Lagrangian Dynamics Project

April 26, 2017

By:

Sean Link | Jinsung Lee | Parker Haiberg

Submitted to Dr. Bradley Wall
Department of Aerospace Engineering
College of Engineering
In Partial Fulfillment
Of the Requirements
Of
ES 204
Dynamics
Spring 2017

Embry-Riddle Aeronautical University
Prescott, AZ

1. CONCEPTUALIZE THE PROBLEM

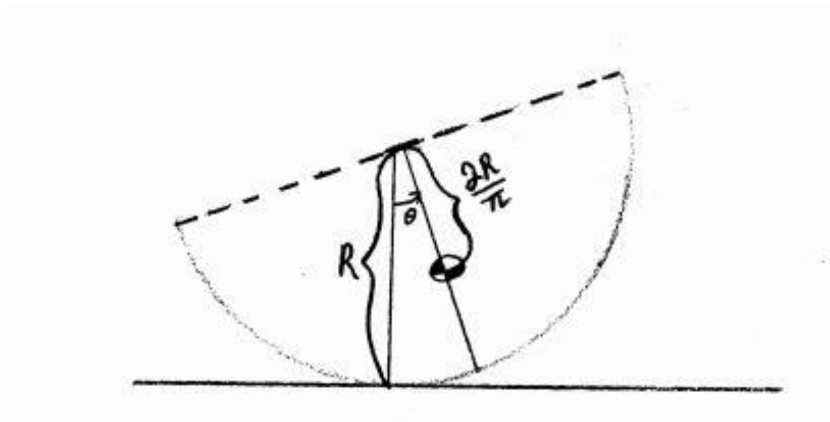


Figure 1: Center of Mass & Moment of Inertia of Half Cylinder

The center of mass of the half cylinder was determined using basic statics principles. Through general theories and looking at the shape of the half cylinder one can assume that the centroid is located closer to the bottom of the object opposed to the top because there is more material rather than the empty space at the open top. As the half cylinder is put into motion the kinetic energy of the half cylinder is converted into gravitational potential energy of the center of mass. Given the oscillatory motion of the half cylinder an oscillatory graph can be expected to describe the angular displacement of the center of mass of the half cylinder, θ .

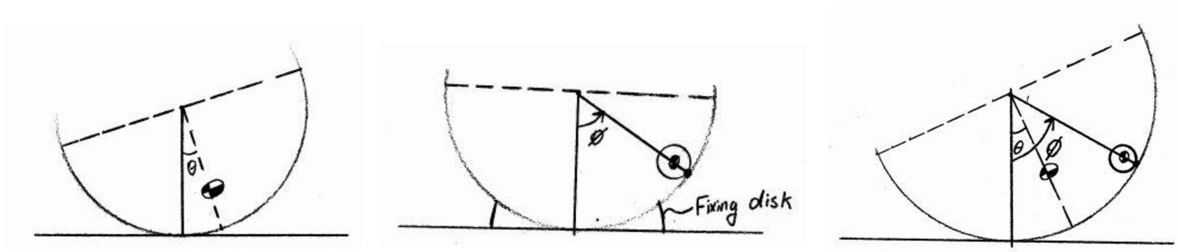


Figure 2: Initial Setups (A,B,C)

Table 1: Givens

Dimensions	Half Cylinder	Disk
Mass	578 grams	162 grams
Radius	15.625 cm	3.8 cm

A)

Initially the half cylinder is rotating along the ground by itself and its motion is only described by theta(θ), as shown in Figure 2A. There is a surface constraint between the half cylinder and the ground. This surface constraint, which is known to be rolling without slipping, allows for a relationship between the center point of the cylinder and the ground using the relative velocity equation. Given the geometry of the half cylinder, the velocity of its center point must only be in the x direction only. This is due to the fact that the distance between the ground and the center point of the half cylinder will always be the radius of the half cylinder R. Since there is no change is vertical displacement the y velocity is zero. Using theses constraints, the lagrangian will be used to find the equation of motion of the half cylinder $\ddot{\theta}$. Using differential equations and the given initial conditions, $\theta(t)$ can be solved.

B)
In this variation, a disk is added to the system and the half cylinder is in a fixed position as depicted by the supports in Figure 2B. The half cylinder has a theoretical θ of 0 degrees and the disk is associated with an angular displacement of phi(ϕ), which measures from the center of mass of the half cylinder to the center of mass of the disk. In this instance, our angular position function ϕ, was to be solved using a similar process. A rolling without slipping constraint in tandem with the relative velocity equation allowed for the lagrangian of the disk to be solved for in terms of all knowns and state variables. Using the lagrangian, the equation of motion of the disk $\ddot{\phi}$ could be determined. Using differential equations and the given initial conditions, $\phi(t)$ could be determined.

C)
In the last setup, both the cylinder and disk are free to rotate, as shown in Figure 2C. Conceptually the two objects in motion are going to affect one another. The motion relating the half cylinder and the disk are related to each using the relative velocity equation. Using constraints derived in the relative velocity equation, the lagrangian for the system can be expressed in terms of all knowns and state variables. Using differentials, the equation of motions $\ddot{\phi}\ and\ \ddot{\theta}$ can be derived. Using both equations of motion and the given initial conditions, $\phi(t)\ and\ \theta(t)$ can be numerically approximated.

2. FREE-BODY DIAGRAM (NO FBD)

3. COORDINATE FRAME

For the entire project a standard Cartesian coordinate frame was used. Figure 3 shows the setup of the half cylinder and disk at variable angular positions based on θ and ϕ. A positive deflection and angular velocity was assumed to help eliminate difficulty in the matrices with sign errors. The enlarged portion in Figure 3 at the right of the system emphasizes the relative velocity relationship at the contact point between the interior surface of the half cylinder and the exterior surface of the disk. Since the disk is a uniform shape, its center of mass is located at the exact center.

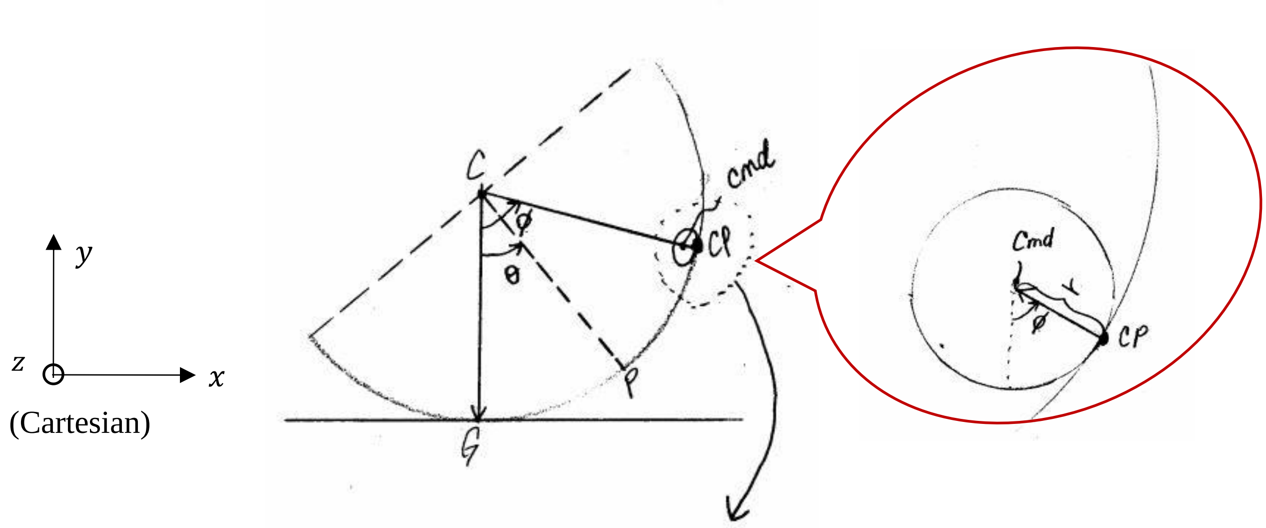


Figure 3: Diagram of System

4. - 6. EQUATIONS OF MOTION: LAGRANGIAN, KNOWNS, UNKNOWNS, AND CONSTRAINTS

The following figures are analytical solutions to the various problems. The center of mass and moment of inertia can be solved for using simple statics equations. It should be noted that while solving for the center of mass and moment of inertia, the half cylinder was assumed to have infinitesimally small thickness.

The moment of inertia about the center of mass was found using the parallel axis theorem.

$$Y_{cm} = \frac{\int y\,dM}{\int dM} = \frac{\int_0^\pi R\sin\theta R\,d\theta}{\int_0^\pi R\,d\theta} = \frac{R^2\int_0^\pi \sin\theta\,d\theta}{R\int_0^\pi d\theta} = -R\,\frac{\cos\theta\big|_0^\pi}{\theta\big|_0^\pi}$$

$$= -R\,\frac{[-1-1]}{\pi} = \frac{2R}{\pi}$$

$$x_{cm} = \frac{\int x\,dM}{\int dM} = \frac{\int_0^\pi R^2\cos\theta\,d\theta}{\int_0^\pi R\,d\theta} = R\,\frac{\int_0^\pi \cos\theta\,d\theta}{\int_0^\pi d\theta} = R\,\frac{[\sin\theta]_0^\pi}{[\theta]_0^\pi}$$

$$= R\,\frac{0}{\pi} = 0$$

$$\boxed{\langle x_{cm}, Y_{cm}\rangle = \left\langle 0, \frac{2R}{\pi}\right\rangle}$$

$$I_O = I_{zcm} + mY_{cm}^2$$

$$I_{zcm} = I_O - mY_{cm}^2$$

$$= \int_0^M R^2\,dM - M\left(\frac{4R^2}{\pi^2}\right)$$

$$= R^2[M]_0^M - M\left(\frac{4R^2}{\pi^2}\right)$$

$$= R^2M - M\left(\frac{4R^2}{\pi^2}\right)$$

$$= R^2M\left(1 - \frac{4}{\pi^2}\right)$$

The pages below contain all of the analytical work done that relates to the Lagrangian solution of the system. Throughout the project $(Q_j)_{non}$ was zero due to assumptions of no drag, friction, or losses.

In Part 2, the system is solely the half cylinder with an angular velocity of $\dot{\theta}$. The knowns were found to be M, I_{zcm}, $\dot{\theta}$, g, R, and θ. In this part the only unknown variable is v_{cm}, which can be solved for with relative velocity equations/relationships. The velocity of the ground(v_G) is known to be zero given the rolling without slipping constraint.

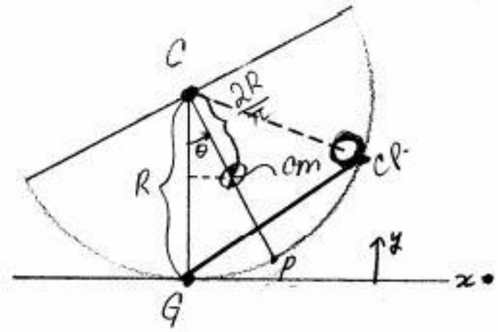
2. Determine the lagrangian of the half of the cylinder

$KE = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{zcm} \dot{\theta}^2$

$PE = mgh$, where $h = R - \frac{2R}{\pi}\cos\theta$

Known: $M, I_{z_{cm}}, \dot{\theta}, g, R, \theta$

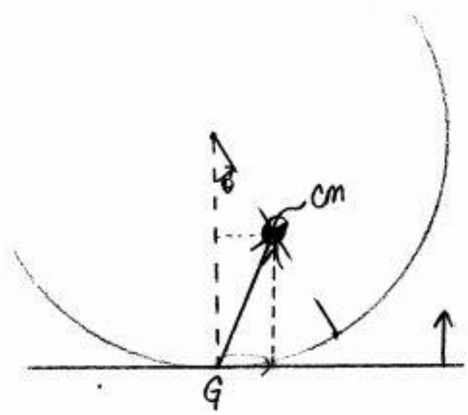
unknown: V_{cm}



$$V_{cm} = \cancel{V_G}^{\,0} + \dot{\theta} \times \vec{r}_{cm/G}$$

$$r_{cm/G} = \left\langle \frac{2R}{\pi}\sin\theta,\ R - \frac{2R}{\pi}\cos\theta,\ 0 \right\rangle$$

$$V_{cm} = \begin{vmatrix} x & y & z \\ 0 & 0 & \dot{\theta} \\ \frac{2R}{\pi}\sin\theta & R\left(1 - \frac{2}{\pi}\cos\theta\right) & 0 \end{vmatrix}$$



$$= \begin{bmatrix} -\dot{\theta} R\left(1 - \frac{2}{\pi}\cos\theta\right) \\ \frac{2R\dot{\theta}}{\pi}\sin\theta \end{bmatrix} = \boxed{\dot{\theta} R \begin{bmatrix} \frac{2}{\pi}\cos\theta - 1 \\ \frac{2}{\pi}\sin\theta \end{bmatrix}}$$

$$\mathcal{L} = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{zcm}\dot{\theta}^2 - mgR\left(1 - \frac{2}{\pi}\cos\theta\right)$$

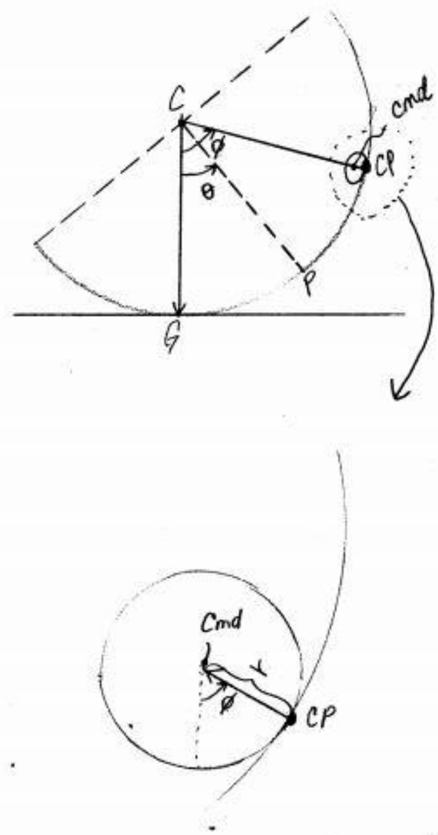
- Lagrangian, all known

In Part 3, we solved for the lagrangian of the disk in terms of all knowns given the following relative velocity equations.

Part 3, Find Lagrangian for disk

$KE = \frac{1}{2} m V_{cm}^2 + \frac{1}{2} I_{zcm} \overline{\omega}_d^2$

$PE = mgh, \; h = R - (R-r)\cos\phi$



$V_c = \cancel{V_G}^{0} + \dot{\theta} \times \vec{r}_{c/G}$

$$V_c = \begin{vmatrix} x & y & z \\ 0 & 0 & \dot{\theta} \\ 0 & R & 0 \end{vmatrix} = \begin{bmatrix} -\dot{\theta}R \\ 0 \\ 0 \end{bmatrix}$$

• Find V_{cm}

$V_{cmd} = V_c + \dot{\phi} \times \vec{r}_{cmd/c}$

$$V_{cmd} = \begin{bmatrix} -\dot{\theta}R \\ 0 \\ 0 \end{bmatrix} + \begin{vmatrix} x & y & z \\ 0 & 0 & \dot{\phi} \\ (R-r)\sin\phi & (R-r)\cos\phi & 0 \end{vmatrix}$$

$$V_{cmd} = \begin{bmatrix} -\dot{\theta}R \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} (R-r)\cos\phi\,\dot{\phi} \\ (R-r)\sin\phi\,\dot{\phi} \end{bmatrix} = \boxed{\begin{bmatrix} (R-r)\dot{\phi}\cos\phi - \dot{\theta}R \\ (R-r)\dot{\phi}\sin\phi \\ 0 \end{bmatrix}} \quad (1)$$

$V_{cp} = V_c + \dot{\theta} \times \vec{r}_{cp/c}$

$$= \begin{bmatrix} -\dot{\theta}R \\ 0 \\ 0 \end{bmatrix} + \begin{vmatrix} x & y & z \\ 0 & 0 & \dot{\theta} \\ R\sin\phi & -R\cos\phi & 0 \end{vmatrix} = \begin{bmatrix} -\dot{\theta}R \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\theta}R\cos\phi \\ \dot{\theta}R\sin\phi \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\theta}R\cos\phi - \dot{\theta}R \\ \dot{\theta}R\sin\phi \\ 0 \end{bmatrix}$$

• Find ω_d

$$V_{cmd} = V_{cp} + \overline{\omega}_d \times \vec{r}_{cmd/cp} = \begin{bmatrix} (R-r)\dot{\phi}\cos\phi - \dot{\theta}R \\ (R-r)\dot{\phi}\sin\phi \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}R\cos\phi - \dot{\theta}R \\ \dot{\theta}R\sin\phi \\ 0 \end{bmatrix} + \begin{vmatrix} x & y & z \\ 0 & 0 & \overline{\omega}_d \\ -r\sin\phi & r\cos\phi & 0 \end{vmatrix}$$

$$= \begin{bmatrix} (R-r)\dot{\phi}\cos\phi - \dot{\theta}R \\ (R-r)\dot{\phi}\sin\phi \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}R\cos\phi - \dot{\theta}R \\ \dot{\theta}R\sin\phi \\ 0 \end{bmatrix} + \begin{bmatrix} -\overline{\omega}_d r\cos\phi \\ -\overline{\omega}_d r\sin\phi \\ 0 \end{bmatrix}$$

By using x-component

$R\dot{\phi}\cos\phi - r\dot{\phi}\cos\phi - \cancel{\dot{\theta}R} = \dot{\theta}R\cos\phi - \cancel{\dot{\theta}R} - \overline{\omega}_d r\cos\phi$

$\overline{\omega}_d r\cos\phi = \dot{\theta}R\cos\phi + r\dot{\phi}\cos\phi - R\dot{\phi}\cos\theta$

$$\omega_d = \frac{\cos\phi\,(\dot{\theta}R + r\dot{\phi} - R\dot{\phi})}{r\cos\phi} = \boxed{\frac{\dot{\theta}R + r\dot{\phi} - R\dot{\phi}}{r}} \quad (2)$$

By using y component

$$R\dot{\phi}\sin\phi - r\dot{\phi}\sin\phi = \dot{\theta}R\sin\phi - \overline{\omega}_d r\sin\phi$$

$$\overline{\omega}_d r\sin\phi = \dot{\theta}R\sin\phi + r\dot{\phi}\sin\phi - R\dot{\phi}\sin\phi$$

$$\overline{\omega}_d = \frac{\sin\phi\,(\dot{\theta}R + r\dot{\phi} - R\dot{\phi})}{r\sin\phi} \Rightarrow \boxed{\frac{\dot{\theta}R + r\dot{\phi} - R\dot{\phi}}{r}} \; ②$$

KE = all known , PE = all known

$$\mathcal{L} = \frac{1}{2}m①^2 + \frac{1}{2}I_{z_{cm}}②^2 - mg\left(R - (R-r)\cos\phi\right)$$

For part 5, we solved the lagrangian of the system assuming the disk does not exist. To account for the disk not being a part of the system we substituted zero for the mass, angular velocity and displacement of the disk. Note that our equation of motion of the half cylinder is identical to the equation of motion solved by Mathematica.

Part 5

Simplify $\ddot{\theta}$ Such that the disk doesn't exist

$m = \dot{\phi} = \phi = \ddot{\phi} = 0$

⓪ $\mathcal{L} = \frac{1}{2} m V_{cm}^2 + \frac{1}{2} I_{z_{cm}} \dot{\theta}^2 - mgR\left(1 - \frac{2}{\pi}\cos\theta\right)$

① $V_{cm}^2 = \left(\dot{\theta} R \frac{2}{\pi}\cos\theta - \dot{\theta} R\right)^2 + \left(\frac{2\dot{\theta} R}{\pi}\sin\theta\right)^2$

$= \frac{4\dot{\theta}^2 R^2}{\pi^2} - \frac{4\dot{\theta}^2 R^2}{\pi}\cos\theta + \dot{\theta} R^2$

$= \dot{\theta}^2 R^2 \left(\frac{4}{\pi^2} - \frac{4}{\pi}\cos\theta + 1\right)$

② $I_{zcm} = MR^2\left(1 - \frac{4}{\pi^2}\right)$

①,② → ⓪

$$\mathcal{L} = \frac{1}{2} m \dot{\theta}^2 R^2 \left(\frac{4}{\pi^2} - \frac{4}{\pi}\cos\theta + 1\right) + \frac{1}{2} MR^2\left(1 - \frac{4}{\pi^2}\right)\dot{\theta}^2 - mgR\left(1 - \frac{2}{\pi}\cos\theta\right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2} m \dot{\theta}^2 R^2 \left(\frac{4}{\pi}\sin\theta\right) - mgR\left(\frac{2}{\pi}\sin\theta\right)$$

$$= \frac{2mR}{\pi}\sin\theta\left(\dot{\theta}^2 R - g\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\dot{\theta}R^2\left(\frac{4}{\pi^2} - \frac{4}{\pi}\cos\theta + 1\right) + mR^2\left(1 - \frac{4}{\pi^2}\right)\dot{\theta}$$

$$= m\dot{\theta}R^2\left(2 - \frac{4}{\pi}\cos\theta\right)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m\dot{\theta}R^2\left(\frac{4}{\pi}\sin\theta\right)\dot{\theta} + \left(2 - \frac{4}{\pi}\cos\theta\right)\left(m\ddot{\theta}R^2\right)$$

$$= mR^2\left(\dot{\theta}^2\left(\frac{4}{\pi}\sin\theta\right) + \ddot{\theta}\left(2 - \frac{4}{\pi}\cos\theta\right)\right)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = mR^2\left(\dot{\theta}^2\left(\frac{4}{\pi}\sin\theta\right) + \ddot{\theta}\left(2 - \frac{4}{\pi}\cos\theta\right)\right) - \frac{2mR}{\pi}\sin\theta\left(\dot{\theta}^2 R - g\right)$$

$$mR^2\dot{\theta}^2\left(\frac{4}{\pi}\sin\theta\right) + \ddot{\theta}MR^2\left(2 - \frac{4}{\pi}\cos\theta\right) = \frac{2MR}{\pi}\sin\theta\left(\dot{\theta}^2 R - g\right)$$

$$\ddot{\theta} = \frac{\frac{2mR}{\pi}\sin\theta\left(\dot{\theta}^2 R - g\right) - MR^2\dot{\theta}^2\left(\frac{4}{\pi}\sin\theta\right)}{MR^2\left(2 - \frac{4}{\pi}\cos\theta\right)} = \frac{\frac{2}{\pi}\sin\theta\left(\dot{\theta}^2 R - g\right) - \dot{\theta}^2 R\left(\frac{4}{\pi}\sin\theta\right)}{R\left(2 - \frac{4}{\pi}\cos\theta\right)}$$

$$\ddot{\theta} = \frac{\sin\theta\left[-\frac{2}{\pi}\left(\dot{\theta}^2 R - g\right) - \dot{\theta}^2 R\left(\frac{4}{\pi}\right)\right]}{R\left(2 - 4/\pi \cdot \cos\theta\right)}$$

$$\ddot{\theta} = \frac{\sin\theta\left[\dot{\theta}^2 R + g\right]}{R\left(-\pi + 2\cos\theta\right)}$$

In part 6, we solved the lagrangian of the system assuming that the half cylinder is fixed. Since the linear velocity and angular velocity for the center of mass of the half cylinder is zero, we substituted $\theta, and\ \dot{\theta}$ equals to zero. Again, the simplified case solved for analytically matches what was derived in Mathematica.

Step 6

Simplify ϕ, such that the half of the cylinder doesn't exist.

Condition: $\dot{\theta} = \theta = 0$

$$\mathcal{L} = \frac{1}{2} m v_{cmd}^2 + \frac{1}{2} I_{zcm} \omega_d^2 - mg\left(R - (R-r)\cos\phi\right)$$

Find unknown: V_{cmd}

from part 3

$$V_{cmd} = \begin{bmatrix} (R-r)\dot{\phi}\cos\phi - \dot{\theta}R \\ (R-r)\dot{\phi}\sin\phi \\ 0 \end{bmatrix}$$

$$V_{cmd}^2 = \left[(R-r)\dot{\phi}\cos\phi - \cancel{\dot{\theta}R}^{\,0}\right]^2 + \left[(R-r)\dot{\phi}\sin\phi\right]^2$$

$$= (R-r)^2\dot{\phi}^2\cos^2\phi + (R-r)^2\dot{\phi}^2\sin^2\phi$$

$$= (R-r)^2\dot{\phi}^2\underbrace{\left(\cos^2\phi + \sin^2\phi\right)}_{1}$$

$$= (R-r)^2\dot{\phi}^2$$

$$I_{zcm} = \frac{1}{2} m r^2$$

from part 3

$$\omega_d = \left(\frac{\dot{\theta}R + r\dot{\phi} - R\dot{\phi}}{r}\right)$$

$$\mathcal{L} = \frac{1}{2} m (R-r)^2 \dot{\phi}^2 + \frac{1}{4} m r^2 \left(\frac{r\dot{\phi} - R\dot{\phi}}{r}\right)^2 - mg\left(R - (R-r)\cos\phi\right)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mg\left(0 - (R-r)(-\sin\phi)\right)$$

$$= -mg(R-r)\sin\phi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\dot{\phi}(R-r)^2 + \frac{1}{2} m r^2 \left(\frac{r\dot{\phi} - R\dot{\phi}}{r}\right)\left(\frac{r-R}{r}\right)$$

$$= m\dot{\phi}(R-r)^2 + \frac{1}{2} m (r\dot{\phi} - R\dot{\phi})(r-R)$$

$$\frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right\} = m\ddot{\phi}(R-r)^2 + \frac{1}{2} m (r\ddot{\phi} - R\ddot{\phi})(r-R)$$

$$= \ddot{\phi}\left(m(R-r)^2 + \frac{1}{2} m (r-R)^2\right)$$

$$= \ddot{\phi}\left(\frac{3}{2} m (R-r)^2\right)$$

$$\frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right\} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\ddot{\phi}\left(\frac{3}{2} m (R-r)^2\right) + mg(R-r)\sin\phi$$

$$\ddot{\phi} = \frac{-2mg(R-r)\sin\phi}{3m(R-r)^2}$$

$$\ddot{\phi} = \frac{-2g\sin\phi}{3(R-r)}$$

For Part 7, we determined the lagrangian of the entire system adding our previously derived lagrangians of the disk and the half cylinder. The lagrangian of the entire system is as follows.

$$L = \frac{1}{2}M\left(\dot{\theta}^2 R^2\right)\left(\frac{4}{\pi^2} - \frac{4}{\pi}\cos(\theta) + 1\right) + \frac{1}{2}MR^2\left(1 - \frac{4}{\pi^2}\right)\dot{\theta}^2 - MgR\left(1 - \frac{2}{\pi}\cos(\theta)\right) + \frac{1}{2}m(R-r)^2\dot{\phi}^2 + \frac{1}{4}mr^2\left(\frac{\left(\dot{\theta}R + r\dot{\phi} - R\dot{\phi}\right)}{r}\right)^2 - mg(R - (R-r)\cos(\phi))$$

7. SOLVE FOR THE EQUATION(S) OF MOTION

The full equations of motion, $\ddot{\theta}\ and\ \ddot{\phi}$, were solved for in Mathematica.

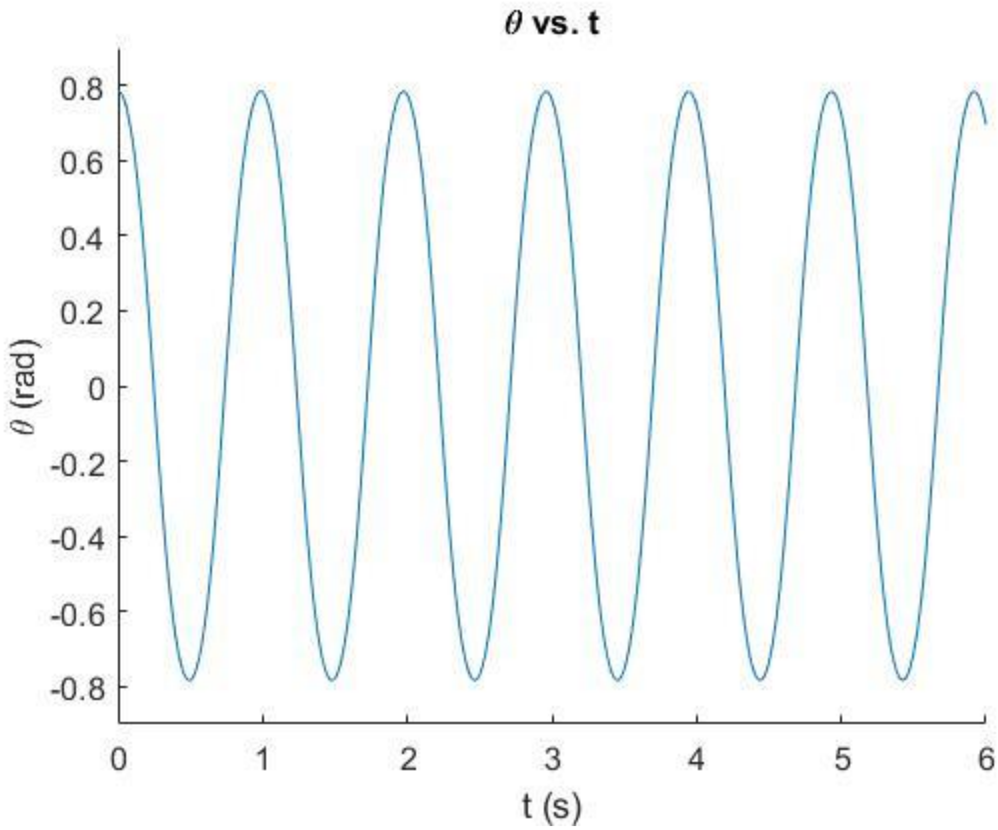
8. SOLVE THE EQUATION OF MOTION, SOLVE THE PROBLEM

1. ANALYTICAL

The analytical work above was used to create relationships and Lagrangian equations that were used to solve for the EOMs $\ddot{\theta}$ and $\ddot{\phi}$.

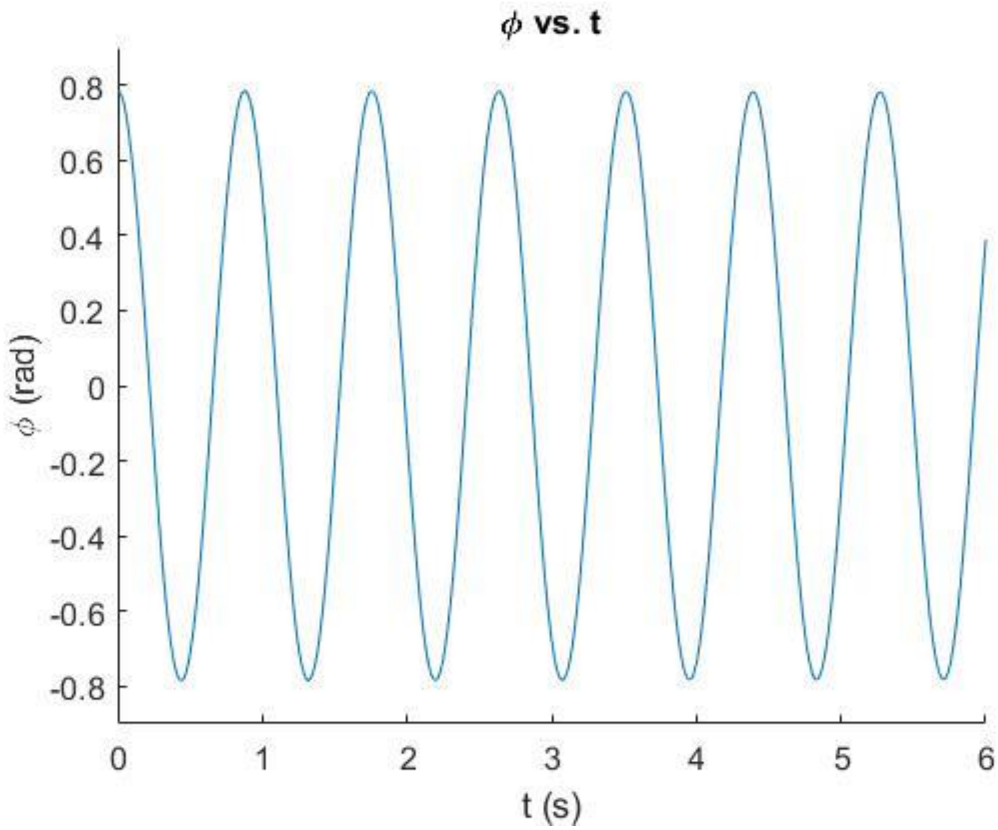
2. NUMERICAL

For parts 5 and 6, the natural frequency is solved for with basic physics equations from the chapters concerning waves. After further inspection of the graphs found, the sinusoidal wave's natural frequency can be solved with the equation: $natural\ frequency = \frac{2\pi}{\lambda}$, where λ is the time difference between two local minimum or maximum peaks. For part 7, the angular displacements $\theta\ and\ \phi$ were numerically approximated using Mathematica's built in function NDSolveValue[].



Graph 1: Natural Frequency of Half Cylinder

Graph 1 depicts angular displacement of the center of mass of the half cylinder as a function of time. The initial conditions were $\theta = \frac{\pi}{4}\ rad$ and $\dot{\theta} = 0\frac{rad}{sec}$. There are no non-conservative forces in the system which allows the sinusoid to have a consistent amplitude. The natural frequency of the half cylinder is $6.370\frac{rad}{s}$

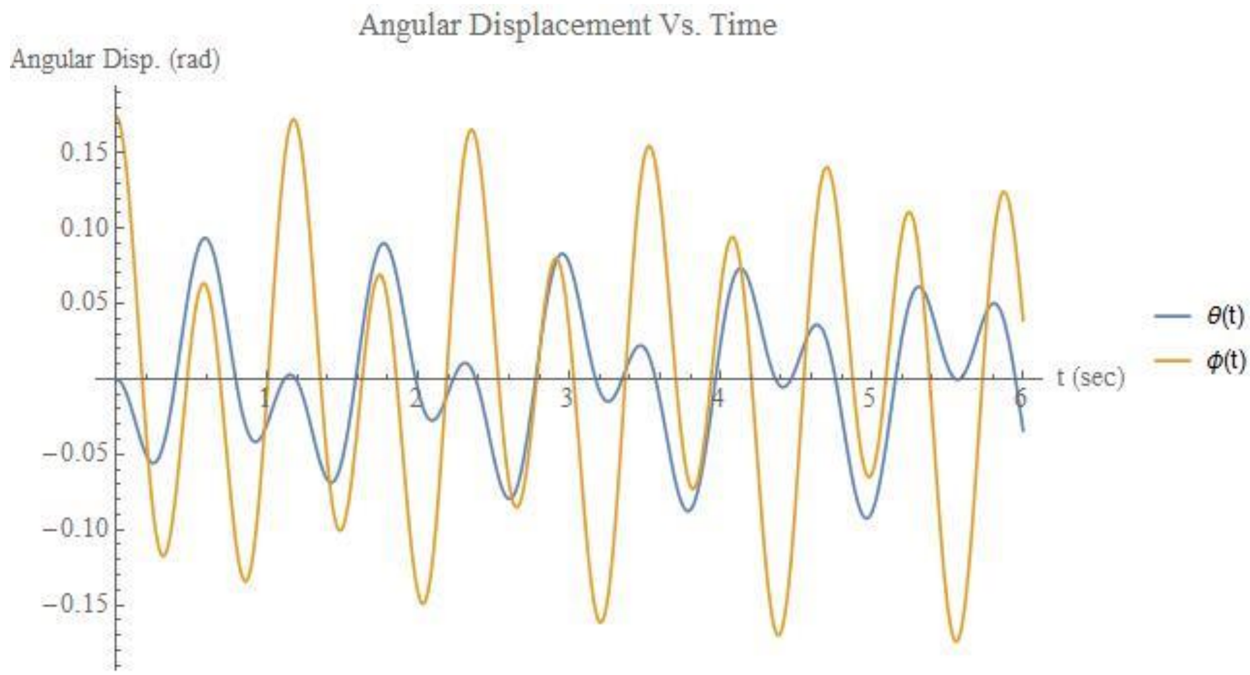


Graph 2: Natural Frequency of Disk

Graph 2 illustrates the angular displacement of the disk relative to point G. The initial condition of the graph is as follows: $\phi = \frac{\pi}{4}\ rad$ and $\dot{\phi} = 0\,\frac{rad}{sec}$. With no non-conservative forces, the disk can oscillate back and forth and create the sinusoidal curve. The natural frequency of the disk is $7.153\,\frac{rad}{s}$.

Graphs 3(a,b,c) represent three different sets of initial conditions for θ, $\dot{\theta}$, ϕ, and $\dot{\phi}$. The equations of motion are integrated over 6 second time intervals. The graphs depict an accurate representation of the angular displacement of the half cylinder and the disk as a single system.

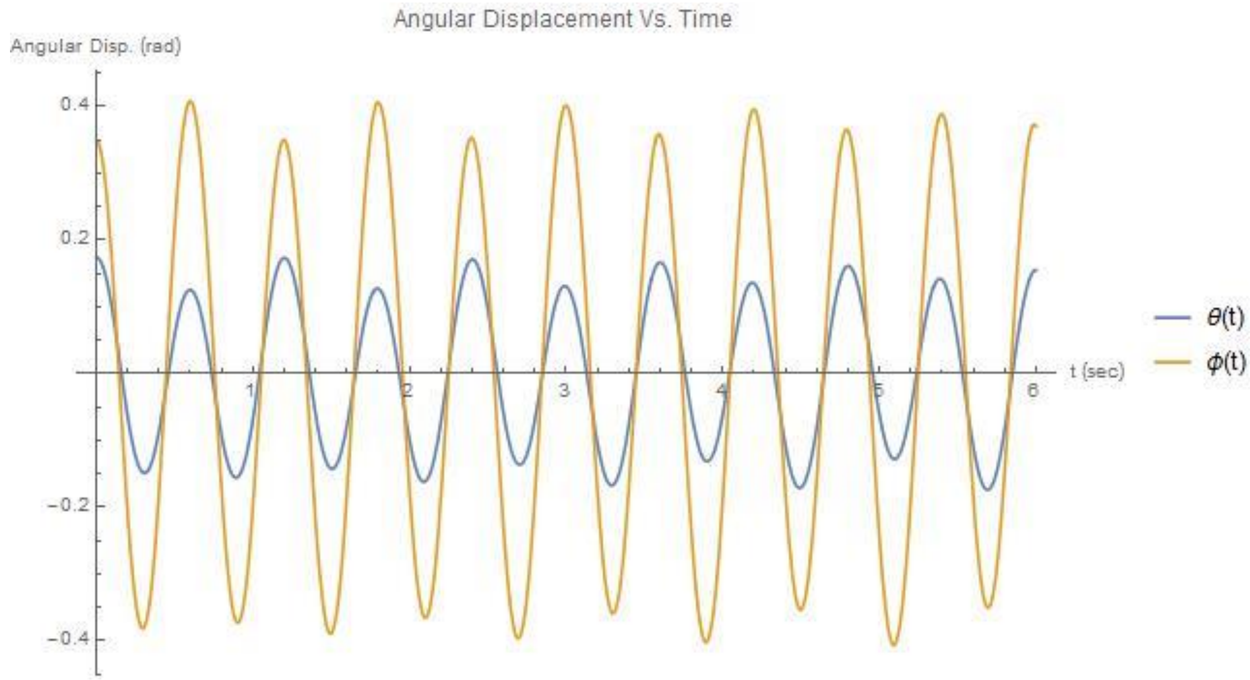
Graph 3a has initial conditions: $\theta = 0\,rad$, $\dot{\theta} = 0\,rad/sec$, $\phi = \frac{\pi}{18}\,rad$, and $\dot{\phi} = 0\,rad/sec$.



Graph 3a: Integration of full EOMs

Graph 3a shows oscillatory motion. This makes sense given the videos posted on canvas. In the video, the system is given the same initial conditions that produced Graph 3a. In the video, the velocity of the half cylinder and the disk change direction at different times. Similarly, in Graph 3a the velocity of the half cylinder and the disk change at different times given that the local maxima of the two angular position curves do not occur at the same time.

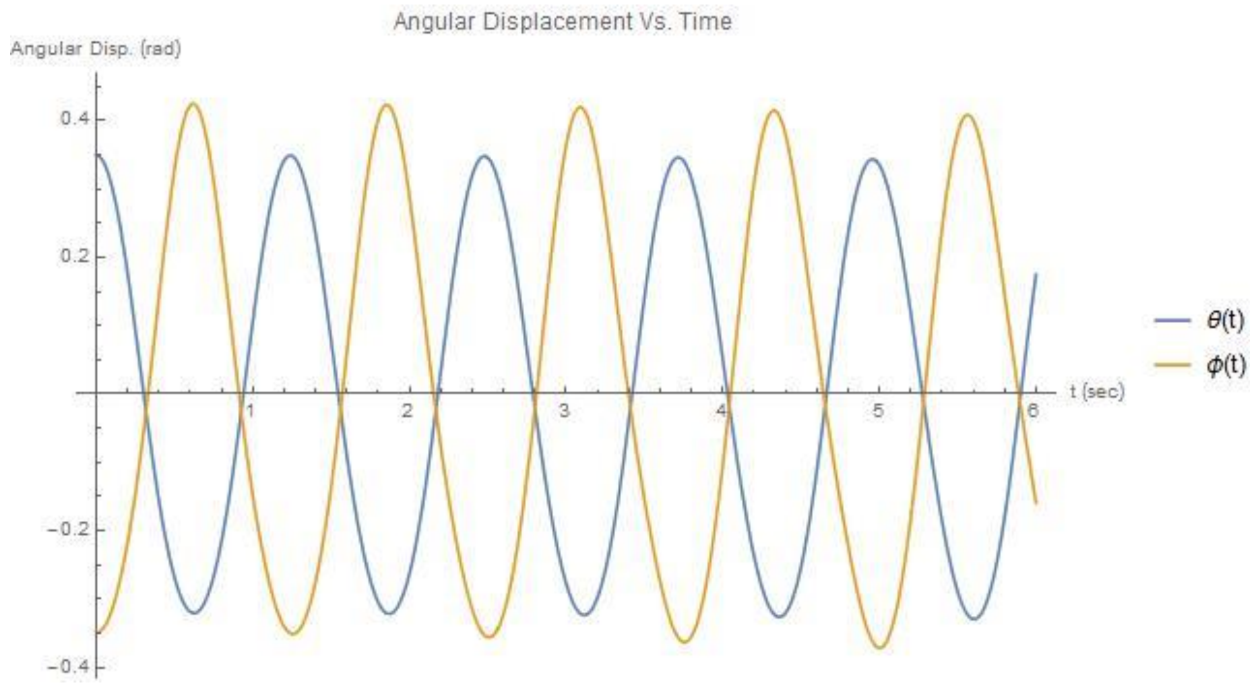
Graph 3b has initial conditions: $\theta = \frac{\pi}{18}\ rad$, $\dot{\theta} = 0\ rad/sec$, $\phi = \frac{\pi}{9}\ rad$, and $\dot{\phi} = 0\ rad/sec$.



Graph 3b: Integration of full EOMs

Graph 3b illustrates the motion of the half cylinder and disk with the previously stated initial conditions. In both the graph and the video posted on canvas, the two objects change direction at the same time. This is consistent with Graph 3b given that the local minima and maxima align for both solutions curves of $\theta(t)\ and\ \phi(t)$. In addition, Graph 3b demonstrates that energy is conserved. For every case, the magnitude of the sum of the amplitudes of the two solution curves are equal at every local maxima and minima. This means that the gravitation potential energy of the system is the same every time the system's velocity is zero.

Graph 3c has initial conditions: $\theta = \frac{\pi}{9}\ rad$, $\dot{\theta} = 0\ rad/sec$, $\phi = \frac{-\pi}{9}\ rad$, and $\dot{\phi} = 0\ rad/sec$.



Graph 3c: Integration of full EOMs

Graph 3c represents the motion when the half cylinder and disk have initial angular displacements of equal magnitude and opposite direction. Again, Graph 3c conforms to video posted on canvas. In both cases, the direction of the angular velocity of the half cylinder and the disk are changing at the same time. Also, the frequency of the solution curves is greatly diminished given the opposite signs in the initial conditions.

9. DOES IT MAKE SENSE

Through conservation of energy, the variations of the system make sense. The net amplitude of the two solution curves in Graph 3b and 3c remain constant at each local maxima and minima. This means the gravitation potential energy of the system remains constant throughout the given intervals when the velocity of the system is zero. This gives us reason to believe that the overall energy is conserved.

Unit analysis is as follows.

Part 1

$$< x_{cm}\,, y_{cm} > = < 0\,, \frac{2r}{\pi} >$$

$$< m\ , m > = < m\ , m >$$

$$Iz_{cm} = mR^2(1 - \frac{4}{\pi^2})$$

$$Mass\ moment\ of\ inertia = kg * m^2$$

Part 2

$$V_{cm=}\ \dot{\theta}R \begin{bmatrix} \frac{2}{\pi}\cos(\theta) - 1 \\ \frac{2}{\pi}\sin(\theta) \end{bmatrix}$$

$$\frac{m}{s} = \frac{1}{s} * m$$

$$h_{hc} = R\left(1 - \frac{2}{\pi}\cos(\theta)\right)$$

$$m = m$$

Part 3

$$V_c = \begin{bmatrix} -\dot{\theta}R \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{m}{s} = \frac{1}{s} * m$$

$$V_{cmd} = \begin{bmatrix} -\dot{\theta}R \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} (R - r)\dot{\phi}\cos(\phi) \\ (R - r)\dot{\phi}\sin(\phi) \\ 0 \end{bmatrix}$$

$$\frac{m}{s} = \left[\frac{1}{s} * m\right] + \left[m * \frac{1}{s}\right]$$

$$V_{cp} = \begin{bmatrix} \dot{\theta}Rcos(\phi) \ -\dot{\theta}R \\ \dot{\theta}Rsin(\phi) \\ 0 \end{bmatrix}$$

$$\frac{m}{s} = \begin{bmatrix} \frac{1}{s} * m = \frac{1}{s} * m \\ \frac{1}{s} * m \\ 0 \end{bmatrix}$$

$$w_d = \frac{\dot{\theta}R + r\dot{\phi} - R\dot{\phi}}{r}$$

$$\frac{1}{s} = \frac{m(\frac{1}{s} + \frac{1}{s} - \frac{1}{s})}{m}$$

Part 5

$$\ddot{\theta} = \frac{\sin(\theta)\,(\dot{\theta}^2 R + g)}{R(2\cos(\theta) - \pi)}$$

$$\frac{1}{s^2} = \frac{rad\ \left[\frac{1}{s^2} * m + \frac{m}{s^2}\right]}{m * rad}$$

Part 6

$$\ddot{\phi} = \frac{-2gsin(\theta)}{3(R-r)}$$

$$\frac{1}{s^2} = \frac{const * \frac{m}{s^2} * rad}{const * m}$$

APPENDICES:

A. ATTRIBUTIONS

Team Member	Attributions/Responsibilities
Sean Link	Main contributor for the code and helped with the math and writeup.
Jinsung Lee	Main contributor for the math and helped with the code, and writeup.
Parker Haiberg	Main contributor for the writeup.

B. NUMERICAL SOLUTION

1. MATHEMATICA

Team 2

Sean Link

Jinsung Lee

Parker Hailberg

ES204-PC01

Project 3

Equation and Variable Definition

Equations

LagrangianS = Lagrangian of the entire system
Lagrangianhc = Lagrangian of the half cylinder
LagrangianDisk = Lagrangian of the disk

Variables

M = mass of the half cylinder (kg)
R = radious of the half cylinder (m)
Izcmhc = Moment of inertia of the half cylinder about its center of mass (kg m^2)
Vcmhc = velocity of the center of mass of the half cylinder
θ [t] = angular displacement of the half cylinder (rad)
θ'[t] = angular velocity of the half cylinder (rad/s)
g = acceleration due to gravity $\left(\frac{m}{s^2}\right)$
m = mass of the disk (kg)
VcmDisk = velocity of the center of mass of the disk (m/s)
IzcmDisk = moment of inertia of the disk about its center of mass (kg m^2)
wd = rotiational velocity of the disk (rad/s)
r = radious of the disk (m)
ϕ [t] = angular displacement of the disk relative to the contact point between the half cylinder and the ground (rad)
ϕ'[t] = rate of change of the angular displacement of the half cylinder with respect to time (rad/s)

Part 1: storing moment of inertia of the half cylinder about its center of mass

In[1]:= $\text{Izcmhc} = \text{M}\,\text{R}^2 \left(1 - \frac{4}{\pi^2}\right);$

Part 2: Lagrangian of Half Cylinder

In[2]:= $\text{Lagrangianhc} = 1/2\,\text{M}\,\text{Vcmhc}^2 + 1/2\,\text{Izcmhc} * (\theta'[\text{t}])^2 - \text{M}\,\text{g}\,\text{R}\left(1 - \frac{2}{\pi}\text{Cos}[\theta[\text{t}]]\right);$

Storing velocity of the center of mass of the half cylindar

In[3]:= $\text{Vcmhc} = \sqrt{\left(\left(\theta'[\text{t}]\,\text{R}\,\frac{2}{\pi}\text{Cos}[\theta[\text{t}]] - \theta'[\text{t}]\,\text{R}\right)^2 + \left(\theta'[\text{t}]\,\text{R}\,\frac{2}{\pi}\text{Sin}[\theta[\text{t}]]\right)^2\right)};$

Displaying Lagrangianhc

In[4]:= `Simplify[Lagrangianhc] // TraditionalForm`

Out[4]//TraditionalForm=

$$\frac{M\,R\,(\pi - 2\cos(\theta(t)))\left(R\,\theta'(t)^2 - g\right)}{\pi}$$

Part 3: Lagrangian of the Disk

In[5]:= $\text{LagrangianDisk} = 1/2\,\text{m}\,\text{VcmDisk}^2 + 1/2\,\text{IzcmDisk}\,\text{wd}^2 - \text{m}\,\text{g}\,(\text{R} - (\text{R} - \text{r})\,\text{Cos}[\phi[\text{t}]]);$

storing wd in terms of knowns

In[6]:= $\text{wd} = \frac{\theta'[\text{t}]\,\text{R} + \text{r}\,\phi'[\text{t}] - \text{R}\,\phi'[\text{t}]}{\text{r}};$

storing VcmDisk in terms of knows

In[7]:= $\text{VcmDisk} = \sqrt{\left(((\text{R} - \text{r})\,\phi'[\text{t}]\,\text{Cos}[\phi[\text{t}]] - \theta'[\text{t}]\,\text{R})^2 + ((\text{R} - \text{r})\,\phi'[\text{t}]\,\text{Sin}[\phi[\text{t}]])^2\right)};$

storing IzcmDisk in terms of knowns

In[8]:= $\text{IzcmDisk} = 1/2\,\text{m}\,\text{r}^2;$

Displaying Lagrangian of the Disk

```
In[9]:= Simplify[LagrangianDisk] // TraditionalForm
```

Out[9]//TraditionalForm=

$$\frac{1}{4} m \left(-4 g ((r-R)\cos(\phi(t))+R)+((r-R)\phi'(t)+R\theta'(t))^2+2\left(((r-R)\phi'(t)\cos(\phi(t))+R\theta'(t))^2+(r-R)^2\phi'(t)^2\sin^2(\phi(t))\right)\right)$$

Creating new equation for the Lagrangian of the system

```
In[10]:= LagrangianS = LagrangianDisk + Lagrangianhc;
```

Part 4: Finding Equations of motion

```
In[11]:= equation1 = D[D[LagrangianS, θ'[t]], t] - D[LagrangianS, θ[t]] == 0;

In[12]:= equation2 = D[D[LagrangianS, ϕ'[t]], t] - D[LagrangianS, ϕ[t]] == 0;

In[13]:= EOMs = Solve[{equation1, equation2}, {θ''[t], ϕ''[t]}] // Simplify

Out[13]= {{θ''[t] → (g (6 M Sin[θ[t]] + m π (Sin[ϕ[t]] + Sin[2 ϕ[t]])) +
            6 M R Sin[θ[t]] θ'[t]^2 - 3 m π (r - R) Sin[ϕ[t]] ϕ'[t]^2) /
          (R (12 M Cos[θ[t]] + π (-3 m - 6 M + 2 m Cos[ϕ[t]] + m Cos[2 ϕ[t]]))),
       ϕ''[t] → -((g (2 M (1 + 2 Cos[ϕ[t]]) Sin[θ[t]] + (3 m π + 4 M π - 8 M Cos[θ[t]]) Sin[ϕ[t]]) +
              2 M R (1 + 2 Cos[ϕ[t]]) Sin[θ[t]] θ'[t]^2 -
              m π (r - R) (Sin[ϕ[t]] + Sin[2 ϕ[t]]) ϕ'[t]^2) /
            ((r - R) (12 M Cos[θ[t]] + π (-3 m - 6 M + 2 m Cos[ϕ[t]] + m Cos[2 ϕ[t]]))))}}
```

Part 5: Finding EOM such that the disk does not exist

substituting m = 0 in original equation $\ddot{\theta}$

```
In[14]:= EOMs[[1]][[1]] /. m → 0 // Simplify
```

Out[14]= $\theta''[t] \to \frac{\text{Sin}[\theta[t]]\,(g + R\,\theta'[t]^2)}{R\,(-\pi + 2\,\text{Cos}[\theta[t]])}$

Note: Analytical work matches this equation

Part 6: Finding EOM such that the half cylinder is fixed

In[15]:=
```
Solve[D[D[LagrangianDisk, ϕ'[t]], t] - D[LagrangianDisk, ϕ[t]] == 0, ϕ''[t]] /.
  {θ'[t] → 0, θ''[t] → 0} // Simplify
```

Out[15]= $\left\{\left\{\phi''[t] \rightarrow \frac{2\, g\, \mathrm{Sin}[\phi[t]]}{3\, r - 3\, R}\right\}\right\}$

Note: Analytical work matches this equation

Note: Numberical approximations to find the peroid of the two equations in Part 5 and Part 6 are done in MATLAB

Part 7: Integrating and plotting equations of motion

Initializing Constants

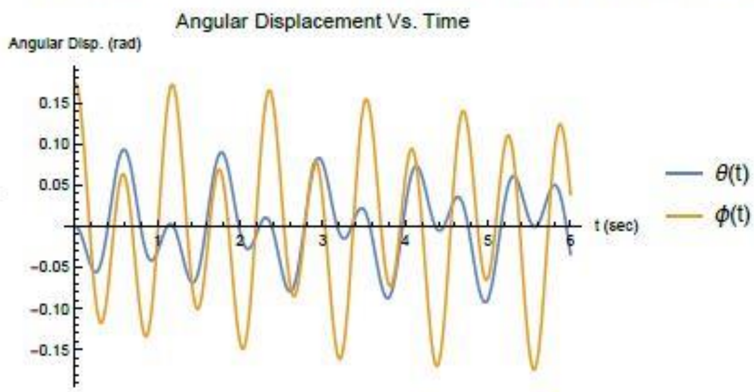
In[16]:=
```
M = 0.578; (*kg*)
m = 0.162; (*kg*)
R = 0.15625; (*m*)
r = 0.038; (*m*)
g = 9.81; (*m/s^2*)
```

Plotting EOMs with following initial conditions

$\theta(0) = 0,\ \dot{\theta}(0) = 0,\ \phi(0) = \pi/18,\ \dot{\phi}(0) = 0$

In[21]:=
```
solutionCurves1 = NDSolveValue[{equation1, equation2,
    θ[0] == 0, θ'[0] == 0, ϕ[0] == π / 18, ϕ'[0] == 0}, {θ, ϕ}, {t, 0, 6}];
```

In[22]:=
```
Plot[{solutionCurves1[[1]][t], solutionCurves1[[2]][t]},
 {t, 0, 6}, PlotLabel → "Angular Displacement Vs. Time",
 AxesLabel → {"t (sec)", "Angular Disp. (rad)"}, PlotLegends → {"θ(t)", "ϕ(t)"}]
```

Out[22]=


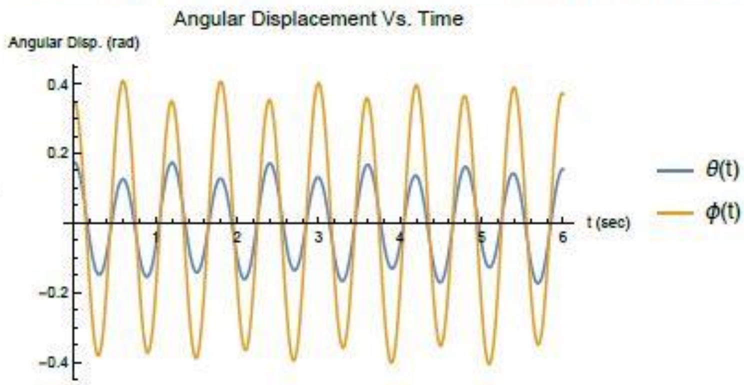
Plotting EOMs with following initial conditions

$\theta(0) = \pi/18,\ \dot{\theta}(0) = 0,\ \phi(0) = \pi/9,\ \dot{\phi}(0) = 0$

In[23]:=
```
solutionCurves2 = NDSolveValue[{equation1, equation2,
    θ[0] == π / 18, θ'[0] == 0, ϕ[0] == π / 9, ϕ'[0] == 0}, {θ, ϕ}, {t, 0, 6}];
```

In[24]:=
```
Plot[{solutionCurves2[[1]][t], solutionCurves2[[2]][t]},
 {t, 0, 6}, PlotLabel → "Angular Displacement Vs. Time",
 AxesLabel → {"t (sec)", "Angular Disp. (rad)"}, PlotLegends → {"θ(t)", "ϕ(t)"}]
```

Out[24]=



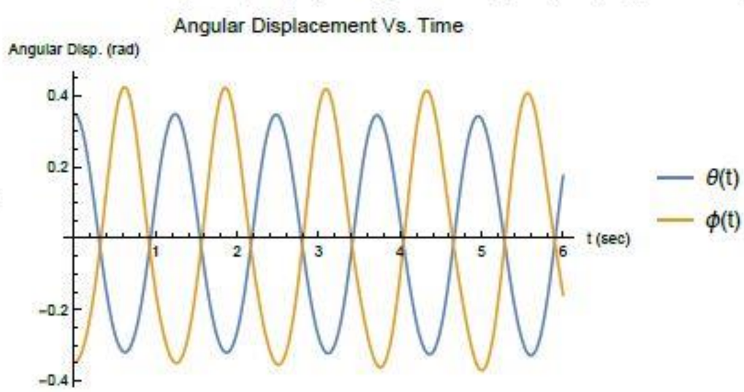
Plotting EOMs with following initial conditions

$\theta(0) = \pi/9,\ \dot{\theta}(0) = 0,\ \phi(0) = -\pi/9,\ \dot{\phi}(0) = 0$

In[25]:=
```
solutionCurves3 = NDSolveValue[{equation1, equation2,
    θ[0] == π / 9, θ'[0] == 0, ϕ[0] == -π / 9, ϕ'[0] == 0}, {θ, ϕ}, {t, 0, 6}];
```

In[26]:=
```
Plot[{solutionCurves3[[1]][t], solutionCurves3[[2]][t]},
 {t, 0, 6}, PlotLabel → "Angular Displacement Vs. Time",
 AxesLabel → {"t (sec)", "Angular Disp. (rad)"}, PlotLegends → {"θ(t)", "ϕ(t)"}]
```

Out[26]=



2. MATLAB

Part 5 and 6: Determining the frequency of the two simplified cases

```
function thetaddotAndPhiddotSimplifications
% Determining the natural frequency for theta(0) = pi/4rad, thetadot = 0
% initializing given variables
c.M = 0.578; %kg
c.m = 0.162; %kg
c.R = 0.15625; %m
c.r = 0.038; %m
c.g = 9.81; %m/s^2

 options = odeset('Events',@eventFunction);

 % Begining integration for the simplified thetaddot
 [T,S,TE,~,~] = ode45(@(t,s)simpleThetaddot(t,s,c),...
     linspace(0,6,6001),[double(pi/4),0],options);

 % Plotting theta vs t
 figure(1)
 hold on
 set(gca,'fontsize',12)
 xlabel('t (s)')
 ylabel('\theta (rad)')
 title('\theta vs. t')
 ylim([-0.9 0.9])
 plot(T,S(:,1))
 fprintf('The natural frequency of the cylinder is %0.3f 1/s.\n',...
     (2*pi)/(TE(2)-TE(1)))
 clear('T','S')

 % Begining integration for the simplified phiddot
 [T,S,TE,~,~] = ode45(@(t,s)simplePhiddot(t,s,c),...
     linspace(0,6,6001),[double(pi/4),0],options);

 % plotting phi vs t
 figure(2)
 hold on
 set(gca,'fontsize',12)
 xlabel('t (s)')
 ylabel('\phi (rad)')
 title('\phi vs. t')
 ylim([-0.9 0.9])

 plot(T,S(:,1))
 fprintf('The natural frequency of the hockey puck is %0.3f 1/s.\n',...
```

```
    (2*pi)/(TE(2)-TE(1)))
end

function [value,isterminal,direction] = eventFunction(~,s)
    value(1) = s(2);
    direction(1) = -1;
    isterminal(1) = 0;
end

function ds = simpleThetaddot(~,s,c)
    ds = zeros(2,1);
    ds(1) = s(2);
    ds(2) = (sin(s(1))*(c.g+c.R*(s(2))^2))/(c.R*(-pi+2*cos(s(1))));
end

function ds = simplePhiddot(~,s,c)
    ds = zeros(2,1);
    ds(1) = s(2);
    ds(2) = -2/3*(c.g*sin(s(1)))/(c.R-c.r);
end
```

```
The natural frequency of the cylinder is 6.370 1/s.
The natural frequency of the hockey puck is 7.153 1/s.
```