The ultraspherical spectral method: Higher-order operators

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$C_k^{(\lambda)}$ = degree k ultraspherical polynomial (a generalization of T_k)

Differentiation

$$\frac{d^\lambda}{dx^\lambda} T_k(x) = 2^{\lambda-1}(\lambda-1)! k C_{k-\lambda}^{(\lambda)}(x)$$

$$\mathcal{D}_\lambda = 2^{\lambda-1}(\lambda-1)! \begin{pmatrix} \overbrace{0 \quad \cdots \quad 0}^{\lambda \text{ times}} & \lambda & & & \\ & & \lambda+1 & & \\ & & & \lambda+2 & \\ & & & & \ddots \end{pmatrix}$$

Conversion

$$C_k^{(\lambda)}(x) = \frac{\lambda}{\lambda+k}\left(C_k^{(\lambda+1)}(x) - C_{k-2}^{(\lambda+1)}(x)\right)$$

$$\mathcal{S}_\lambda = \begin{pmatrix} 1 & 0 & -\frac{\lambda}{\lambda+2} & & \\ & \frac{\lambda}{\lambda+1} & 0 & -\frac{\lambda}{\lambda+3} & \\ & & \frac{\lambda}{\lambda+2} & 0 & \ddots \\ & & & \frac{\lambda}{\lambda+3} & \ddots \\ & & & & \ddots \end{pmatrix}$$

Multiplication

Multiplication in U or $C^{(\lambda)}$ is more involved, but still great for computations. See "A fast and well-conditioned spectral method", SIAM Review, 2013. Multiplication of $a(x)$ in the $C^{(\lambda)}$ basis is denoted by $\mathcal{M}_\lambda[a]$.

Higher-order linear ODE

$$a_N(x)u^{(N)}(x) + a_{N-1}(x)u^{(N-1)}(x) + \ldots + a_0(x)u(x) = f(x)$$

$$\left(\mathcal{M}_N[a_N]\mathcal{D}_N + \mathcal{S}_{N-1}\mathcal{M}_{N-1}[a_{N-1}]\mathcal{D}_{N-1} + \ldots + \mathcal{S}_{N-1}\cdots\mathcal{S}_0\mathcal{M}_0[a_0]\right)\mathbf{u} = \mathcal{S}_{N-1}\cdots\mathcal{S}_0\mathbf{f}$$