Stochastic Mathematical Equations for Galaxy Filaments

Modeling the Cosmic Web Structure

1. Introduction

Galaxy filaments are the largest known structures in the universe, forming thread-like formations that connect galaxy clusters and superclusters. These structures, typically 50-80 megaparsecs in length, create the cosmic web—a vast network surrounding enormous voids. The following stochastic equations describe the formation, evolution, and statistical properties of these filamentary structures.

2. Stochastic Density Field Evolution

The cosmic density field evolves according to stochastic differential equations driven by primordial quantum fluctuations.

2.1 Langevin Equation for Density Perturbations

The density contrast $\delta(x,t) = (\rho - \rho\blacksquare)/\rho\blacksquare$ evolves according to:

$$d\delta(x,t) = [D(t)\nabla^2\delta + f(\delta)]dt + \sigma(\delta)dW(x,t)$$

where $D(t)$ is the growth factor, $f(\delta)$ represents nonlinear gravitational collapse, $\sigma(\delta)$ is the noise amplitude, and $W(x,t)$ is a Wiener process representing primordial fluctuations.

2.2 Stochastic Continuity Equation

$$\partial\delta/\partial t + \nabla\cdot[(1+\delta)v] = \eta(x,t)$$

where v is the peculiar velocity field and $\eta(x,t)$ is a Gaussian white noise term with correlation: $\blacksquare\eta(x,t)\eta(x',t')\blacksquare = Q\cdot\delta\_D(x-x')\cdot\delta\_D(t-t')$

2.3 Fokker-Planck Equation for Density PDF

The probability density function $P(\delta,t)$ of the density field evolves as:

$$\partial P/\partial t = -\partial/\partial\delta[A(\delta)P] + (1/2)\partial^2/\partial\delta^2[B(\delta)P]$$

$A(\delta) = \blacksquare d\delta/dt\blacksquare$ represents gravitational drift toward overdensities

$B(\delta) = \sigma^2(\delta)$ represents diffusion from primordial fluctuations

3. Hessian-Based Filament Detection

Filaments are identified as ridges in the smoothed density field using the Hessian matrix.

3.1 Smoothed Density Field

$$\delta_s(x) = \int \delta(x')W_R(|x-x'|)d^3x'$$

where W_R is a Gaussian smoothing kernel with scale R:

$$W_R(r) = (2\pi R^2)^{(-3/2)} \exp(-r^2/2R^2)$$

3.2 Hessian Matrix

$$H_{ij}(x) = \partial^2\delta_s/\partial x_i\partial x_j$$

with eigenvalues $\lambda_\blacksquare \leq \lambda_\blacksquare \leq \lambda_\blacksquare$ and eigenvectors $e_\blacksquare$, $e_\blacksquare$, $e_\blacksquare$

3.3 Filament Classification Criterion

A point x lies on a filament spine if:

$$\lambda_\blacksquare < 0,\ \lambda_\blacksquare < 0,\ |\lambda_\blacksquare| << |\lambda_\blacksquare|,\ |\lambda_\blacksquare| << |\lambda_\blacksquare|$$

$$\text{Filamentarity index: } F = (\lambda_\blacksquare - \lambda_\blacksquare)/(\lambda_\blacksquare + \lambda_\blacksquare)$$

3.4 Stochastic Filament Width Distribution

The filament width w follows a log-normal distribution:

$$P(w) = (1/w\sigma\sqrt{(2\pi)}) \exp[-(\ln w - \mu)^2/(2\sigma^2)]$$

with mean width $\blacksquare w\blacksquare = 5\text{-}8\ h^{\blacksquare 1}$ Mpc depending on smoothing scale

4. Stochastic Point Processes for Galaxy Distribution

4.1 Cox Process (Doubly Stochastic Poisson)

Galaxy positions follow a Cox process with random intensity:

$$\Lambda(x) = \Lambda_\blacksquare \cdot \exp[b\cdot\delta(x) + \varepsilon(x)]$$

where b is the galaxy bias parameter and $\varepsilon(x)$ is a stochastic field

$$P(N \text{ galaxies in } V) = \int P_{Poisson}(N|\Lambda)P(\Lambda)d\Lambda$$

4.2 Marked Point Process on Filaments

Galaxies on filaments form a marked point process:

$$\Phi = \{(x_i, m_i) : x_i \in \text{filament}, m_i = \text{galaxy mass}\}$$

with intensity along filament spine s:

$$\lambda(s) = \lambda\blacksquare \cdot [1 + A \cdot \exp(-|s-s\_node|/\blacksquare)]$$

where s_node is the nearest cluster node and ■ is a characteristic length

4.3 Two-Point Correlation Function

$$\xi(r) = \blacksquare\delta(x)\delta(x+r)\blacksquare = (r/r\blacksquare)^{\wedge}(-\gamma)$$

with $r\blacksquare \approx 5\ h\blacksquare^{1}$ Mpc and $\gamma \approx 1.8$ for galaxies

5. Stochastic Network/Graph Models

5.1 Random Geometric Graph

Nodes (clusters) connected if separation < r_c:

$$P(\text{edge}|d_{ij}) = \Theta(r_c - d_{ij}) \cdot \exp(-d_{ij}/\xi_{connect})$$

where Θ is the Heaviside function and $\xi_{connect}$ is the connection scale

5.2 Erd■s-Rényi-Gilbert Cosmic Web

Modified random graph with mass-dependent connection probability:

$$p_{ij} = p_\blacksquare \cdot (M_i \cdot M_j/M_*^2)^\alpha \cdot \exp(-d_{ij}/\lambda_{fil})$$

where M_i, M_j are cluster masses and $\lambda_{fil} \approx$ 10-30 $h^{\blacksquare 1}$ Mpc

5.3 Minimum Spanning Tree Statistics

Edge length distribution follows:

$$P(L) = (\beta/L_\blacksquare)(L/L_\blacksquare)^{(\beta-1)} \exp[-(L/L_\blacksquare)^\beta] \text{ (Weibull)}$$

Degree distribution of filament network:

$$P(k) \propto k^{(-\gamma_{net})} \cdot \exp(-k/k_{cut})$$

6. Dark Matter Halo Connection

6.1 Excursion Set Theory (Bond-Cole-Kaiser)

First-crossing distribution for halo formation:

$$f(S) = (\delta_c/\sqrt{(2\pi S^3)}) \exp(-\delta_c^2/2S)$$

where $S = \sigma^2(M)$ is variance and $\delta_c \approx 1.686$ is collapse threshold

6.2 Stochastic Halo Bias

$$b(M,z) = 1 + (\nu^2 - 1)/\delta_c + 2p/(\delta_c(1 + \nu^{(2p)}))$$

where $\nu = \delta_c/\sigma(M,z)$ is the peak height

6.3 Conditional Mass Function in Filaments

$$n(M|\text{filament}) = n_{field}(M) \cdot [1 + b_{fil}(M) \cdot \delta_{fil}]$$

with filament overdensity $\delta_{fil} \sim$ 5-20

7. Void-Filament Boundary Dynamics

7.1 Shell-Crossing Condition

Filament forms when eigenvalues satisfy:

$$\lambda_1(t_{sc}) = -1/D(t_{sc}) \text{ (first axis collapse)}$$

7.2 Stochastic Void Growth

Void radius evolution:

$$dR_v/dt = H(t) \cdot R_v \cdot [1 - (R_v/R_*)^{(-3\delta_v)}] + \eta_v(t)$$

where $\delta_v \approx -0.8$ is the void underdensity and η_v is a noise term

8. Power Spectrum and Correlation Functions

8.1 Matter Power Spectrum

$$P(k) = A\cdot k^{n_s}\cdot T^2(k)\cdot D^2(z)$$

Transfer function: $T(k) \approx \ln(1+2.34q)/(2.34q) \cdot [1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^{\blacksquare}]^{(-1/4)}$

where $q = k/(\Omega_m h^2 \text{ Mpc}^{\blacksquare 1})$

8.2 Filament-Specific Power Spectrum

$$P_{fil}(k_\blacksquare, k_\perp) = P(k)\cdot W_{fil}(k_\blacksquare, k_\perp)$$

Anisotropic window: $W_{fil} = \exp(-k_\perp^2 R_{fil}^2/2)\cdot \mathrm{sinc}^2(k_\blacksquare L_{fil}/2)$

8.3 Bispectrum for Non-Gaussianity

$$B(k_\blacksquare,k_\blacksquare,k_\blacksquare) = \blacksquare\delta(k_\blacksquare)\delta(k_\blacksquare)\delta(k_\blacksquare)\blacksquare$$

For gravitational evolution: $B = 2F_\blacksquare(k_\blacksquare,k_\blacksquare)P(k_\blacksquare)P(k_\blacksquare) + \text{cyclic}$

$$F_\blacksquare(k_\blacksquare,k_\blacksquare) = 5/7 + (1/2)(k_\blacksquare/k_\blacksquare + k_\blacksquare/k_\blacksquare)\cos\theta + (2/7)\cos^2\theta$$

9. Redshift Space Distortions

9.1 Kaiser Formula with Stochastic Velocities

$$\delta_s(k) = \delta_r(k)\cdot[1 + \beta\cdot\mu^2]\cdot D_{FoG}(k\cdot\mu\cdot\sigma_v)$$

$\beta = f/b$ where $f = d \ln D/d \ln a \approx \Omega_m^{(0.55)}$

Fingers-of-God damping: $D_{FoG} = \exp(-k^2\mu^2\sigma_v^2/2)$ or $1/(1+k^2\mu^2\sigma_v^2/2)$

9.2 Stochastic Velocity Field

$$v(x) = (aHf/4\pi) \int \delta(x')(x-x')/|x-x'|^3 \, d^3x' + v_{rand}(x)$$

$$\blacksquare v_{rand,i}\, v_{rand,j}\blacksquare = \sigma_v^2\ \delta_{ij}$$

10. Stochastic Geometry of the Cosmic Web

10.1 Minkowski Functionals

$V_\blacksquare$ = Volume (void fraction)

$V_\blacksquare = (1/6)\int(1/R_\blacksquare + 1/R_\blacksquare)dA$ (mean curvature)

$V_\blacksquare = (1/6)\int(1/R_\blacksquare R_\blacksquare)dA$ (Gaussian curvature $\to$ Euler characteristic)

$$V_{■} = (1/4\pi)\iint \kappa \cdot dA \quad \text{(Genus)}$$

10.2 Genus Curve for Gaussian Random Fields

$$g(\nu) = A \cdot (1 - \nu^2) \cdot \exp(-\nu^2/2)$$

where $\nu = \delta/\sigma$ is the threshold in units of standard deviation

10.3 Shapefinders

Thickness: $T = 3V/S$

Width: $W = S/(3C)$

Length: $L = C/(4\pi)$

Planarity: $P = (W-T)/(W+T)$, Filamentarity: $F = (L-W)/(L+W)$

11. Temporal Evolution Equations

11.1 Zel'dovich Approximation with Noise

$$x(q,t) = q + D(t)\cdot\Psi(q) + \int^{\blacksquare\blacksquare} \eta(q,t')dt'$$

$$\text{Displacement field: } \Psi(q) = -\nabla\phi(q)/4\pi G\rho^{\blacksquare}a^2$$

11.2 Stochastic Adhesion Model

$$\partial v/\partial t + (v\cdot\nabla)v = -\nabla\phi + \nu\nabla^2 v + F(x,t)$$

Burgers equation with stochastic forcing F(x,t) representing mergers

11.3 Filament Lifetime Distribution

$$P(\tau) = (\tau/\tau^{\blacksquare 2})\cdot\exp(-\tau/\tau^{\blacksquare})$$

with characteristic lifetime $\tau^{\blacksquare}$ ~ few Gyr, after which filaments merge or dissipate

12. Observable Predictions

12.1 Filament Number Density

$$n\_fil(L)\ dL = n^{\blacksquare}\cdot(L/L\_*)^{(-\alpha)}\cdot\exp(-L/L\_*)\cdot dL$$

with L_* ~ 10-20 $h^{\blacksquare 1}$ Mpc and $\alpha \approx 2$

12.2 Mass Function of Filaments

$$dn/dM\_fil = (\rho^{\blacksquare}/M\_fil)\cdot f(\sigma\_fil)\cdot|d\ \ln\ \sigma\_fil/d\ \ln\ M\_fil|$$

$$\text{with Press-Schechter-like multiplicity: } f(\sigma) = \sqrt{(2/\pi)}\cdot(\delta\_c/\sigma)\cdot\exp(-\delta\_c^2/2\sigma^2)$$

12.3 Angular Correlation of Filament Orientations

$$\blacksquare\cos^2\theta(r)\blacksquare - 1/3 = C(r) = C^{\blacksquare}\cdot\exp(-r/r\_align)$$

where r_align ~ 30-50 $h^{\blacksquare 1}$ Mpc is the alignment correlation length

Summary of Key Parameters

Parameter	Symbol	Typical Value	Physical Meaning
Filament length	L_fil	10-80 Mpc	Extent between clusters
Filament width	w_fil	5-8 h■^{1} Mpc	Cross-sectional diameter
Galaxy bias	b	1-2	Galaxy-matter clustering ratio
Correlation length	r■	5 h■^{1} Mpc	Clustering scale
Growth rate	f	Ω_m^0.55	Structure growth parameter
Collapse threshold	δ_c	1.686	Spherical collapse overdensity
Velocity dispersion	σ_v	300-500 km/s	Random velocities in clusters
Void underdensity	δ_v	-0.8	Typical void contrast

References

Bond, J. R., Kofman, L., & Pogosyan, D. (1996). How filaments of galaxies are woven into the cosmic web. Nature, 380, 603-606.

Sousbie, T. (2011). The persistent cosmic web and its filamentary structure. MNRAS, 414, 350-383.

Cautun, M., van de Weygaert, R., & Jones, B. J. T. (2013). NEXUS: Tracing the cosmic web connection. MNRAS, 429, 1286-1308.

Libeskind, N. I., et al. (2018). Tracing the cosmic web. MNRAS, 473, 1195-1217.