B565 Spring 2018: Homework #1

Due on Saturday, February 3, 11:59PM

Dr. Predrag Radivojac

Pulkit Maloo (maloop)

February 1, 2018

Contents

Problem 1 3

Problem 2 4

Problem 3 5

Problem 4 10

Problem 5 11

Problem 6 12

Problem 1

Problem 2

Problem 3

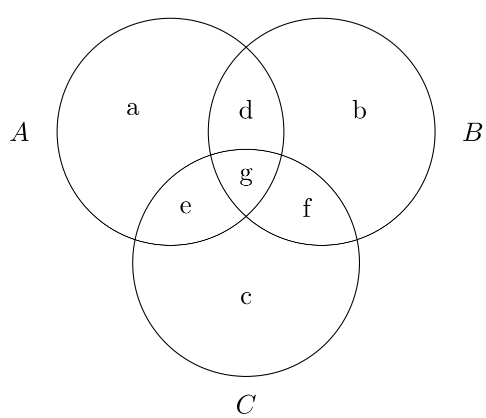
(a) $\mathbf{d_1(A, B) = |A - B| + |B - A|}$

$|A - B| = |A| - |A \cap B|$
$d_1(A, B) = |A| - |A \cap B| + |B| - |B \cap A|$
$d_1(A, B) = |A| + |B| - 2|A \cap B|$
$d_1(A, B) = |A \cup B| - |A \cap B|$

To prove our distance function is a metric we need to prove that it satisfies the following properties of a metric:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

Property 1:
Since $|A \cup B| \geqslant |A \cap B|$, property 1 is always true
Property 2:
If A = B, then $|A \cup B| = |A \cap B|$, hence $d_1(A, B) = 0$
Property 3:
Because intersection and union of sets is commutative
$d_1(A, B) = |A \cup B| - |A \cap B|$ is equivalent to $d_1(B, A) = |B \cup A| - |B \cap A|$, hence property 3 is satisfied
Property 4:
$d(A, C) \leq d(A, B) + d(B, C)$
$|A \cup C| - |A \cap C| \leq |A \cup B| - |A \cap B| + |B \cup C| - |B \cap C|$
$|A| + |C| - 2|A \cap C| \leq |A| + |B| - 2|A \cap B| + |B| + |C| - 2|B \cap C|$
$|A| + |C| - 2|A \cap C| \leq |A| + 2|B| + |C| - 2|A \cap B| - 2|B \cap C|$
$0 \leq |A| + 2|B| + |C| - 2|A \cap B| - 2|B \cap C| - (|A| + |C| - 2|A \cap C|)$
$0 \leq 2|B| - 2|A \cap B| - 2|B \cap C| + 2|A \cap C|$
$0 \leq |B| - |A \cap B| - |B \cap C| + |A \cap C|$
$|B| - |A \cap B| - |B \cap C| + |A \cap C| \geq 0$



$(b + d + f + g) - (d + g) - (f + g) + (e + g) \geq 0$
$b + d \geq 0$
`Hence Proved`
$d_1(A, B) = |A - B| + |B - A|$ is a metric

(b) $\mathbf{d_2(A, B)} = \dfrac{|\mathbf{A} - \mathbf{B}| + |\mathbf{B} - \mathbf{A}|}{|\mathbf{A} \cup \mathbf{B}|}$

We know from last question, $|A - B| + |B - A| = |A \cup B| - |A \cap B|$

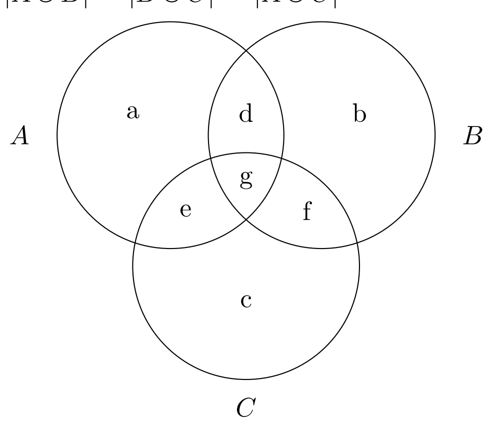
$$d_2(A, B) = \frac{|A \cup B| - |A \cap B|}{|A \cup B|}$$
$$d_2(A, B) = 1 - \frac{|A \cap B|}{|A \cup B|}$$

To prove our distance function is a metric we need to prove that it satisfies the following properties of a metric:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

Property 1:
Since $|A \cup B| \geqslant |A \cap B|$, property 1 is always true
Property 2:
If A = B, then $|A \cup B| = |A \cap B|$, hence $d_2(A, B) = 1 - \dfrac{|A \cap B|}{|A \cup B|} = 1 - 1 = 0$
Property 3:
Because intersection and union of sets is commutative
$d_2(A, B) = 1 - \dfrac{|A \cap B|}{|A \cup B|}$ is equivalent to $d_2(B, A) = 1 - \dfrac{|B \cap A|}{|B \cup A|}$, hence property 3 is satisfied
Property 4:
$d(A, C) \leq d(A, B) + d(B, C)$

$$1 - \frac{|A \cap C|}{|A \cup C|} \leq 1 - \frac{|A \cap B|}{|A \cup B|} + 1 - \frac{|B \cap C|}{|B \cup C|}$$
$$\frac{|A \cap B|}{|A \cup B|} + \frac{|B \cap C|}{|B \cup C|} - \frac{|A \cap C|}{|A \cup C|} \leq 1$$



Let $s = a + b + c + d + e + f + g$
$$\frac{d + g}{s - c} + \frac{f + g}{s - a} - \frac{e + g}{s - c} \leq 1$$
$$(d + g)(s - a)(s - b) + (f + g)(s - b)(s - c) - (e + g)(s - a)(s - c) \leq (s - a)(s - b)(s - c)$$

Problem 3 continued on next page...

Solving the above equation, we get
$2es^2 + s(ab + ac + bc + ad + bd + bf + cf + 2bg - ae - ce) + (ace + acg - abc - abd - abg - bcf - bcg) \geq 0$
$2es^2 - aes - ces + s(ab + ac + bc + ad + bd + bf + cf + 2bg) + (ace + acg - abc - abd - abg - bcf - bcg) \geq 0$
$es(2s - a - c) + s(ac + ad + bd + bf + cf + 2bg) + (ace + acg) + abs + bcs - abc - abd - abg - bcf - bcg \geq 0$
$s(2s - a - c) + s(ac + ad + bd + bf + cf + 2bg) + ace + acg + ab(s - c - d - g) + bc(s - f - g) \geq 0$

Substituting $s = a + b + c + d + e + f + g$ for terms in inside the bracket, we get
$s(s+b+d+e+f+g)+s(ac+ad+bd+bf+cf+2bg)+ace+acg+ab(a+b+e+f)+bc(a+b+c+d+e) \geq 0$

Since, all terms are non-negative, we have established the triangle inequality
`Hence Proved`

$d_2(A, B) = \dfrac{|A - B| + |B - A|}{|A \cup B|}$ is a metric

(c) $\mathbf{d_3(A, B) = 1 - \left(\frac{1}{2}\frac{|A \cap B|}{|A|} + \frac{1}{2}\frac{|A \cap B|}{|B|}\right)}$

$d_3(A, B) = 1 - \frac{|A \cap B|}{2}\left(\frac{1}{|A|} + \frac{1}{|B|}\right)$

To prove our distance function is a metric we need to prove that it satisfies the following properties of a metric:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

Let $setA = \{1, 2, 3\}$, $setB = \{4, 5, 6\}$ and $setC = \{1, 2, 3, 4, 5, 6\}$

Problem 3 continued on next page. . .

(d) $\mathbf{d_4(A, B)} =$

To prove our distance function is a metric we need to prove that it satisfies the following properties of a metric:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

Problem 4

Problem 5

Problem 6

References