The ultraspherical spectral method: First-order operators

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$$T_k = \text{degree } k \text{ Chebyshev polynomial of the 1st kind}$$

$$U_k = \text{degree } k \text{ Chebyshev polynomial of the 2nd kind}$$

First-order differentiation

$$\frac{d}{dx}T_k(x) = kU_{k-1}(x), \qquad \mathcal{D}_1 = \begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$

Conversion

$$T_k(x) = \frac{1}{2}\left(U_k(x) - U_{k-2}(x)\right), \qquad \mathcal{S}_0 = \frac{1}{2}\begin{pmatrix} 2 & 0 & -1 & & \\ & 1 & 0 & -1 & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

Multiplication

$$T_j(x)T_k(x) = \frac{1}{2}\left(T_{j+k}(x) + T_{|j-k|}\right)$$

$$\mathcal{M}_0[a] = \frac{1}{2}\left[\begin{pmatrix} 2a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & 2a_0 & a_1 & a_2 & \ddots \\ a_2 & a_1 & 2a_0 & a_1 & \ddots \\ a_3 & a_2 & a_1 & 2a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \iddots \\ a_3 & a_4 & a_5 & a_6 & \iddots \\ \vdots & \iddots & \iddots & \iddots & \iddots \end{pmatrix}\right]$$

First-order linear ODE

$$u'(x) + a(x)u(x) = f(x)$$

$$\left(\mathcal{D}_1 + \mathcal{S}_0\mathcal{M}_0[a]\right)\mathbf{u} = \mathcal{S}_0\mathbf{f}$$