1 Derivation of Solutions to the Wave Equation

1.1 General Solution to the 1D Wave Equation

The one-dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u(x,t)$ is the wave function and c is the wave speed.

The three-dimensional case is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Here, we can apply use our new strategy to apply the correct change of variables:

$$\xi = x - ct, \quad \eta = x + ct$$

This gives us $u(\xi, \eta)$. Now, we need to apply the chain rule to determine the new form of the PDE. The first derivatives are:

With respect to x:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

With respect to t:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}$$

Next, we need to compute the second derivatives:

With respect to x:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x}$$

Note that by Clairaut's theorem, the order of the partial derivatives does not matter for continuous functions. We expect from experiment that the electromagnetic field is continuous and therefore enforce this.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

With Respect to t:

$$\frac{\partial^2 u}{\partial t^2} = -c\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial t} + c\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial t} + c\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial t} + -c\frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial t}$$

Again, with Clairaut's theorem:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \xi^2} + c^2\frac{\partial^2 u}{\partial \eta^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

From here, we can plug back into the initial PDE:

$$c^2\frac{\partial^2 u}{\partial x^2} = c^2[\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}] = \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \xi^2} + c^2\frac{\partial^2 u}{\partial \eta^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

$$2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = -2c^2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

$$4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

Now we can solve. First, we divide out the constant term:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

Now, we can integrate with respect to ξ. Integration adds a value constant with respect to the integration variable, but since this is a multivariable function the constant can still be a function of non-integration variables, in our case η:

$$\int \frac{\partial^2 u}{\partial \xi \partial \eta}\ d\xi = \frac{\partial u}{\partial \eta} = h(\eta)$$

Now, we integrate one more time. This will give us another function of η and a function constant in eta but potentially variable in ξ:

$$\int\ du = \int h(\eta)\ d\eta = g(\eta) + f(\xi)$$

Now, all we have to do is substitute back in our initial variables:

$$u(x,t) = g(x+ct) + f(x-ct) \tag{2}$$

This is the general solution to the 1D wave equation.

1.2 Solution for Plane Waves

We want to use this general solution to solve specifically for the case of plane waves. However, this requires some specification of initial conditions and boundary conditions. Let's consider our wave starting at $t = 0$. Then, we want some initial spatial profile $u_0(x) = u(x, 0)$. Second, we need $\frac{\partial u}{\partial t}|_{t=0} \equiv v_0(x)$. As for the boundary conditions, we will not restrict x, as electromagnetic waves can propagate freely through vacuums. Now, we plug in the general solution:

$$u_0(x) = f(x) + g(x) \tag{3}$$

$$\frac{\partial u}{\partial t}|_{t=0} = cf'(x) - cg'(x) \tag{4}$$

Next, integrate from x_0 to x:

$$\int_{x_0}^{x} v_0(x)\ dx = \int_{x_0}^{x} cf'(x) - cg'(x)\ dx = c[(f(x) - g(x)) - (f(x_0) - g(x_0))]$$

$$(f(x_0) - g(x_0) + \frac{1}{c}\int_{x_0}^{x} v_0(x)\ dx = \frac{1}{c}[\int_{x_0}^{x} cf'(x) - cg'(x)\ dx] + (f(x_0) - g(x_0) = f(x) - g(x)$$

$$f(x_0) - g(x_0) + \frac{1}{c}\int_{x_0}^{x} v_0(x)\ dx = f(x) - g(x)$$

Now, add equation (3)

$$u_0(x) + (f(x_0) - g(x_0)) + \frac{1}{c}\int_{x_0}^{x} v_0(x)\ dx = 2f(x)$$

Since x is arbitrary to this point, we can change the bounds of integration $x \to x + ct$. This also shifts the argument of f, g, and u, since they all depend on the integration bound. Now we have:

$$u_0(x + ct) + (f(x_0) - g(x_0)) + \frac{1}{c}\int_{x_0}^{x+ct} v_0(x)\ dx = 2f(x + ct)$$

Finally:

$$\frac{1}{2}u_0(x) + \frac{1}{2}((f(x_0) - g(x_0)) + \frac{1}{2c}\int_{x_0}^{x+ct} v_0(x)\ dx = f(x + ct) \tag{5}$$

We can use this same trick as above by subtracting $u_0(x)$ to get $g(x - ct)$, noting that this time we want $x \to x - ct$:

$$\frac{1}{2}u_0(x - ct) + \frac{1}{2}((g(x_0) - f(x_0)) - \frac{1}{2c}\int_{x_0}^{x-ct} v_0(x)\ dx = g(x - ct) \tag{6}$$

Now comes the last step. $u(x,t) = f(x+ct) + g(x-ct)$, so we add (5) and (6):

$$\frac{1}{2}u_0(x)+\frac{1}{2}u_0(x)+\frac{1}{2}((f(x_0)-g(x_0))+\frac{1}{2}((g(x_0)-f(x_0))+\frac{1}{2c}\int_{x_0}^{x+ct} v_0(x)\ dx-\frac{1}{2c}\int_{x_0}^{x-ct} v_0(x)\ dx = u(x,t)$$

$$\frac{1}{2}[u_0(x+ct)+u_0(x-ct)] - \frac{1}{2c}\int_{x_0}^{x-ct} v_0(x)\ dx + \frac{1}{2c}\int_{x_0}^{x+ct} v_0(x)\ dx = u(x,t)$$

We now switch the bounds on the first integral, which induces a factor of -1:

$$\frac{1}{2}[u_0(x+ct)+u_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x_0} v_0(x)\ dx + \frac{1}{2c}\int_{x_0}^{x+ct} v_0(x)\ dx = u(x,t)$$

Finally, by the fundamental theorem of calculus:

$$\frac{1}{2}[u_0(x+ct)+u_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(x)\ dx = u(x,t) \tag{7}$$

2 Application to Electromagnetism

2.1 Deriving the Electromagnetic Wave Equation

Now that we've done all this work solving a particular system, we need to ensure it arises in electromagnetism. Let us start with Maxwell's laws. Firstly, we restrict ourselves to a vacuum, wherein the charge and current densities are 0, as we are primary interested in observing the wave properties of the electromagnetic field rather than effects from different mediums.

$$\nabla \cdot E = 0 \tag{8}$$

$$\nabla \times E = \frac{-1}{c}\frac{\partial B}{\partial t} \tag{9}$$

$$\nabla \cdot B = 0 \tag{10}$$

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} \tag{11}$$

Additionally, we will need the following identity:

$$\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E \tag{12}$$

Now, from (9):

$$\nabla \times \nabla \times E = \nabla \times \ \frac{-1}{c}\frac{\partial B}{\partial t}$$

$$\nabla \times \nabla \times E = \frac{-1}{c}\frac{\partial}{\partial t}\nabla \times \; B$$

Next, plug in (12) on the left and 11 on the right:

$$\nabla(\nabla \cdot E) - \nabla^2 E = \frac{-1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Finally, use (8):

$$-\nabla^2 E = \frac{-1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \tag{13}$$

This same procedure can be done for B as follows, starting from (11):

$$\nabla \times \nabla \times B = \nabla \times \; \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\nabla \times \nabla \times B = \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \; E$$

Now plug in (12) on the left and (9) on the right:

$$\nabla(\nabla \cdot B) - \nabla^2 B = \frac{-1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

Finally, use (10):

$$-\nabla^2 B = \frac{-1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

$$\nabla^2 B = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} \tag{14}$$

From here, we focus on the electric field E. First, we apply the spatial profile of a plane wave $E_0 e^{ikx}$. Furthemore, note the use of Gaussian units to make clear the role of the wave speed c and to de-emphasize other constants. Plugging this into our solution to the wave equation (7) with corresponding constants gives:

$$E(x,t) = \frac{1}{2}[E_0 e^{ik(x+ct)} + E_0 e^{ik(x-ct)}] + \frac{1}{2c}\int_{x-ct}^{x+ct} \pm i\omega E_0 e^{ikx} \; dx$$

Note that we were able to acquire the expression for $v_0(x)$ by assuming the integral returns a function of the same form as $u_0(x)$. This can be determined by the application of (9), (11), and (12) and (14). Loosely, since E and B co-vary in time and space, we assign the temporal derivative to function like the spatial derivative with a different constant.

The integral can be easily evaluated with the u-substitution:

$$u = \pm ikx, \pm\frac{du}{ik} = dx$$

For $+$, this gives:

$$E(x,t) = \frac{E_0}{2}[e^{ik(x+ct)} + e^{ik(x-ct)} + \frac{\omega}{ck}(e^{ik(x+ct)} - e^{ik(x-ct)})]$$

Now, we use $c = \frac{\omega}{k} \to ck = \omega$ to simplify:

$$E(x,t) = \frac{E_0}{2}[e^{ik(x+ct)} + e^{ik(x-ct)} + (e^{ik(x+ct)} - e^{ik(x-ct)})]$$

$$E(x,t) = \frac{E_0}{2}[2e^{ik(x+ct)}]$$

$$E(x,t) = E_0 e^{ik(x+ct)}$$

Once again, using $c = \frac{\omega}{k}$, we get:

$$E(x,t) = E_0 e^{i(kx+\omega t)} \tag{15}$$

For $-$, this gives:

$$E(x,t) = \frac{E_0}{2}[e^{ik(x+ct)} + e^{ik(x-ct)} - \frac{\omega}{ck}(e^{ik(x+ct)} - e^{ik(x-ct)})]$$

Now, we use $c = \frac{\omega}{k} \to ck = \omega$ to simplify:

$$E(x,t) = \frac{E_0}{2}[e^{ik(x+ct)} + e^{ik(x-ct)} - (e^{ik(x+ct)} - e^{ik(x-ct)})]$$

$$E(x,t) = \frac{E_0}{2}[2e^{ik(x-ct)}]$$

$$E(x,t) = E_0 e^{ik(x-ct)}$$

Once again, using $c = \frac{\omega}{k}$, we get:

$$E(x,t) = E_0 e^{i(kx-\omega t)} \tag{16}$$

To extend this to three dimensions, note that because there is no coupling between spatial dimensions the three-dimensional solution can be made simply by substituting x for a 3D vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and by extending k so that it is $\vec{k} = \vec{k_x} + \vec{k_y} + \vec{k_z}$

$$E(x,t) = E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{17}$$

3 Simulating Solutions to the Wave Equation

3.1 Discretization

In order to make our results legible to computers, we need to find a way to discretize these continuous functions so that computers can process them algorithmically. In this section, for brevity, note that u_{mn} denotes a second derivative of u, once with respect to m and once with respect to n. To approximate, we discretize space in the following way:

$$x = x_0 + i\Delta x; \quad i = 0, 1, 2, ...$$

Next, we Taylor expand to acquire the second-order centered-difference approximation:

$$u(x + \Delta x) = u(x) + u_x(x)\Delta x + \frac{1}{2}u_{xx}(x)\Delta x^2 + O(\Delta x^3)$$

$$u(x - \Delta x) = u(x) - u_x(x)\Delta x + \frac{1}{2}u_{xx}(x)\Delta x^2 + O(\Delta x^3)$$

Now add the two:

$$u(x + \Delta x) + u(x - \Delta x) = 2u(x) + u_{xx}(x)\Delta x^2 + O(\Delta x^3)$$

Now, rearrange for u_{xx}:

$$u_{xx}(x) = \frac{u(x) + \Delta x - 2u(x) + u(x - \Delta x)}{\Delta x^2} + O(\Delta x^3)$$

This gives, for discrete points:

$$u_{xx}(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

For time, we can use a similar approach. Consider time step t^n: $t^n = t_0 + n\Delta t$. Then, using the same Taylor-expansion approach, we will arrive at:

$$u_{tt}(t) \approx \frac{u(t) + \Delta t - 2u(t) + u(t - \Delta t)}{\Delta t^2}$$

and:

$$u_{tt}(t^n) \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}$$

Now consider our discretized wave equation:

$$u_{tt} = c^2 u_{xx} \to \frac{u(t) + \Delta t - 2u(t) + u(t - \Delta t)}{\Delta t^2} = c^2 \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

Next, rearrange to:

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \frac{c^2 \Delta t^2}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

This is our final form. We note that we bind $c\frac{\Delta t}{\Delta x} \leq 1$. This is know as the Courant-Friedrichs-Lewy condition, and it ensures stability for numerical wave simulations.

3.2 Simulating Waves

For the purpose of simulating waves, we used the Gaussian pulse:

$$u_0(x) = e^{-\frac{(x-x_c)^2}{w^2}} \tag{18}$$

Here, x_c denotes the center x. We note that this initial spatial profile is notably not of the same form as the derived solution in the theoretical results. However, we made this tradeoff on account of the ease of use of the Gaussian pulse in preparation of the simulations, as well as the similar visual results. Gifs of the simulations are available following the Github link provided in the next section.

4 Additional Information

4.1 References

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4.2 Github

`https://github.com/seanzero7/Wave-Equation`Link