# Physics of disordered systems

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Disordered systems:disorder in the laws of motion,

I will not discuss real glasses, where the disorder is in the configurations, not in the Hamiltonian).

New phenomena, that are not present for ordered systems.

New collective behaviour.

Very strong slowing down of the dynamics $\Longrightarrow$ glassy behaviour.

Many applications beyond physics.

- Ferromagnets in a random field.
- Electrons in a random potential.
- Spin glasses

The key point is *emergent* collective behaviour.

We want to get insight, i.e. quantitative and qualitative understanding.

Universal quantities.

The example of homogenous Heisenberg ferromagnets.

$$\chi \approx \frac{A}{(T - T_c)^\gamma}$$

The exponent $\gamma$ does not depends on the material, but only only the dimension of the space.

$A$ and $T_c$ depend on the material.

Field theory approach:

$D \geq 4$: mean field theory $\gamma = 1$.

$D = 4 - \epsilon$: perturbative renormalization group $\gamma = 1 + \frac{5}{11}\epsilon$.

$D = 3$:

perturbative renormalization group $\gamma = 1.382(9)$,

resummed $\epsilon$ expansion $\gamma = 1.388(1)$ fixed dimension analysis,

Experiments 1.38-1.39.

$D = 2 + \epsilon$: perturbation renormalization group around the non linear-sigma model.

$D = 2$: $\chi \approx \exp(4\pi/T)$ perturbation renormalization group around the non linear-sigma model.

$D < 2$: No transition.

$D = 4$: upper critical dimension. $D = 2$: lower critical dimension.

Computer based analysis:

real space renormalization group, finite size scaling...

## A difficult simple case. The random magnetic field Ising ferromagnet.

In the nutshell, the probability of the random magnetic field is

$$P(h) = \exp\left(\frac{1}{2g}\int dx h^2(x)\right) \ .$$

The field $\phi$ is the minimum of

$$S = \int dx \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\tau\phi^2 + \frac{1}{4}\phi^4 \right).$$

If the solution of the following stochastic equation is unique

$$-\nabla\phi + \tau\phi + \phi^3 = h(x) \ .$$

then

$$\overline{\phi(x)\phi(0)}_D = \langle\phi(x)\phi(0)\rangle_{D-2} \ .$$

$\langle\cdot\rangle$ is the statistical expectation value with weight $\exp(-S/g)$ in two dimensions less.

Dimensional reduction that follow from an hidden supersymmetry of the stochastic differential equation.

Dimensional reduction implies that

$$\gamma_{RF}(D) = \gamma(D-2)$$

The lower critical dimension is 2, not 3=1+2.

Dimensional reduction does not work!

Non-perturbative effects destroy dimensional reduction.

A possible conjecture

- $D > 6$: mean field.
- $D = 6 - \epsilon$: $\gamma_{RF}(D) = \gamma(D-2) + O(\exp(-A/\epsilon))$

It is not clear how to compute $\gamma_{RF}(D) - \gamma(D-2)$ .

There are some suggestions (e.g. instantons), but no consensus.

Hidden symmetry. Non perturbative-phenomena.

## Schrödinger equations for non-interacting electrons

$$(-\nabla + V(x)\psi(x) = E\psi(x) \qquad \mathbf{H} = -\nabla + V$$

Periodic potentials, Block theorem, band structure.

Random potentials

$$P(V) = \exp\left(\frac{1}{2g}\int dx V^2(x)\right)$$

For large negative $E$ localized states. Elsewhere extended states.

Localized-extended transition. The conductivity is related to

$$G_2(x,E) = \overline{|R(x|E)|^2} \qquad R(x|E) = \langle x|\frac{1}{E-\mathbf{H}}|0\rangle$$

$E$ is complex and eventually goes to 0.

$E = E_R + i\epsilon$ (like $\frac{1}{p^2-m^2-i\epsilon}$)

The conductivity vanishes at zero frequency in the localized phase.

## Field theory representation.

$n$ real fields $\phi_a$ and $n$ real fields $\overline{\phi}_a$.

The action is invariant under the non-compact group $O(n,n)$

$$\sum_{a=1,n} \left(\phi_a^2 - \overline{\phi}_a^2\right) \qquad \text{is an invariant}$$

for real $E$ and the symmetry is broken when $E$ is complex.

Eventually $n \to 0$ (like in quenched QCD)

Exact symmetry in the localized phase.

Broken symmetry in the extend phase.

Peculiar behaviour: non-compact symmetry group.

No extended states in $D = 2$ (a variation of the no-go theorem).

Exponents know in $2 + \epsilon$ dimensions.

The mean field theory is complicated and the upper critical dimension in not yet known (may be it is infinite!)

Many progresses have recently done in spin glasses: theory, experiments, simulations and even theorems!

The simplest Ising spin glasses has the following Hamiltonian:

$$H_J = \sum_{i,k=1,N} J_{i,k}\sigma_i\sigma_k$$

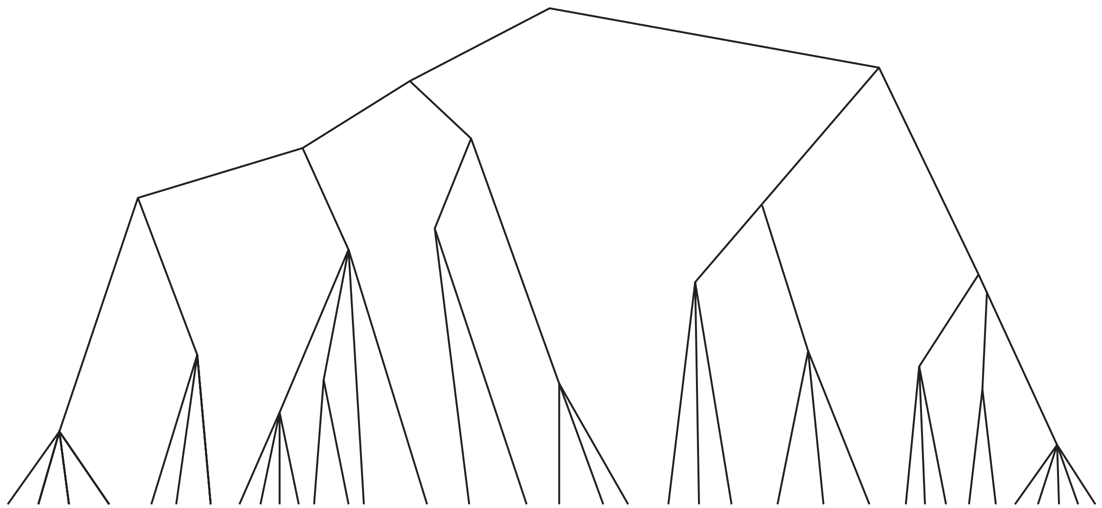
$\sigma = \pm 1$, $i$ and $k$ are *neighbours*. The $J$ are random (e.g. $\pm C$).

- All points are connected: Sherrington Kirkpatrick model: Mean field theory and infinite dimensions limit.
- Nearest neighbour on a regular lattice: Edwards Anderson model in finite dimensions.

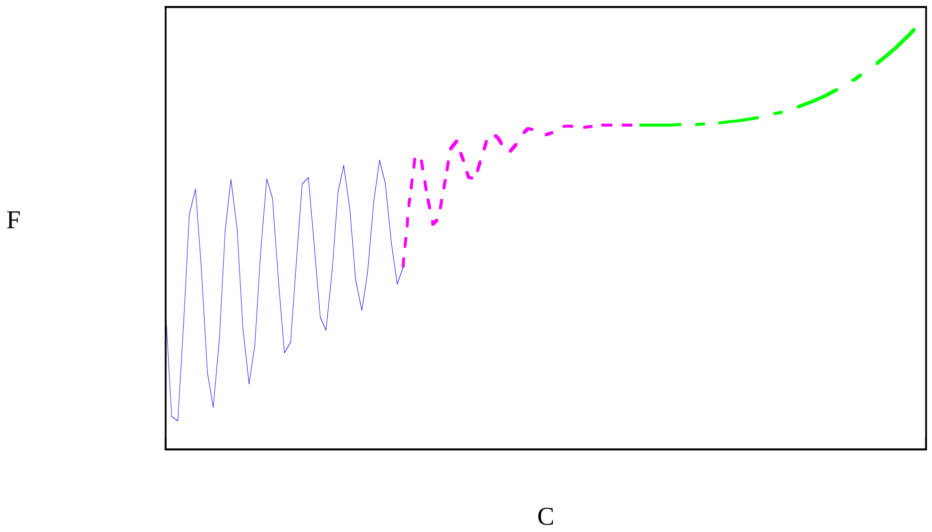
The mean field theory was solved using two non rigorous approaches. (Firstly with replicas and the later probabilistic). After 25 (20) years this solution was proved to be correct.

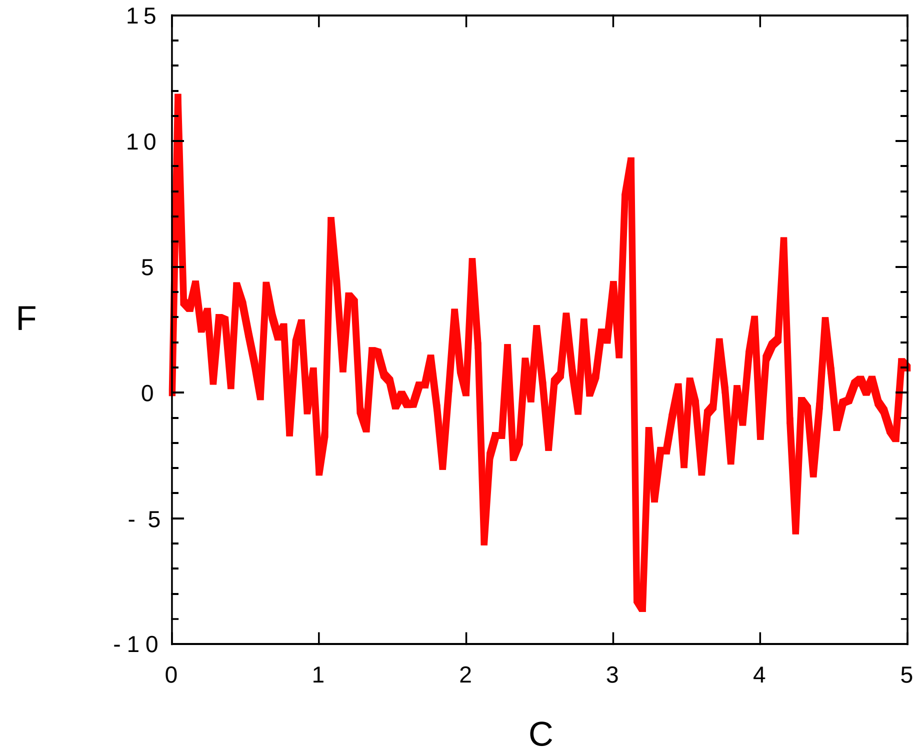
We do the following hypothesis (we can prove the first twos)

- The Hamiltonian has many quite different configurations of nearly minimal energy an the energy difference of these configurations is of order 1.
- At finite low temperature (i.e. at $T < T_c$) the phase space divides in many equilibrium states whose free energy difference is of order 1.
- The states are organized on a tree.



## Free energy landscape




At the left: a schematic view of the free energy of a complex system as function of the configuration space.

At the right: a magnification of the low energy region.

The system has a slow approach to equilibrium.

There is a fast dynamics inside the states and a slow dynamics: jumps between the states. (Punctuated equilibria)

If we add a small magnetic field ($\Delta h$) at times 0, the magnetic susceptibilities is

$$\chi(t) \equiv \frac{\Delta m(t)}{\Delta h}$$

At short times $t << t^*$ we get$\chi_{LR}$. i.e. the response inside one state.

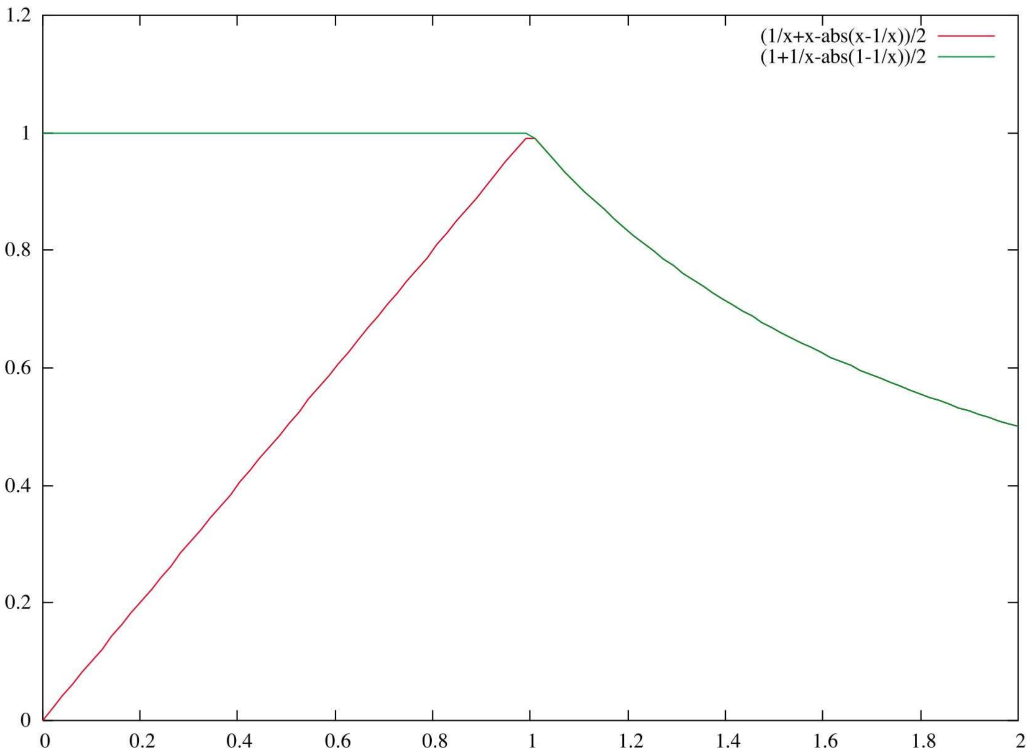
At long times $t >> t^*$ $\chi_{eq}$.i.e. the equilibrium response inside one state.

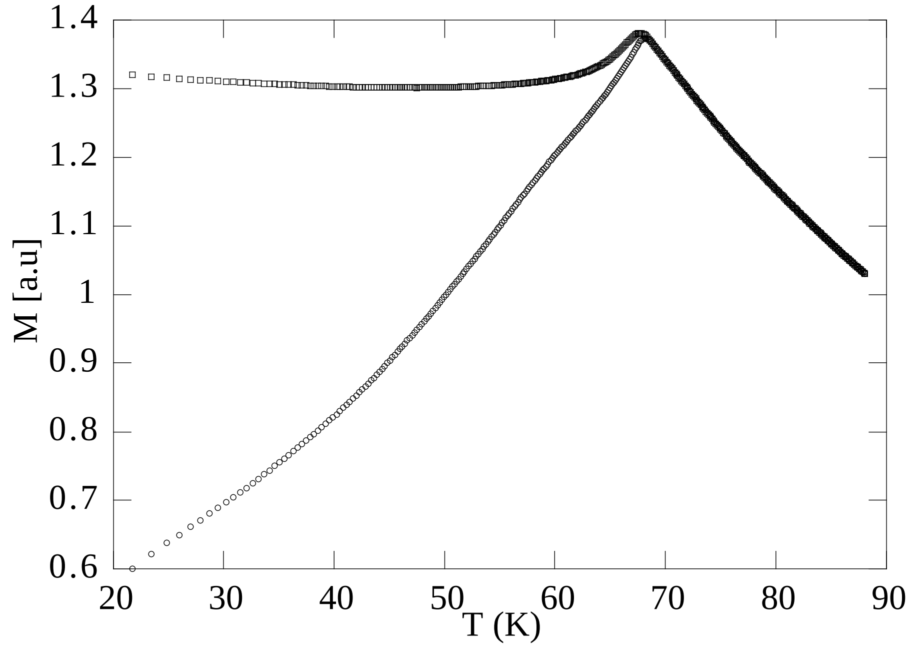
If the system was at equilibrium at time 0, $t^*$ goes to infinity when $\Delta h$ goes to zero.

If the system has an age $t_w$ at time 0, $t^* = O(t_w)$.

Field cooled susceptibility is an approximate way to compute $\chi_{eq}$.

We cool the systems in a field starting from high temperature. We suppose that the system chooses the good valley and that the magnetization is near to equilibrium.




At the left we show the results for the SK model, at the right we have experimental data on metallic spin glasses. The similarities among the two figures are striking.

Experimentally there are still some small dependence on the time on a log scale, so the extrapolation to the infinite time limit is not easy.

The mean field theory was solved by replica approach.

We need to compute

$$F \equiv \overline{\ln(Z)}$$

while it is simple to compute

$$F_n = \frac{\log(\overline{Z^n})}{n} \qquad \lim_{n \to 0} F_n = F$$

This lead to a strange mathematics: one introduces a is $n \times n$ matrix $Q$ and the symmetry group is $S_n$: eventually $n \to 0$.

$$F = F[Q^*] \qquad \frac{\partial F}{\partial Q}\Big|_{Q=Q^*} = 0$$

Replica symmetry breaking $S_n \to S_{n/m} \otimes (S_m)^{n/m}$ .

When $n \to 0$ $S_{n/m} \to S_0$ so that $S_0 \supset S_0$ .

The matrix $Q$ is parametrized in terms of a function $q(x)$ defined on the interval $0 - 1$ .

Replicas are an elegant approach. But what do they mean? Physical equivalent approach.

Description of the phases at low temperature

Ising at zero field: two phases (+ and -).

$$q_{++} = q_{--} = m^2 \qquad q_{+-} = q_{+-} = m^2 \qquad w_+ = w_- = \frac{1}{2}$$

where

$$\langle \cdot \rangle_{Gibbs} = w_+ \langle \cdot \rangle_+ + w_- \langle \cdot \rangle_-$$

$Z_p$ spontaneous symmetry breaking. There are $p$ phases

$$q_{\alpha.\alpha} = m^2 \qquad q_{\alpha,\gamma} = -\frac{m^2}{p-1} \qquad w_\alpha = \frac{1}{p}$$

For a given system the descriptor $\mathcal{D}$ is given by the following set

$$\{w_\alpha, q_{\alpha,\gamma}\}$$

In the SK model for each sample we have a different descriptor, so we must introduce a probability distribution of the descriptors

$$\mathcal{P}(\mathcal{D}) \ .$$

We can compute a free energy $\mathcal{F}[\mathcal{P}]$ such that one should have

$$\frac{\delta \mathcal{F}}{\delta \mathcal{P}} = 0$$

It took some effort to understand that the matrix $Q$ of the replica approach is a very compact to code the probability $\mathcal{P}(\mathcal{D})$. The following two equations are equivalent:

$$\frac{\partial F}{\partial Q} = 0 \qquad \frac{\delta \mathcal{F}}{\delta \mathcal{P}} = 0$$

## Rigorous results.

$$F = \max_{\mathcal{P}} F[\mathcal{P}]$$

Please notice that we maximize the free energy, we do not minimize it!

Let us call $\mathcal{P}^*$ the probability distribution of the descriptor making the hierarchical structure of replica symmetry breaking (i.e. the original solution).

Talagrand's theorem:

$$F = \mathcal{F}[\mathcal{P}^*]$$

i.e. the hierarchical solution gives the correct free energy for the SK model.

Open problem: are there other probability distributions that give the same free energy?

Some progress have been done in this direction, but a final theorem is still missing.

In this period we are doing many progresses in spin glasses for different reasons:

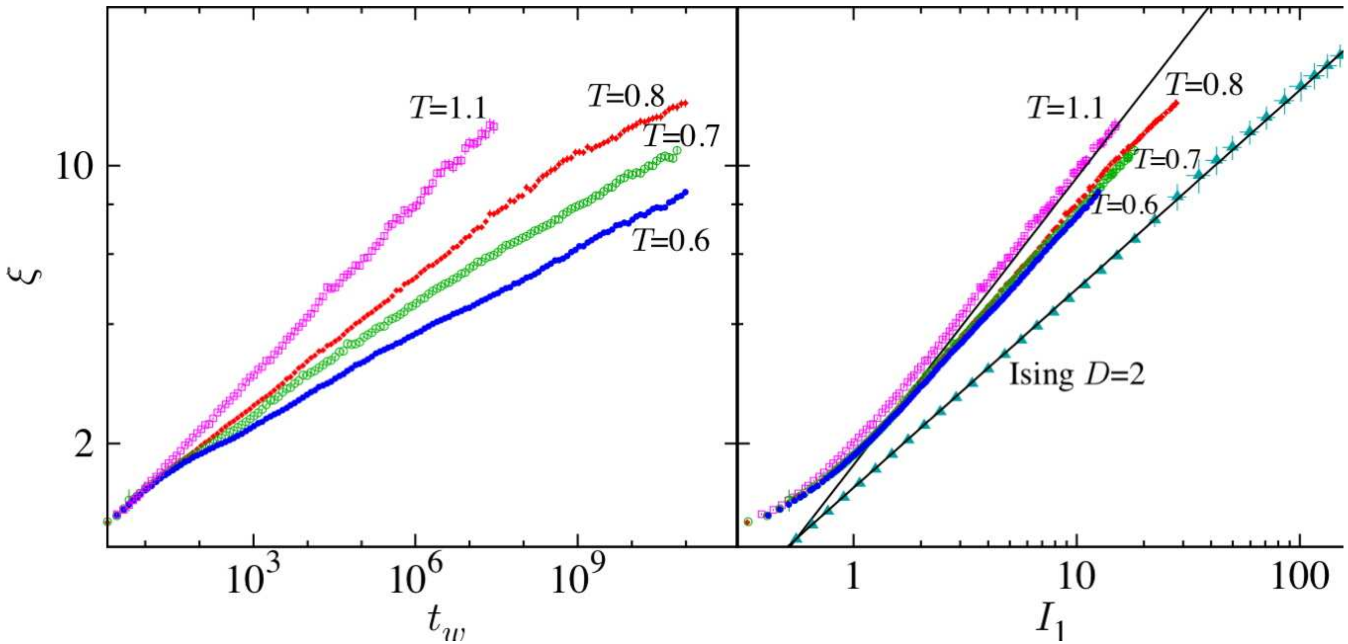
- We are hopefully near to prove that the features of the the analytical solution using the replica method are correct (Aizenman's and Guerra's talks).
- New observables have been measured in numerical simulations.
- A new models have introduced (Levy Lattice), that helps us to understand the behaviour near the upper critical dimension.
- New hardware has been constructed that allows the simulations of large lattices for $O(10^{12})$ Montecarlo sweeps, i.e about one second in physical scale.
- There are new non-perturbative computations.
- There are very interesting new experiments (rejuvenation, memory, effective temperatures).

When you cool the systems you start to forms domains that becomes larger and larger and larger. Different copies of the systems (with different initial conditions) will form different domains.

The size of the domains ($\xi$) diverges when the time goes to infinity.

The first data has be fitted as $t^{\lambda(T)}$, with $\lambda(T) = O(0.1)$. Quite a slow decrease. We need to have a large range of times.

Janus is a special purpose computer that allows the update of about $10^{13}$ spins per second. The data are for systems of size $80^3$.



## Correlations and aging (in spin glasses).

You start at $t = 0$ from an high temperature configuration and you cool the system (may be in 1 step , i.e. $10^{-12}s$).

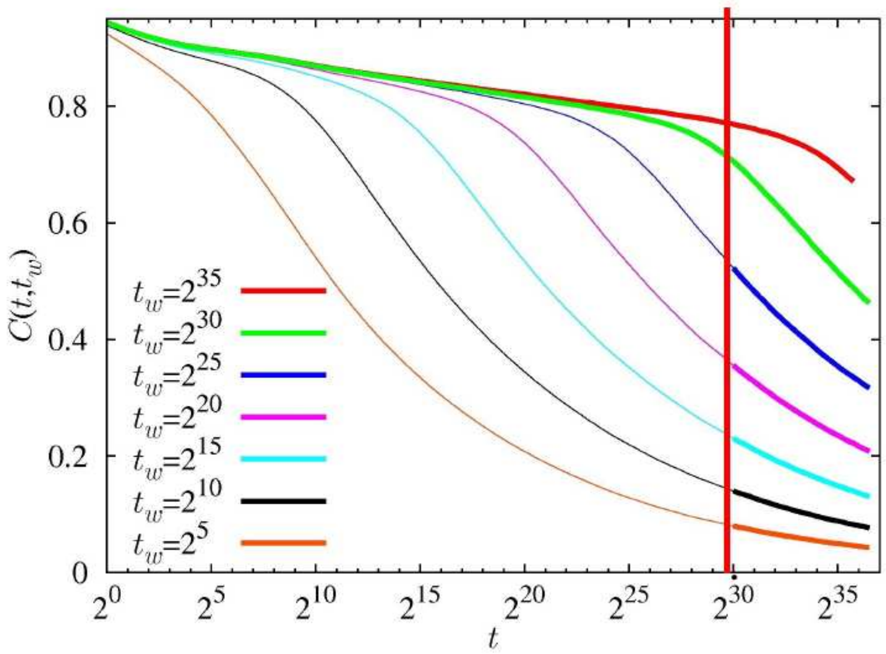
You measure the correlation

$$C(t_w, t) = \frac{\sum_i \langle \sigma_i(t_w) \sigma_i(t) \rangle}{L^3}$$

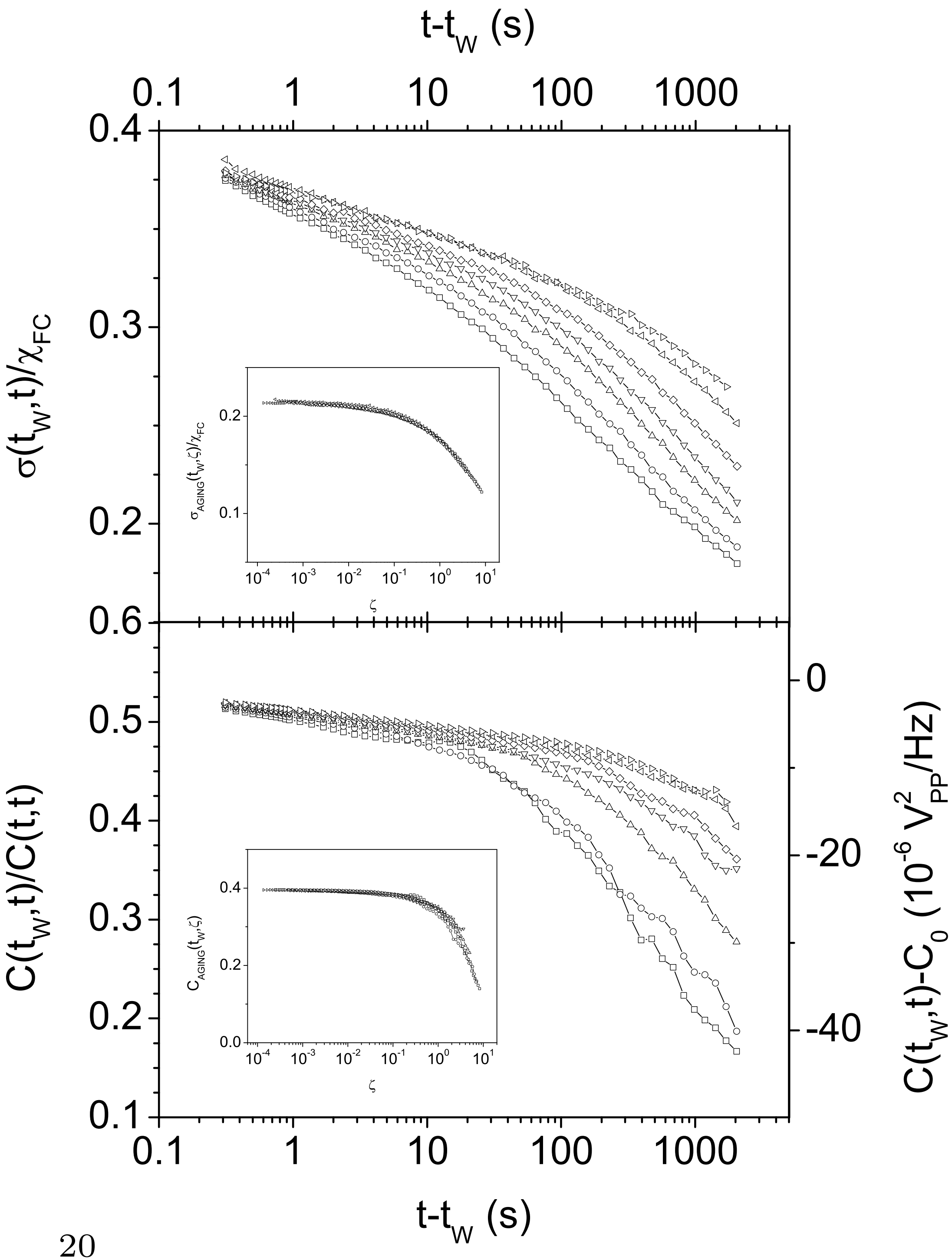
The correlations should decay with a time that increases with $t_w$ This is related to what is measured in experiments.

You have an observation range that goes form $10^{-12}s$ to $O(1)s$ and the possibility of cooling very very fast!

With a small hardware effort we could reach $10^2 s$ now!




$$C(t,t_w)=L^{-3}\overline{\sum_x \sigma_x(t+t_w)\sigma_x(t)}$$



Thermodynamics for slightly off equilibrium systems (adiabatically off-equilibrium).

Cugliandolo Kurchan effective temperatures.

In the simplest case there are two temperatures:

- For measurements on a short time scale, the temperature is the usual one (the one measured with a standard thermometer).
- For measurements on a long time scale(i.e. equal or larger of a times depending on how much the system is off-equilibrium), the temperature takes a different value and it can be measured with a thermometer which coupled only to low frequency modes).

New fluctuation dissipation relations.

Numerical simulations give ample evidence of this two (or many) temperature phases. The same behaviour has been observed in some difficult experiments (notably by Herisson and Ocio in spin glasses).

## Applications beyond physics.

Collective behaviour of a large heterogenous system of agents that interact.

Morphology of river basins, study of the behaviour of optimization algorithms, neural networks (e. g. associative memory), structure of networks in quite different contests (internet, food, finance, proteins), species dynamics, evolution…

The disorder has often two origins:

- Variations in the agent.
- Interaction with the environment

## Open problems and perspectives

There are many simple systems where we do not have a good control of physics (e.g. we do know the behaviour near the upper critical dimension, if any):

- Equilibrium systems, e.g. ferromagnets with a random field, localization transition, directed polymers, spin glasses in a field.

- Non-equilibrium systems, e.g. surface growth (KPZ equation), diffusion limit aggregation (DLA).

Many of these problems stand for twenty years or more.

We badly need new ideas, new tools, techniques and also smart people trying to solve them.