

Begin with the pure system. Let the correlation length ξ be large. The volume of a coherence region is

$$V \sim \xi^d, \tag{85}$$

in d dimensions. Since the correlation length

$$\xi \sim t^{-\nu}, \tag{86}$$

we have

$$V_\xi \sim t^{-\nu d}. \tag{87}$$

dictating the size of the coherence volume. Permissible fluctuations in the distance to the transition, the t variable, then scale as

$$t \sim V_\xi^{-1/\nu d}. \tag{88}$$

The disorder may be expected to give rise to fluctuations of the local transition temperature. Given a volume V, this fluctuation should scale as the square root of the number of degrees of freedom in that volume, from the central limit theorem. Thus, disorder-induced fluctuations away from the true T_c scale as $V_\xi^{1/2}$. Comparing terms, we get $\nu d < 2$ as a condition for disorder-induced critical fluctuations to dominate, driving the system away from the critical fixed point corresponding to the pure system.

We can now make use of the the hyperscaling relation due to Josephson

$$d\nu = 2 - \alpha. \tag{89}$$

Inserting this yields,

$$\alpha > 0, \tag{90}$$

as the condition for disorder to be a relevant perturbation, a result first obtained by Harris.

Note that for a sharp phase transition in a disordered system to occur, one must necessarily have $\alpha < 0$. That is, the specific heat must have the form of a cusp.

5.2 The Random Field Problem and the Imry-Ma argument

Consider a model with the Hamiltonian

$$\mathcal{H} = -\sum_{\langle ij \rangle} J\sigma_i\sigma_j - \sum_i h_i\tau_i. \tag{91}$$

Note that the exchange interaction is uniform, but that disorder enters in the form of a random field coupling independently to spins at each site. The spins above are Ising spins but could, in general, be be n-component spins, with n=1 the Ising case, n=2 the XY model and n=3 the Heisenberg model. The pure problem has a transition with a lower critical dimension of 1 (Ising) and of 2 ($n \geq 2$). The question of whether the transition survives the introduction of weak random-field disorder can be posed in terms of the stabiity of the ground state.

Imry and Ma formulated a simple yet elegant argument for this. We illustrate this first for the Ising case. Consider the aligned ground state in the absence of a random field. Introduce the random field. Over a given region of dimension L, the field is, on average zero, but fluctuates about this average. Thus the value of the $h_i\sigma_1$ term, fluctuates to order $L^{d/2}\Delta$, where Δ denotes the width of the field distribution. If this

fluctuation is such that the system can lower its energy by flipping all spins in that region, it will do so, while paying a cost for creation of an interface at the boundary. If a single length L dominates, then this interface cost is $\propto L^{d-1}$ in $d-$ dimensions for the Ising case.

Combining these two terms into a free energy cost for flipping the domain yields the two competing terms

$$\Delta F = -\Delta L^{d/2} + J L^{d-1}. \tag{92}$$

The minimum of the free energy can be obtained as

$$\frac{\partial F}{\partial L} = 0 = -\Delta L^{d/2-1} + J L^{d-2}. \tag{93}$$

This then yields

$$L^* \sim [J/\Delta]^{2/(2-d)} \tag{94}$$

One can see that a change in behaviour occurs when $d \geq 2$, For dimensions $d < 2$, there is a finite L^* at which the energy gain from flipping the domain outweighs the energy cost of creating a domain wall. It is always favourable to flip domains of dimension larger than L^*. For dimensions $d > 2$, the cost of creating the domain wall dominates at large scales and only a few small domains are formed, insufficient in number to fully destroy the long-ranged order in the ground state of the pure model.

This can be generalized to the case of continuous spins, with the only modification being in the domain wall energy term. Since, in continuous spin systems, the energy cost of the domain wall can be spread out over a region of size L, the ordered ground state is stable for $d > 4$ and unstable below that. A rigorous proof of the lower critical dimension $d_c = 2$ for the random field Ising model was provided by Spencer. A detailed discussion of the physics of random field models, extending and clarifying the simple argument above, is contained in the chapter of P. Shukla.

6 Spin Glass Basics

6.1 Experimental Phenomenology

Studies of the spin glass state were initially motivated by the desire to understand the low-temperature behaviour of certain magnetic alloys. While early work concentrated on understanding the fate of isolated magnetic impurities in a non-magnetic, typically metallic, host, it was realized that understanding the interactions between impurities posed a challenging modeling problem. Such systems of dilute magnetic impurities coupled to each other behaved as though there was a low-temperature freezing of the moments associated with the magnetic ions. The field of spin glasses was born out of these early studies. There are a variety of different experimental
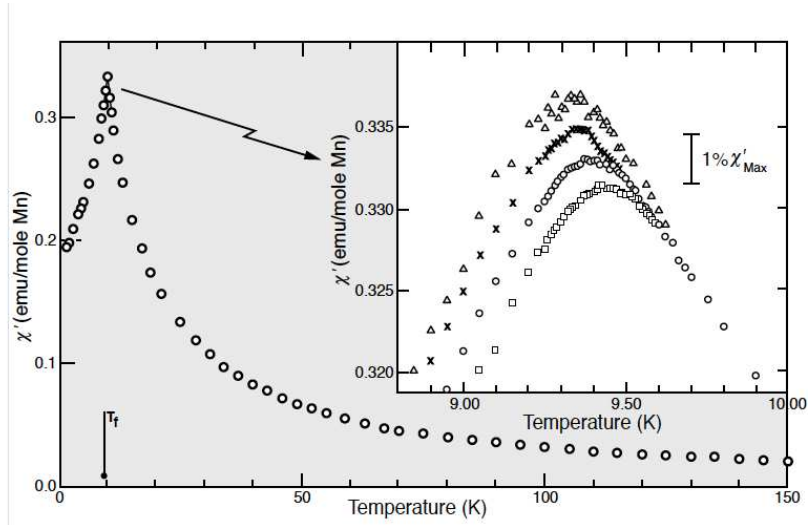
Figure 1. AC susceptibility of CuMn (1 at% Mn) showing the cusp at the freezing temperature. The inset shows the frequency dependence of the cusp from 2.6 Hz (triangles) to 1.33 kHz

Fig. 5 Illustration of the cusp in the real part of the ac susceptibility, incicating the spin glass transition and its broadening with increasing frequency. Figure from C.A.M Mulder, A.J. van Duyneveldt and J.A. Mydosh, *Phys. Rev. B* **23** *1384 (1981)*

systems which exhibit spin glass behaviour: (I) stable spins (Mn, Fe Gd, Eu) diluted into non-magnetic metals (Cu,Au) such as $Cu_{1-x}Mn_x$, $Au_{1-x}Fe_x$, (II) nonmagnetic compounds with some magnetic ions, such as $Eu_xSr_{1-x}S$ or La_{1-x} Ga_x Al_2 or (III) amorphous intermetallics such as $GdAl_2$, YFe_2. Spin glass samples exhibit irreversible behaviour below T_f and reversible behaviour above it: cooling down in constant field (field-cooled or FC) leads to smooth reversible curves below T_f while zero-field cooling (ZFC) leads to irreversible behaviour which relaxes towards FC behaviour very slowly. Typical plots are shown in Fig. 6.

The principal experimental signatures of spin glasses are the following: the specific heat at zero applied field shows a broad maximum at about 1.4 T_f, coupled with a slow decrease at high T. The peak is smeared by the application of a field. The dc susceptibility shows a sharp cusp at T_f; this cusp is very strongly affected by magnetic fields with even very small fields capable of rounding out the cusp, as in Fig. 5. The AC magnetic susceptibility peak shows a shift depending on frequency, in contrast to behaviour in conventional ferromagnets or antiferromagnets. There is evidence for a large distribution of relaxation times in ac susceptibility. Typically there are no clear signs of freezing in resistivity.

Over 500 different experimental systems exhibiting spin glass behavior have now been identified, including transition-metals in noble metal solutes, rare earth combinations, amorphous metallic spin glasses, semiconducting spin glasses and insulating spin glasses.

7 Spin Glass Theory

To understand the spin glass problem, we must understand the fate of magnetic impurities in metals. Indirect exchange, mediated through conduction electrons in metallic systems, couples moments over large distances. It is the dominant exchange interaction in metals where there is little or no direct overlap between neighbouring magnetic electrons.

In a conventional metallic system with dilute magnetic impurities, the impurities are far apart. They can thus be modeled by what are called single-impurity Anderson or Kondo Hamiltonians, reflecting the model problem of an isolated magnetic impurity coupled to a sea of conduction electrons. The properties of such a system are by now fairly well understood at a quantitative level.

When the density of magnetic impurities is high, we can no longer ignore spin-ordering tendencies arising from the interaction between impurities. The two competing effects that lead to two different tendencies of the system to interact with conduction electrons, known as the Kondo effect and the Ruderman-Kittel-Kasuya-Yoshida (RKKY) interaction, have a crucial influence on the impurity magnetism. The conduction electron-mediated, RKKY indirect-exchange interaction favors magnetic ordering of impurities, whereas the Kondo effect tends to quench individual impurity spins.

We briefly describe the interactions between two magnetic impurities in a metal. Taking the simplest case of hybridization between the $s - p$ conduction electrons of the host metal and the $d-$ (or $f-$) electrons of the magnetic impurity produces an effective on-site exchange coupling at the impurity site. For $s - p/d$ hybridization, the sign of the interaction is typically antiferromagnetic, as the conduction electrons attempt to screen the spin of the impurity in their midst. Rather than forming a negative spin-1/2 at the impurity site, however, the electrons spin-polarize in concentric rings around the impurity.

This leads to an interaction of the form

$$J(\mathbf{R}_i - \mathbf{R}_j) \sim \cos(2k_F R)/R^3. \tag{95}$$

with k_f the Fermi wave-vector. This is the RKKY interaction.

The source of the rings of alternating polarization is that a true delta-function in space would require, in Fourier space, all $k-$vectors from 0 to infinity to be equally weighted, Since the host is a metal, available $k-$vectors range only from 0 to k_F. The system thus cannot screen the impurity spin in a localized manner. Instead it does the next best thing. It first creates an opposite alignment in the vicinity of
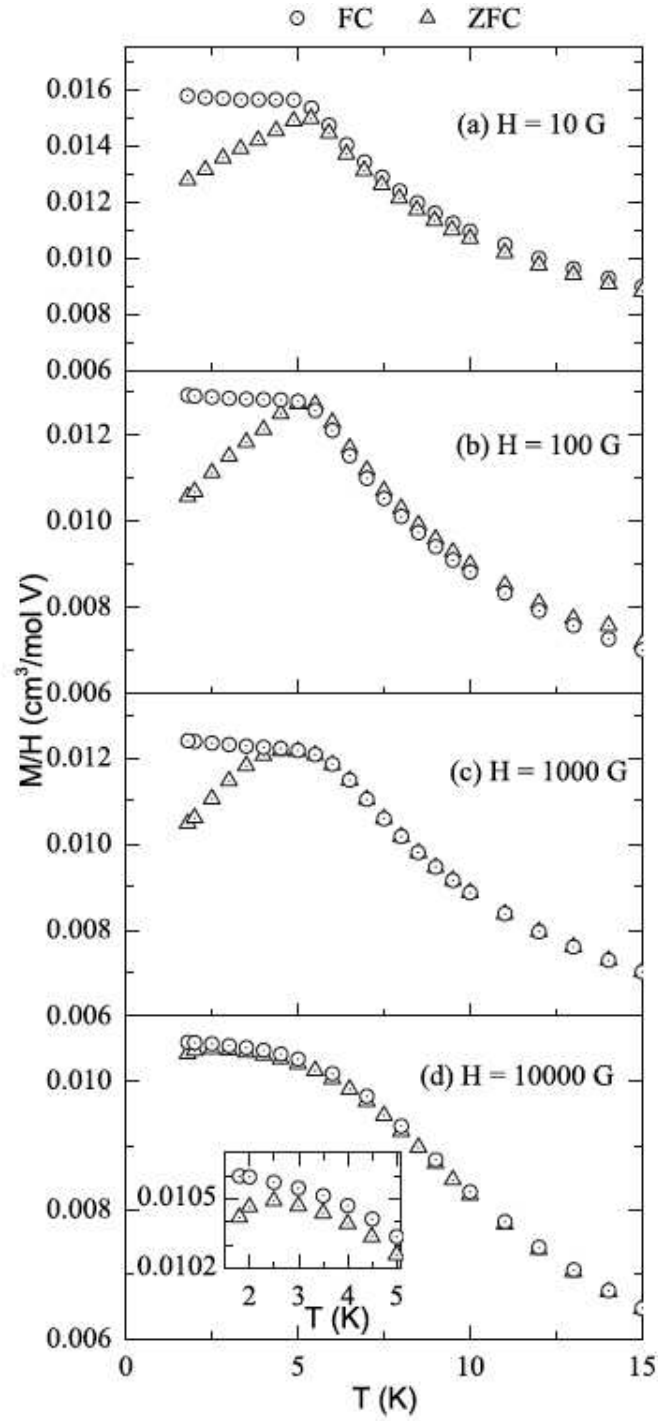
Fig. 6 Splitting of field cooled (FC) and zero-field cooled (ZFC) magnetization with temperature of $(Li_{0.40}V_{0.60})_3B\ O_5$ crystals at low temperatures and fields of $H = 10, 100, 1000$ and 10000 Gauss. The inset is an expanded plot of the low temperature region. Figure taken from X. Zong, A. Niazi, F. Borsa, X. Ma and D.C.Johnston, Phys. Rev. B **76** 054452 (2007)

the impurity which overscreens it, followed by a parallel alignment further away which overcompensates in the opposite direction. This proceeds with decreasing amplitude out to infinity. Thus, depending on the distance between two impurity spins, their coupling could be either ferromagnetic or anti-ferromagnetic, resulting in frustration.

The most realistic model for spin glasses would consider vector spins occupying random sites. The Hamiltonian would then read

$$\mathscr{H} = \sum_{\langle ij \rangle} J(\mathbf{R}_i - \mathbf{R}_j)\mathbf{S}_i\mathbf{S}_j. \tag{96}$$

The origins of this magnetic interactions can be several, including direct exchange, RKKY, superexchange and dipolar. Dipolar interactions can favour various types of ordering, including Ising, XY, and Heisenberg types.

A simpler model for freezing retains some of these ingredients: Only RKKY interactions are considered, there is no Kondo effect or long-range order and one considers only the interactions of single spins. The freezing temperature can be estimated: The RKKY interaction energy is of the form: $J_{RKKY}(r) \sim J_0/r^3$. The average impurity distance scales as $(1/x)^{1/3}$ with x the impurity concentration. Given a thermal energy whose scale is set by $k_B T$, the freezing temperature is then obtained by equating $J_{RKKY} = k_B T_f$, which implies that $T_f = xJ_0/K_B T$.

In the Edwards-Anderson model, the combination of site disorder and the RKKY interaction gives a random set of bonds satisfying a Gaussian distribution. Edwards and Anderson considered a non-dilute system of spins with such random bonds, describing the spin freezing in terms of each spin locking on to a random direction.

If we think of the orientation of the spin vector at time t as $\sigma_i(t)$, Edwards and Anderson proposed the following order parameter

$$q = \lim_{t \to \infty} q(t) = \lim_{t \to \infty} \langle \sigma(0) \rangle \langle \sigma(t) \rangle. \tag{97}$$

This definition involves the projection of the thermally averaged value of a spin at a site i at a particular time with its value at an infinite time later. (If the spin is frozen in orientation, this projection yields a non-zero result. Note that this particular order parameter would be non-zero even for a ordered ferromagnet.) The Edwards-Anderson method involved the first use of what is now called the replica method, which we will discuss in some detail later. The model gave good predictions for susceptibility but disagreed with the behaviour of the specific heat seen in experiments. Later work by Thouless, Anderson and Palmer described a "cavity mean-field theory" which showed the existence of a macroscopic number of metastable states within this model.

The model we discuss here is called the Sherrington-Kirkpatrick model. It is a model with long-range interactions. The pure version of the SK model is what we solved exactly in an earlier section.

We begin with the Hamiltonian. The spins σ_i are Ising spins, taking values ± 1. The sum is over all pairs of spins and the model is infinite-ranged as a consequence.

$$\mathscr{H} = -\sum_{i\langle j} J_{ij}\sigma_i\sigma_j. \tag{98}$$

We do not introduce factors of $1/N$ as we did previously. Such factors will be included in the definition of the distribution for the J_{ij}'s.

The J_{ij}'s are quenched random interactions specified by the probability distribution

$$P[J_{ij}] = \prod_{i<j} \sqrt{\frac{N}{2\pi}} \exp\left(-\frac{J_{ij}^2 N}{2}\right). \tag{99}$$

This distribution, being Gaussian, is completely specified by its mean and standard deviation

$$[J_{ij}] = 0, [J_{ij}^2] = \frac{1}{N}. \tag{100}$$

Averages with respect to this probability distribution are defined through

$$[\cdot] = \int \mathscr{D}J P[J](\cdot) = \prod_{i<j} \sqrt{\frac{N}{2\pi}} \int_{\infty}^{+\infty} \exp\left(-\frac{J_{ij}^2 N}{2}\right)(\cdot). \tag{101}$$

We would like to calculate the disorder-averaged free energy which encodes all thermodynamic properties of this model *i.e.*

$$F = [F_d], \tag{102}$$

where the brackets $[\cdot]$ denote an average over the probability distribution for the quenched variables, the exchange terms J_{ij}'s. Of course the difficulty lies precisely in the fact that a *logarithm* must be averaged. This was circumvented, in a technique invented for this purpose in the original Edwards-Anderson paper, where the following expansion was used

$$Z^n = \exp n\ln Z = 1 + n\ln Z + \dots \tag{103}$$

and thus

$$\ln Z = \lim_{n\to 0} \frac{Z^n - 1}{n.} \tag{104}$$

Note that now what is required is the average of a power of the partition function. This is certainly easier to do than averaging the logarithm. We can now write

$$Z^n = \prod_{\alpha=1}^{n} Tr_{\sigma_\alpha} \exp \sum_{\alpha=1}^{n}\sum_{i<j}^{N} J_{ij}\sigma_i\sigma_j, \tag{105}$$

from which the free energy follows as

$$F_n = -\frac{1}{\beta} \ln Z_n, \tag{106}$$

and we have performed the average

$$Z_n \equiv [Z_J^n]. \tag{107}$$

Now, taking the limit yields

$$\lim_{n\to 0} F_n = -\frac{1}{\beta}[\ln Z_J] = F. \tag{108}$$

We now return to the explicit calculation of the nth power of the partition function

$$Z_n = (Tr_{\sigma_\alpha^i}) \int \mathscr{D}J_{ij} \exp\left\{ \beta \sum_{\alpha=1}^{n} \sum_{i<j}^{N} J_{ij}\sigma_i^\alpha \sigma_j^\alpha - \frac{1}{2}\sum_{i<j} J_{ij}^2 N \right\}. \tag{109}$$

Note that we can write this as

$$Z_n = (Tr_{\sigma_\alpha^i}) \int \mathscr{D}J_{ij} \exp\left\{ \beta \sum_{i<j}^{N} J_{ij} \sum_{\alpha=1}^{n} \sigma_i^\alpha \sigma_j^\alpha - \frac{1}{2}\sum_{i<j} J_{ij}^2 N \right\}, \tag{110}$$

interchanging the order of the summations. Then, averaging over disorder yields

$$Z_n = (Tr_{\sigma_\alpha^i}) \exp\left[\frac{\beta^2}{2N} \sum_{i<j}^{N} \left(\sum_{a=1}^{n} \sigma_i^a \sigma_j^a \right)^2 \right]. \tag{111}$$

We will attempt to simplify the term in

$$Z_n = \sum_{i<j}^{N} \left(\sum_{a=1}^{n} \sigma_i^a \sigma_j^a \right)^2. \tag{112}$$

Fixing $\{ij\}$, we write each term as

$$\sigma_i^1 \sigma_j^1 \sigma_i^1 \sigma_j^1 + \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 + \sigma_i^1 \sigma_j^1 \sigma_i^3 \sigma_j^3 + \sigma_i^2 \sigma_j^2 \sigma_i^2 \sigma_j^2 + \dots \tag{113}$$

There are n terms (for n replica's in this expansion) which contributes unity from terms which involve the same replica index; there are $N^2/2$ terms arising from the sum over ij arising out of all possible pairs. So we can pull this out to get

$$\exp\left[\frac{\beta^2}{2N} \times n \times \frac{N^2}{2} \right]. \tag{114}$$

Now we consider the expansion of the terms left out in doing this

$$\sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 + \sigma_i^1 \sigma_j^1 \sigma_i^3 \sigma_j^3 + \sigma_i^2 \sigma_j^2 \sigma_i^3 \sigma_j^3 + \dots \tag{115}$$

We reorder these as

$$(\sigma_i^1\sigma_i^2)(\sigma_j^1\sigma_j^2) + (\sigma_i^1\sigma_i^3)(\sigma_j^1\sigma_j^3) + (\sigma_i^2\sigma_i^2)(\sigma_j^3\sigma_j^3) + \tag{116}$$

These are terms which turn up in the expansion of

$$\left[\sum_i(\sigma_i^a\sigma_i^b)\right]^2. \tag{117}$$

Now both $a > b$ and $b > a$ appear in the sum above; by symmetry of the terms we can consider just one, say $a < b$ and multiply by a factor of 2. Also, while the sum outside is over $i < j$, we may divide by 2 and retain the term. So factors of 2 cancel from top and bottom. Now we have

$$\mathscr{Z}_n = \sum_{\sigma_i^a}\exp\left[\frac{1}{4}\beta^2 Nn + \frac{\beta^2 N}{2}\sum_{a\langle b}^n(\frac{1}{N}\sum_i^N\sigma_i^a\sigma_i^b)^2.\right] \tag{118}$$

We restrict the sum to $a < b$ because we have already accounted for the $a = b$ terms by pulling them out. Following the same procedures we used earlier in the mean-field theory for the pure problem

$$Z_n = \prod_{a<b}^n\left(\int dQ_{ab}\right)\sum_{\sigma_i^a}\exp\left(\frac{1}{4}\beta^2 Nn - \frac{\beta^2 N}{2}\sum_{a<b}^n Q_{ab}^2 + \beta^2\sum_{a<b}^n\sum_1^N Q_{ab}\sigma_i^a\sigma_i^b\right) \tag{119}$$

Here, the quantities Q_{ab} are related to the correlation functions

$$Q_{ab} = \frac{1}{N}\sum_{i=1}^N\langle\sigma_i^a\sigma_i^b\rangle. \tag{120}$$

Then we have, upon taking the sum over spins inside, and using the fact that now the sites are decoupled from each other,

$$Z_n = \prod_{a<b}^n\left(\int dQ_{ab}\right)\exp(\frac{1}{4}\beta^2 Nn - \frac{\beta^2 N}{2}\sum_{a\langle b}Q_{ab}^2)\prod_i^N\sum_{\sigma_i^a}\exp(\beta^2 Q_{ab}\sigma_i^a\sigma_i^b). \tag{121}$$

The second product involves N independent terms. The value of each term is

$$\left[\sum_{\sigma_a}\exp\beta^2\sum_{a<b}^n Q_{ab}\sigma_a\sigma_b\right]. \tag{122}$$

This can be exponentiated to yield

$$Z_n = \prod_{a<b}^n\left(\int dQ_{ab}\right)\sum_{\sigma_i^a}\exp\left(\frac{1}{4}\beta^2 Nn - \frac{\beta^2 N}{2}\sum_{a\langle b}Q_{ab}^2 + N\log\left[\sum_{\sigma_a}\exp\beta^2\sum_{a<b}^n(Q_{ab}\sigma_a\sigma_b)\right]\right) \tag{123}$$

Note that this can now be represented as

$$Z_n = \int d\hat{Q} \exp\left(-\beta n N f\left[\hat{Q}\right]\right). \tag{124}$$

where

$$f[\hat{Q}] = -\frac{\beta}{4} + \frac{\beta}{2n}\sum_{a<b}^{n} Q_{ab}^2 - \frac{1}{\beta n}\log\left[\sum_{\sigma_a}\exp\left(\beta^2\sum_{a<b}^{n} Q_{ab}\sigma_a\sigma_b\right)\right]. \tag{125}$$

In the large N limit,

$$Z_n \simeq \left[det\frac{\partial^2 f}{\partial\hat{Q}^2}\right]^{-1/2}\exp\left(-\beta n N f\left[\hat{Q}^*\right]\right), \tag{126}$$

where

$$\frac{\partial f}{\partial Q_{ab}} = 0, \tag{127}$$

defines the saddle point value of the matrix Q_{ab}.

Note that we can take the Q matrix to have zero diagonal values, thereby accounting for the fact that we have already separated the diagonal components out.

We will now go on to describing the solutions to this mean field problem. The strategy will be to parametrize the matrix in a simple manner and then to perform the saddle point evaluation.

7.1 Replica Symmetric Solution

It is natural to assume a symmetry between replicas of the system

$$Q_{ab} = q \qquad \forall\, a \neq b \tag{128}$$

(It will turn out to be equivalent to the approximation that there is a unique ground state.) Thus,

$$f(q) = -\frac{1}{4}\beta + \frac{\beta}{2n}\frac{n(n-1)}{2}q^2 - \frac{1}{\beta n}\log\left[\sum_{\sigma_a}\exp\left(\frac{\beta^2}{2}(\sum_{a}^{n}\sigma_a)^2 q - \frac{1}{2}\beta^2 nq\right)\right] \tag{129}$$

where we have pulled out the diagonal terms, n of them, all of the same value. This can now be extracted from the integral as well with

$$\frac{-1}{\beta n}\times\frac{-1}{2}\beta^2 nq = \frac{\beta q}{2}. \tag{130}$$

Now we apply the same methods used earlier, of linearizing the quadratic expression. We use

$$\prod_{\alpha=1}^{n}\left(\sum_{\sigma_a=\pm 1}\exp\beta\sigma_a\sqrt{q}z\right)=([2\cosh(\beta\sqrt{q}z)]^n, \tag{131}$$

inside

$$f(q)=-\frac{\beta}{4}+\frac{\beta q}{2}+\frac{1}{4}(n-1)\beta q^2-\frac{1}{\beta n}\log\left[\int_{-\infty}^{+\infty}\frac{dz}{\sqrt{\pi}}\exp\left(-\frac{1}{2}z^2\right)\prod_{\alpha=1}^{n}\left(\sum_{\sigma_a=\pm 1}\exp\beta\sigma_a\sqrt{q}z\right)\right] \tag{132}$$

Now writing

$$([2\cosh(\beta\sqrt{q}z)]^n=e^{n\ln\left([2\cosh(\beta\sqrt{q}z)\right]}, \tag{133}$$

and expanding in n, the first term is

$$1+n[2\cosh(\beta\sqrt{q}z)]+n^2\ldots \tag{134}$$

The first term goes away on integration since this is a Gaussian integral normalized to 1; taking the logarithm gives zero. The second term is the important contributor and all higher order terms will vanish as $n\to 0$. Thus we may now take the limit $n\to 0$

$$f(q)=-\frac{1}{4}\beta(1-q)^2-\frac{1}{\beta}\int_{-\infty}^{-\infty}\frac{dz}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}z^2\right]\ln[2\cosh(\beta\sqrt{q}z)]. \tag{135}$$

We now take the derivative with respect to q to find the minimum of the replica free energy. One term we obtain is

$$\frac{1}{2}\beta(1-q). \tag{136}$$

The second is more complicated and looks like the integral

$$\frac{1}{2}\int_{-\infty}^{-\infty}\frac{dz}{\sqrt{2\pi q}}\exp\left[-\frac{1}{2}z^2\right][z\tanh\beta\sqrt{q}z)]. \tag{137}$$

We now evaluate this through integration by parts, with $dv=\exp\left[-\frac{1}{2}z^2\right](-z)$ which implies that $v=\exp\left[-\frac{1}{2}z^2\right]$. Then $u=\left[\tanh\beta\sqrt{q}z)\right]$. Because of the exponential falloff, the uv term doesn't contribute; the only contribution is from the $\int vdu$ term. This is then

$$-\frac{1}{2}\int_{-\infty}^{-\infty}\frac{dz}{\sqrt{2\pi q}}\exp\left[-\frac{1}{2}z^2\right](\beta\sqrt{q})\left[sech^2(\beta\sqrt{q}z)\right]. \tag{138}$$

We can now eliminate $\sqrt{q}$ from top and bottom, substitute for

$$sech^2(x) = 1 + \tanh^2(x), \tag{139}$$

and then get

$$\beta(1-q) = -\beta \int_{-\infty}^{-\infty} \frac{dz}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2\right] \left[1 + \tanh^2(\beta\sqrt{q}z)\right], \tag{140}$$

to finally get the saddle point equation in the form

$$q = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2\right] \tanh^2(\beta\sqrt{q}z), \tag{141}$$

where

$$q = \frac{1}{N} \sum_i^N \langle\sigma_i\rangle^2. \tag{142}$$

Note that for $T\rangle 1$ there is only the solution $q = 0$. For smaller T, there is a non-zero solution; if $(1-T) \sim \tau$, then $q(\tau) \sim \tau$. If $T \to 0$, then the solution is $q \to 1$.

The solution obtained for q is the physical order parameter

$$q = \frac{1}{N} \sum_i^N \langle\sigma_i\rangle^2. \tag{143}$$

The fact that q is non-zero relates to the fact that spins are frozen in a random state. Since there is only one solution for q, the ground state is unique.

This solution can be shown to have interesting properties. The specific heat and susceptibility show a cusp, as in the experiments, while the order parameter q rises smoothly at the transition point. Unfortunately, this solution is qualitatively incorrect below the transition. It has an entropy which becomes negative below T_c. This was interpreted by de Almeida and Thouless in terms of the breakdown of the stability of the replica symmetric solution. For some time it was believed that it was the replica trick which gave this erroneous result. It turns out that the answer is more subtle than that.

The error arises because the pattern of symmetry breaking we have discussed is not sophisticated enough. The correct form of the pattern of symmetry breaking was worked out by Parisi in a creative and insightful set of papers from the early 1980's. Essentially, Parisi made an ansatz for $q_{\alpha\beta}$ which goes beyond the simplest parametrization described above. He formulated the calculation in several steps, starting with the simplest, or *one-step* replica symmetry breaking. He then went on to show how an infinite hierarchy of replica symmetry breaking could be accomodated in an ansatz, the *Parisi ansatz*. This calculation, however, is complex enough that it will not be described here.

To understand the physics in some detail, it is useful to introduce the notion of pure states. Consider a ferromagnet. At low temperatures, in the ordered state, the system breaks symmetry to choose one of a large number of possible ordered

states. The barriers between these symmetry broken states is exponentially large. The observable state is then not the *Gibbs* state, which is obtained by summing over all spin configurations but only those configurations which are appropriate to the particular symmetry broken state.

Such a state is called a *pure state* and has the property that connected correlation functions, defined below, vanish at large distances:

$$\langle \sigma_i \sigma_j \rangle_c \equiv \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \langle \rangle \sigma_j \rangle \tag{144}$$

Note that the pure states in non-disordered models are typically related to each other by symmetry, as in the case of the simple ferromagnetic Ising model, which has two pure states.

The structure of (replica-) symmetry breaking in the spin glass state suggests that there may be a very large number of pure states in the thermodynamic limit. Thus, the Gibbs state can be constructed if we knew the weight factor for each of the pure states

$$\langle \sigma_i \rangle = m_i = \sum_\alpha w_\alpha m_i^\alpha. \tag{145}$$

Here the statistical weights w_α can be written as

$$w_\alpha = \exp(-F_\alpha), \tag{146}$$

with F_α the free energy corresponding to state α. A similar relation holds for the correlation functions.

$$\langle \sigma_1 \sigma_2 \rangle = \sum_\alpha w_\alpha \langle \sigma_1 \sigma_2 \rangle_\alpha. \tag{147}$$

To understand how pure states in the spin glass problem might be distinguished from each other, define a quantity measuring the overlap between the two states, defined through

$$q_{\alpha\beta} = \frac{1}{N} m_i^\alpha m_i^\beta. \tag{148}$$

To describe the statistics of overlaps, we introduce

$$P_J(q) = \sum_{\alpha\beta} w_\alpha w_\beta \delta(q_{\alpha\beta} - q). \tag{149}$$

Note that this could depend on the disorder. Once we disorder average we get

$$P(q) = \overline{P_J(q)}. \tag{150}$$

The function $P(q)$ gives the probability of finding two pure states with a given overlap q, conditioned by the fact that these states are weighted by the probability of their appearance in the ensemble.

In a ferromagnet, $P(q)$ is a δ-function peak at $q = 0$ in the high temperature phase and consists of two δ-function peaks at $q = \pm m^2$. In a spin glass with self-evidently a large number of pure states, unrelated by symmetry, at low temperatures, one expects $P(q)$ to have far more structure. It is this function $P(q)$ which turns out to be the physical order parameter. If a disordered system exhibited only two frozen ground states related by symmetry, as in some competing theories of the spin glass transition in finite-dimensional systems, then $P(q)$ would resemble that in the pure ferromagnet.

Here is the picture which emerges of the spin glass state order parameter from a detailed calculation due to Parisi and collaborators.

1. Just below T_c the system can be in a large number of pure states, defined as valleys constituting minima of the free energy. The configurations of the m's are different in each of these states.
2. The value of the self-overlaps $q(T) = \sum_i^N m_i^2$ appears to be the same in all the states
3. The mutual pair overlaps appear to continuously fill the interval $0 \leq q^{\alpha\beta} \leq q(T)$. The distribution of values of $q^{\alpha\beta}$ is described by a probability function $P(q)$
4. If the temperature is decreased slightly , each pure state is divided into numerous new ones. These states have a larger value of the overlap.
5. On a further decrease of the temperature, each pure state is further subdivided. This branching goes on to zero temperature.
6. In a whole temperature interval $T \leq T_c$, there is a continuous sequence of phase transitions.

It is clear that this extremely complex pattern of symmetry breaking has no parallel in the conventional phase transitions describable by Landau theory.

8 Modern Developments

It is now generally believed that Parisi's ansatz for replica symmetry breaking in the Sherrington-Kirkpatrick model is exact. Mathematical results from the French mathematician M. Talagrand and others prove the validity of the expression for the free energy obtained in Parisi's detailed calculation. However, the relevance of the Parisi ansatz to real spin glasses is still a largely open problem. The Sherrington-Kirkpatrick model is, after all, an infinite range model in which all spins are coupled to each other and thus very far from real experimental systems.

From simulations on finite-ranged Ising spin glasses, due principally to Young, Marinari and their collaborators, it appears very likely that there is a spin-glass transition in the three-dimensional nearest neighbour Ising spin glass with Gaussian disorder. There also appears to be a finite temperature transition in the Heisenberg spin glass in three dimensions.

To address the issue of spin glass ordering in finite dimensional systems, a number of workers including Fisher, Huse, Moore, Bray and McMillan, came up with

an alternative view of the spin glass phase. In this picture, there is a single configuration, upto a global flip of all spins, which is the true ground state. There is no net magnetization i.e.

$$m = \frac{1}{N} \sum \langle \sigma_i \rangle = 0, \tag{151}$$

while the Edwards-Anderson order parameter

$$q = \frac{1}{N} \sum_i \langle \sigma_i \rangle^2 \tag{152}$$

is non-zero, signalling the freezing of spins into the disordered ground state.

This view of the spin glass phase is thus very much like the conventional picture of transitions in a pure system. The only difference is in the nature of excitations out of this ground state. Since the system is disordered however, one should expect that excitations out of this ground state should be non-trivial.

Such excitations have been argued to be "droplets". These excitations, obtained by flipping a group of spins across a scale L, are proposed to have a typical excitation energy (or gap) which scales as

$$\Delta E_t(L) \propto L^{\theta}. \tag{153}$$

The exponent θ must be bounded as $0 < \theta < (d-1)/2$.

The kinetics of spin flips is accomodated by assuming that the typical barriers for flipping a region of size L, $B_t(L)$ scale as

$$B(L) \sim L^{\Psi}. \tag{154}$$

Within replica symmetry breaking theory $\theta = 0$ for some excitations, whereas the droplet picture has $\theta > 0$ always.

Huse and Fisher assume that the probability distribution of energy gaps $\rho(\Delta E)$ has a scaling form

$$\rho(\Delta E(L)) = \frac{1}{\Delta E_t(L)} \tilde{\rho}\left(\frac{\Delta E}{\Delta E_t(L)}\right). \tag{155}$$

provided $\tilde{\rho}(x)$ is nonzero as $x \to 0$, there is weight for droplet excitations of arbitrarily low energy, permitting slow dynamics even for T close to zero. It is these excitations which can give the spin glass phase its characteristic complex relaxation.

The picture of Huse and Fisher differs in several important ways from that proposed by Parisi. Parisi's solution indicates that a line, called the Almeida-Thouless line, in the magnetic field-temperature plane, exists across which replica symmetry breaking occurs. This is a non-trivial attribute because the line separates an ergodic phase from a non-ergodic phase. Since the field breaks the up-down symmetry of the Hamiltonian, one has a transition without a change in symmetry. Also, the overlap distribution function of Parisi is a non-trivial one. In the Fisher-Huse picture,

neither of these are true. There is no AT line and the order parameter is a number as opposed to a function.

An alternative scenario which combines some of the ideas of the droplet model with intuition from the Parisi approach is the trivial-non-trivial scenario for finite dimensional spin glasses, due to Krzakala and Martin and Palassini and Young. The TNT scenario suggests that there are different local (at scales much less than the system size) and global exponents (for scales comparable to the system size) for the excitation energy as a function of the excitation length scale.

The local exponent is given by domain wall estimates. The global exponent is possibly consistent with $\theta_g = 0$. In the TNT scenario, one has the coexistence of a droplet model at finite length scales with mean-field behaviour for system-scale excitations. The excitations are argued to be topologically non-trivial, reaching out to the boundaries of the lattice. Physically, this is a situation where excitations look random beyond the scale of a few lattice spacings, have a surface to volume ratio of order unity and are space filling and space spanning.

9 K-SAT and the Spin Glass Problem

In theoretical computer science, a recurring theme is the understanding of the "computational complexity" of a given computational task. Many such tasks are surprisingly difficult to solve and are found in contexts ranging from optimization and hardware design to computational biology.

A large class of such problems belong to the class of what are called NP-complete problems. This class of computational tasks consists of problems where, given a solution, it can be checked for correctness in a time which is polynomial in the size of the input. However, *finding* such a solution *a priori* takes an exponential time in the size of the input in the worst case.

A very large number of problems have been shown to be NP-complete. It is believed that if an efficient algorithm for finding the solution of one such problem is found, one would have an efficient algorithm for all NP-complete problems belonging to the same class. A fundamental conjecture of complexity theory is that no such efficient algorithm exists.

One example of an interesting class of problems is the K-satisfiability problem, or **K-SAT**. It is stated as follows: Consider a set of N boolean variables x_i, with $i = 1 \ldots N$. A *literal* is either a variable x_i or its negation $\neg x_i$. A *clause* is the logical OR ($\vee$) between K distinct literals. It is thus true as soon as one of the literals is true.

A formula is the logical AND ($\wedge$) between M clauses. It is true if and only if all the clauses are true. A formula is said to be **satisfiable** if there is an assignment of the variables such that the formula is true. An example of a 2-SAT formula is

$$(x \vee y) \wedge ((\neg x) \vee (\neg y)). \tag{156}$$

This formula can be satisfied by setting x to true and y to false.

In the general case, the goal is to find out whether an assignment exists. In the worst case, a search through the space of truth assignments is required to determine that no such assignment exists or to find a suitable assignment.

A large number of problems can be shown to be instances of K-SAT. The K-SAT problem is a constraint satisfaction problem. It involves several variables, each taking values in a small domain, and some constraints, each forbidding some of the possible joint values for the variables. For a given instance, the question is whether there are values for the variables that simultaneously satisfy all the constraints.

The spin-glass version of the K-SAT problem is related to a random version of K-SAT. In this version, formulas are drawn in the following way: Choose a triplet of distinct indices in $1 \ldots N$, uniformly on all possible triplets. For each of the three corresponding variables, choose the variable itself or its negation with equal probability. Construct a clause with the chosen literals. Repeating this process M times yields a set of M independently chosen clauses. These can be combined to generate an instance of the formula. The thermodynamic limit is achieved in the limit in which $M = \alpha N$, with both M and N going to infinity.

The link with spin glasses comes from considering a group (pair, triplet etc.) of interacting spins as a constraint. Finding the state of minimal energy amounts to minimizing the number of violated constraints. The general difficulty both with the spin glass and the K-SAT problem arises from frustration. This makes it diffcult to find the global minimum by local manipulations. The connection to statistical mechanics comes from assigning an energy to the system. A satisfactory assignment can be assigned zero energy, with the energy increasing with the number of unsatisfied clauses.

Consider formulas with a fixed ratio of clauses to variables. These exhibit the following remarkable behaviour. When the ratio is small, formulas have many variables and few constraints. There are thus many assignments which are satisfying. When the ratio is large, the variables are highly constrained and formulas almost certainly have no satisfying assignments.

Surprisingly, as the ratio r of clauses to variables grows, the transition from probably satisfiable to probably unsatisfiable is not gradual but abrupt. It is conjectured that there was a single threshold value r_k of r, depending only on k, such that as the number of variables goes to infinity, random formulas with $r < r_k$ are almost surely satisfiable, while random formulas with $r > r_k$ are almost surely unsatisfiable. This change resembles a phase transition. Random K-SAT problems appear to be hardest to solve close to this transition point, from numerical studies.

Inspired by ideas from spin glasses, Mezard, Parisi and collaborators have suggested that the hardness of random K-SAT is connected not to the satisfiability threshold phase transition, but to another phenomenon altogether. For a given formula, the geometry of the space of solutions undergoes a dramatic transformation at a ratio well below the satisfiability threshold. It is this latter transition that appears to be the key to understanding what makes some instances of K-SAT hard.

References

1. S. F. Edwards and P. W. Anderson, J. Phys. F **5** 965 (1975)
2. D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975)
3. K. Binder and A. P. Young, Rev. Mod. Phys. **58** , 801 (1986)
4. K.H. Fischer and J.A. Hertz, *Spin Glasses*, Cambridge University Press (1991)
5. M. Mezard, G. Parisi, M.A. Virasoro *Spin Glass theory and Beyond* Singapore: World Scientific (1987)
6. S. Kirkpatrick , C. D. Gelatt, Jr. and M. P. Vecchi, Science, **220**, 671 (1983).
7. *Spin Glasses and Random Fields*, A.P. Young, Ed., World Scientific, Singapore, (1998).
8. M. Palassini, and A. P. Young, Phys. Rev. Lett. **83** , 5216 (1999);
9. P. Nordblad, L. Lundgren and L. Sandlund, J. Mag. and Mag. Mater. **54**, 185 (1986)
10. V. S. Dotsenko, *Introduction to the theory of spin glasses and neural networks*, World Scientific
11. D. Stauffer: Phys. Rep. **54** 1 (1979)
12. D. Stauffer, *Introduction to Percolation Theory*, Taylor and Francis Inc., London.
13. M. Sahimi, *Applications of Percolation Theory* Taylor and Francis, (1994)