Towards data-driven control of general nonlinear systems with stability and performance guarantees

IfA Coffee Talk, October 5^{th} 2023

Chris Verhoek, Roland Tóth

Control Systems group, Dept. Electrical Engineering, Eindhoven University of Technology, The Netherlands

A few things about me

Eindhoven University of Technology (TU/e)

MSc in Systems & Control (TU/e)

PhD @ Control Systems group (EE) since Feb. ‘21

- Roland Tóth & Sofie Haesaert

A few things about me

Hiking (multi-day trails)

Drumming (jazz)

Swimming

Motivation

Dynamic systems in engineering

- Increasing performance requirements
- Surge of system complexity
- Nonlinear (NL) behavior is becoming dominant

Industrial practice:

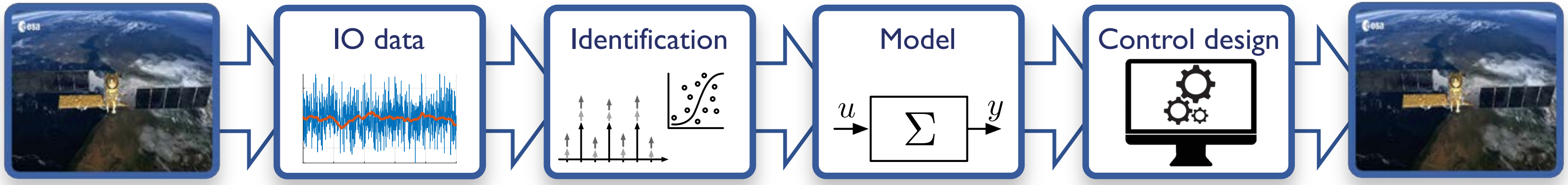
- Linear Time-Invariant (LTI) framework
  - Systematic tools for shaping performance
  - Small operating range
- Need for an NL framework
  - Stability guarantees, but (in general) no performance shaping
  - Non-convex, cumbersome tools

Toolchain based on models

First-principles modeling + model-based control

- Control design with stab. & perf. guarantees (LTI, LPV, etc.)
- Complex, inaccurate, costly modelling
- Effect of unmodelled dynamics on the design

Identify system model + model-based control



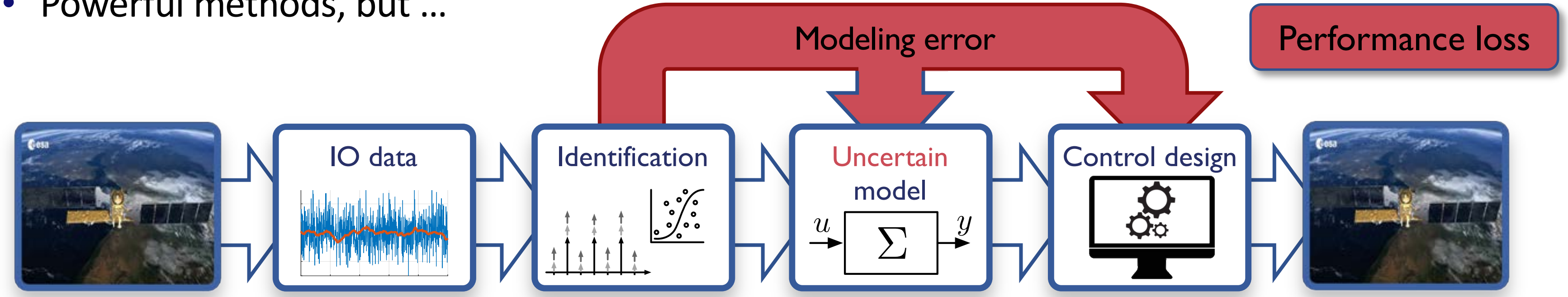
Toolchain based on models

First-principles modeling + model-based control

- Control design with stab. & perf. guarantees (LTI, LPV, etc.)
- Complex, inaccurate, costly modelling
- Effect of unmodelled dynamics on the design

Identify system model + model-based control

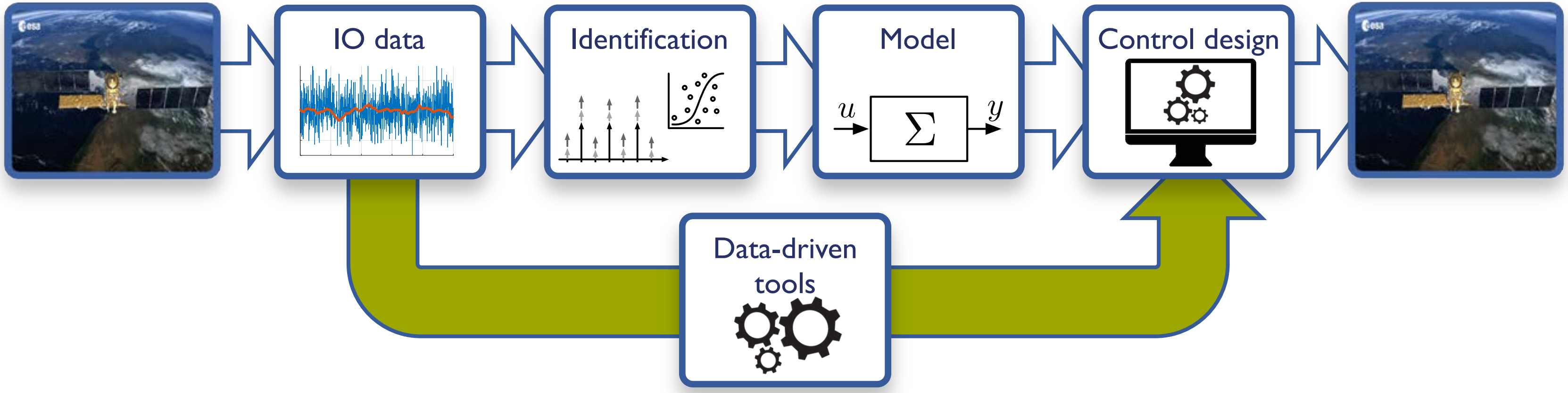
- Powerful methods, but …



Direct data-driven control

Direct data-driven analysis and control design

- Joint design with guarantees
- Promising approaches



Direct data-driven control

LTI approaches

- Frequency-domain methods
  - PID tuning [1]
  - Nyquist stability (conservative) [2]
  - Nyquist stability (necessary & sufficient) [3]
  - MIMO stab. through approximation [4]
- Time-domain methods
  - Virtual-feedback reference tuning (VFRT) [5]
  - Non-iterative correlation-based tuning (CbT) [6]
  - Behavioral methods [7,8]
  - Many more …

LPV approaches

- Time-domain approaches
  - VFRT methods [9]
- Frequency-domain
  - Nyquist-based, conservative [10]
- Behavioral [12]

NL approaches

- Sector bounded static nonlinearites [13]
- Behavioral (LTI+, Wiener & Ham.) [14-16]

How to address NL systems systematically and give guarantees?

[1] K. Aström, et al., ECC, 2013
[2] S. Khadraoui, et al., Automatica, 2014
[3] A. Karimi et al., Int. J. Rob. Cont., 2018
[4] A. Karimi et al., Automatica 2017
[5] M. Campi, et al., Automatica, 2002
[6] van Heusden, et al. Int. J. ACDS, 2011
[7] Markovsky, Dörfler, Ann.R. Cont.,2022
[8] Coulson, et. al., ECC, 2019
[9] Formentin et al. Automatica, 2016
[10] Kunze et. al, ECC , 2007
[11] Bloemers et. al., IEEE-LCSS, 2022
[12] Verhoek et.al., IEEE TAC 2022
[13] Nicoletti et al., J. Rob. Cont., 2018
[14] Alsalti ,et. al., IEEE TAC, 2023
[15] Mishra, et. al., ESPC, 2021
[16] Berberich, et. al., ECC, 2021

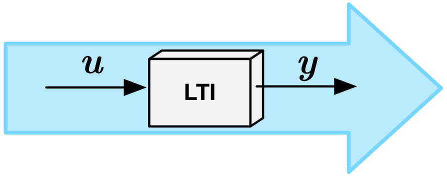
Table of contents

- Behavioral LTI data-driven control
    - LTI behavioral theory
    - Data-driven LTI behavioral representation
    - Data-driven LTI behavioral control
- Behavioral LPV data-driven control
- Behavioral NL data-driven control
- Conclusions

LTI behavioral theory

Behavioral concept (discrete time)

Linearity



Time axis: $\mathbb{T} = \mathbb{Z}$

Behavior: $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$

- Linearity: $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$
- Time invariance: $\mathfrak{B} = q^{\tau}\mathfrak{B}, \forall \tau \in \mathbb{Z}$

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

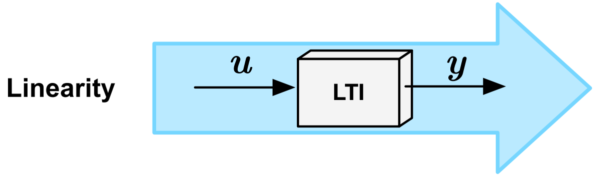
Signal space: $\mathbb{W} = \mathbb{R}^{n_{\mathrm{w}}}$

$$w = \mathrm{col}(u, y)$$

[16] Polderman, Willems: Int. to Mathematical Systems Theory: A Behavioral Approach, *Springer* (1998)

LTI behavioral theory

Kernel representation (discrete time)



Coefficients: $r_i \in \mathbb{R}^{n \times n_\mathrm{w}}$

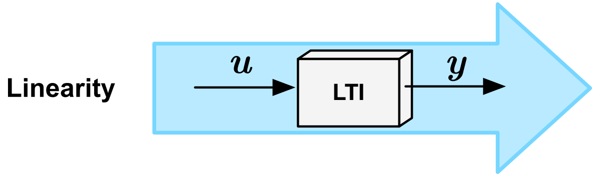
$$R(q)w = \sum_{i=0}^{n_\mathrm{r}} r_i q^i w = 0$$

Forward time-shift: $q^i w_k = w_{k+1}$

[16] Polderman, Willems: Int. to Mathematical Systems Theory: A Behavioral Approach, *Springer* (1998)

LTI behavioral theory

Kernel representation (discrete time)



$$R(q)w = \sum_{i=0}^{n_\mathrm{r}} r_i q^i w = 0$$

Existence of full row-rank kernel representation

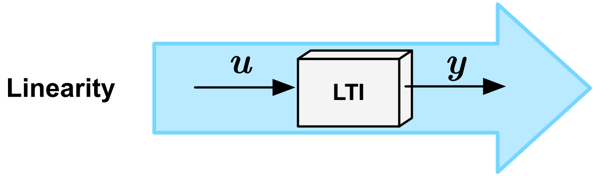
is the representation of the LTI system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ if

$$\mathfrak{B} = \{w \in (\mathbb{R}^{n_\mathrm{w}})^{\mathbb{Z}} \mid R(q)w = 0\}$$

[16] Polderman, Willems: Int. to Mathematical Systems Theory: A Behavioral Approach, *Springer* (1998)

Data-driven LTI behavioral representation

Data-driven representation (discrete time)



$$\mathcal{D}_N = \{\underbrace{u_k^{\mathrm{d}}, y_k^{\mathrm{d}}}_{w_k^{\mathrm{d}}}\}_{k=1}^{N}$$

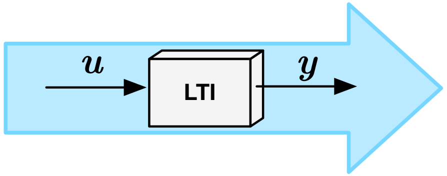
(data dictionary)

[17] Willems et al.: A note on persistency of excitation, *Systems & Control Letters*, (2005).

Data-driven LTI behavioral representation

Data-driven representation (discrete time)

Linearity



Hankel matrix (L-row):

$$\mathcal{H}_L(w^{\mathrm{d}}) = \begin{bmatrix} w_1^{\mathrm{d}} & w_2^{\mathrm{d}} & \cdots & w_{N-L+1}^{\mathrm{d}} \\ w_2^{\mathrm{d}} & w_3^{\mathrm{d}} & \cdots & w_{N-L+2}^{\mathrm{d}} \\ \vdots & \vdots & \ddots & \vdots \\ w_L^{\mathrm{d}} & w_{L+1}^{\mathrm{d}} & \cdots & w_N^{\mathrm{d}} \end{bmatrix}$$

(data dictionary)

Willems' Fundamental Lemma [17]:

$$\mathrm{span}^{\mathrm{col}}\left(\mathcal{H}_L(w^{\mathrm{d}})\right) = \mathfrak{B}|_{[1,L]}$$

if $\mathrm{rank}(\mathcal{H}_L(u^{\mathrm{d}})) = n_{\mathrm{u}}L$

(Persistency of excitation)

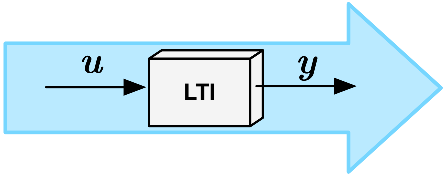
$N \geq (n_{\mathrm{u}} + 1)L - 1$

[17] Willems et al.: A note on persistency of excitation, *Systems & Control Letters*, (2005).

Data-driven LTI behavioral representation

Data-driven representation (discrete time)

Linearity



Hankel matrix (L-row):

$$\mathcal{H}_L(w^{\mathrm{d}}) = \begin{bmatrix} w_1^{\mathrm{d}} & w_2^{\mathrm{d}} & \cdots & w_{N-L+1}^{\mathrm{d}} \\ w_2^{\mathrm{d}} & w_3^{\mathrm{d}} & \cdots & w_{N-L+2}^{\mathrm{d}} \\ \vdots & \vdots & \ddots & \vdots \\ w_L^{\mathrm{d}} & w_{L+1}^{\mathrm{d}} & \cdots & w_N^{\mathrm{d}} \end{bmatrix}$$

(data dictionary)

Data-driven representation:

$$\exists g \in \mathbb{R}^{N-L+1}$$

$$\begin{bmatrix} \mathcal{H}_L(u^{\mathrm{d}}) \\ \mathcal{H}_L(y^{\mathrm{d}}) \end{bmatrix} g = \begin{bmatrix} \mathrm{col}(\bar{u}_{[1,L]}) \\ \mathrm{col}(\bar{y}_{[1,L]}) \end{bmatrix}$$

$$\Updownarrow$$

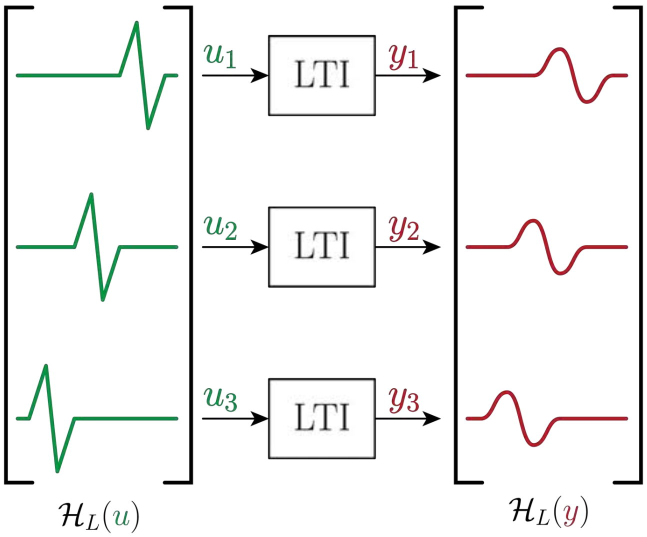
$$\mathrm{col}(\bar{y}_{[1,L]}, \bar{u}_{[1,L]}) \in \mathfrak{B}_{[1,L]}$$

[17] Willems et al.: A note on persistency of excitation, *Systems & Control Letters*, (2005).

Data-driven LTI behavioral representation

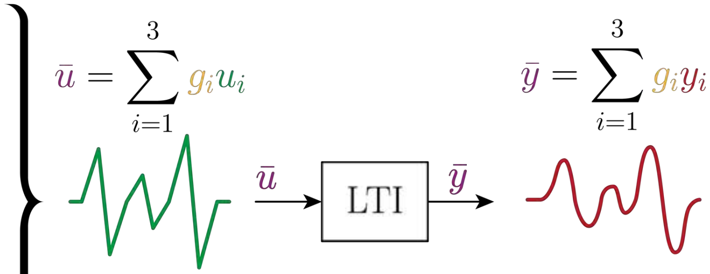
Data-driven representation (discrete time)

Gathered data:



Superposition principle:

$$\bar{u} = \sum_{i=1}^{3} g_i u_i \qquad \bar{y} = \sum_{i=1}^{3} g_i y_i$$



$$\begin{bmatrix} \mathcal{H}_L(u) \\ \mathcal{H}_L(y) \end{bmatrix} g = \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}$$

(Data-driven representation)

Data-driven LTI behavioral control

Direct data-driven analysis and control design

- Analysis
  - Simulation (Data spans the full behavior of length L) [7]
  - Stability & performance analysis (dissipativity, quadratic perf., etc.) [18]
- Control
  - Predictive control schemes (e.g., DeePC [8])
  - State-feedback control [7]
  - Noise handling & robustness guarantees [19]

[7] Markovsky, et.al.: Data-driven simulation and control, *Int. Journal of Control,* (2008)
[8] Coulson, et.al.: Data-Enabled Predictive Control: In the Shallows of the DeePC, in *Proc. of the ECC,* (2019)
[18] Romer et al.: One-shot verification of dissipativity properties from input-output data, *Control Systems Letters,* (2019)
[19] Berberich et al.: Data-Driven Model Predictive Control With Stability and Robustness Guarantees, *IEEE TAC,* (2021)

Table of contents

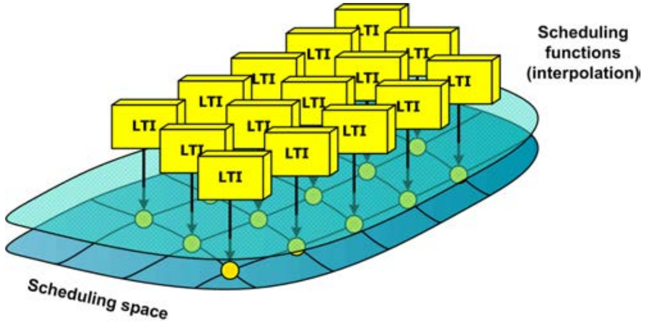
Table of contents

Linear parameter-varying framework

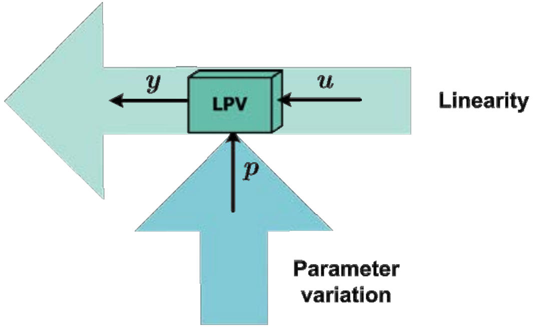
The Engineers' Dream:

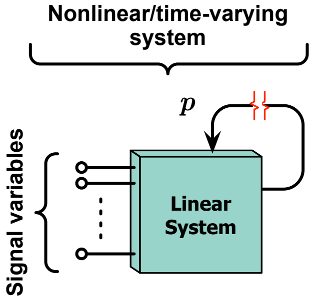
How to use "simple" linear control for NL systems with performance guarantees?

Linear parameter-varying framework

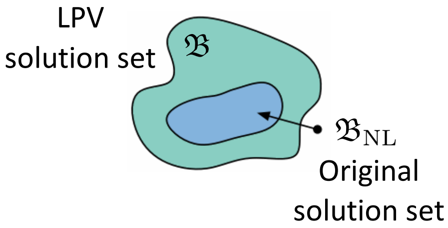


Local approximation principle

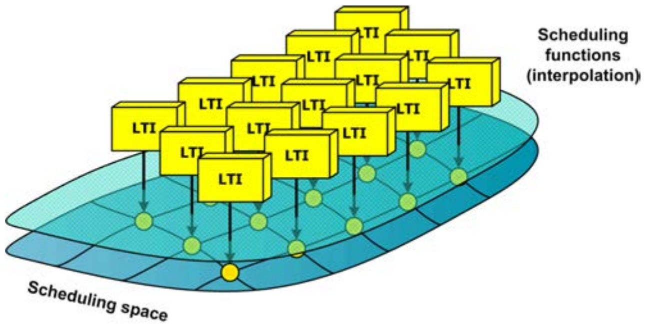




Global embedding principle

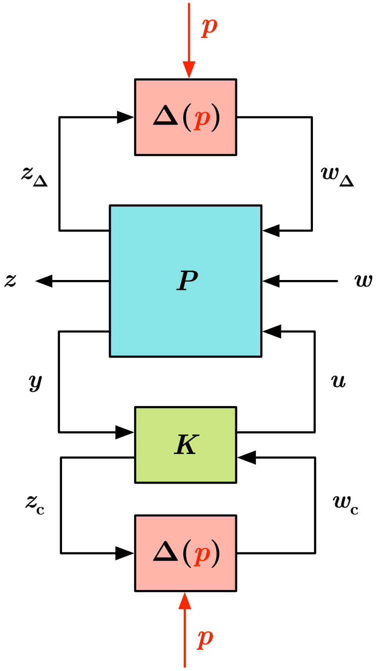


Linear parameter-varying framework

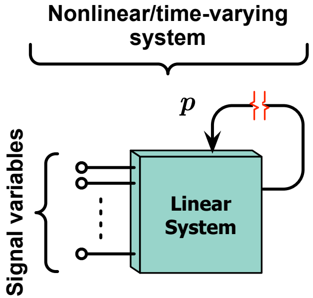


Local approximation principle

Local synthesis:
Gain scheduling
(interpolated LTI control)



Controller Synthesis

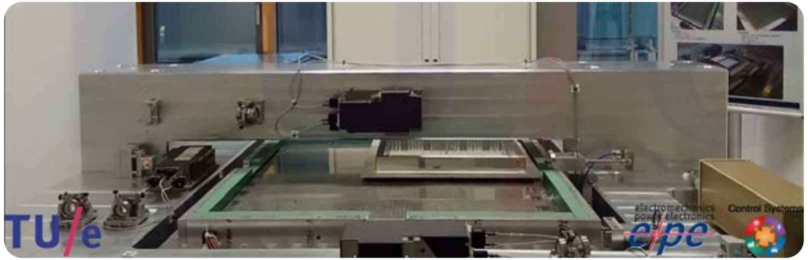


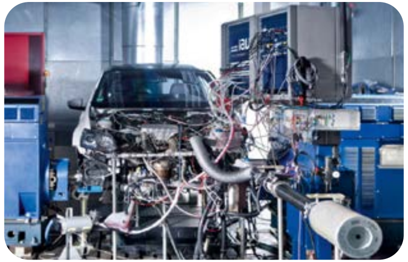
Global embedding principle

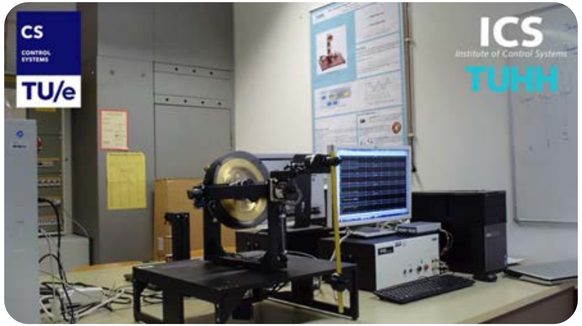
Global synthesis:
Optimal LPV control
(NL control)

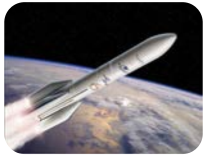
Linear parameter-varying framework

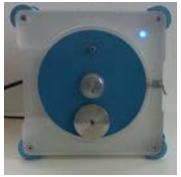
A plethora of success stories via model-based control









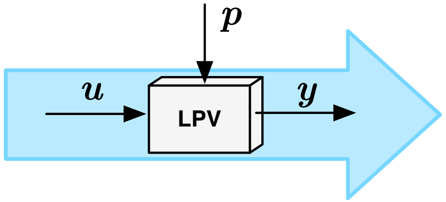


Pending question:
How to achieve data-driven control with guarantees?

LPV behavioral theory

Behavioral concept (discrete time)

Linearity



Time axis: $\mathbb{T} = \mathbb{Z}$

Behavior: $\mathfrak{B} \subseteq (\mathbb{W} \times \mathbb{P})^{\mathbb{T}}$

- Linearity: $\mathfrak{B}$ is a linear subspace
- Time invariance: $\mathfrak{B} = q^{\tau}\mathfrak{B}, \forall \tau \in \mathbb{Z}$

$$\Sigma = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$$

Scheduling space: $\mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$

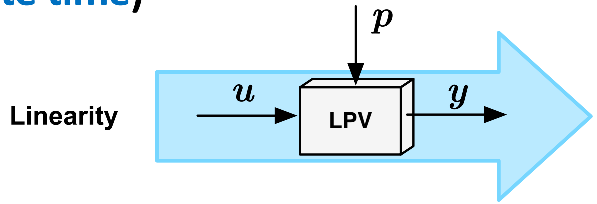
Signal space: $\mathbb{W} = \mathbb{R}^{n_\mathrm{w}}$

$w = \mathrm{col}(u, y)$

[20] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer*, (2010)

LPV behavioral theory

Behavioral concept (discrete time)



$$\Sigma = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$$

- Projected scheduling behavior:

$$\mathfrak{B}_{\mathbb{P}} = \pi_p \mathfrak{B} := \{p \in \mathbb{P}^{\mathbb{T}} \mid \exists w \in \mathbb{W}^{\mathbb{T}} \text{ s.t. } (w, p) \in \mathfrak{B}\}$$

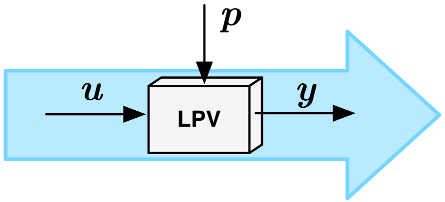
- Projected behavior for a given: $p \in \mathfrak{B}_{\mathbb{P}}$

$$\mathfrak{B}_p = \{w \in \mathbb{W}^{\mathbb{T}} \mid (w, p) \in \mathfrak{B}\}$$

[20] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer,* (2010)

LPV behavioral theory

Linearity



Kernel representation (discrete time)

Coefficient functions:

$r_i : \mathbb{R}^{n_\mathrm{p}} \to \mathbb{R}^{n \times n_\mathrm{w}}$

Types (static dep.):

- Affine/linear functions
- Polynomial functions
- Rational functions
- Meromorphic functions

$r(\cdot) = \dfrac{g(\cdot)}{h(\cdot)}$ holomorphic $h \neq 0$

Meromorphic field

$r_i \in \mathcal{R}^{n \times n_\mathrm{w}}$

Shift operator:

$q^i w_k = w_{k+1}$

$$\sum_{i=0}^{n_\mathrm{r}} r_i(p_k) q^i w_k = 0$$

Confined in

$\mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$

Signals:

$w : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{w}}$

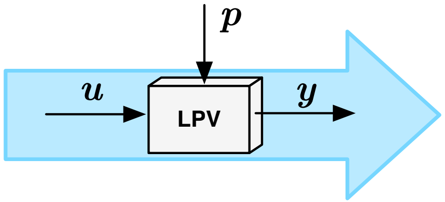
Commutation problem (dynamic dependence):

$$q^i r_i(p_k) w_k \neq r_i(p_k) w_{k+i}$$

[20] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer*, (2010)

LPV behavioral theory

Linearity



Kernel representation (discrete time)

Coefficient functions:

$$r_i : \mathbb{R}^{n_\mathrm{p}} \rightarrow \mathbb{R}^{n \times n_\mathrm{w}}$$

Types (static dep.):

- Affine/linear functions
- Polynomial functions
- Rational functions
- Meromorphic functions

$$r(\cdot) = \frac{g(\cdot)}{h(\cdot)}$$

holomorphic

$h \neq 0$

Meromorphic field

$$r_i \in \mathcal{R}^{n \times n_\mathrm{w}}$$

$$\sum_{i=0}^{n_\mathrm{r}} r_i(p_k) q^i w_k = 0$$

Shift operator:

$$q^i w_k = w_{k+1}$$

Confined in

$$\mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$$

Signals:

$$w : \mathbb{Z} \rightarrow \mathbb{R}^{n_\mathrm{w}}$$

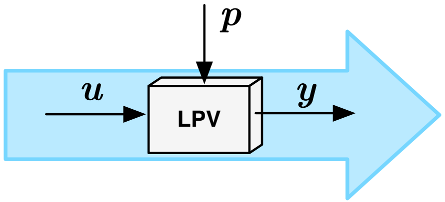
Commutation problem (dynamic dependence):

$$q^i r_i(p_k) w_k = r_i(p_{k+i}) w_{k+i}$$

[20] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer,* (2010)

LPV behavioral theory

Linearity



Kernel representation (discrete time)

Coefficient functions with finite dynamic dependence

Features:

- Causal
  $r_i(p_k, p_{k-1}, p_{k-2}, \ldots)$
- Non-causal
  $r_i(\ldots, p_{k+1}, p_k, p_{k-1}, \ldots)$

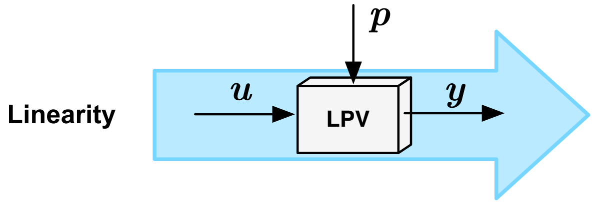
Shorthand for evaluation over dynamic dependence

$$\underbrace{\sum_{i=0}^{n_\mathrm{r}} (r_i \diamond p)_k \, q^i w_k}_{R(q) \diamond p} = 0$$

Polynomials over $\mathcal{R}$
$R \in \mathcal{R}[\xi]^{n \times n_\mathrm{w}}$

[20] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer*, (2010)

LPV behavioral theory



Kernel representation (discrete time)

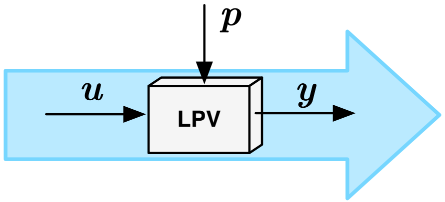
$$\underbrace{\sum_{i=0}^{n_\mathrm{r}} (r_i \diamond p)_k \, q^i w_k}_{R(q)\diamond p} = 0$$

is the representation of the LPV system $\Sigma = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$ if

$$\mathfrak{B} = \{(w, p) \in (\mathbb{R}^{n_\mathrm{w}} \times \mathbb{P})^{\mathbb{Z}} \mid (R(q) \diamond p) w = 0\}$$

[20] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer,* (2010)

Data-driven LPV behavioral representation



Data-driven representation (discrete time)

$$\underbrace{\sum_{i=0}^{n_\mathrm{r}} (r_i \diamond p)_k \, q^i w_k}_{R(q)\diamond p} = 0$$

Data-dictionary:

$$\mathcal{D}_N = \{u_k^\mathrm{d}, y_k^\mathrm{d}, p_k^\mathrm{d}\}_{k=1}^{N}$$

Complex condition

Can we simplify this to an easily computable form / representation?

LPV Fundamental Lemma:

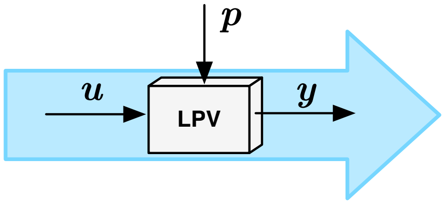
$$\mathrm{span}^{\mathrm{col}}_{\mathcal{R},p}(\mathcal{H}_L(w^\mathrm{d})) = \mathfrak{B}_p|_{[1,L]}$$

(Persistency of excitation)

existence of a “unique“ R w.r.t $\mathcal{D}_N$.

[21] Verhoek, et. al: Fundamental Lemma for Data-Driven Analysis of Linear Parameter-Varying Systems, *CDC*, (2021)

Data-driven LPV behavioral representation



Data-driven representation (discrete time, simple case)

Consider the IO form (partitioned kernel rep.):

$$y_k + \sum_{i=1}^{n_\mathrm{a}} a_i(p_{k-i}) y_{k-i} = \sum_{i=1}^{n_\mathrm{b}} b_i(p_{k-i}) u_{k-i}$$

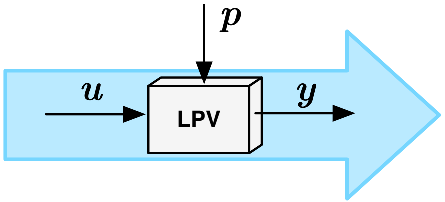
Restricted, but useful subclass of LPV systems

with shifted-affine scheduling dependence:

$$a_i(p_{k-i}) = a_{i,0} + \sum_{j=1}^{n_\mathrm{p}} a_{i,j} p_{j,k-i}, \quad b_i(p_{k-i}) = b_{i,0} + \sum_{j=1}^{n_\mathrm{p}} b_{i,j} p_{j,k-i}$$

[21] Verhoek, et. al: Fundamental Lemma for Data-Driven Analysis of Linear Parameter-Varying Systems, *CDC,* (2021)

Data-driven LPV behavioral representation



Data-driven representation (discrete time, simple case)

$$y_k + \sum_{i=1}^{n_\mathrm{a}} a_i \underbrace{(p_{k-i}) y_{k-i}}_{p_{k-i} \otimes y_{k-i}} = \sum_{i=1}^{n_\mathrm{b}} b_i \underbrace{(p_{k-i}) u_{k-i}}_{p_{k-i} \otimes u_{k-i}}$$

Data-dictionary:

$$\mathcal{D}_N = \{u_k^\mathrm{d}, y_k^\mathrm{d}, p_k^\mathrm{d}\}_{k=1}^N$$

Data-driven representation:

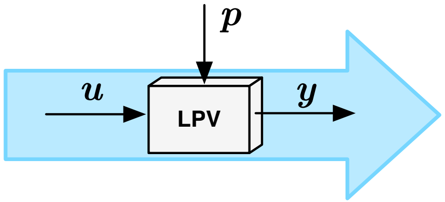
$$\begin{bmatrix} \mathcal{H}_L(u^\mathrm{d}) \\ \mathcal{H}_L(p^\mathrm{d} \otimes u^\mathrm{d}) - \bar{\mathcal{P}}^{n_\mathrm{u}} \mathcal{H}_L(u^\mathrm{d}) \\ \mathcal{H}_L(y^\mathrm{d}) \\ \mathcal{H}_L(p^\mathrm{d} \otimes y^\mathrm{d}) - \bar{\mathcal{P}}^{n_\mathrm{y}} \mathcal{H}_L(y^\mathrm{d}) \end{bmatrix} g = \begin{bmatrix} \mathrm{col}(\bar{u}_{[1,L]}) \\ 0 \\ \mathrm{col}(\bar{y}_{[1,L]}) \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\mathrm{col}(\bar{y}_{[1,L]}, \bar{u}_{[1,L]}, \bar{p}_{[1,L]}) \in \mathfrak{B}_{[1,L]}$$

[21] Verhoek, et. al: Fundamental Lemma for Data-Driven Analysis of Linear Parameter-Varying Systems, *CDC,* (2021)

Data-driven LPV behavioral representation



Simplified LPV Fundamental Lemma (discrete time)

Given $\mathcal{D}_N = \{u_k^{\mathrm{d}}, p_k^{\mathrm{d}}, x_k^{\mathrm{d}}\}_{k=1}^{N}$ and let

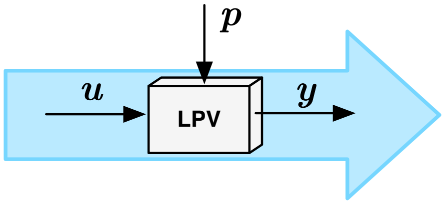
$$\mathcal{N}_{\bar{p}} := \operatorname{nullspace}\left\{\begin{bmatrix} \mathcal{H}_L(p^{\mathrm{d}} \otimes u^{\mathrm{d}}) - \bar{\mathcal{P}}^{n_{\mathrm{u}}}\mathcal{H}_L(u^{\mathrm{d}}) \\ \mathcal{H}_L(p^{\mathrm{d}} \otimes y^{\mathrm{d}}) - \bar{\mathcal{P}}^{n_{\mathrm{y}}}\mathcal{H}_L(y^{\mathrm{d}}) \end{bmatrix}\right\}, \qquad \mathcal{S} := \operatorname{span}^{\mathrm{col}}\left\{\begin{bmatrix} \mathcal{H}_L(u^{\mathrm{d}}) \\ \mathcal{H}_L(y^{\mathrm{d}}) \end{bmatrix}\right\}$$

Then, for all scheduling signals $\bar{p} \in \mathfrak{B}_{\mathbb{P}}$

$$\operatorname{Proj}_{\mathcal{N}_{\bar{p}}}(\mathcal{S}) = \mathfrak{B}_{\bar{p}}|_{[1,L]} \iff \dim\left\{\operatorname{Proj}_{\mathcal{N}_{\bar{p}}}(\mathcal{S})\right\} = n_{\mathrm{x}} + n_{\mathrm{u}}L$$

[22] Verhoek, et. al: Data-driven Dissipativity Analysis of Linear Parameter-Varying Systems, *arXiv:2303.10031,* (2023)
LPV CORE erc TU/e

Data-driven LPV behavioral representation



Data-driven representation (discrete time, state-feedback case)

Consider the SS form:

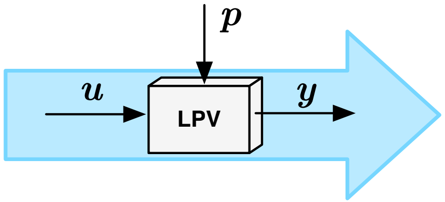
$$
\begin{aligned}
x_{k+1} &= A(p_k)x_k + B(p_k)u_k \\
y_k &= x_k
\end{aligned}
$$

with static-affine scheduling dependence:

$$
A(p_k) = A_0 + \sum_{i=1}^{n_\mathrm{p}} A_i p_{i,k}, \quad B(p_k) = B_0 + \sum_{i=1}^{n_\mathrm{p}} B_i p_{i,k}
$$

[23] Verhoek, et. al: Data-driven Direct Data-Driven State-Feedback Control of Linear Parameter-Varying Systems, *arXiv:2211.17182*, (2022)

Data-driven LPV behavioral representation



Data-driven representation (discrete time, state-feedback case)

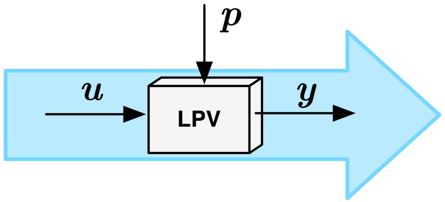
$$
\begin{aligned}
x_{k+1} &= A(p_k)x_k + B(p_k)u_k, \\
y_k &= x_k
\end{aligned}
$$

Data-dictionary:

$$\mathcal{D}_N = \{u_k^{\mathrm{d}}, x_k^{\mathrm{d}}, p_k^{\mathrm{d}}\}_{k=1}^{N}$$

[23] Verhoek, et. al: Data-driven Direct Data-Driven State-Feedback Control of Linear Parameter-Varying Systems, *arXiv:2211.17182,* (2022)

Data-driven LPV behavioral representation



Data-driven representation (discrete time, state-feedback case)

$$x_{k+1} = A(p_k)x_k + B(p_k)u_k,$$
$$y_k = x_k$$

Multi-step: Use direct realization of the shifted-affine IO form

Data-dictionary:

$$U = \begin{bmatrix} u_1^{\mathrm{d}} & \cdots & u_{N-1}^{\mathrm{d}} \end{bmatrix}$$
$$X = \begin{bmatrix} x_1^{\mathrm{d}} & \cdots & x_{N-1}^{\mathrm{d}} \end{bmatrix}$$
$$\overrightarrow{X} = \begin{bmatrix} x_2^{\mathrm{d}} & \cdots & x_N^{\mathrm{d}} \end{bmatrix}$$
$$U^{\mathrm{p}} = \begin{bmatrix} p_1^{\mathrm{d}} \otimes u_1^{\mathrm{d}} & \cdots & p_{N-1}^{\mathrm{d}} \otimes u_{N-1}^{\mathrm{d}} \end{bmatrix}$$
$$X^{\mathrm{p}} = \begin{bmatrix} p_1^{\mathrm{d}} \otimes x_1^{\mathrm{d}} & \cdots & p_{N-1}^{\mathrm{d}} \otimes x_{N-1}^{\mathrm{d}} \end{bmatrix}$$

Data-driven representation:

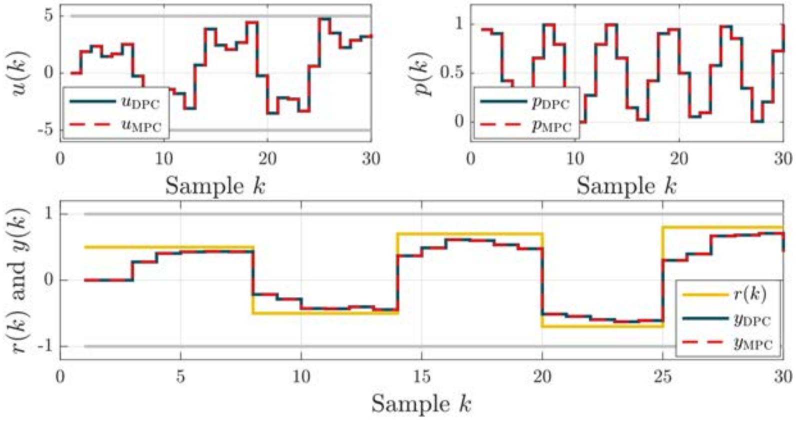
$$\bar{x}_{k+1} = \overrightarrow{X} \begin{bmatrix} X \\ X^{\mathrm{p}} \\ U \\ U^{\mathrm{p}} \end{bmatrix}^{\dagger} \begin{bmatrix} \bar{x}_k \\ \bar{p}_k \otimes \bar{x}_k \\ \bar{u}_k \\ \bar{p}_k \otimes \bar{u}_k \end{bmatrix}$$

[23] Verhoek, et. al: Data-driven Direct Data-Driven State-Feedback Control of Linear Parameter-Varying Systems, *arXiv:2211.17182*, (2022)

Data-driven LPV behavioral control

Direct data-driven analysis and control design

- Analysis
  - Simulation [24]
  - Stability & performance analysis [22] (dissipativity, quadratic perf., etc.)
- Control
  - Predictive control [24, 25]
  - State-feedback control [23]
  - Noise handling & robustness guarantees (coming soon, initial results in [25])

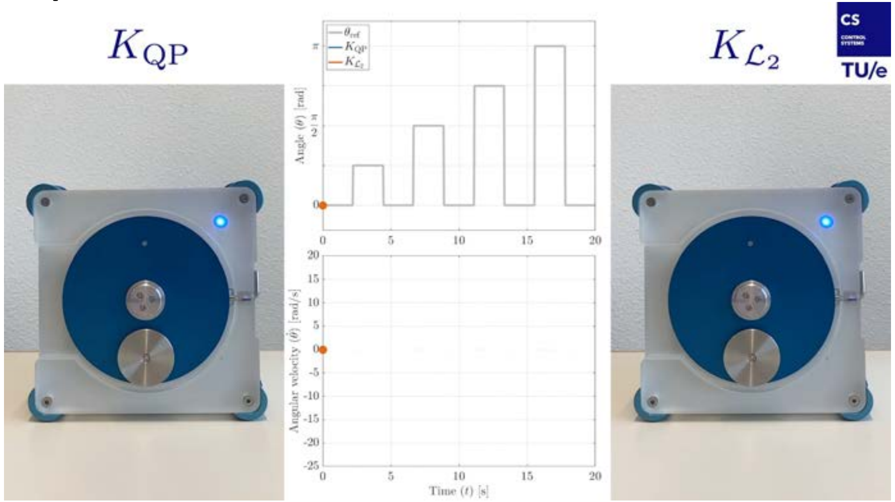


Data-driven vs. model based predictive control

[22] Verhoek, et. al: Data-driven Dissipativity Analysis of Linear Parameter-Varying Systems, *arXiv:2303.10031,* (2023)
[23] Verhoek, et. al: Direct Data-Driven State-Feedback Control of Linear Parameter-Varying Systems, *arXiv:2211.17182,* (2022)
[24] Verhoek, et. al: Data-Driven Predictive Control for Linear Parameter-Varying Systems, *LPVS,* (2021)
[25] Verhoek, et. al: A Linear Parameter-Varying Approach to Data Predictive Control, *about to be submitted* (2023)

Data-driven LPV behavioral control

Example (unbalanced disc):



Optimal state-feedback design

Data-driven advantage
LPV controller synthesized using **7** data-points (70 milliseconds)!

[26] Verhoek, et. al: Direct data-driven LPV control of nonlinear systems: An experimental result, *IFAC WC,* (2023)

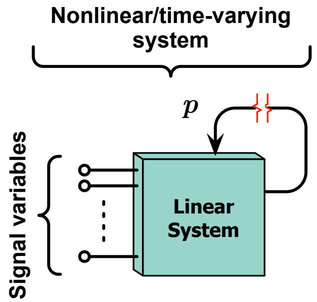
Towards nonlinear data-driven control?

Now developed a data-driven behavioral framework for LPV systems

- When underlying system is NL → possible unexpected stability restrictions [27]

Data-driven control for nonlinear systems:

- Feedback/online linearizations [28, 29]
- Nonlinearity cancellation [30]
- Koopman-based [31]
- Polynomial systems [32]



Mostly rely on / go back to **LTI** Fundamental Lemma...

→ Local guarantees

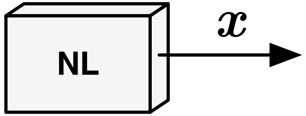
Can we get a bit more general?

[27] Koelewijn, et al. ECC (2020).
[29] Berberich, et al., *IEEE-TAC* (2022).
[31] Lian, et al., *arXiv:2102.05122* (2021).
[28] De Persis, et al., *arXiv:2308.11229* (2023).
[30] De Persis, et al., *IEEE-TAC* (2023).
[32] Guo, et al., *IEEE-TAC* (2021).

Table of contents

Concept of global stability

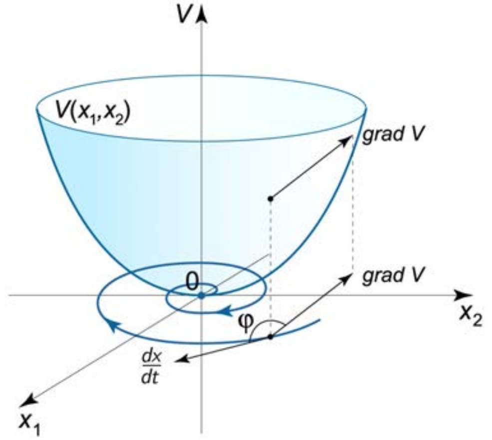
Nonlinear system (autonomous, discrete time)

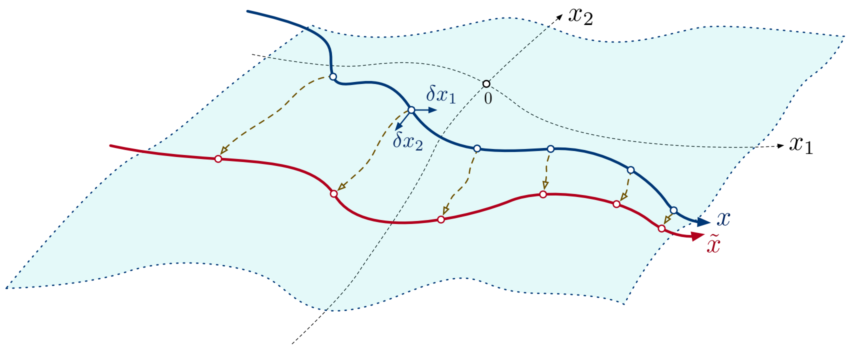


$$x_{k+1} = f(x_k)$$

$$f \in \mathcal{C}^1$$

Concept of global stability





Lyapunov Stability

Core stability concept in the NL/LPV context

Incremental Stability

Convergence of trajectories

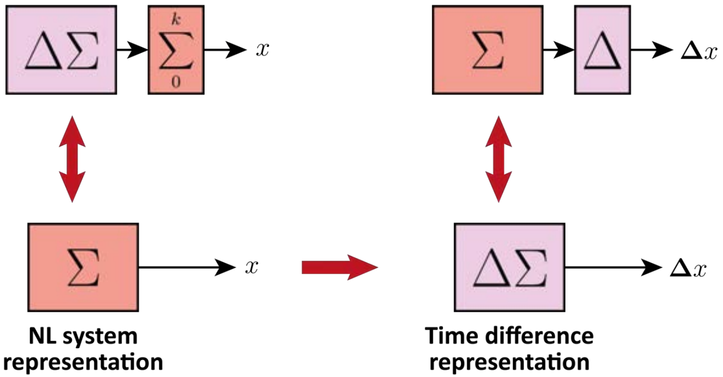
(equilibrium free stability)

Can fail in case of tracking!

Closed-loop NL guarantees can be lost.

Concept of global stability

- **Krasovskii type of condition**
  - Consider the time difference form



$$\Delta x_{k+1} = f(x_k) - f(x_{k-1}) \qquad \Delta x_0 \in \mathbb{R}^{n_\mathrm{x}}$$

$$\Delta x_{k+1} = \int_0^1 \frac{\partial f}{\partial x}(\breve{x}_k(\lambda))\, d\lambda \cdot \Delta x_k \qquad \breve{x}_k(\lambda) = x_{k-1} + \lambda(x_k - x_{k-1})$$

$$\Delta x_{k+1} = \mathcal{A}(x_k, x_{k-1})\Delta x_k$$

[27] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the Linear Parameter-Varying Framework, *CDC* (2021)
[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

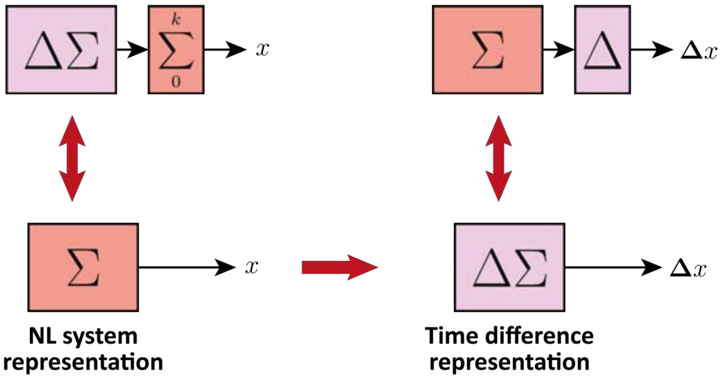
- **Krasovskii type of condition**
  - Consider the time difference form

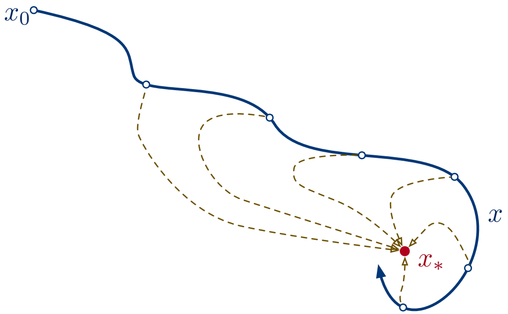
$$\Delta x_{k+1} = \mathcal{A}(x_k, x_{k-1})\Delta x_k$$
$$\Delta x_0 \in \mathbb{R}^{n_\mathrm{x}}$$

Shifted Stability
(Asymptotic)

There exists a KL function β such that for any $x_0 \in \mathbb{X}$, there is a $x_* \in \mathbb{X}$ s.t.:

$$\|x_k - x_*\|_2 \leq \underbrace{\beta(\|x_0 - x_*\|_2, k)}_{\kappa e^{-ct}\|x_0 - x_*\|_2}$$



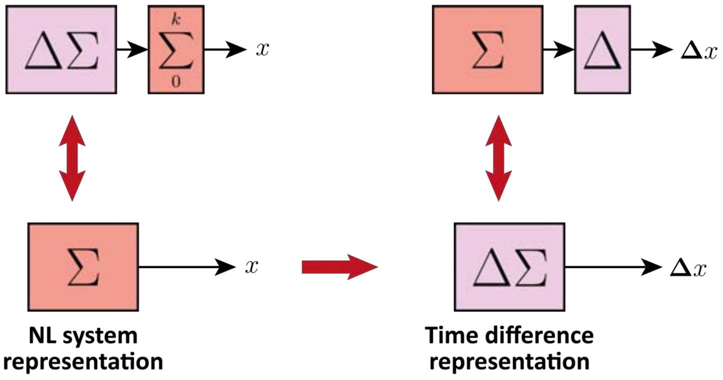


[27] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the Linear Parameter-Varying Framework, *CDC* (2021)
[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Krasovskii type of condition**
  - Consider the time difference form

$$\Delta x_{k+1} = \mathcal{A}(x_k, x_{k-1})\Delta x_k$$
$$\Delta x_0 \in \mathbb{R}^{n_\mathrm{x}}$$



Shifted Stability
(Asymptotic)

There exists a KL function β such that for any $x_0 \in \mathbb{X}$, there is a $x_* \in \mathbb{X}$ s.t.:

$$\|x_k - x_*\|_2 \leq \underbrace{\beta(\|x_0 - x_*\|_2, k)}_{\kappa e^{-ct}\|x_0 - x_*\|_2}$$

Shifted Stability
(Sufficiency condition)

If there exists a $\mathcal{X} \succ 0$ such that $\forall \bar{x}, \bar{\bar{x}} \in \mathbb{X}$

$$\mathcal{A}^\top(\bar{x}, \bar{\bar{x}})\mathcal{X}\mathcal{A}(\bar{x}, \bar{\bar{x}}) - \mathcal{X} \prec 0$$

[27] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the Linear Parameter-Varying Framework, *CDC* (2021)
[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Krasovskii type of condition**
  - Consider the time difference form

$$\Delta x_{k+1} = \mathcal{A}(x_k, x_{k-1})\Delta x_k$$
$$\Delta x_0 \in \mathbb{R}^{n_\mathrm{x}}$$

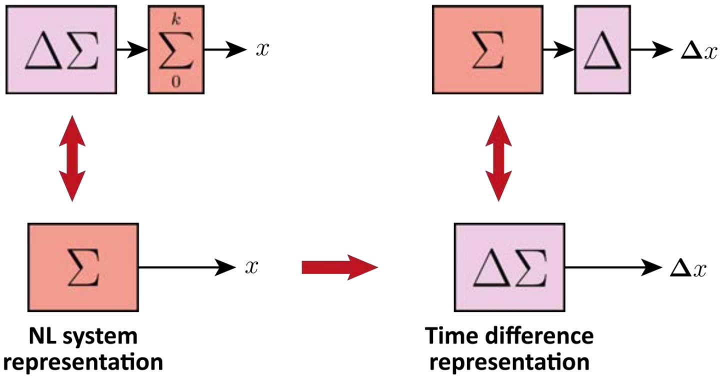
  - Quadratic stability

$$V(x) = \underbrace{(f(x) - x)^\top}_{\Delta x} \mathcal{X} \underbrace{(f(x) - x)}_{\Delta x} \qquad \mathcal{X} \succ 0$$

$$\Delta x^\top \mathcal{A}^\top(qx, x)\mathcal{X}\mathcal{A}(qx, x)\Delta x - \Delta x^\top \mathcal{X}\Delta x \prec 0$$

$$\Downarrow$$

$$(f(x) - x)^\top \mathcal{A}^\top(f(x), x)\mathcal{X}\mathcal{A}(f(x), x)(f(x) - x) - (f(x) - x)^\top \mathcal{X}(f(x) - x) \prec 0$$



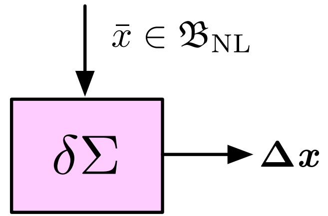
Shifted Stability
(Sufficiency condition)

If there exists a $\mathcal{X} \succ 0$ such that $\forall \bar{x}, \bar{\bar{x}} \in \mathbb{X}$

$$\mathcal{A}^\top(\bar{x}, \bar{\bar{x}})\mathcal{X}\mathcal{A}(\bar{x}, \bar{\bar{x}}) - \mathcal{X} \prec 0$$

[27] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the Linear Parameter-Varying Framework, *CDC* (2021)
[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability



Velocity system representation

- **Velocity stability**
  - Enough to consider the stability of the velocity form

$$\Delta x_{k+1} = \mathcal{A}(\bar{x}_k, \bar{x}_{k-1}) \cdot \Delta x_k$$

Velocity stability

$$\mathcal{X} \succ 0$$

$$\mathcal{A}^\top(\bar{x}, \bar{\bar{x}})\mathcal{X}\mathcal{A}(\bar{x}, \bar{\bar{x}}) - \mathcal{X} \prec 0$$

$$\forall \bar{x}, \bar{\bar{x}} \in \mathbb{X}$$

Shifted stability

$$V(x) \succ 0$$

$$\Delta V(x) \prec 0$$

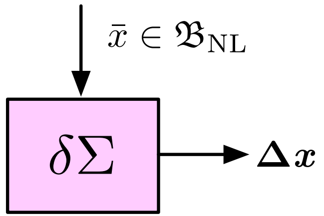
$$\forall x \in \mathbb{X}_{x_*} \subseteq \mathbb{X}$$

[27] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the Linear Parameter-Varying Framework, *CDC* (2021)
[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Velocity stability**
  - Enough to consider the stability of the velocity form

$$\Delta x_{k+1} = \mathcal{A}(\bar{x}_k, \bar{x}_{k-1}) \cdot \Delta x_k$$



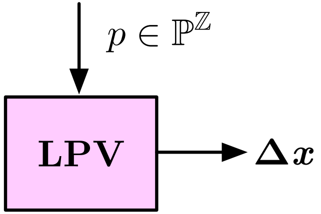
Velocity system representation

Looks like an LPV form!

[27] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the Linear Parameter-Varying Framework, *CDC* (2021)
[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Velocity stability**
  - Enough to consider the stability of the velocity form



LPV velocity system representation

$$\Delta x_{k+1} = A(p_k) \cdot \Delta x_k$$

Looks like an LPV form!

Quadratic LPV stability

$$\mathcal{X} \succ 0$$

$$A^\top(p)\mathcal{X}A(p) - \mathcal{X} \prec 0$$

$$\forall p \in \mathbb{P}$$

Shifted stability

$$V(x) \succ 0$$

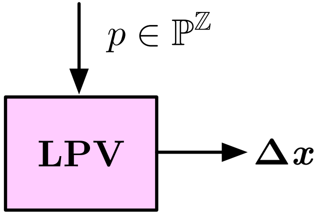
$$\Delta V(x) \prec 0$$

$$\forall x \in \mathbb{X}_{x_*} \subseteq \mathbb{X}$$

[27] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the Linear Parameter-Varying Framework, *CDC* (2021)
[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Velocity stability**
  - Enough to consider the stability of the velocity form



LPV velocity system representation

$$\Delta x_{k+1} = A(p_k) \cdot \Delta x_k$$

Looks like an LPV form!

Quadratic LPV stability

$$\mathcal{X} \succ 0$$

$$A^\top(p)\mathcal{X}A(p) - \mathcal{X} \prec 0$$

$$\forall p \in \mathbb{P}$$

Shifted stability

$$V(x) \succ 0$$

$$\Delta V(x) \prec 0$$

$$\forall x \in \mathbb{X}_{x_*} \subseteq \mathbb{X}$$

LPV embedding

We can guarantee stability via an LPV embedding of the velocity form

[27] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the Linear Parameter-Varying Framework, *CDC* (2021)

[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

Theory in pictures

$\delta\Sigma$

Stability

$\downarrow$

$\Delta\Sigma$

Stability

$\downarrow$

Σ

Stability

Velocity stability

Stability w.r.t. time-difference dynamics

$\Downarrow$

Shifted stability

Stability w.r.t. arbitrary equilibrium point

$\Downarrow$

Lyapunov stability

Stability w.r.t. the origin

[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023
[29] Verhoek, et. al: Convex Incremental Dissipativity Analysis of Nonlinear Systems, *Automatica*, (2023)

Concept of global performance

Theory in pictures

$\delta\Sigma$

Dissipativity

$\Delta\Sigma$

Dissipativity

Σ

Dissipativity

Velocity dissipativity

Stability w.r.t. time-difference dynamics

$$\Downarrow \quad V(\Delta x_{k_1}) \leq V(\Delta x_{k_2}) + \sum_{t=k_1}^{k_2} s(\Delta u_t, \Delta y_t)$$

Shifted dissipativity*

Dissipativity w.r.t. arbitrary storage point

$$\Downarrow \quad V(x_{k_1}, x_*) \leq V(x_{k_2}, x_*) + \sum_{t=k_1}^{k_2} s(u_t, u_*, y_t, y_*)$$

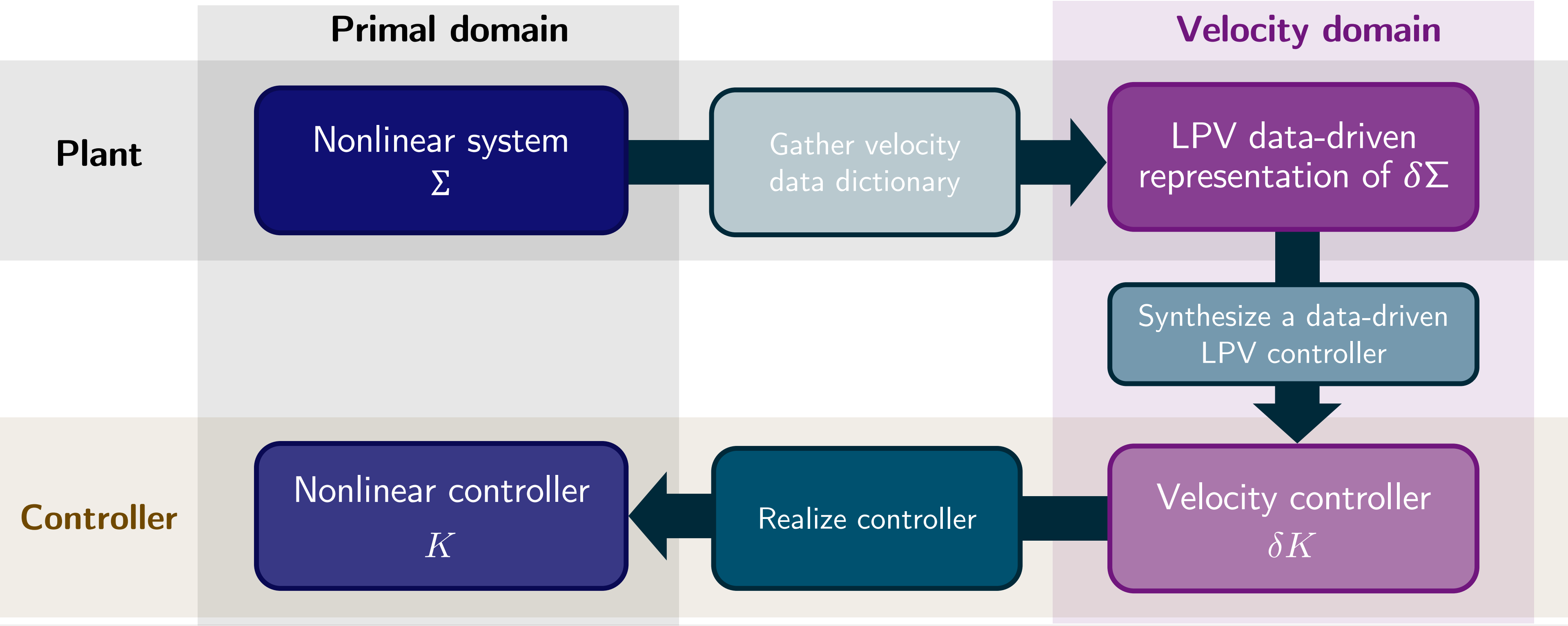
General stability

Dissipativity w.r.t. the origin

$$V(x_{k_1}) \leq V(x_{k_2}) + \sum_{t=k_1}^{k_2} s(u_t, y_t)$$

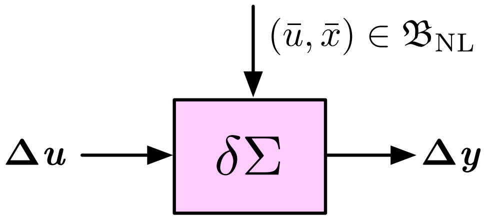
[28] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023
[29] Verhoek, et. al: Convex Incremental Dissipativity Analysis of Nonlinear Systems, *Automatica,* (2023)

Data-driven NL controller synthesis



[30] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *arXiv:2303.10648,* (2023)

Data-driven NL controller synthesis



Velocity system representation

- **State-feedback synthesis**
  - Consider the time difference form ($w_k = \mathrm{col}(u_k, x_k)$)

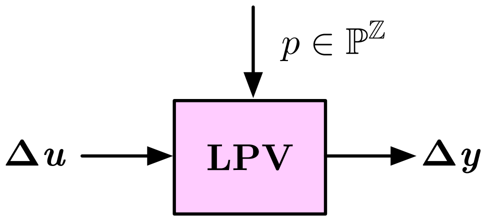
$$\begin{aligned} \Delta x_{k+1} &= \mathcal{A}(w_k, w_{k-1})\Delta x_k + \mathcal{B}(w_k, w_{k-1})\Delta u_k, \\ \Delta y_k &= \Delta x_k \end{aligned}$$

  - Assume a given set of basis functions $\psi_1, \ldots, \psi_{n_\mathrm{p}}$, such that

$$\begin{aligned} \mathcal{A}(w_k, w_{k-1}) &= A_0 + \textstyle\sum_{i=1}^{n_\mathrm{p}} A_i \psi_i(w_k, w_{k-1}) \\ \mathcal{B}(w_k, w_{k-1}) &= B_0 + \textstyle\sum_{i=1}^{n_\mathrm{p}} B_i \psi_i(w_k, w_{k-1}) \end{aligned}$$

[30] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, a*rXiv:2303.10648,* (2023)

Data-driven NL controller synthesis



- **State-feedback synthesis**
  - LPV embedding

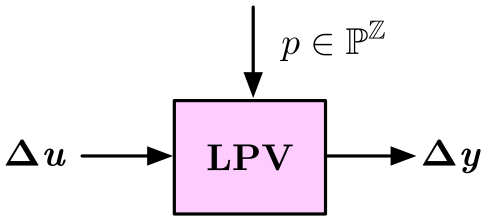
$$
\begin{aligned}
\Delta x_{k+1} &= A(p_k)\Delta x_k + B(p_k)\Delta u_k,\\
\Delta y_k &= \Delta x_k
\end{aligned}
$$

  - Scheduling is defined as

$$
p_k := \psi(x_k, u_k, x_{k-1}, u_{k-1})
$$

[30] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, a*rXiv:2303.10648,* (2023)

Data-driven NL controller synthesis



LPV velocity representation

- **State-feedback synthesis**

$$\begin{aligned} \Delta x_{k+1} &= A(p_k)\Delta x_k + B(p_k)\Delta u_k, \\ \Delta y_k &= \Delta x_k \end{aligned}$$

Data-dictionary:

$$\mathcal{D}_{N+1}^{\mathrm{NL}} = \{u_k^{\mathrm{d}}, x_k^{\mathrm{d}}\}_{k=0}^{N}$$

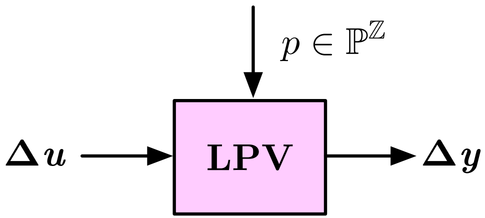
$$\downarrow$$

$$\mathcal{D}_N^{\Delta} = \{\Delta u_k^{\mathrm{d}}, \Delta x_k^{\mathrm{d}}, p_k^{\mathrm{d}}\}_{k=1}^{N}$$

[30] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, a*rXiv:2303.10648,* (2023)

Data-driven NL controller synthesis

- **State-feedback synthesis**



LPV velocity representation

$$\begin{aligned} \Delta x_{k+1} &= A(p_k)\Delta x_k + B(p_k)\Delta u_k, \\ \Delta y_k &= \Delta x_k \end{aligned}$$

Apply LPV state feedback synthesis

Data-dictionary:

$$\begin{aligned} U_\Delta &= \begin{bmatrix} \Delta u_1^{\mathrm{d}} & \cdots & \Delta u_{N-1}^{\mathrm{d}} \end{bmatrix}, \\ X_\Delta &= \begin{bmatrix} \Delta x_1^{\mathrm{d}} & \cdots & \Delta x_{N-1}^{\mathrm{d}} \end{bmatrix}, \\ \overrightarrow{X}_\Delta &= \begin{bmatrix} \Delta x_2^{\mathrm{d}} & \cdots & \Delta x_N^{\mathrm{d}} \end{bmatrix}, \\ U_\Delta^{\mathrm{p}} &= \begin{bmatrix} p_1^{\mathrm{d}} \otimes \Delta u_1^{\mathrm{d}} & \cdots & p_{N-1}^{\mathrm{d}} \otimes \Delta u_{N-1}^{\mathrm{d}} \end{bmatrix}, \\ X_\Delta^{\mathrm{p}} &= \begin{bmatrix} p_1^{\mathrm{d}} \otimes \Delta x_1^{\mathrm{d}} & \cdots & p_{N-1}^{\mathrm{d}} \otimes \Delta x_{N-1}^{\mathrm{d}} \end{bmatrix}. \end{aligned}$$

Data-driven representation:

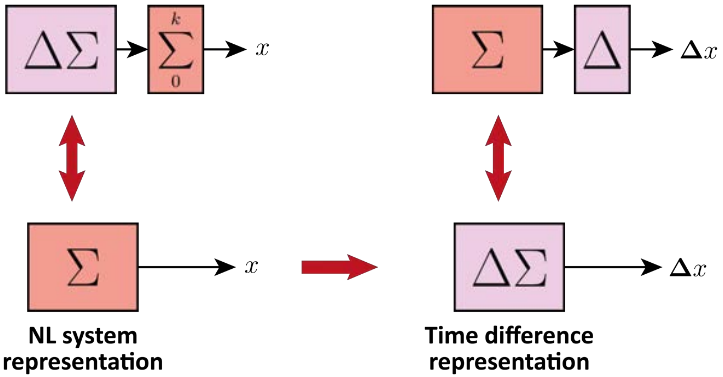
$$\Delta \bar{x}_{k+1} = \overrightarrow{X}_\Delta \begin{bmatrix} X_\Delta \\ X_\Delta^{\mathrm{p}} \\ U_\Delta \\ U_\Delta^{\mathrm{p}} \end{bmatrix}^{\dagger} \begin{bmatrix} \Delta \bar{x}_k \\ \bar{p}_k \otimes \Delta \bar{x}_k \\ \Delta \bar{u}_k \\ \bar{p}_k \otimes \Delta \bar{u}_k \end{bmatrix}$$

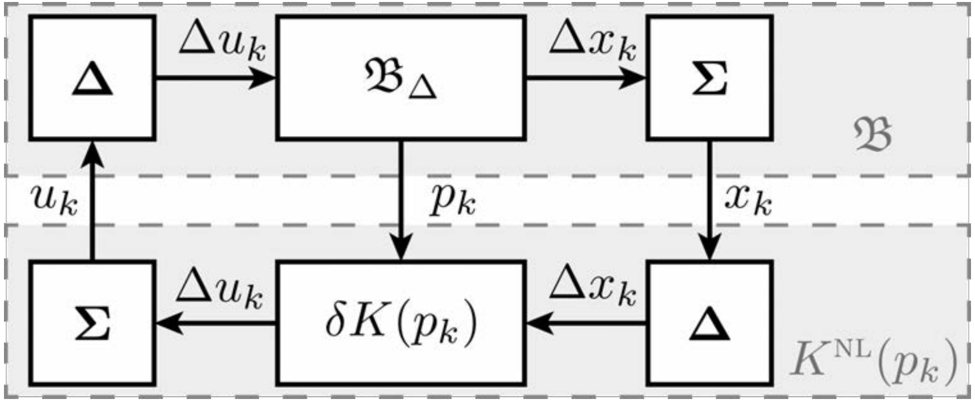
[30] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, a*rXiv:2303.10648,* (2023)

Controller realization

- **Controller realization**
  - Velocity form (state-feedback case):

$$\begin{aligned}\Delta u_k &= \delta K(p_k)\Delta x_k \\ &= \delta K(\psi(x_k, u_k, x_{k-1}, u_{k-1}))\Delta x_k\end{aligned}$$





[30] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *arXiv:2303.10648,* (2023)

Controller realization

- **Controller realization**
  - Velocity form (state-feedback case):

$$\begin{aligned}\Delta u_k &= \delta K(p_k)\Delta x_k \\ &= \delta K(\psi(x_k, u_k, x_{k-1}, u_{k-1}))\Delta x_k\end{aligned}$$

  - Primal form (realization):

$$K^{\mathrm{NL}} \begin{cases} \chi_{k+1} = \begin{bmatrix} 0 & 0 \\ -\delta K(p_k) & I \end{bmatrix} \chi_k + \begin{bmatrix} I \\ \delta K(p_k) \end{bmatrix} x_k \\ u_k = \begin{bmatrix} -\delta K(p_k) & I \end{bmatrix} \chi_k + \delta K(p_k) x_k \\ p_k = \psi(x_k, u_k, \chi_k) \\ \chi_k = \begin{bmatrix} x_{k-1}^\top & u_{k-1}^\top \end{bmatrix}^\top \end{cases}$$

Preservation of guarantees

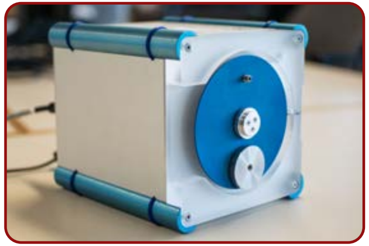
Realization preserves shifted stability & dissipativity!

Achievement:

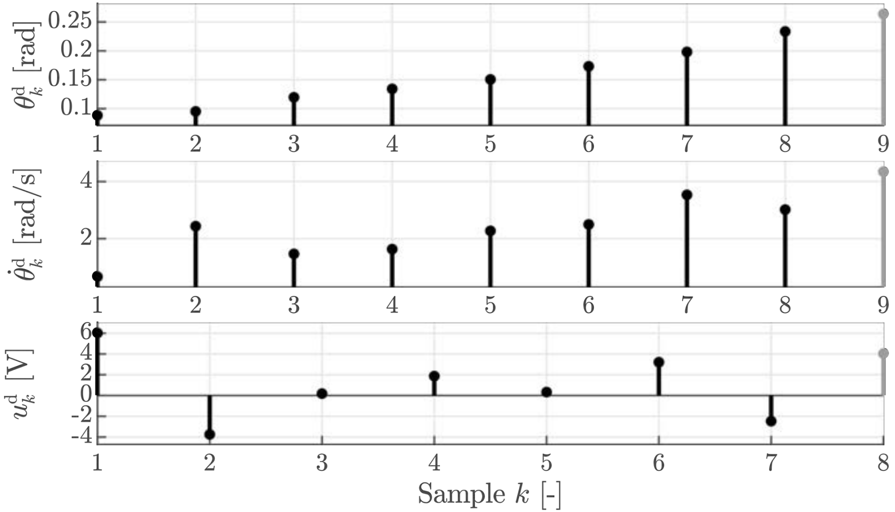
The LPV synthesis is used as a surrogate tool for designing an NL controller with perf. guarantees.

[30] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, a*rXiv:2303.10648,* (2023)

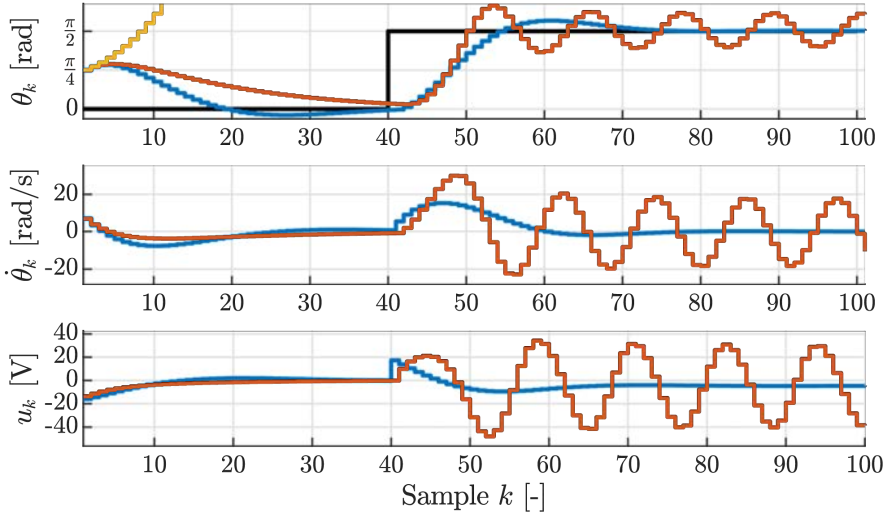
Controller realization



- **Unbalanced disc system (simulation):**
    - Basis functions chosen based on a priori knowledge



Data-dictionary



Data-driven nonlinear, LPV and LTI controller

[30] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, arXiv:2303.10648, (2023)

Table of contents

- Behavioral LTI data-driven control
- Behavioral LPV data-driven control
- Behavioral NL data-driven control
- Conclusions

Conclusions

Effective tools for direct synthesis NL controllers from time-domain data

- Using tools in behavioral data-driven LPV framework
- Generalization to output-feedback and predictive control case
- General performance objectives (passivity, $\mathcal{L}_2$, generalized $\mathcal{H}_2$, etc.)

Outlooks

- Data-driven learning of the basis functions
- Scaling up to incremental stability and performance (reference tracking)
- Handling noise and stochastic aspects
- Integration into LPVcore (off-the-shelf software solution)