can compute by DCT

$\cos(k \cos^{-1}(x))$

$$u(x) \approx \sum_{k=0}^{n-1} u_k T_k(x), \quad x \in [-1,1]$$

Interested in $Lu = f$, $Bu = c$

Start easy: <u>First order</u> $\quad u'(x) + a(x)u(x) = f(x)$

$u(-1) = c$

<u>3 Ingredients</u>: ① Differentiation

$$\frac{d}{dx}(u(x)) \approx \sum_{k=0}^{n-1} u_k \frac{d}{dx}(T_k(x))$$

$$= \sum_{k=1}^{n-1} u_k \, k \, U_{k-1}(x) \quad \text{(after some trig)}$$

Cheb. poly of 2^{nd} kind

$$= \underbrace{\begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \\ & & & & n-1 \\ 0 & & & & 0 \end{pmatrix}}_{D_1} \begin{pmatrix} u_0 \\ \vdots \\ u_{n-1} \end{pmatrix}$$

Coefficients in U basis now!

Need to be able to convert between, for other terms

② Multiplication

Find coeffs for a, u from $\{a_k\}, \{u_k\}$

$$a(x)u(x) = \left(\sum_{i=0}^{n-1} a_i T_i(x)\right)\left(\sum_{j=0}^{n-1} u_j T_j(x)\right)$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} a_i u_j T_i(x) T_j(x)$$

Looks dense, actually banded

Trig: $\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B))$, $A = i\cos^{-1}(x)$, $B = j\cos^{-1}(x)$

$$\Rightarrow T_i T_j = \frac{1}{2}(T_{i+j} + T_{i-j}) \Rightarrow M_a^{[a]} \; \frac{1}{2}\left[\quad + \quad \right]$$

Hankel $\quad$ Symmetric Toeplitz

③ Conversion: $T_k(x) = \frac{1}{2}\left[U_k(x) - U_{k-2}(x)\right]$

Substitute...

$$S_0 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & & \\ & \frac{1}{2} & 0 & \ddots & \\ & & \ddots & \ddots & -\frac{1}{2} \\ & & & \ddots & 0 \\ & & & & \frac{1}{2} \end{bmatrix}$$

$$\underset{U \leftarrow T}{u'(x)} + \underset{U \leftarrow T \leftarrow T}{a(x)\,u(x)} = \underset{U \leftarrow T}{f(x)}$$

$$D_1 \underline{u} + S_0 M_0[a] \underline{u} = S_0 \underline{f}$$

Boundary conditions: $u(-1) \approx \sum u_k T_k(-1) = c$

$$= \left[T_0(-1) \cdots T_{n-1}(-1)\right] \underline{u}$$

$$\Rightarrow \begin{bmatrix} D_0 + S_0 M_a \\ T_0(-1) \cdots T_{n-1}(-1) \end{bmatrix} \underline{u} = \begin{bmatrix} S_0 \underline{f} \\ c \end{bmatrix}$$

<u>Higher order?</u>

Ultraspherical poly.

$$\frac{d^2}{dx^2} T_k(x) = k \frac{dU_{k-1}}{dx} = k \frac{dC^{(1)}_{k-1}}{dx} = 2k\, C^{(2)}_{k-2}$$

$$\Rightarrow D_2 = \begin{pmatrix} 0 & 0 & 2 & & & \\ & & & 4 & & \\ & & & & 6 & \\ & & & & \ddots & 2(n-2) \\ 0 & & & \cdots & & 0 \\ 0 & & \cdots & & & 0 \end{pmatrix} \longrightarrow D_\lambda = 2^{\lambda-1}(\lambda-1)! \begin{pmatrix} \overbrace{0 \cdots 0}^{\lambda \text{ times}} & \lambda & & \\ & & \lambda+1 & \\ & & & \lambda+2 \end{pmatrix}$$

Use more formulas to get

$$S_\lambda = \begin{pmatrix} 1 & 0 & \frac{-\lambda}{\lambda+2} & & \\ \frac{\lambda}{\lambda+1} & 0 & \frac{-\lambda}{\lambda+3} & & \\ & \frac{\lambda}{\lambda+2} & \ddots & & \end{pmatrix}$$

$M_\lambda[a] \to$ more complicated

Note, B.C.'s can be complicated

Neumann, $u'(-1) = c \Rightarrow [T_0'(-1) \cdots T_{n-1}'(-1)] = [0 \; 1 \; -4 \cdots (-1)^{k+1} k^2 \cdots]$

Integral: $\int_{-1}^{1} u(x)\,dx = \sum u_k \int_{-1}^{1} T_k(x)$

$$\Rightarrow \left[\int_{-1}^{1} T_0(x) \cdots \int_{-1}^{1} T_{n-1}(x)\right]$$

Just replace last rows in system

<u>Example</u> $c(x)u''(x) + a(x)u'(x) + b(x)u(x) = f(x)$

$u(-1) = u(1) = 0$

$C^{(2)} \leftarrow C^{(2)} \leftarrow T \quad C^{(2)} \leftarrow U \leftarrow U \leftarrow T \quad C^{(2)} \leftarrow U \leftarrow T \leftarrow T \quad C^{(2)} \leftarrow U \leftarrow T$ ✓

$M_2[c] D_2 \underline{u} + S_1 M_1[a] D_1 \underline{u} + S_1 S_0 M_0[b] \underline{u} = S_1 S_0 \underline{f}$

$$\Rightarrow \begin{bmatrix} M_2[c]D_2 + S_1 M_1[a] D_1 + S_1 S_0 M_0[b] \\ T_0(-1) \cdots\cdots T_{n-1}(-1) \\ T_0(1) \cdots\cdots T_{n-1}(1) \end{bmatrix} \underline{u} = \begin{bmatrix} S_1 S_0 \underline{f} \\ 0 \\ 0 \end{bmatrix}$$

Nonlinear ODEs ?

$u'(x) + u(x)^2 = f(x), \quad u(-1) = 0$

$$\begin{bmatrix} D_1 + S_0 M_{\underline{u}} \\ T_0(-1) \cdots T_{n-1}(-1) \end{bmatrix} \underline{u} = \begin{bmatrix} S_0 \underline{f} \\ 0 \end{bmatrix} \quad (*)$$

Iterate: u_0 = guess (to satisfy BC)

for $k = 0, 1, 2, \cdots$

solve $\begin{bmatrix} D_1 + S_0 M_{\underline{u}_k} \\ T_0(-1) \cdots T_{n-1}(-1) \end{bmatrix} \underline{u}_{k+1} = \begin{bmatrix} S_0 \underline{f} \\ 0 \end{bmatrix}$ for u_{k+1}

end

Let $N(u) = u'(x) + u(x)^2 - f(x)$. Want $N(u) = 0$

Newton's method: $f'(x_k)\underbrace{(x_{k+1} - x_k)}_{\text{update}} + f(x_k) = 0$

$$\Rightarrow x_{k+1} = x_k - [f'(x_k)]^{-1} f(x_k)$$

Newton's method for operators:

u_0 = guess (BC)

for $k = 0, 1, 2, \cdots$

solve $N'(u_k) \overset{\text{update}}{v} = -N(u_k)$ for v

$u_{k+1} = u_k + v$

end

Example $N(u) = u'' + \sin(u)$

$$N'(u) \approx N(u + u_h) - N(u) = u'' + u_h'' + \sin(u + u_h) - u'' - \sin(u)$$

$$= u_h'' + \sin(u) + u_h \cos(u) + O(\|u_h\|^2) - \sin(u)$$

$$\approx u_h'' + u_h \cos(u)$$

u_0 = guess

$\Rightarrow$ for $k = 0, 1, 2, \ldots$

solve $v'' + v\cos(u_k) = -u_k'' + \sin(u_k)$

$v(-1) = v(1) = 0$

$u_{k+1} = u_k + v$

end

Can also do PDEs!