

# A Linear Parameter-Varying Approach to Data Predictive Control

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**Abstract**—By means of the *linear parameter-varying* (LPV) Fundamental Lemma, we derive novel *data-driven predictive control* (DPC) methods for LPV systems. In particular, we present output-feedback and state-feedback-based LPV-DPC methods with terminal ingredients, which guarantee exponential stability and recursive feasibility. We provide methods for the data-based computation of these terminal ingredients. Furthermore, an in-depth analysis of the application and implementation aspects of the LPV-DPC schemes is given, including application for nonlinear systems and handling noisy data. We compare and demonstrate the performance of the proposed methods in a detailed simulation example involving a nonlinear unbalanced disc system.

**Index Terms**—Data-Driven Control, Linear Parameter-Varying Systems, Behavioral systems, Predictive control.

## I. INTRODUCTION

EVER-INCREASING performance requirements in engineering are pushing practical control design problems to become increasingly more complex. In particular, controlling systems in operating regions with dominant nonlinear behavior is becoming important with still the need to guarantee stability and performance of the closed-loop operation. A powerful framework to systematically deal with complex nonlinear control problems and obtain such guarantees is the *linear parameter-varying* (LPV) framework [1]. In this framework, system descriptions are considered with a linear signal relation between the input, state and output signals, while this relation itself is varying along a measurable, time-varying signal  $p$ . The signal  $p$  is referred to as the *scheduling signal*, capturing nonlinear, time-varying and/or exogenous effects in the original system behavior, and hence characterizing its embedding in the solution set of an LPV description. Linearity

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of the resulting LPV surrogate representation has allowed the extension of many *linear-time invariant* (LTI) control design methods, widely used in the industry, to address nonlinear systems.

In many engineering problems, actuator and operational constraints are required to be satisfied in closed-loop operation. For such problems, *model predictive control* (MPC) is a well-suited solution with theoretical guarantees of recursive feasibility, stability and performance [2]. Combining the LPV concept and MPC techniques, i.e., LPV-MPC methods such as [3]–[5], have been shown to be advantageous for a large range of complex problems [6]. However, often it is cumbersome or even impossible with first-principles modeling to obtain accurate models in practice on which these MPC methods can be deployed. While data-driven modeling in terms of LPV model identification, has been developed to supply reliable models for control [1], it still requires an elaborate toolchain and human expertise to successfully accomplish it. Hence, as an alternative, *data-driven predictive control* (DPC) methods have been developed to design predictive controllers from data.

In existing LPV *data-driven predictive control* (LPV-DPC) schemes, often a *two-step* approach is still used under-the-hood, which is based on an LPV model identification scheme integrated with a model-based LPV-MPC design, cf. [7], [8]. This two-step approach is also applied in *continuous-time* in [9], while the work in [10] uses a Koopman-based identification scheme in combination with the LPV framework to obtain the predictor. In [11], data-driven (predictive) controller design for SISO systems is considered using a two-component hierarchical structure, using a reference model from [12]. Data-driven predictive control in the form of learning an LPV model that is used in an LPV-MPC scheme is presented in [13] with formal safety and stability guarantees. However, for all the aforementioned approaches, the quality of the identified LPV model still *governs* the performance of the predictive controller, while the identification objective often significantly deviates from the control objective. Also many choices in terms of the used model structure are still required to be accomplished by the user, similarly to regular system identification. In the LTI case, these observations lead to the idea of designing predictive controllers *directly* from data, without the need<sup>1</sup> for a model identification step.

A key result in direct DPC design for LTI systems, which also allows for theoretical guarantees, is Willems' Fundamen-

<sup>1</sup>See [14] for a detailed discussion on *when* and *when not* to use a model.

tal Lemma [15]. With this result, the behavior of an LTI system can be characterized using only data. The use of the Fundamental Lemma to obtain direct LTI-DPC schemes has been initiated by [16], [17], where the name ‘DeePC’ was coined. From these works, a multitude of papers on LTI-DPC have been published, providing deeper analysis of the DPC scheme, stability/performance/robustness guarantees, and applications of the paradigm in practice [18]–[21]. DPC reaches beyond the class of LTI systems by means of (extensions of) the Fundamental Lemma with respect to linear time-periodic systems [22], Koopman-based surrogates for nonlinear systems [23], [24], nonlinear systems by means of implicit online linearization [25] and LPV systems [26]. In this paper, we build on the preliminary results of the conference paper [26], where an LPV-DPC scheme has been proposed for LPV systems in *input-output* (IO) form. In the present work, we *extend* the results of [26] by (i) providing stability and feasibility guarantees for this LPV-IO-DPC scheme, which were not present in [26]; (ii) introducing a novel state-feedback formulation in terms of an LPV-SS-DPC scheme; and (iii) analyzing and extending the obtained LPV-DPC schemes to support their effective implementation. More specifically, our contributions in this paper are:

- C1: Development of a direct data-driven output-feedback LPV-DPC scheme<sup>2</sup> that uses only a measured sequence of input-scheduling-output data. We show that the method is recursively feasible and guarantees constraint satisfaction and exponential stability of the closed-loop without a model or extensive prior knowledge on the system.
- C2: Development of a direct data-driven state-feedback variant of the LPV-DPC scheme with recursively feasibility and stability guarantees, ensured by novel terminal ingredients computed purely based on data.
- C3: Providing effective approaches for handling noise and disturbances in the data together with determining the scheduling sequence in case the proposed DPC approach is applied for nonlinear systems or systems dependent on exogenous effects.
- C4: Extensive comparison of the developed DPC schemes with nonlinear, LPV, and LTI MPC solutions, as well as a comparison with the LTI DeePC scheme.

The paper is structured as follows: A detailed formulation of the considered problem setting is given in Section II, while, in Section III, the fully data-based predictors used in the LPV-DPC schemes are derived. Section IV provides Contribution C1 by developing the LPV-IO-DPC scheme, while the LPV-SS-DPC is derived in Section V corresponding to Contribution C2. Analysis of the LPV-DPC schemes together with robustification against noise and computation of scheduling sequences used in the predictor is given in Section VI, constituting to Contribution C3. Comparison and demonstration of the effectiveness of the approaches (Contribution C4) are given in Section VII. Conclusions on the

<sup>2</sup>The main difference between the methods in this paper and LTI-DPC methods, such as DeePC [17], is the considered system class. In fact, our work can be seen as a generalization of the DeePC method to the class of LPV systems, meaning that the two methods coincide if the scheduling signal is taken to be *constant* for all times (i.e., there is no parameter variation).

presented results are drawn in Section VIII.

**Notation:** The set of positive integers is denoted as  $\mathbb{N}$ , while  $\mathbb{R}$  denotes the set of real numbers. The  $p$ -norm of a vector  $x \in \mathbb{R}^{n_x}$  is denoted by  $\|x\|_p$  and the Moore-Penrose (right) pseudo-inverse of a matrix is denoted by  $\dagger$ . The Kronecker product of two matrices  $A$  and  $B$  is  $A \otimes B$ . We use  $(*)$  for a symmetric term in a quadratic expression, e.g.  $(*)^\top Ax = x^\top Ax$  for  $A \in \mathbb{R}^{n \times n}$  and  $x \in \mathbb{R}^n$ . The identity matrix of size  $n$  is denoted as  $I_n$  and  $0_{n \times m}$  denotes the  $n \times m$  zero-matrix, while  $1_n$  denotes the vector  $[1 \dots 1]^\top \in \mathbb{R}^n$ . For  $\mathbb{A}$  and  $\mathbb{B}$ ,  $\mathbb{B}^\mathbb{A}$  indicates the collection of all maps from  $\mathbb{A}$  to  $\mathbb{B}$ . The projection of  $\mathbb{D} \subseteq \mathbb{A} \times \mathbb{B}$  onto the elements of  $\mathbb{A}$  is denoted by  $\pi_a \mathbb{D} = \{a \in \mathbb{A} \mid (a, b) \in \mathbb{D}\}$ . We denote the interior of a set by  $\text{int}(\mathbb{A})$ . The notation  $A > 0$  and  $A < 0$  ( $A \succeq 0$  and  $A \preceq 0$ ) stands for positive/negative (semi) definiteness of  $A \in \mathbb{R}^{n \times n}$ . A block-matrix of the form  $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$  is denoted by  $\text{blkdiag}(A, B)$ . For a parameter-varying matrix  $X : \mathbb{P} \rightarrow \mathbb{R}^{n_x \times n_x}$ , we denote the maximum and minimum eigenvalue of  $X(p)$  over all possible  $p \in \mathbb{P}$  by  $\bar{\lambda}_{\mathbb{P}}(X)$  and  $\underline{\lambda}_{\mathbb{P}}(X)$ , respectively. For a given signal  $w \in (\mathbb{R}^{n_w})^{\mathbb{Z}}$  and a compact set  $[t_1, t_2] \subset \mathbb{Z}$ , the notation  $w_{[t_1, t_2]}$  corresponds to the truncation of  $w$  to the time interval  $[t_1, t_2]$ . For  $w_{[1, N]}$ , we denote the Hankel matrix of depth  $L$  associated with it as

$$\mathcal{H}_L(w_{[1, N]}) = \begin{bmatrix} w_1 & w_2 & \cdots & w_{N-L+1} \\ w_2 & w_3 & \cdots & w_{N-L+2} \\ \vdots & \vdots & \ddots & \vdots \\ w_L & w_{L+1} & \cdots & w_N \end{bmatrix},$$

while the vectorization of  $w_{[1, N]}$  is given as  $\text{vec}(w_{[1, N]})$ . The block-diagonal Kronecker operator is denoted by  $\odot$ , i.e., we have  $w_{[1, N]} \odot I_n := \text{blkdiag}(w_1 \otimes I_n, \dots, w_N \otimes I_n)$ . Moreover,  $w_{[1, N]}^{\mathbb{P}}$  denotes the sequence  $\{p_k \otimes w_k\}_{k=1}^N$ . Finally,  $\mathbb{I}_{\tau_1}^{\tau_2} = \{\tau \in \mathbb{Z} \mid \tau_1 \leq \tau \leq \tau_2\}$  is an index set for  $\tau_1 \leq \tau_2$ .

## II. PROBLEM STATEMENT

### A. System definition

Consider *discrete-time* (DT) LPV systems that can be represented by *state-space* (SS) representations of the form:

$$x_{k+1} = A(p_k)x_k + B(p_k)u_k, \quad (1a)$$

$$y_k = C(p_k)x_k + D(p_k)u_k, \quad (1b)$$

where  $x_k \in \mathbb{R}^{n_x}$ ,  $u_k \in \mathbb{R}^{n_u}$ ,  $y_k \in \mathbb{R}^{n_y}$  and  $p_k \in \mathbb{P} \subseteq \mathbb{R}^{n_p}$  are the state, input, output and scheduling signals at time moment  $k \in \mathbb{Z}$ , respectively. The matrix functions  $A : \mathbb{P} \rightarrow \mathbb{R}^{n_x \times n_x}$ ,  $B : \mathbb{P} \rightarrow \mathbb{R}^{n_x \times n_u}$ ,  $C : \mathbb{P} \rightarrow \mathbb{R}^{n_y \times n_x}$ ,  $D : \mathbb{P} \rightarrow \mathbb{R}^{n_y \times n_u}$  are considered to have affine dependence<sup>3</sup> on  $p_k$ :

$$\begin{aligned} A(p_k) &= A_0 + \sum_{i=1}^{n_p} p_{k,i} A_i, & B(p_k) &= B_0 + \sum_{i=1}^{n_p} p_{k,i} B_i, \\ C(p_k) &= C_0 + \sum_{i=1}^{n_p} p_{k,i} C_i, & D(p_k) &= D_0 + \sum_{i=1}^{n_p} p_{k,i} D_i, \end{aligned} \quad (1c)$$

where  $\{A_i, B_i, C_i, D_i\}_{i=0}^{n_p}$  are real matrices with appropriate dimensions. The scheduling signal  $p$  is varying in a compact, convex set  $\mathbb{P} \subset \mathbb{R}^{n_p}$ . The solution set of (1) is defined as

$$\begin{aligned} \mathfrak{B} = \{(u, p, x, y) \in (\mathbb{R}^{n_u} \times \mathbb{P} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y})^{\mathbb{Z}} \mid p \in \mathbb{P} \\ \text{and (1) holds } \forall k \in \mathbb{Z}\}. \end{aligned} \quad (2)$$

<sup>3</sup>We will discuss the generality of affine dependence in terms of LPV embeddings of nonlinear systems in Section VI.

which is called the behavior of (1). Here,  $\mathcal{P} \subseteq \mathbb{P}^{\mathbb{Z}}$  corresponds to the set of admissible trajectories of  $p$ , e.g., rate bounded scheduling trajectories. We can also define the projected behaviors  $\mathcal{B}^{ss} = \pi_{(u,p,x)}\mathcal{B}$  and  $\mathcal{B}^{io} = \pi_{(u,p,y)}\mathcal{B}$ , where the latter is called the manifest or IO behavior. It is assumed that the system has got no autonomous dynamics, which means that (1) can be chosen such that it is *structurally* state observable and controllable, see [1], and the associated  $\mathcal{B}^{io} = \pi_{(u,p,y)}\mathcal{B}$  represents all possible solution trajectories of the system. Such a representation is called *minimal* and, without loss of generality, (1) is chosen to be such in the rest of this paper. Furthermore, we consider that the LPV system (1) is subject to (pointwise-in-time) input and output constraints. Hence, for all  $k \in \mathbb{Z}$ ,

$$u_k \in \mathbb{U}, \quad y_k \in \mathbb{Y}, \quad (3)$$

where  $\mathbb{U} \subset \mathbb{R}^{n_u}$  and  $\mathbb{Y} \subset \mathbb{R}^{n_y}$  are compact and convex input and output constraint sets.

In this work, we consider two predictive control design scenarios for (1):

**Output-feedback case:** For the output-feedback data-driven predictive controller design, we assume that only input-scheduling-output measurements from (1) are available. These measurements are collected in the *data-dictionary*

$$\mathcal{D}_{N_d}^{io} = \{\check{u}_k, \check{p}_k, \check{y}_k\}_{k=1}^{N_d}, \quad (4)$$

where the notation  $\check{\cdot}$  is used throughout the paper for measured signals in the data-dictionary. To make the data-driven design tractable, we assume that the following condition is satisfied:

**Condition 1.** *The IO map of (1) admits a shifted-affine LPV-IO realization. That is, the manifest behavior  $\mathcal{B}^{io}$  is characterized by*

$$y_k + \sum_{i=1}^{n_a} a_i(p_{k-i})y_{k-i} = \sum_{i=1}^{n_b} b_i(p_{k-i})u_{k-i}, \quad (5a)$$

with  $n_a, n_b \geq 1$  and  $a_i : \mathbb{P} \rightarrow \mathbb{R}^{n_y \times n_y}$  and  $b_i : \mathbb{P} \rightarrow \mathbb{R}^{n_y \times n_u}$  that are affine functions of the time-shifted values of  $p_k$ , i.e.,

$$a_i(p_{k-i}) = a_{i,0} + \sum_{j=1}^{n_p} a_{i,j} p_{k-i,j}, \quad (5b)$$

$$b_i(p_{k-i}) = b_{i,0} + \sum_{j=1}^{n_p} b_{i,j} p_{k-i,j}. \quad (5c)$$

Fulfillment of this condition is often assumed in LPV modeling and identification as it enables a direct state-space realization of the IO map [27]. In general, an LPV-SS representation (1) does not necessarily have an LPV-IO realization that satisfies Condition 1. On the other hand, for a given LPV-IO representation in the form of (5), there always exists an affine LPV-SS representation (1), such that their manifest behaviors coincide.

**Remark 1.** (1) admits an IO form (5), if it has a companion observability canonical form with static-affine dependence [27]. When  $n_u = n_y = 1$ , (5) can be equivalently represented by a state-minimal (1), where  $C = [1 \ 0 \ \cdots \ 0]$ ,  $D = 0$  and

$$A(p_k) = \begin{bmatrix} a_1(p_k) & 1 & 0 \\ \vdots & \ddots & \vdots \\ a_{n-1}(p_k) & 0 & 1 \\ a_n(p_k) & 0 & \cdots & 0 \end{bmatrix}, \quad B(p_k) = \begin{bmatrix} b_1(p_k) \\ \vdots \\ b_n(p_k) \end{bmatrix},$$

where  $n = \max(n_a, n_b)$  and  $a_i = 0$  and  $b_j = 0$  for all  $i > n_a$  and  $j > n_b$ . When  $n_u, n_y > 1$ , realization of (5) in terms

of (1) follows a similar scheme with the matrix coefficients  $a_i$  and  $b_i$  used to form the above given  $A$  and  $B$ . Based on the independent columns of the observability matrix of the resulting state-space form, a  $T \in \mathbb{R}^{nn_y \times m}$  state-transformation can be constructed that brings the LPV-SS representation to a state-minimal form with  $m \leq nn_y$  states.  $\square$

**State-feedback case:** If state measurements are directly available, then (1) simplifies with  $C(p) = I$  and  $D(p) = 0$ . In this case, we consider the data-dictionary

$$\mathcal{D}_{N_d}^{ss} = \{\check{u}_k, \check{p}_k, \check{x}_k\}_{k=1}^{N_d}. \quad (6)$$

In this case, output constraints such as (3) directly translate to state constraints, i.e., for all  $k \in \mathbb{Z}$ ,  $x_k \in \mathbb{X} := \mathbb{Y}$ , where  $\mathbb{X} \subset \mathbb{R}^{n_x}$ . Moreover, we want to highlight that Condition 1 is trivially satisfied for the state-feedback case, as

$$y_k - A(p_{k-1})y_{k-1} = B(p_{k-1})u_{k-1}, \quad y_k = x_k, \quad (7)$$

is simply the time-shifted version of (1a).

### B. Problem formulation

In this work, we aim to solve the purely data-driven predictive control problem for unknown LPV systems under the the output-feedback and state-feedback cases. More precisely, without knowing the model of system (1), we aim to design a predictive controller, based on a measured data-dictionary  $\mathcal{D}_{N_d}$ , which can stabilize a desired (forced) equilibrium of (1).

**Problem 1.** Consider the *unknown* LPV system (1) with behavior  $\mathcal{B}$  from which an  $N_d$ -length data-dictionary  $\mathcal{D}_{N_d}$  is measured. Based only on  $\mathcal{D}_{N_d}$ , design a predictive controller  $K_{PC}$  that stabilizes the data-generating system with a priori specified performance. Furthermore,  $K_{PC}$  must ensure constraint satisfaction for all  $k \in \mathbb{Z}$ .

In this paper, we provide a solution to this problem for both the output-feedback and state-feedback case.

## III. DATA-DRIVEN LPV REPRESENTATIONS AND PREDICTORS

For the derivation of the data-driven predictive controllers, we need the concept of data-based representations of the LPV system behavior. In this section, we will introduce such representations based on [28]–[30]. These representations are instrumental for the solutions to Problem 1, i.e., Contributions C1 and C2, which are presented in Sections IV and V.

### A. LPV Fundamental Lemma

The data-driven representations will serve as the *predictors* in the LPV-DPC schemes. For this purpose, an essential result is the LPV Fundamental Lemma [28], which provides a data-driven representation of a general LPV system. In case Condition 1 is satisfied, the representation is directly *computable* from data and enables the prediction of the IO response for  $N_c$  steps in the future using only the data-dictionary and a given  $N_c$ -long scheduling sequence. To compute this prediction, we take the following assumption:

**Assumption 1.** At time step  $k$ , the scheduling trajectory  $p_{[k,k+N_c-1]}$  is known up to time step  $k + N_c - 1$ .

Similar to other methods used for LPV prediction problems and LPV-MPC for nonlinear systems, such as [5], [31], [32], we first take this assumption for technical reasons to formulate our results. As  $p$  acts as a free variable in an LPV system, we require Assumption 1 to predict the future trajectories of the system up to a prediciton horizon. Section VI-A shows how Assumption 1 can be reliably overcome in practice. Similar to *sequential quadratic programming* (SQP) methods, the future scheduling trajectory  $p_{[k,k+N_c-1]}$  is iteratively synthesized (see Algorithm 4 later) using an a priori selected scheduling map that defines the LPV embedding of the nonlinear system in the form of (1).

Before we present the data-driven representations, we first introduce some required concepts. The *state-cardinality*  $\mathbf{n}(\mathfrak{B})$  is the minimal state-dimension among all possible LPV-SS realizations (1) of  $\mathfrak{B}$ . We assume that the LPV-SS realization we consider to define  $\mathfrak{B}$  is minimal, hence  $n_x = \mathbf{n}(\mathfrak{B})$ . The *lag*  $\mathbf{L}(\mathfrak{B})$  is the minimum lag over all possible kernel realizations of  $\mathfrak{B}^{10}$ , i.e.,  $\mathbf{L}(\mathfrak{B}) = \max\{n_a, n_b\}$  if (5) is minimal. Note that  $\mathbf{L}(\mathfrak{B}) \leq \mathbf{n}(\mathfrak{B})$ . Finally,  $\mathfrak{B}|_{[t_1,t_2]}$  contains the truncation of all the trajectories in  $\mathfrak{B}$  to the interval  $[t_1, t_2] \subset \mathbb{Z}$ . Now we can give the following key result from [30], which, in essence, gives a condition on when the data can serve as an  $N_c$ -step predictor for the LPV system.

**Proposition 1** (Simplified<sup>4</sup> LPV Fundamental Lemma [30]). *Given a data set  $\mathcal{D}_{N_d} \in \mathfrak{B}^{10}|_{[1,N_d]}$  from an LPV system represented by (1) that satisfies Condition 1, and let  $N_c \geq \mathbf{L}(\mathfrak{B})$ . Then, for any  $(u, p, y)|_{[1,N_c]} \in \mathfrak{B}^{10}|_{[1,N_c]}$ , there exists a vector  $g \in \mathbb{R}^{N_d-N_c+1}$  such that*

$$\begin{bmatrix} \mathcal{H}_{N_c}(\check{u}_{[1,N_d]}) \\ \mathcal{H}_{N_c}(\check{y}_{[1,N_d]}) \\ \mathcal{H}_{N_c}(\check{u}_{[1,N_d]}^p) - \mathcal{P}^{n_u} \mathcal{H}_{N_c}(\check{u}_{[1,N_d]}) \\ \mathcal{H}_{N_c}(\check{y}_{[1,N_d]}^p) - \mathcal{P}^{n_y} \mathcal{H}_{N_c}(\check{y}_{[1,N_d]}) \end{bmatrix} g = \begin{bmatrix} \text{vec}(u_{[1,N_c]}) \\ \text{vec}(y_{[1,N_c]}) \\ 0 \\ 0 \end{bmatrix}, \quad (8)$$

where  $\mathcal{P}^\bullet = p_{[1,N_c]} \odot I_\bullet$ , if and only if

$$\text{rank} \left( \begin{bmatrix} \mathcal{H}_{N_c}(\check{u}_{[1,N_d]}) \\ \mathcal{H}_{N_c}(\check{u}_{[1,N_d]}^p) \\ \mathcal{H}_{N_c}(\check{y}_{[1,N_d]}) \\ \mathcal{H}_{N_c}(\check{y}_{[1,N_d]}^p) \end{bmatrix} \right) = (n_p(n_y + n_u) + n_u) N_c + n_x. \quad (9)$$

Equation (9) can be considered as a form of a *persistence of excitation* (PE) condition for LPV systems. In line with [28], we can see that PE of a data set coming from an LPV system is, next to the input signal, also dependents on the scheduling signal. As (9) also involves the output signal of the data set, we could refer to it as a *generalized* LPV persistence of excitation condition, similar to [33]. In the sequel, we will refer to  $\mathcal{D}_{N_d}$  satisfying (9) as  $\mathcal{D}_{N_d}$  being PE of order  $(N_c, n_x)$ . Note that (9) also defines the minimum number of data points, i.e., a lower bound for  $N_d$ , that is required to represent the  $N_c$ -length behavior of an LPV system (1) that satisfies Condition 1:

$$N_d \geq (1 + n_p(n_y + n_u) + n_u) N_c + n_x - 1.$$

<sup>4</sup>We refer to this result as the *simplified* LPV Fundamental Lemma, because, under Condition 1, it gives an efficiently *computable* representation compared to the general LPV Fundamental Lemma of [28]. The latter does not lead to a numerically computable form as the corresponding  $g$  can be potentially any meromorphic function of  $p_{[1,N_c]}$ .

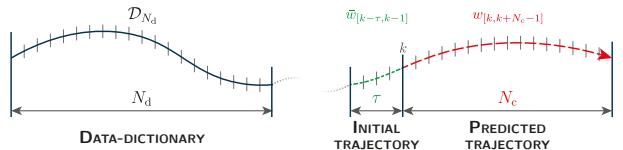


Fig. 1. Prediction problem for a given data-dictionary. The signals of the system are collected here in variable  $w$ . Figure adopted from [26].

Proposition 1 can be applied for both the output-feedback case and the state-feedback case (via (7)). Moreover, this result provides, based on the single data-sequence in  $\mathcal{D}_{N_d}$ , all possible trajectories that are compatible with any given scheduling sequence  $p_{[1,N_c]}$ . Hence, we now can formulate the data-based predictors for the output-feedback and state-feedback LPV-DPC schemes. We derive them through the simulation problem, cf. [26], [28], [34], depicted in Fig. 1. Based on an a priori<sup>5</sup> measured sequence  $\mathcal{D}_{N_d}$ , called the data-dictionary, we want to predict the continuation of the evolution of the system trajectories at time step  $k$ . For this we need the initial condition of the trajectory that we want to predict, which corresponds to the previous observations of the system in the time interval  $[k-\tau, k-1]$  (a  $\tau$ -long window) up to the current time step  $k$ . We denote these recorded signals as  $u_{[k-\tau, k-1]}$  (similar for  $x, y, p$ ), while the predicted trajectory at  $k$  is  $\bar{u}_{[0, N_c-1]|k}$ . We will use the notation  $\bullet$  for all predicted trajectories. Note that  $u_k = \bar{u}_{0|k}$  with this notation.

According to Proposition 1 and [30], the initial trajectory must have a length  $\tau \geq \mathbf{L}(\mathfrak{B})$ . We can then apply (8) for both the initial trajectory and the predicted trajectory, such that  $g$  at time step  $k$  is restricted to the subspace that relates to all the predicted trajectories that are possible continuations of the initial trajectory. Note that the Hankel matrices must have a depth of at least  $\mathbf{L}(\mathfrak{B}) + N_c$ . We are now ready to present the data-driven predictors for  $\mathfrak{B}^{10}$  and  $\mathfrak{B}^{ss}$ .

### B. Predictor formulation for $\mathfrak{B}^{10}$

For the formulation of the predictor for the LPV-IO-DPC scheme, we assume that (an upper bound on) the lag of the system is known. Then, along a given trajectory of  $p_{[k,k+N_c-1]}$  that defines  $\bar{p}_{[0, N_c-1]|k}$ , the following  $N_c$ -step ahead predictor for the LPV-IO-DPC scheme is obtained:

$$g_k = \begin{bmatrix} \mathcal{H}_\tau(\check{u}_{[1,N_\tau]}) \\ \mathcal{H}_\tau(\check{y}_{[1,N_\tau]}) \\ \mathcal{H}_\tau(\check{u}_{[1,N_\tau]}^p) - \mathcal{P}_k^{n_u} \mathcal{H}_\tau(\check{u}_{[1,N_\tau]}) \\ \mathcal{H}_\tau(\check{y}_{[1,N_\tau]}^p) - \mathcal{P}_k^{n_y} \mathcal{H}_\tau(\check{y}_{[1,N_\tau]}) \\ \mathcal{H}_{N_c}(\check{u}_{[\tau+1,N_d]}) \\ \mathcal{H}_{N_c}(\check{y}_{[\tau+1,N_d]}) \\ \mathcal{H}_{N_c}(\check{u}_{[\tau+1,N_d]}^p) - \bar{\mathcal{P}}_k^{n_u} \mathcal{H}_{N_c}(\check{u}_{[\tau+1,N_d]}) \\ \mathcal{H}_{N_c}(\check{y}_{[\tau+1,N_d]}^p) - \bar{\mathcal{P}}_k^{n_y} \mathcal{H}_{N_c}(\check{y}_{[\tau+1,N_d]}) \end{bmatrix} \begin{bmatrix} \text{vec}(u_{[k-\tau, k-1]}) \\ \text{vec}(y_{[k-\tau, k-1]}) \\ 0 \\ 0 \\ \text{vec}(\bar{u}_{[0, N_c-1]|k}) \\ \text{vec}(\bar{y}_{[0, N_c-1]|k}) \\ 0 \\ 0 \end{bmatrix}, \quad (10)$$

<sup>5</sup>It is assumed that the data-dictionary  $\mathcal{D}_{N_d}$  was measured offline in the past and (9) was verified a posteriori. Note that input design for guaranteeing the satisfaction of (9) is still an open question. Alternatively, the theory and schemes discussed in this paper can be modified such that the data-dictionary is updated online, which results in an adaptive scheme, similar to the online adaptation schemes for LTI-DPC, such as [25], [35].

where  $N_\tau = N_d - N_c$ ,  $\mathcal{P}_k^\bullet = p_{[k-\tau, k-1]} \odot I_\bullet$  and  $\bar{\mathcal{P}}_k^\bullet = \bar{p}_{[0, N_c-1]|k} \odot I_\bullet$ . Note that the predicted  $y$  and the required  $u$  trajectories are completely determined by  $g_k$ , which is hence essentially the only required decision variable. Finally, note that for a full data-driven representation of  $\mathfrak{B}^{10}|_{[1, N_c+\tau]}$ ,  $\mathcal{D}_{N_d}^{10}$  must be PE of order  $(N_c + \tau, n_x)$ .

### C. Predictor formulation for $\mathfrak{B}^{ss}$

The lag of  $\mathfrak{B}^{ss}$  is equal to 1, which can clearly be observed from (7). This implies that all the information to advance the trajectory is in the measured  $x_k = x_{0|k}$ . In terms of the predictor, we aim to predict  $x_{[1, N_c]|k}$ , where  $x_{N_c|k}$  will be used in the terminal ingredients. Moreover, because there is no feed-through in (7),  $\bar{u}_{N_c|k}$  and  $\bar{p}_{N_c|k}$  are obsolete, while  $u_k = u_{0|k}$  itself can be taken as a decision variable. With these considerations, we end up with the following predictor for the LPV-SS-DPC scheme:

$$\begin{bmatrix} \mathcal{H}_1(\check{x}_{[1, N_d - N_c]}) \\ \mathcal{H}_{N_c}(\check{x}_{[2, N_d]}) \\ \mathcal{H}_{N_c}(\check{u}_{[1, N_d - 1]}) \\ \mathcal{H}_{N_c}(\check{x}_{[1, N_d - 1]}^{\check{p}}) - \bar{\mathcal{P}}_k^{n_x} \mathcal{H}_{N_c}(\check{x}_{[1, N_d - 1]}) \\ \mathcal{H}_{N_c}(\check{u}_{[1, N_d - 1]}^{\check{p}}) - \bar{\mathcal{P}}_k^{n_u} \mathcal{H}_{N_c}(\check{u}_{[1, N_d - 1]}) \end{bmatrix} g_k = \begin{bmatrix} x_k \\ \text{vec}(\bar{x}_{[1, N_c]|k}) \\ \text{vec}(\bar{u}_{[0, N_c-1]|k}) \\ 0 \\ 0 \end{bmatrix}, \quad (11)$$

where  $\bar{\mathcal{P}}_k^\bullet = \bar{p}_{[0, N_c-1]|k} \odot I_\bullet$  and  $\mathcal{D}_{N_d}^{ss}$  must be PE of order  $(N_c + 1, n_x)$ . As with (10), the predicted  $\bar{x}$  and the required  $\bar{u}$  trajectories are completely determined by  $g_k$ .

## IV. LPV-DPC WITH INPUT-OUTPUT MEASUREMENTS

With the data-driven predictors defined, we now present the solution to Problem 1 for the output-feedback case. We formulate the LPV-IO-DPC scheme using only a PE data set  $\mathcal{D}_{N_d}^{10}$  from an unknown LPV system and show that we can get guarantees on stability and recursive feasibility under mild conditions. We want to highlight that this section generalizes the preliminary results of [26]. In contrast to [26], we provide an LPV-IO-DPC scheme that guarantees recursive feasibility, constraint satisfaction and closed-loop exponential stability.

### A. LPV-IO-DPC scheme

We formulate the LPV-IO-DPC scheme in a regulation scenario for a IO setpoint reference  $(u^r, p^r, y^r)$ . Because we work with IO data and thus do not have access to the state, it is difficult to formulate the LPV-IO-DPC scheme with terminal ingredients that are based on an internal  $x$  associated with the trajectories in  $\mathfrak{B}^{10}$ . Therefore, we follow the lines of [18] by considering a terminal equality constraint that can guarantee exponential stability of the closed-loop. For this reason, we consider the setpoint to be a forced equilibrium point of  $\mathfrak{B}^{10}$ :

**Definition 1.** A  $(u^r, p^r, y^r) \in \mathbb{U} \times \mathbb{P} \times \mathbb{Y}$  is a forced equilibrium of  $\mathfrak{B}^{10}$ , if  $\{u_k, p_k, y_k\}_{k=1}^{\mathbf{L}(\mathfrak{B})+1}$  with  $(u_k, p_k, y_k) = (u^r, p^r, y^r)$  for all  $k \in \mathbb{I}_1^{\mathbf{L}(\mathfrak{B})+1}$  is a trajectory of  $\mathfrak{B}^{10}$ .  $\square$

In Remark 2, we will discuss how to obtain this setpoint equilibrium for an unknown system directly from data. For an equilibrium  $(u^r, p^r, y^r)$ , we denote by  $u_n^r$  the stacked column vector containing  $n$  times  $u^r$ , similarly for  $p_n^r, y_n^r$ . Note that for

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### Algorithm 1: LPV-DPC under output-feedback

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- 1: **initialization:** set  $k \leftarrow k_0$  (starting time)
  - 2: **loop**
  - 3:   **measure**  $(u_{[k-\tau, k-1]}, p_{[k-\tau, k-1]}, y_{[k-\tau, k-1]})$
  - 4:   **get**  $\bar{p}_{[0, N_c-1]|k}$
  - 5:   **solve** the QP (12)
  - 6:   **apply**  $u_k = \bar{u}_{0|k}$
  - 7:   **set**  $k \leftarrow k + 1$
  - 8: **end loop**
- 

our LPV-IO-DPC scheme, Assumption 1 implies that at  $k \in \mathbb{Z}$ , we have  $p_{[-\tau, N_c-1]|k}$  available (with  $p_{[-\tau, -1]|k} = p_{[k-\tau, k-1]}$  corresponding to the measured past of  $p$  and  $\bar{p}_{[0, N_c-1]|k}$  being its assumed future). The cost function used for the LPV-IO-DPC scheme is as follows:

$$J_{N_c}((u, p, y)_{[k-\tau, k-1]}, \bar{p}_{[0, N_c-1]|k}, g_k) = \sum_{i=0}^{N_c-1} \ell(\bar{u}_{i|k}, \bar{y}_{i|k}),$$

where  $\ell$  is the quadratic stage cost that penalizes the distance w.r.t. the setpoint equilibrium:

$$\ell(\bar{u}_{i|k}, \bar{y}_{i|k}) = (*)^\top Q(\bar{y}_{i|k} - y^r) + (*)^\top R(\bar{u}_{i|k} - u^r),$$

with  $Q \succ 0, R \succ 0$  being performance tuning matrices, similar to those in LQR control design. With this, we propose the LPV-IO-DPC scheme with terminal equality constraints as

$$\min_{g_k} \sum_{i=0}^{N_c-1} \ell(\bar{u}_{i|k}, \bar{y}_{i|k}) \quad (12a)$$

$$\text{s.t. } (10) \text{ holds,} \quad (12b)$$

$$\bar{u}_{i|k} \in \mathbb{U}, \bar{y}_{i|k} \in \mathbb{Y}, \forall i \in \mathbb{I}_0^{N_c-1}, \quad (12c)$$

$$\begin{bmatrix} u_\tau^r \\ y_\tau^r \end{bmatrix} = \begin{bmatrix} \text{vec}(\bar{u}_{[N_c-\tau, N_c-1]|k}) \\ \text{vec}(\bar{y}_{[N_c-\tau, N_c-1]|k}) \end{bmatrix}, \quad (12d)$$

where  $\tau \geq \mathbf{L}(\mathfrak{B})$ , and (12d) are the terminal equality constraints that force the stage cost to 0 in the last  $\tau$  steps of the control horizon. This guarantees stability and recursive feasibility, and is common in MPC [2, Sec. 2.5.6]. Note that, in terms of (10),  $\bar{u}_{i|k}, \bar{y}_{i|k}$  are completely determined by  $g_k$ , which is the decision variable in (12). Problem (12), is solved as a *quadratic program* (QP) and applied in a receding horizon fashion (see Step 6 in Algorithm 1). By comparing (12) to LTI-DPC (e.g., DeePC), the data-driven representation in (12b) is varying along the scheduling signal, representing an LPV behavior that is richer than the one in DeePC, where the behavior is inherently restricted to be LTI. Hence, (12) can be seen as a generalization of the celebrated DeePC scheme.

### B. Stability and recursive feasibility

Let us for the remainder of this section abbreviate the optimal cost at time step  $k$  as  $J_{N_c}^*(k)$

$$J_{N_c}^*(k) := J_{N_c}((u, p, y)_{[k-\tau, k-1]}, \bar{p}_{[0, N_c-1]|k}, g_k^*),$$

where  $g_k^*$  is the minimizer of (12). For notational brevity, we assume  $(u^r, y^r) = (0, 0)$ , which is without loss of generality due to linearity of LPV systems. Moreover, let  $x_k$  be the state of the unknown minimal LPV-SS representation (1) of the data-generating system, which is induced by the initial trajectory  $(u, p, y)_{[k-\tau, k-1]}$ . To show recursive feasibility and

exponential stability of the proposed LPV-IO-DPC scheme, we require the following standard weak controllability condition (see [2, Assum. 2.17]).

**Condition 2.** *The optimal cost  $J_{N_c}^*(k)$  of (12) is quadratically upper bounded, i.e., there exists a  $c_u > 0$  such that, for any initial condition  $x_k$  corresponding to the feasible trajectory  $(u, p, y)_{[k-\tau, k-1]}$ , it holds that*

$$J_{N_c}^*(k) \leq c_u \|x_k\|_2^2. \quad (13)$$

Note that Condition (2) is satisfied under convex polytopic constraints as the resulting optimal cost of Problem (12) is continuous and piece-wise quadratic [36].

One of the main technical challenges for proving exponential stability and recursive feasibility of the LPV-IO-DPC scheme is characterizing and ensuring LPV input-output-to-state-stability (IOSS), which is an extension of the LTI result in [37]. We have summarized the solution to this problem in the following lemma:

**Lemma 1.** *Consider an LPV system defined by a state-minimal (1) with behavior  $\mathfrak{B}$ . Then,  $\mathfrak{B}$  is controllable and there exists an IOSS Lyapunov function  $W(x_k, p_k) = x_k^\top P(p_k) x_k$  with  $P(\mathbf{p}) \succ 0$  for all  $\mathbf{p} \in \mathbb{P}$  such that, for suitable  $c_1, c_2 > 0$ ,*

$$\begin{aligned} W(x_{k+1}, p_{k+1}) - W(x_k, p_k) &\leq -\frac{1}{2} \|x_k\|_2^2 \\ &\quad + c_1 \|u_k\|_2^2 + c_2 \|y_k\|_2^2, \end{aligned} \quad (14)$$

for all  $(u, p, x, y) \in \mathfrak{B}$  and  $k \in \mathbb{Z}$ .

See Appendix A for the proof. The following result shows that the proposed LPV-IO-DPC scheme controlling (1) is recursively feasible and ensures closed-loop constraint satisfaction as well as exponential stability.

**Theorem 1.** *Given a  $\mathcal{D}_{N_d}^{\text{IO}}$  from the data-generating system defined by a state-minimal (1) with behavior  $\mathfrak{B}$ . Let  $\mathcal{D}_{N_d}^{\text{IO}}$  be PE of order  $(N_c + \tau, n_x)$  with  $\tau \geq \mathbf{L}(\mathfrak{B})$ . If Condition 2 is satisfied and the LPV-IO-DPC problem (12) is feasible at  $k_0$ , then for all  $p \in \mathcal{P}$  and  $k > k_0$*

- (i) *the LPV-DPC problem (12) is feasible,*
- (ii) *the closed-loop system satisfies the constraints:  $u_k \in \mathbb{U}, y_k \in \mathbb{Y}$ ,*
- (iii)  *$(u^r, y^r) = (0, 0)$  is an exponentially stable equilibrium of the closed-loop system.*

*Proof.* In order to prove recursive feasibility (i), we define a feasible candidate input at time  $k+1$  by shifting the previously optimal solution and appending it with 0, i.e.,  $u_{i|k+1} = u_{i+1|k}^*$ ,  $i \in \mathbb{I}_0^{N_c-2}$  and  $u_{N_c-1|k+1} = 0$ . This input and the corresponding output trajectory satisfy the constraints of Problem (12). Moreover, by Proposition 1 and the PE assumption, there exists a variable  $g_{k+1}$  such that all constraints of Problem (12) are fulfilled. This also implies constraint satisfaction (ii).

For showing exponential stability, we use standard sub-optimality arguments, cf. [2], combined with the result of Lemma 1. The above-defined candidate solution implies

$$J_{N_c}^*(k+1) - J_{N_c}^*(k) \leq -\ell(\bar{u}_{0|k}, \bar{y}_{0|k}). \quad (15)$$

Lemma 1 implies the existence of an *input-output-to-state stability* (IOSS) Lyapunov function  $W(x_k, p_k) = x_k^\top P(p_k) x_k$ ,  $P(\mathbf{p}) \succ 0$  for all  $\mathbf{p} \in \mathbb{P}$  and  $k \geq 0$  satisfying (14) with  $c_1, c_2 > 0$  defined as in (40). With  $W$ , we can define the Lyapunov function candidate  $V(k) = J_{N_c}^*(k) + \gamma W(x_k, p_k)$  for some  $\gamma > 0$ , which will show exponential stability of the closed-loop system. Note that, under Condition 2,  $V$  has trivial quadratic lower and upper bounds for all feasible  $x_k$ :

$$\gamma \underline{\lambda}(P(p_k)) \|x_k\|_2^2 \leq V(k) \leq (c_u + \gamma \bar{\lambda}(P(p_k))) \|x_k\|_2^2.$$

Combining (15) and (14) and choosing  $\gamma = \frac{\underline{\lambda}(Q, R)}{\max\{c_1, c_2\}}$ , we obtain  $V(k+1) - V(k) \leq -\gamma \|x_k\|_2^2$ . Then it follows from standard Lyapunov arguments with Lyapunov function  $V$  that the origin of the closed-loop system is exponentially stable. ■

The proof of this result follows the line of reasoning in the proof of the LTI case, see [18, Thm. 2]. This is possible due to linearity of the LPV system along a scheduling signal  $p$ . The main difference compared to the LTI case, is that the detectability argument used in [18, Thm. 2] is needed to be recast for the LPV case, which requires the nontrivial technical derivations of Lemma 1. As the considered realization of the data-generating LPV system (5) admits a stabilizable state-minimal LPV-SS representation that has a static-affine scheduling dependence [27], [38], we could exploit this property to formulate an IOSS Lyapunov function for the LPV system. As in [18], the LPV IOSS Lyapunov function is used to prove stability of the origin of the closed-loop.

*Remark 2.* In practice, the setpoint equilibrium  $(u^r, p^r, y^r)$  is often not given in full, but usually only in terms of a desired output at a given scheduling, e.g., a desired speed (output) for a given altitude (scheduling) in an aircraft control problem. A possible method to find the corresponding  $u^r$ , such that  $(u^r, p^r, y^r)$  satisfies Definition 1, is to use Proposition 1 to compute  $u^r$  via, e.g., the quadratic program:

$$u^r = \arg \min \|g\|_2^2 \text{ subject to}$$

$$\left[ \begin{array}{c} \mathcal{H}_\tau(\check{u}_{[1, N_d-1]}) \\ \mathcal{H}_\tau(\check{y}_{[2, N_d]}) \\ \mathcal{H}_\tau(\check{u}_{[1, N_d-1]}^\ddot{p}) - \mathcal{P}^{n_u} \mathcal{H}_\tau(\check{u}_{[1, N_d-1]}) \\ \mathcal{H}_\tau(\check{y}_{[2, N_d]}^\ddot{p}) - \mathcal{P}^{n_y} \mathcal{H}_\tau(\check{y}_{[2, N_d]}) \end{array} \right] g = \begin{bmatrix} 1_\tau \otimes u^r \\ 1_\tau \otimes y^r \\ 0 \\ 0 \end{bmatrix},$$

where  $\mathcal{P}^\bullet = I_\tau \otimes (p^r \otimes I_\bullet)$ , or using an artificial equilibrium as in [39], [40]. Alternatively, the need for  $u^r$  can be alleviated by formulating a so-called offset-free cost [19] in terms of  $\Delta \bar{u}_{i|k} := \bar{u}_{i|k} - \bar{u}_{i-1|k}$  and have  $\Delta \bar{u}_{[N_c-\tau, N_c-1]|k} = 0$  as terminal equality constraint. □

*Remark 3.* We want to highlight that in the LTI case it is possible to avoid the use of terminal equality constraints, and formulate an LTI-IO-DPC scheme with a terminal cost and a terminal set constraint by defining the extended state vector  $\xi_k = [y_{k-1}^\top \cdots y_{k-n_a}^\top u_{k-1}^\top \cdots u_{k-n_b}^\top]^\top$ , and computing the terminal cost and terminal set using the data-driven representation of the associated state-space realization. For the LPV case, this is also possible when the IO representation (5) is considered with coefficients that are *statically*

dependent on  $p_k$ , i.e.,  $a_i(p_k)$  and  $b_i(p_k)$ . In that case, the LPV-IO representation has a nominal LPV-SS realization with static dependence as it is shown in [27]. This gives an alternative solution of the LPV-DPC problem under output-feedback and, in principle, would provide a similar DPC scheme that we will discuss in Section V, see also [41]. However, multiple problems exist with this formulation: (i) non-minimality of this representation makes it hard to satisfy the PE condition in practice [42] and (ii) the corresponding data-driven driven representation grows much faster in size with increasing  $N_c$  than in the LTI case, due to the lower two blocks in (11).  $\square$

## V. LPV-DPC WITH STATE MEASUREMENTS

In this section, we propose the LPV-SS-DPC scheme to solve Problem 1 for the *state-feedback* case. Again, we show that the proposed DPC scheme guarantees recursive feasibility, exponential stability and constraint satisfaction of the closed-loop operation, while only using a data set  $\mathcal{D}_{N_d}^{\text{ss}}$  from an unknown LPV system.

### A. LPV-SS-DPC scheme

We formulate the LPV-SS-DPC scheme for regulation of a setpoint reference  $(u^r, p^r, x^r)$ , which is not required to be an equilibrium point, contrary to the output-feedback case. The input setpoint can be seen as, e.g., an a priori known feedforward signal. Alternatively, its value can also be directly synthesized by the predictive control problem, see Remark 2. Consider time moment  $k \in \mathbb{Z}$  and, in line with the state-feedback setting,  $x_k$  measured from the system, representing its current state. To drive  $x_k$  to the setpoint, we consider the cost function  $J_{N_c}$  of the LPV-SS-DPC scheme as:

$$J_{N_c}(x_k, p_k, \bar{p}_{[1, N_c-1]|k}, g_k) = \sum_{i=0}^{N_c-1} \ell(\bar{u}_{i|k}, \bar{x}_{i|k}), \quad (16)$$

where for  $\ell$ , a quadratic stage cost that penalizes the distance w.r.t. the setpoint reference is chosen:

$$\ell(\bar{u}_{i|k}, \bar{x}_{i|k}) = (*)^\top Q(\bar{x}_{i|k} - x^r) + (*)^\top R(\bar{u}_{i|k} - u^r), \quad (17)$$

with  $Q, R \succ 0$ . Note that  $J_{N_c}$  does not depend on  $\bar{u}_{i|k}, \bar{x}_{i|k}$ , as these are implicitly defined by  $g_k$  through the predictor (11). In general, it is well-known that directly implementing a predictive control scheme in terms of minimization of (16) does not ensure recursive feasibility, and can even destabilize an already stable system. Therefore, we propose an LPV-SS-DPC scheme for  $\mathfrak{B}^{\text{ss}}$  with a terminal cost  $V_f$  and a terminal set  $\mathbb{X}_f$ , which allow to prove closed-loop stability and recursive feasibility of the optimization problem:

$$\min_{g_k} V_f(\bar{x}_{N_c|k} - x^r) + \sum_{i=0}^{N_c-1} \ell(\bar{u}_{i|k}, \bar{x}_{i|k}) \quad (18a)$$

$$\text{s.t. } (11) \text{ holds with } \bar{x}_{0|k} = x_k, \quad (18b)$$

$$\bar{u}_{i|k} \in \mathbb{U}, \bar{x}_{i+1|k} \in \mathbb{X}, \forall i \in \mathbb{I}_0^{N_c-1}, \quad (18c)$$

$$\bar{x}_{N_c|k} \in \mathbb{X}_f. \quad (18d)$$

As in standard model-based predictive control, (18) is solved in a receding horizon fashion, as summarized in Algorithm 2. In order to prove stability of the proposed DPC scheme, it must be ensured that the terminal set  $\mathbb{X}_f$  is *positively invariant* (PI). The conditions on  $\mathbb{X}_f$  and  $V_f$  that allow to prove stability and recursive feasibility will be discussed in the next section.

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### Algorithm 2: LPV-DPC under state-feedback

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- 1: **initialization:** set  $k \leftarrow k_0$  (starting time)
  - 2: **loop**
  - 3:   **measure**  $x_k$  and  $p_k$
  - 4:   **get**  $\bar{p}_{[1, N_c-1]|k}$
  - 5:   **solve** the QP (18)
  - 6:   **apply**  $u_k = \bar{u}_{0|k}$
  - 7:   set  $k \leftarrow k + 1$
  - 8: **end loop**
- 

### B. Stability and recursive feasibility

In this section, we prove recursive feasibility of the optimization problem (18) as well as exponential stability of the data-generating system  $\mathfrak{B}^{\text{ss}}$  under the proposed LPV-SS-DPC control law. Again, we assume without loss of generality that the setpoint reference is  $(x^r, u^r) = (0, 0)$ . Let us denote the optimal cost of (18) by  $J_{N_c}^*(x_k, p_k, \bar{p}_{[1, N_c-1]|k})$  and define the closed-loop state transition map of (1) with the LPV-DPC scheme (18) by  $\phi_{\text{cl}}$  such that  $x_{k+1} = \phi_{\text{cl}}(x_k, p_k)$ .

We assume that the following conditions hold, which are standard when considering terminal ingredients, see, e.g., [2, Assum. 2.14]. We comment on satisfying these conditions using only  $\mathcal{D}_{N_d}^{\text{ss}}$  in Section V-C.

**Condition 3.** *The following conditions are satisfied:*

- a) *The data-generating system with behavior  $\mathfrak{B}^{\text{ss}}$  is quadratically stabilizable, i.e., there exist a positive definite function  $V_f(x) = x^\top Z x$ , where  $Z \succ 0$ , and an associated control law  $u = K(p)x$  such that*

$$V_f(\phi_{\text{cl}}(x, p)) - V_f(x) \leq -\|x\|_{Q+K^\top(p)RK(p)}^2, \quad (19)$$

*for all  $x \in \pi_x \mathfrak{B}^{\text{ss}}, p \in \mathcal{P}$ .*

- b) *The set  $\mathbb{X}_f \subset \mathbb{X}$  is PI for the data-generating system (1) with behavior  $\mathfrak{B}^{\text{ss}}$  under  $u = K(p)x$ , i.e.,  $\phi_{\text{cl}}(\mathbb{X}_f, \mathbb{P}) \subseteq \mathbb{X}_f$  and  $0 \in \text{int}(\mathbb{X}_f)$ .*

If the system with behavior  $\mathfrak{B}^{\text{ss}}$  is quadratically stabilizable, then there exists a state feedback law  $K(p)$  such that  $x^r = 0$  is globally exponentially stable for the resulting closed-loop system  $\forall p^r \in \mathbb{P}$ . For  $x^r \neq 0$ , global exponential stability is guaranteed if  $p_k \rightarrow p^r$  sufficiently fast. We will show in Section V-C that, for Condition 3, if a stabilizing  $K(p)$  exists, then it can be directly designed based on only  $\mathcal{D}_{N_d}^{\text{ss}}$ .

The following result shows the desired recursive feasibility and stability properties of the closed-loop under Algorithm 2.

**Theorem 2** (LPV-SS-DPC recursive feasibility and exponential stability). *Given a  $\mathcal{D}_{N_d}^{\text{ss}}$  from the data-generating system with behavior  $\mathfrak{B}^{\text{ss}}$  that is PE of order  $(N_c + 1, n_x)$ . If Condition 3 is satisfied and the LPV-SS-DPC problem (18) is feasible at  $k_0$ , then for all  $p \in \mathcal{P}$  and  $k > k_0$*

- (i) *the LPV-SS-DPC problem (18) is feasible,*
- (ii) *the closed-loop system satisfies the constraints, i.e.,  $u_k \in \mathbb{U}, x_k \in \mathbb{X}$ ,*
- (iii) *the origin of  $\mathfrak{B}^{\text{ss}}$  is an exponentially stable equilibrium of the closed-loop system.*

*Proof.* See Appendix B. ■

The proof of Theorem 2 follows the same line of reasoning as that of standard MPC stability theory, thanks to the existence of the data-driven predictor (11) and solvability of the data-based state-feedback design problem (see Section V-C). Hence, the technical challenges for this result lie in the efficient formulation of (11), and in the computation of the terminal ingredients in a fully data-driven setting.

### C. Computation of terminal components

In this section, we give methods that allow the construction of the terminal ingredients of the LPV-SS-DPC approach by computationally efficient linear or quadratic programs using only the data set  $\mathcal{D}_{N_d}^{\text{ss}}$ .

**1) Computation of  $K(p)$  and  $V_f$ :** We first discuss the design of a terminal state-feedback controller  $K(p)$  that satisfies Condition 3.a. We consider  $K(p)$  as an LPV controller with affine scheduling dependence:

$$u_k = K(p_k)x_k, \quad K(p_k) = K_0 + \sum_{i=1}^{n_p} p_{k,i} K_i. \quad (20)$$

In [29, Thm. 4], a direct data-driven method for the design of stabilizing state-feedback controllers has been derived. We give here a brief summary of it, and refer to [29] for further details. Alternatively, data-driven synthesis of terminal ingredients using set-theoretical approaches such as in [43], [44] can be used.

For the synthesis, we only need  $Q, R$  (defining the quadratic performance (16)) and a data dictionary  $\mathcal{D}_{N_d}^{\text{ss}}$  that satisfies the simplified form of the PE condition (9) in the state-feedback case, i.e., when  $y_k = x_k$  and  $N_c = 1$ :

$$\text{rank} \left( \begin{bmatrix} \mathcal{H}_1(\check{x}_{[1, N_d-1]} \\ \mathcal{H}_1(\check{x}_{[1, N_d-1]}^\text{p} \\ \mathcal{H}_1(\check{u}_{[1, N_d-1]} \\ \mathcal{H}_1(\check{u}_{[1, N_d-1]}^\text{p} \end{bmatrix} \right) = (1 + n_p)(n_x + n_u). \quad (21)$$

**Proposition 2** (Data-driven LPV state-feedback synthesis [29]). *Given a  $\mathcal{D}_{N_d}^{\text{ss}}$  that satisfies (21). If there exist  $\tilde{Z} \succ 0$ ,  $\Xi$ ,  $F_Q$ ,  $\mathcal{F}$  and  $\mathcal{Y}$ , such that*

$$\begin{bmatrix} * \\ * \end{bmatrix}^\top \begin{bmatrix} \Xi & 0 \\ 0 & -W \end{bmatrix} \begin{bmatrix} L_{1,1} & L_{1,2} \\ I & 0 \\ L_{2,1} & L_{2,2} \end{bmatrix} \prec 0, \quad (22a)$$

$$\begin{bmatrix} * \\ * \end{bmatrix}^\top \underbrace{\begin{bmatrix} \Xi_{1,1} & \Xi_{1,2} \\ \Xi_{1,2}^\top & \Xi_{2,2} \end{bmatrix}}_{\Xi} \begin{bmatrix} I \\ -\Delta_p \end{bmatrix} \succeq 0, \quad \Xi_{2,2} \prec 0, \quad (22b)$$

$$\begin{bmatrix} \tilde{Z} & 0 & 0 \\ 0 & I_{n_p} \otimes \tilde{Z} & 0 \\ Y_0 & \bar{Y} & 0 \\ 0 & I_{n_p} \otimes Y_0 & I_{n_p} \otimes \bar{Y} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_1(\check{x}_{[1, N_d-1]} \\ \mathcal{H}_1(\check{x}_{[1, N_d-1]}^\text{p} \\ \mathcal{H}_1(\check{u}_{[1, N_d-1]} \\ \mathcal{H}_1(\check{u}_{[1, N_d-1]}^\text{p} \end{bmatrix} \mathcal{F}, \quad (22c)$$

$$\begin{bmatrix} I_{N_d} \\ p \otimes I_{N_d} \end{bmatrix}^\top F_Q \begin{bmatrix} I_{n_x} \\ p \otimes I_{n_x} \end{bmatrix} = \mathcal{F} \begin{bmatrix} I_{n_x} \\ p \otimes p \otimes I_{n_x} \end{bmatrix}, \quad (22d)$$

are satisfied for all  $p \in \mathbb{P}$ , where

$$\mathcal{Y} = [Y_0 \ \bar{Y}], \quad Y_0 \in \mathbb{R}^{n_u \times n_x}, \quad \bar{Y} \in \mathbb{R}^{n_u \times n_p n_x}, \quad (23a)$$

$$\Delta_p = \text{blkdiag}(p_1 I_{2n_x}, \dots, p_{n_p} I_{2n_x}), \quad (23b)$$

$$L_{1,1} = 0_{2n_x n_p \times 2n_x n_p}, \quad (23c)$$

$$L_{1,2} = [1_{n_p} \otimes I_{2n_x} \ 0_{2n_x n_p \times (n_x + n_u)}], \quad (23d)$$

$$L_{2,1} = \begin{bmatrix} 0_{n_x \times 2n_x n_p} \\ I_{n_p} \otimes \Gamma_1 \\ 0_{n_x \times 2n_x n_p} \\ I_{n_p} \otimes \Gamma_2 \\ 0_{(n_x + n_u) \times 2n_x n_p} \end{bmatrix}, \quad \Gamma_1 = [I_{n_x} \ 0], \quad \Gamma_2 = [0 \ I_{n_x}], \quad (23e)$$

$$L_{2,2} = \begin{bmatrix} \Gamma_1 & 0 \\ 1_{n_p} \otimes 0_{n_x \times 2n_x} & 0 \\ \Gamma_2 & 0 \\ 1_{n_p} \otimes 0_{n_x \times 2n_x} & 0 \\ 0 & I_{(n_x + n_u)} \end{bmatrix}, \quad (23f)$$

$$W = \begin{bmatrix} \tilde{Z}_0 & F_Q^\top \vec{\mathcal{X}}^\top & \begin{bmatrix} \tilde{Z} Q^{\frac{1}{2}} \\ 0 \end{bmatrix} & \mathcal{Y}^\top R^{\frac{1}{2}} \\ \vec{\mathcal{X}} F_Q & \tilde{Z}_0 & 0 & 0 \\ \begin{bmatrix} Q^{\frac{1}{2}} \tilde{Z} & 0 \end{bmatrix} & 0 & I_{n_x} & 0 \\ R^{\frac{1}{2}} \mathcal{Y} & 0 & 0 & I_{n_u} \end{bmatrix}, \quad (23g)$$

with  $\tilde{Z}_0 = \text{blkdiag}(\tilde{Z}, 0_{n_x n_p \times n_x n_p})$ , and  $\vec{\mathcal{X}} = \text{blkdiag}(\mathcal{H}_1(\check{x}_{[2, N_d]}), I_{n_p} \otimes \mathcal{H}_1(\check{x}_{[2, N_d]}))$ , then with

$$K_0 = Y_0 \tilde{Z}^{-1}, \quad [K_1 \ \dots \ K_{n_p}] = \bar{Y} (I_{n_p} \otimes \tilde{Z})^{-1}, \quad (24)$$

the LPV state-feedback controller (20) stabilizes  $\mathfrak{B}^{\text{ss}}$ . If  $\tilde{Z}$  is such that  $\text{trace}(\tilde{Z})$  is minimal among all feasible solutions of (22), then (20) minimizes the supremum of  $J_\infty(x, u) = \sum_{k=0}^{\infty} \ell(u_k, x_k)$  along all solutions of the resulting closed-loop system.

In Proposition 2, conditions (22a), (22b) are the result of the full-block  $S$ -procedure, for which the variables  $\Delta_p$  and  $L_{i,j}$  are introduced. Accordingly,  $L(p)^\top W L(p)$ , with  $L(p) = L_{2,2} + L_{2,1} \Delta_p (I - L_{1,1} \Delta_p)^{-1} L_{1,2}$ , results in an LMI that is quadratic in  $p$ , which one may recognize from the design of an LPV-LQR controller, with the decision variables  $\tilde{Z}^{-1}$  and  $\mathcal{Y}$  as the Lyapunov matrix and the transformed controller gains, respectively. The closed-loop LPV system is characterized in a fully data-driven way by  $\vec{\mathcal{X}} F_Q$  and conditions (22c), (22d), see [29] for further details on handling noisy data and the conservatism of Proposition 2. Hence, solving the LMI conditions (22) yields, via (24), a terminal data-driven LPV state-feedback controller  $K(p)$ . Moreover, the controller satisfies Condition 3.a with  $V(x) = x^\top (\tilde{Z}^{-1}) x$  as a Lyapunov function for the closed-loop system. Let  $Z = \tilde{Z}^{-1}$  for the remainder of this section. We now present data-based methods for the computation of the terminal set, by exploiting and extending Proposition 2. That is, we present a method to *jointly* synthesize a terminal controller, a  $V_f$ , and an ellipsoidal  $\mathbb{X}_f$  with an extension of Proposition 2 (Section V-C.2), as well as a method to compute a polyhedral  $\mathbb{X}_f$  by exploiting the resulting  $K(p)$  coming from Proposition 2 (Section V-C.3). For this we need the following definition:

**Definition 2** (PC-sets). A compact and convex set with a non-empty interior that contains the origin is called a *PC-set*.  $\square$

Moreover, for the data-based computation of  $\mathbb{X}_f$ , we assume throughout this section that the following properties hold:

**Assumption 2.** *The following properties are satisfied:*

- a) The scheduling set  $\mathbb{P} \subset \mathbb{R}^{n_p}$  is a convex polytope, generated by a finite number of vertices, i.e.,  $\mathbb{P} := \text{co}(\{\mathbf{p}_i^v\}_{i=1}^{n_v})$ , where  $\text{co}$  denotes the convex hull.
- b) The state and input constraint sets  $\mathbb{X}$  and  $\mathbb{U}$  are polyhedrons that are described by

$$\mathbb{X} := \{x \in \mathbb{R}^{n_x} \mid g_i^{x^\top} x \leq h_i^x, i = 1, \dots, n_{gx}\}, \quad (25a)$$

$$\mathbb{U} := \{u \in \mathbb{R}^{n_u} \mid g_i^{u^\top} u \leq h_i^u, i = 1, \dots, n_{gu}\}, \quad (25b)$$

where,  $g_i^x \in \mathbb{R}^{n_x}$ ,  $g_i^u \in \mathbb{R}^{n_u}$ , and  $h_i^x, h_i^u \in \mathbb{R}$ .

This assumption allows to use computationally efficient *linear programming* (LP) or *quadratic programming* (QP) for constructing the terminal ingredients. When we choose the terminal cost  $V_f$  as the Lyapunov function that guarantees stability of the closed-loop with  $K(p)$ , i.e.,  $V_f(\bar{x}_{N_c|k}) = \bar{x}_{N_c|k}^\top Z \bar{x}_{N_c|k}$ , we can guarantee exponential stability in case  $\mathbb{X}_f$  satisfies Condition 3. We will now give two methodologies for the computation of a *maximum positively invariant* (MPI) set  $\mathbb{X}_f$ , i.e., the computation of  $\mathbb{X}_f$  that maximizes the domain of attraction of the LPV-SS-DPC. The first method assumes an ellipsoidal  $\mathbb{X}_f$ , while the second method assumes a polyhedral  $\mathbb{X}_f$ . As in the model-based case, each method has its own advantages and disadvantages [45].

2) *Joint computation of  $V_f$  and an ellipsoidal  $\mathbb{X}_f$* : Given a  $K(p)$  computed using Proposition 2 from which  $\mathcal{Y}$  and  $Z$  are obtained. A candidate ellipsoidal  $\mathbb{X}_f$  that satisfies Condition 3 and Assumption 2 is a sub-level set of  $V_f$  [2], i.e.,

$$\Omega = \{x \in \mathbb{X} \mid x^\top Z x \leq \alpha^2, K(\mathbb{P})x \subseteq \mathbb{U}\}.$$

The scalar  $\alpha$  is a parameter that is used to enlarge the set  $\Omega$ . The MPI ellipsoidal  $\mathbb{X}_f$ , i.e., the maximum  $\Omega$ , for a given  $Z$  and  $K(p)$  can be obtained by maximizing the value of  $\alpha^2$  subject to the constraints  $\Omega \subseteq \mathbb{X}$  and  $K(p)\Omega \in \mathbb{U}$  for all  $p \in \mathbb{P}$ . The maximization can be carried out jointly with the computation of  $Z$  and the terminal controller  $K(p)$  under the maximization of the determinant or trace of  $Z$ , as the volume of the ellipsoid  $\Omega$  is determined by the determinant or the trace of  $Z$ . Hence, we merge the LPV state-feedback synthesis problem of Proposition 2 with the computation of an MPI ellipsoidal  $\mathbb{X}_f$ . Using the Schur complement, the problem can be recasted in terms of the decision variable  $\tilde{Z} (= Z^{-1})$ . We consider here to maximize the logarithm of the determinant of  $\tilde{Z}$  to preserve the convexity of the associated optimization problem [46]. Merging the two problems together with the incorporation of (25), results in the optimization problem:

$$\max_{\tilde{Z}, p \in \mathbb{P}} \log \det(\tilde{Z}) \quad (26a)$$

$$\text{s.t. data-driven synthesis conditions (22),} \quad (26b)$$

$$g_i^{x^\top} \tilde{Z} g_i^x \leq (h_i^x)^2, \quad \forall i \in \mathbb{I}_1^{n_{gx}}, \quad (26c)$$

$$\begin{bmatrix} (h_i^u)^2 & g_i^{u^\top} \mathcal{Y} \begin{bmatrix} I_{n_x} \\ \tilde{Z} \otimes I_{n_x} \end{bmatrix} \\ (*)^\top & \tilde{Z} \end{bmatrix} \succeq 0, \quad \forall i \in \mathbb{I}_1^{n_{gu}}. \quad (26d)$$

Note that (26b) ensures invariance, whereas (26c) and (26d) ensure constraint satisfaction.

We have now presented a fully data-based method for the computation of the terminal ingredients, with an ellipsoidal  $\mathbb{X}_f$ , which is in-fact rather simple to compute. Note that in the

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### Algorithm 3: Data-based polyhedral MPI set

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1 initialization: Set  $\nu = 0$  and  $\Omega_0 = \Omega$  as in (27)
2 repeat
3    $\Omega_{\nu+1} \leftarrow \text{pre}_1(\Omega_\nu) \cap \Omega_\nu$ 
4    $\nu \leftarrow \nu + 1$ 
5 until  $\Omega_\nu \supseteq \Omega_{\nu-1}$ 
6 return  $\mathbb{X}_f = \Omega_\nu$ 

```

---

model-based case, this is a common approach [45]. However, in that case, full model knowledge is required for the computation, and the input matrix  $B$  must be scheduling *independent*. Contrary, solving (26) only requires a single sequence of data from an unknown LPV system, whose input matrix can be scheduling *dependent*. The simplicity of the SDP (26) comes however at the cost of conservatism by restricting  $\mathbb{X}_f$  to be ellipsoidal. For this reason, we also discuss the computation of the terminal ingredients with a more flexible terminal set in terms of a *polyhedral* MPI.

3) *Computation of a polyhedral  $\mathbb{X}_f$* : The computation of a polyhedral MPI  $\mathbb{X}_f$  is based on the data-driven LPV representation of the closed-loop, i.e., the data-driven representation of  $\phi_{cl}$ , see also [29, Thm. 1]. Polyhedral invariant sets are more favorable for LP/QP-based predictive control problems, as they are generally more flexible than ellipsoidal sets, leading to a larger domain of attraction. This is, however, at the expense of an increased representation complexity in terms of the number of constraints. Furthermore, contrary to the computation of ellipsoidal MPI sets, computing a polyhedral MPI set is often carried out with LP tools [45]. This implies that the computation algorithm cannot handle quadratic scheduling dependency in  $\phi_{cl}(x, p)$ . This can be avoided by designing a scheduling independent controller  $K(p) = K_0$  (see [29, Rem. 1.ii]). Let us denote the data-driven representation of  $\mathcal{B}^{ss}$  in closed-loop with  $K_0$  by  $\phi_{cl,r}(x, p)$ . Next, for computing the polyhedral MPI set, let the state and input constraints in (25) be rewritten in the compact form  $G_x x \leq h_x$ , and  $G_u u \leq h_u$ , respectively, where  $G_x, h_x, G_u, h_u$  collect the respective vectors  $g_i^x, h_i^x, g_i^u, h_i^u$ . Therefore, the state constraint set of the closed-loop system  $x_{k+1} = \phi_{cl,r}(x_k, p_k)$  ensuring invariance is defined as

$$\Omega = \{x \in \mathbb{R}^{n_x} \mid Gx \leq h\}, \quad (27)$$

where  $G = [G_x^\top \quad (G_u K_0)^\top]^\top$  and  $h = [h_x^\top \quad h_u^\top]^\top$ . Inspired by [45], we define now the data-driven extension of a *one-step admissible pre-image set* for a given  $\Omega$ , which is a PC-set used for the computation of the polyhedral MPI  $\mathbb{X}_f$ :

$$\text{pre}_1(\Omega) := \{x \in \mathbb{R}^{n_x} \mid G\phi_{cl,r}(x, p_i^v) \leq h, \forall i \in \mathbb{I}_1^{n_v}\}.$$

The model-based computation of a polyhedral set for time-varying systems in [45] is extended to the data-driven LPV setting in Algorithm 3, which computes a polyhedral MPI  $\mathbb{X}_f$  for the LPV-SS-DPC scheme using only  $\mathcal{D}_{N_d}^{ss}$ . Note that each iteration of Algorithm 3 requires to remove redundant inequalities of  $\Omega_{\nu+1}$  and subset testing, meaning that complexity of these calculations grows with the number of constraints of the obtained  $\Omega_{\nu+1}$ .

## VI. APPLYING THE LPV-DPC SCHEMES

In this section<sup>6</sup>, we discuss how to effectively apply the proposed LPV-DPC schemes to nonlinear systems or systems with exogenous effects, together with the implementation of the methods under noisy measurements.

### A. External and internal scheduling scenarios

In the LPV framework, the scheduling signal  $p$  is considered to be an *independent* signal, a free variable of the system. This allows linearity of the underlying scheduling-dependent behavior, and this property has been key in the establishment of powerful convex control synthesis and analysis methods on which the LPV framework builds on.

In some applications of LPV systems,  $p$  is indeed an independent signal, such as outside temperature, precipitation, or the effect of an other subsystem in terms of a reference signal or control input in an upper control layer. In these cases, the used LPV model provides an exact representation of the original system and the scheduling is called *external*.

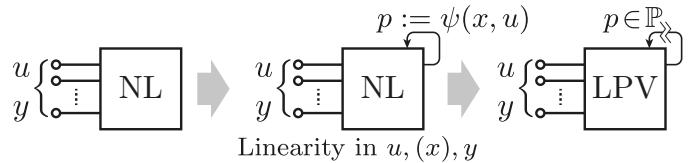
However, in the majority of applied LPV control, LPV descriptions are used as surrogate models of an underlying nonlinear system, where  $p$  is actually a function  $\psi$  of the output, state, and input signals associated with the system. We refer to  $\psi$  as the *scheduling map*. Although, in these cases, the models are sometimes warningly labeled to be *quasi-LPV*, in fact, this *internal* relation is intentionally neglected in the LPV representation, assuming  $p$  to be independently varying in a set containing all possible trajectories of  $\psi(x, u)$  that can occur during operation of the system. This results in the *embedding* of the original nonlinear system in a linear behavior, where the assumed freedom of  $p$  only introduces conservativeness of the representation, i.e., increased size of the solution set, and conservativeness of LPV analysis or synthesis approaches, e.g., upper bounding the true  $\ell_2$ -gain of the system.

In this section, we provide methods to manage Assumption 1 in the LPV-DPC design for the cases when  $p$  is an external signal or when  $p$  is dependent on internal signals. We want to highlight that the *gain-scheduling* approach is applicable for both cases, i.e., when  $p$  is taken to be a constant over the prediction horizon of the LPV-DPC. We consider the cases where  $p$  is not known or measurable outside of the scope of this paper.

**1) The external scheduling scenario:** In this scenario, the future of the scheduling  $p_{[k, k+N_c-1]}$  in the prediction horizon (denoted by  $\bar{p}_{[0, N_c-1] | k}$  in the derivations) is either (i) exactly known upfront (in case of, e.g., the reference signal of the upper control layer), i.e., Assumption 1 is trivially satisfied, or (ii) predicted for  $N_c$  steps in the future based on the current measured value  $p_k$  and its past.

There are many available scheduling prediction methods in the LPV-MPC literature. The majority of these methods can in fact be used in our LPV-DPC schemes. We give a short review of some of the available methods. In [47], [48], an offline-identified linear prediction model is employed to predict the scheduling, while in [32] a recursive least-squares

<sup>6</sup>See the extended version of this paper (on arXiv) for a derivation of the recursive implementations of the LPV-DPC schemes.



**Fig. 2.** Global LPV embedding of a nonlinear system with inputs  $u$  and outputs  $y$  and state realization  $x$ . A scheduling variable  $p := \psi(x, u)$  is defined such that if the trajectory of  $p$  is known, then the remaining signal relations of  $y$  and  $u$  are linear. To obtain an LPV representation, the connection between  $\psi(x, u)$  and  $p$  is severed and  $p$  is assumed to be varying independently from  $u$  in a bounded set  $\mathbb{P} \supseteq \psi(\mathbb{X}, \mathbb{U})$ .

method is proposed. In [49], Gaussian process regression is used to provide effective prediction of the future scheduling trajectory, which allows to characterize the uncertainty or prediction error in the scheduling as well. To take uncertainty of the predicted<sup>7</sup> scheduling trajectories into account, in [50], a tube-based robust LPV-MPC control method is proposed with formal stability and recursive feasibility guarantees. While the former approaches provide prediction methods for externally scheduled setting, the latter approach can be used to incorporate prediction error or uncertainty in our LPV-DPC schemes, which is the subject of future work.

**2) The internal scheduling scenario:** When LPV control is applied to a nonlinear system in the model-based case, an LPV surrogate model of the system is developed based on a *global* LPV embedding of the NL behavior, depicted in Fig. 2, or *local* LPV modeling by means of interpolating local linearizations of the nonlinear system, see, e.g., [1] for more details. For applying our DPC scheme, we consider global embedding of a nonlinear system (Step 1 in Fig. 2) given by

$$x_{k+1} = f(x_k, u_k), \quad y_k = x_k. \quad (28)$$

in the solution set of an LPV representation. The first step is to write (28) as

$$x_{k+1} = f_A(x_k, u_k)x_k + f_B(x_k, u_k)u_k, \quad y_k = x_k. \quad (29)$$

which can always be achieved if  $f$  is continuously differentiable and  $f(0, 0) = 0$ , see [51], where the latter is often ensured by appropriate state-transformation. Then, a scheduling map  $\psi : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{P}$  is defined that gives

$$p_k := \psi(x_k, u_k), \quad (30)$$

such that the resulting  $A(p) = A(\psi(x, u)) = f_A(x, u)$  and  $B(p) = B(\psi(x, u)) = f_B(x, u)$  have a chosen class of functional dependency: affine, polynomial, rational, etc. This allows to write (29) as

$$x_{k+1} = A(p_k)x_k + B(p_k)u_k. \quad (31)$$

corresponding to Step 2 in Fig. 2. Note that (31) and (28) still exhibit the same behavior through (30). To obtain an LPV representation, all allowed trajectories of  $p$  are restricted to a compact (often convex) set  $\mathbb{P} \supseteq \psi(\mathbb{X}, \mathbb{U})$ . Then, it is assumed that  $p$  varies *independently* in the set  $\mathbb{P}$ , which means

<sup>7</sup>The method in [50] can be also used when no prediction of the future trajectory is available, considering a growing tube around the last observed  $p_k$  in the prediction interval, corresponding to the classical, but often rather conservative, setting of LPV-MPC control.

**Algorithm 4:** LPV-SS-DPC for nonlinear systems

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```

1 initialization: set  $k=0$  and  $\{\bar{x}_{i+1|0} = x_0, \bar{u}_{i|0} = 0\}_{i=0}^{N_c-1}$ 
2 loop
3 repeat
4 update  $\bar{p}_{[0,N_c-1]|k}$  by  $\psi(\bar{x}_{[0,N_c-1]|k}, \bar{u}_{[0,N_c-1]|k})$ 
5 solve (18) to obtain  $\bar{x}_{[1,N_c]|k}^*, \bar{u}_{[0,N_c-1]|k}^*$ 
6 set  $\bar{x}_{[1,N_c-1]|k} \leftarrow \bar{x}_{[1,N_c-1]|k}^*$ 
7  $\bar{u}_{[0,N_c-1]|k} \leftarrow \bar{u}_{[0,N_c-1]|k}^*$ 
8 until  $\bar{u}_{[0,N_c-1]|k}$  has converged
9 apply  $u_k = u_{0|k}$  and observe  $x_{k+1}$ 
10 set  $k \leftarrow k + 1$ 
11 set  $\bar{x}_{[1,N_c-1]|k} \leftarrow \bar{x}_{[2,N_c]|k-1}$  and  $\bar{x}_{0|k} = x_k$ 
12  $\bar{u}_{[0,N_c-2]|k} \leftarrow \bar{u}_{[1,N_c-1]|k-1}$  and  $\bar{u}_{N_c-1|k} = 0$ .
13 end loop
```

---

that the actual relation between  $\psi(x, u)$  and  $p$  in (30) is disregarded. Due to the assumed independence of  $(u, p, x)$ , the behavior  $\mathfrak{B}_{NL}$  associated with (31) satisfies  $\mathfrak{B}_{NL} \subseteq \pi_{(x,u)}\mathfrak{B}$ , corresponding to conservativeness of the representation as a price for linearity, but the original behavior  $\mathfrak{B}_{NL}$  is always contained in  $\mathfrak{B}$ .

Based on a given  $\psi$ , we will now provide a method to tackle Assumption 1, i.e., obtain  $\bar{p}_{[0,N_c-1]|k}$  for the prediction horizon. A simple approach is known as *gain-scheduled* predictive control, where  $\bar{p}_{[0,N_c-1]|k}$  is set to be equal with  $p_k$  for all  $k \in \mathbb{I}_0^{N_c-1}$ . While this method often works unexpectedly well in practice, it disregards variation of  $A$  and  $B$  through the  $\psi$ -represented connection with the optimization variables. An approach that is in line with the concept of the global embedding is to iteratively synthesize the scheduling trajectory as proposed in [5]. The approach in [5] calculates the scheduling trajectory over  $N_c$  based on the solution of  $\bar{x}$  and  $\bar{u}$  obtained in a previous iteration, iteratively synthesizing  $(\bar{u}, \bar{p}, \bar{x})$  through a sequence of QPs. The core idea of this scheme is extended to the data-driven setting in terms of Algorithm 4. By seeing the iterative procedure as inexact Newton steps for a root finding problem, [52] shows under assumptions similar to those for standard SQP methods that the sequence of QPs has a local contraction property around a feasible, suboptimal solution of the LPV-DPC problem.

Additionally, for the IO scheme under LPV embedding of a nonlinear system, the setpoint equilibrium  $(u^r, p^r, y^r)$  can be determined based on only  $y^r$  by combining Remark 2 and a simplified version of Algorithm 4, as  $p^r$  is determined by  $y^r$  through  $\psi$ .

### B. Working with noisy data

In practical applications of data-driven control, measurement noise inevitably affects the behavior of closed-loop systems. Next, we propose a *robust* modification of the LPV-DPC scheme using slack variables and regularization terms, which makes it possible to handle disturbances in the data-driven LPV representation. We only discuss the modification of the state-feedback case, but the same arguments are applicable to the output-feedback case.

Suppose that the system is affected by a bounded disturbance signal in an ARX-type setting<sup>8</sup>, both in the offline collected data  $\check{x}_{[1,N_d]}$  entering the Hankel matrices in Proposition 1, as well as in the online measured  $x_k$  – used as initial condition. More precisely, we can only measure

$$\check{x}_{[1,N_d]} = \check{x}_{[1,N_d]} + \check{\varepsilon}_{[1,N_d]}, \quad \text{and} \quad z_k = x_k + \varepsilon_k$$

with a  $\varepsilon_k$  with uniform distribution  $\varepsilon_k \sim \mathcal{U}(-\varepsilon_{\max}, \varepsilon_{\max})$  for all  $k \in \mathbb{Z}$ , which gives that  $\|\check{\varepsilon}_{[1,N_d]}\|_\infty \leq \varepsilon_{\max}$  and  $\|\varepsilon_k\|_\infty \leq \varepsilon_{\max}$ . We assume that the measurements of the scheduling  $\check{p}_k$  in  $\mathcal{D}_{N_d}$ , as well as the online measurements  $p_k$ , are noise-free.

In the literature, two main modifications have been proposed to cope with noise in LTI-DPC. First, adding an additional slack variable to the data-dependent equality constraints in the prediction model (11) was used for deriving closed-loop guarantees [18]. Inspired by this, we replace the nominal data-driven predictor from (11) by the following robust one:

$$\begin{aligned} & \begin{bmatrix} \mathcal{H}_1(\check{z}_{[1,N_d-N_c]}) \\ \mathcal{H}_{N_c}(\check{z}_{[2,N_d]}) \\ \mathcal{H}_{N_c}(\check{u}_{[1,N_d-1]}) \\ \mathcal{H}_{N_c}(\check{z}_{[1,N_d-1]}^{\check{p}}) - \mathcal{P}_k^{n_x} \mathcal{H}_{N_c}(\check{z}_{[1,N_d-1]}) \\ \mathcal{H}_{N_c}(\check{u}_{[1,N_d-1]}^{\check{p}}) - \mathcal{P}_k^{n_u} \mathcal{H}_{N_c}(\check{u}_{[1,N_d-1]}) \end{bmatrix} g_k \\ &= \begin{bmatrix} z_k + \sigma_{0|k} \\ \text{vec}(\bar{x}_{[1,N_c]|k}) + \sigma_{[1,N_c]|k} \\ \text{vec}(\bar{u}_{[0,N_c-1]|k}) \\ \sigma_{[N_c+1,2N_c]|k} \\ 0 \end{bmatrix}. \quad (32) \end{aligned}$$

Here, with a slight misuse of notation, the vector  $\sigma_{[0,2N_c+1]|k} \in \mathbb{R}^{n_x(2N_c+1)}$  is a slack variable that is optimized online in order to relax the equality constraints and ensure feasibility. The main goal of adding the slack variable is to compensate the influence of the noise. To avoid an overly large prediction error due to the noise, the slack variable is regularized in the cost, i.e., we add a term  $+\lambda_\sigma \|\sigma_{[0,2N_c+1]|k}\|_2^2$ . In the LTI case, the regularization parameter  $\lambda_\sigma > 0$  is required to scale inversely with the noise level in order to prove practical stability, and we conjecture a similar connection in the LPV case. The second modification is an additional regularization of  $g_k$  that robustifies the DPC scheme against noise [33]. Thus, for the proposed robust LPV-DPC scheme, we add the term  $\lambda_g \|g_k\|_2^2$  to the cost. Conversely to the regularization of the slack variable, the regularization parameter  $\lambda_g > 0$  needs to scale directly (and not inversely) with the noise level to ensure theoretical guarantees. In particular, for zero noise, the regularization is not required. Note that the proposed regularization can be further refined. For example, it can be modified to the form of the  $\Pi$ -regularization [54], which further reduces bias in the predictor.

In summary, the following optimization problem defines the robust LPV-DPC scheme to control unknown LPV systems

<sup>8</sup>Following the reasoning in [53], we can also handle other noise structures, e.g., Box-Jenkins, by increasing the order, i.e.,  $\tau$ .

based on noisy data.

$$\begin{aligned} \min_{g_k, \sigma_{[0,2N_c+1]k}} & V_f(\bar{z}_{N_c|k} - x^r) + \sum_{i=0}^{N_c-1} \ell(\bar{u}_{i|k}, \bar{z}_{i|k}) \\ & + \lambda_g \|g_k\|_2^2 + \lambda_\sigma \|\sigma_{[0,2N_c+1]k}\|^2 \end{aligned} \quad (33a)$$

$$\text{s.t. } (32) \text{ holds and } \bar{z}_{0|k} = z_k, \quad (33b)$$

$$\bar{u}_{i|k} \in \mathbb{U}, \quad \forall i \in \mathbb{I}_0^{N_c-1}, \quad \bar{z}_{N_c|k} \in \mathbb{X}_f. \quad (33c)$$

Note that problem (33) does not contain state constraints, which would require an additional constraint tightening due to the uncertain predictions (cf. [55] for an output constraint tightening in robust LTI-DPC). Furthermore, [29] also presents an extension of Proposition 2 with noisy data, allowing for data-driven computation of the terminal ingredients with noisy data. In the LTI case with noisy data, DPC with terminal cost and terminal region constraints can be shown to provide practical exponential stability guarantees (see [56] for the main argument based on inherent robustness). Here, practical exponential stability means that the closed-loop exponentially converges to a set around the setpoint, the size of which depends on the noise level [57]. Using an analogous approach to prove practical exponential stability of the proposed robust LPV-DPC scheme as well as handling noisy scheduling data are important objectives of future research.

## VII. EXAMPLE: UNBALANCED DISC SYSTEM

### A. Setup

We demonstrate the applicability<sup>9</sup> of our results on a simulator of an unbalanced disc system. This system consists of a disc containing an off-centered mass, whose angle can be controlled by a DC motor. The nonlinear dynamics, discretized using the Euler method with a sampling-time of  $T_s = 0.02$  seconds, are given by the state-space realization

$$\theta_{k+1} = \theta_k + T_s \omega_k, \quad (34a)$$

$$\omega_{k+1} = (1 - \frac{T_s}{\tau_m})\omega_k + \frac{T_s m g l}{J_m} \sin(\theta_k) + \frac{T_s K_m}{\tau_m} u_k, \quad (34b)$$

where  $\theta_k$  is the angular position in radians,  $\omega_k$  is the angular speed in radians per second and  $u_k$  the voltage in volts supplied to the DC motor. Moreover,  $m, g, l, J_m, K_m$  and  $\tau_m$  are the model parameters<sup>10</sup>. Note that  $\theta_k \equiv 0$  corresponds to the upright position of the unbalanced disc, which is an unstable equilibrium. We can embed (34) as an LPV-SS or LPV-IO representation by choosing<sup>11</sup>  $p_k = \text{sinc}(\theta_k)$ , from which we immediately see that  $\mathbb{P} := [-0.22, 1]$ . For this system, we will compare our LPV-IO-DPC method with model-based IO methods and DeePC for noiseless and noisy data. To determine the future scheduling trajectory in the prediction horizon we compare the *gain-scheduling* (GS) approach with the *iterative scheduling estimation* (ISE) approach via Algorithm 4. Furthermore, we take  $\theta_k$  to be the measured

<sup>9</sup>Software implementation of the methods and the simulation environment are available at: <https://gitlab.com/releases-c-verhoeck/lpvdp>

<sup>10</sup>The model (34) of the system and parameter values taken from [58] are only used for the model-based designs, data-generation, and validation of the controllers and not for the design of the data-driven controllers.

<sup>11</sup>Note that the underlying (considered to be unknown) system is not an LPV system, but a nonlinear system, which requires that the map  $\psi$  in (30) is assumed to be known.

output of the system. The input and output constraints are defined as  $u_k \in [-10, 10] = \mathbb{U}$  and  $\theta_k \in [-\pi, \pi] = \mathbb{Y}$ . The objective is to regulate the system to  $(\theta_r, u_r) = (\frac{\pi}{8}, -1.77)$ , where  $u_r$  is obtained as in Remark 2. We first show the comparison on noise-free data, and finalize this example by demonstrating the performance of the LPV-IO-DPC with noise corrupted measurements. We want to emphasize that all the LPV-DPC designs discussed in this section are accomplished using only  $\mathcal{D}_{N_d}$  and a set of tuning parameters.

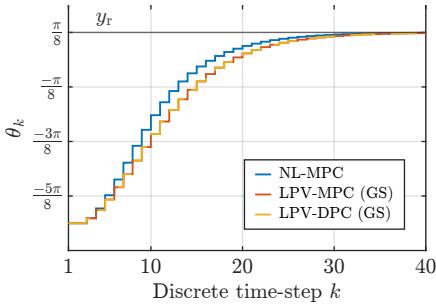
### B. Noise-free data

1) *Comparison with model-based methods:* We first compare our LPV-IO-DPC methods to nonlinear IO-MPC (NL-MPC) and LPV-IO-MPC methods. For a fair comparison, we consider the same predictive control setup, except for the system representation. Hence, we use the same design parameters, and use the terminal equality constraints (i.e., (12d)) as terminal ingredients for all the schemes. The LPV schemes are solved with GUROBI, while the NL-MPC is solved with IPOPT. The design parameters are as follows. We choose  $N_c = 20$ ,  $\tau = 2$  and the cost function is parametrized with  $Q = R = 1$ . The terminal equality constraints are such that  $(y_r, u_r) = (\theta_r, u_r)$ . Because the underlying system is nonlinear, the scheduling in the LPV schemes is *dependent* on  $y_r$ . To account for a possible mismatch in the prediction such that the terminal equality constraint yields the optimization problem infeasible, we have added a slack variable to the terminal output equality constraint, i.e.,  $\bar{y}_{N_c-i|k} = y_r + \sigma_i$ ,  $i \in \mathbb{I}_1^T$ . The slack variable is penalized by adding  $+10\|\sigma\|^2$  in the cost function. For the LPV schemes with the GS case, the scheduling signal is computed with the current measurement, i.e.,  $\bar{p}_{i|k} = \text{sinc}(y_k)$  for  $i \in \mathbb{I}_0^{N_c-1}$ , while for the ISE case we follow Algorithm 4.

The data-dictionary for the LPV-IO-DPC scheme is generated by applying  $u_k \sim \mathcal{U}(\mathbb{U})$  to the system for  $N_d$  time steps. Based on Proposition 1, we have that for  $N_c = 20$ ,  $\tau = 2$ ,  $N_d \geq 89$ , and thus chose  $N_d = 89$ . A posteriori verification of (9) gives that the obtained  $\mathcal{D}_{N_d}^{\text{IO}}$  fully represents  $\mathfrak{B}^{\text{IO}}|_{[1, N_c+\tau]}$ . Solving the predictive control problems yields the simulation results in Fig. 3.

From the simulation results, we can conclude that, for noise-free data, the LPV-MPC and LPV-DPC schemes are equivalent. This is expected because  $\mathcal{D}_{N_d}^{\text{IO}}$  is able to fully represent the system behavior over the prediction horizon. However, one big advantage of the data-based scheme over the model-based scheme is that the former only requires a data-set for the formulation of the predictive control scheme, while the latter requires an accurate model with exact knowledge of the model parameters  $m, g, l, J_m, K_m, \tau_m$ . Furthermore, when compared to NL-MPC, which is solved as a nonlinear optimization problem, we see that the LPV-DPC achieves a similar performance. In fact, when the ISE method is used, the performance is even slightly better. The difference in performance between the GS and ISE approach is because the ISE method *synthesizes* a scheduling sequence that corresponds to the planned solution of  $y$  and  $u$  in the prediction horizon.

2) *Comparison with DeePC:* We will now demonstrate the advantages of our methods over DeePC [17] by means of



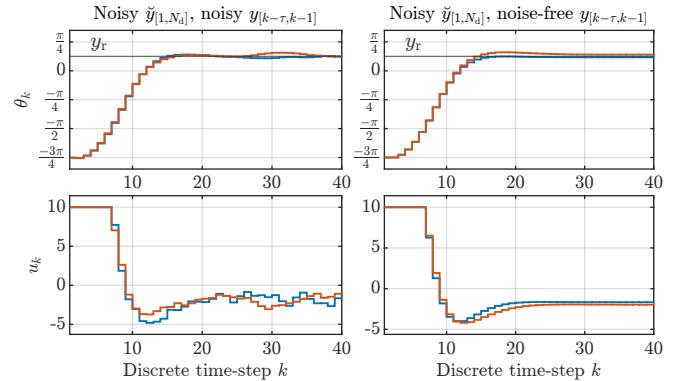
**Fig. 3.** Simulation results for the comparison of predictive controllers that use IO measurements on the unbalanced disc system. We compare here a NL-MPC (—), an LPV-MPC (—) and an LPV-DPC (—). The left plot shows the results where the LPV predictive controllers use GS to determine the scheduling in the prediction horizon, while the right plot shows the result where the LPV predictive controllers use ISE for this purpose. The simulation results show that our LPV-DPC method results in a similar performance as nonlinear and LPV model-based approaches.

**Fig. 4.** Comparison of the simulation results on the unbalanced disc system with a DeePC controller (—) and our LPV-IO-DPC controller (---) with different reference points. The further away  $\theta_r$  is from the stable equilibrium  $\theta_k = 0$ , the more the DeePC performance degrades.

a comparison on the unbalanced disc system with IO measurements. Note that DeePC has been introduced for LTI systems, hence when we tried to directly design a DeePC controller with  $\mathcal{D}_{N_d}^{\text{IO}}$  used in Section VII-B.1, we did not manage to obtain a stabilizing DeePC controller. Therefore, we generated a new data-set with  $\theta_k \equiv 0$  as the downward vertical position, i.e., the stable equilibrium, and with  $\bar{u}_k \in [-2, 2]$ . This setting resulted in  $\dot{\theta}$  staying close to 0. For the controller design, we added a regularization on  $g_k$  with  $\lambda = 0.01$  to the cost function and modified the stage cost to  $\ell = (*)^\top Q(\bar{y}_{i|k} - y^r) + (*)^\top R(\bar{u}_{i|k} - \bar{u}_{i-1|k})$  with  $Q = 1$ ,  $R = 0.1$ . For a fair comparison, we used the same stage cost in the LPV-IO-DPC controller (the extra regularization on  $g$  was not included). The scheduling prediction over the control horizon, i.e., Assumption 1, is achieved with GS. We simulated the closed-loop system for the reference points  $\theta_r = \{\frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\}$ , i.e., increasingly further away from the origin. The simulation results are depicted in Fig. 4, where the dotted lines represent the reference points, the dashed lines represent the simulations with the LPV-IO-DPC controller and the solid lines represent the simulations with the DeePC controller. The plot shows that close to  $\theta = 0$ , both controllers have a similar performance. Further away from the origin, however, the DeePC controller cannot handle the nonlinear effects of the unbalanced system behavior and even becomes unstable for  $\theta_r \geq \frac{3\pi}{8}$ . On the other hand, the LPV-IO-DPC controller effortlessly regulates the nonlinear system to the setpoint, further emphasizing the advantages of using LPV-DPC over DeePC for data-driven control of nonlinear systems.

### C. Handling noisy measurements

We also demonstrate the performance with noisy measurements for a GS LPV-IO-DPC with the modifications discussed in Section VI-B and compare it with the classic two-step approach, i.e., identification of an LPV-IO model followed by LPV-MPC design. For the data-generation, we add a white noise signal  $e_k \sim \mathcal{U}(-\varepsilon_{\max}, \varepsilon_{\max})$  to the output in an LPV-ARX setting. We choose  $\varepsilon_{\max} = 0.01$ , which corresponds to an angular error of at most 3.6 degrees and a signal-to-noise ratio of 10 dB, calculated w.r.t. the noise process. We again take  $N_d = 89$ . Note that the noisy output signal



**Fig. 5.** Comparison of the simulation results with the indirect two-step approach (—) and the LPV-IO-DPC (—). For the left two plots, both  $\bar{y}$  and  $y_{[k-\tau, k-1]}$  are affected by noise, while for the right two plots only  $\bar{y}$  is noisy and  $y_{[k-\tau, k-1]}$  is measured noise free.

also propagates through the model equations, corresponding to an ARX model structure. Additionally, for illustration, we consider  $p_k := \text{sinc}(y_k + e_k)$ , i.e., a noisy measurement of  $p_k$ .

For the two-step approach, we identify an LPV-ARX model with  $n_a = n_b = \tau$ . Using the standard settings in LPVCORE<sup>12</sup>, we identify an LPV-ARX model based on  $\mathcal{D}_{N_d}^{\text{IO}}$  ( $N_d = 89$ ) with `lpvarx`. The identified model is used as a  $N_c$ -step-ahead predictor in a standard LPV-MPC scheme with the same terminal equality constraints. We take  $Q = 10$  and  $R = 0.05$ , while all other design parameters are the same as for the noise-free design. Additionally, for the LPV-IO-DPC design, we tuned the regularization parameters to  $\lambda_\sigma = 10^9$  and  $\lambda_g = 0.01$ .

We compare the two designs in two scenarios where we regulate the unbalanced disc to  $(y_r, u_r) = (\frac{\pi}{8}, -1.77)$ . In the first scenario, we have next to the noisy  $\mathcal{D}_{N_d}^{\text{IO}}$ , also noisy online measurements of  $y$ . The simulation results for this scenario are shown in the left two plots of Fig. 5. To compare the performance of the two designs for handling a noisy data-dictionary, we have a second scenario, where the online measurements of  $y$  are *noise-free*. These results are shown in the right two plots of Fig. 5. On average, the added

<sup>12</sup>LPVCORE is an open-source MATLAB toolbox, see [lpvcore.net](http://lpvcore.net).

regularization terms and variables increased the computation time by 6 milliseconds, which corresponds to a 20% increase compared to the noise-free case.

The results show that both the two-step design and the LPV-IO-DPC design can regulate the unbalanced disc to a setpoint close to the upright position. What can be observed is that for the LPV-IO-DPC design, the regularization component seems to pull the input to zero. This phenomenon has been reported in multiple works on noise handling in LTI-DPC. Whether this effect can be negated by using more advanced regularization, e.g., [33], for LPV-DPC requires further investigation.

## VIII. CONCLUSIONS

In this work, we have derived novel output-feedback and state-feedback-based direct data-driven LPV predictive control schemes using the LPV Fundamental Lemma, which allows to construct a fully data-based predictor of an unknown LPV system. Methods for computation of terminal ingredients purely based on data are also provided, ensuring exponential stability and recursive feasibility of the closed-loop system regulated by the proposed DPC designs. Furthermore, effective approaches for handling noise and disturbances in the data are derived, together with methods for determining the scheduling sequence in case the approach is applied for nonlinear systems or systems dependent on exogenous effects. Through comparison of the developed DPC schemes with nonlinear, LPV and LTI MPC solutions and the LTI DeePC scheme on an unbalanced disc system, we have demonstrated that our proposed method, without the need of a modeling step, can achieve the same high performance as LPV and nonlinear predictive controllers which use an exact model of the system, while the LTI DeePC scheme was not able to stabilize the nonlinear system. As a future work, we aim to develop statistically efficient handling of noise in the proposed LPV DPC schemes and, via the concepts detailed in [59], provide reference tracking under global stability and performance guarantees for nonlinear systems.

## APPENDIX

### A. Proof of Lemma 1

For notational brevity, we denote  $\|\cdot\|_2$  by  $\|\cdot\|$ ,  $p_k$  by  $p$ , and  $p_{k+1}$  by  $p^+$  throughout this proof, similarly for  $x$ .

*Proof.* As we consider behaviors for which there exists a state-minimal LPV-SS representation, we know that the corresponding behavior is controllable [38]. Due to minimality and affine scheduling dependence, the LPV-SS realization is detectable, in fact, it is completely state-observable [1], [38]. This implies that there exists a parameter-varying gain matrix  $L : \mathbb{P} \rightarrow \mathbb{R}^{n_x \times n_y}$  and a positive definite  $P : \mathbb{P} \rightarrow \mathbb{R}^{n_x \times n_x}$ , i.e.,  $P(p) \succ 0$  for all  $p \in \mathbb{P}$ , such that

$$P(p) - (A(p) + L(p)C(p))^\top P(p^+) (A(p) + L(p)C(p)) = I, \quad (35)$$

for all  $p, p^+ \in \mathbb{P}$ . Then, with the candidate IOSS Lyapunov function  $V(x, p) = x^\top P(p)x$ , we have the analog of [37, Eq. (12)]:

$$V(x^+, p^+) - V(x, p) = (x^+)^{\top} P(p^+) x^+ - x^\top P(p)x \quad (36)$$

$$\begin{aligned} &= (*)^\top P(p^+) ((A(p) + L(p)C(p))x + B(p)u - L(p)y) \\ &\quad - x^\top P(p)x \end{aligned} \quad (37)$$

$$\begin{aligned} &= x^\top (A(p) + L(p)C(p))^\top P(p^+) (A(p) + L(p)C(p))x \\ &\quad + u^\top B^\top (p)P(p^+)B(p)u + y^\top L^\top (p)P(p^+)L(p)y \\ &\quad - x^\top P(p)x + 2x^\top (A(p) + L(p)C(p))^\top P(p^+)B(p)u \\ &\quad - 2x^\top (A(p) + L(p)C(p))^\top P(p^+)L(p)y \\ &\quad - 2u^\top B^\top (p)P(p^+)L(p)y \end{aligned} \quad (38)$$

$$\begin{aligned} &\leq \|u\|^2 \|B(p)\|^2 \|P(p^+)\| + \|y\|^2 \|L(p)\|^2 \|P(p^+)\| \\ &\quad - \|x\|^2 + 2\|u\| \|B(p)\| \|P(p^+)\| \|L(p)\| \|y\| \\ &\quad + 2\|x\| \|A(p) + L(p)C(p)\| \|P(p^+)\| \|B(p)\| \|u\| \\ &\quad + 2\|x\| \|A(p) + L(p)C(p)\| \|P(p^+)\| \|L(p)\| \|y\|. \end{aligned} \quad (39)$$

We can upper bound the left-hand side of the inequality using (35) and  $2ab \leq \frac{1}{4}a^2 + 4b^2$  by defining the constants:

$$\begin{aligned} \mu_A &= \max_{p \in \mathbb{P}} \|A(p) + L(p)C(p)\|, & \mu_B &= \max_{p \in \mathbb{P}} \|B(p)\|, \\ \mu_P &= \max_{p \in \mathbb{P}} \|P(p)\| = \max_{p^+ \in \mathbb{P}} \|P(p^+)\|, & \mu_L &= \max_{p \in \mathbb{P}} \|L(p)\|, \end{aligned}$$

such that an upper bound of (39) and thus (36) is of the form:

$$\begin{aligned} (36) \leq (39) &\leq \mu_B^2 \mu_P \|u\|^2 + \mu_L^2 \mu_P \|y\|^2 \\ &\quad + 2\mu_A \mu_B \mu_P \|x\| \|u\| + 2\mu_A \mu_L \mu_P \|x\| \|y\| \\ &\quad - \|x\|^2 + 2\mu_B \mu_P \mu_L \|u\| \|y\| \\ &\leq -\frac{1}{2}\|x\|^2 + (4\mu_A^2 \mu_P^2 + \frac{5}{4}\mu_P) \mu_B^2 \|u\|^2 \\ &\quad + (4\mu_A^2 \mu_P^2 + 5\mu_P) \mu_L^2 \|y\|^2 \\ &= -\frac{1}{2}\|x\|^2 + c_1 \|u\|^2 + c_2 \|y\|^2, \end{aligned} \quad (40)$$

which is the upper bound in (14). ■

### B. Proof of Theorem 2

*Proof.* For proving recursive feasibility (i), we define a candidate input for the optimization problem (18) by shifting the previously optimal solution and appending it by the terminal controller from Condition 3:  $\bar{u}_{i|k+1} = \bar{u}_{i+1|k}^*$  with  $i \in \mathbb{I}_0^{N_c-2}$  and  $\bar{u}_{N_c-1|k+1} = K(\bar{p}_{N_c-1|k})\bar{x}_{N_c|k}^*$ . By Assumption 2, this input and the corresponding state trajectory  $\bar{x}_{i|k+1}$  with  $i \in \mathbb{I}_1^{N_c}$  satisfy constraints (18c)–(18d). Furthermore, by Proposition 1 and the PE assumption, there exists  $g_{k+1}$  satisfying (11) such that all constraints of Problem (18) are satisfied. Constraint satisfaction (ii) follows trivially from recursive feasibility.

Finally, to prove exponential stability (iii), we use the above considered candidate solution to arrive at

$$\begin{aligned} J_{N_c}^*(k+1) - J_{N_c}^*(k) &\leq -\ell(\bar{u}_{0|k}^*, \bar{x}_{0|k}) + V_f(\bar{x}_{N_c|k+1}) \\ &\quad + \ell(K(\bar{p}_{N_c|k})\bar{x}_{N_c|k}^*, \bar{x}_{N_c|k}) - V_f(\bar{x}_{N_c|k}^*), \end{aligned}$$

where  $J_{N_c}^*(k) := J_{N_c}^*(x_k, p_k, \bar{p}_{[1, N_c]|k})$ . Using Condition 3, we obtain  $J_{N_c}^*(k+1) - J_{N_c}^*(k) \leq -\ell(\bar{u}_{0|k}^*, \bar{x}_{0|k})$ . Furthermore, the following lower bound trivially holds:  $J_{N_c}^*(k) \geq \underline{\lambda}(Q)\|\bar{x}_{0|k}\|^2$ , for any  $\bar{x}_{0|k} \in \mathbb{R}^{n_x}$ . Finally, for any  $x \in \mathbb{X}_f$ , the following upper bound holds  $J_{N_c}^*(k) \leq V_f(\bar{x}_{0|k}) \leq \bar{\lambda}(Z)\|x\|^2$ . This can be extended to a quadratic upper bound on the set of all feasible initial states, see [2, Prop. 2.16]. In conclusion, using standard Lyapunov arguments, the origin is exponentially stable for the closed-loop system. ■

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