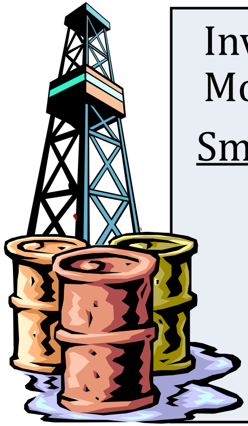
Session 9: Probability

Module BUSN9690

Business Statistics with Python

Example 1: Mick's Investments

Mick has invested in two stocks, Smith Oil and David Mining. He estimated that the possible outcomes of these investments three months from now were as follows.



Investment Gain or Loss in 3 Months (in £1,000)

Smith Oil	David Mining
10	8
5	-2
0	
-20	

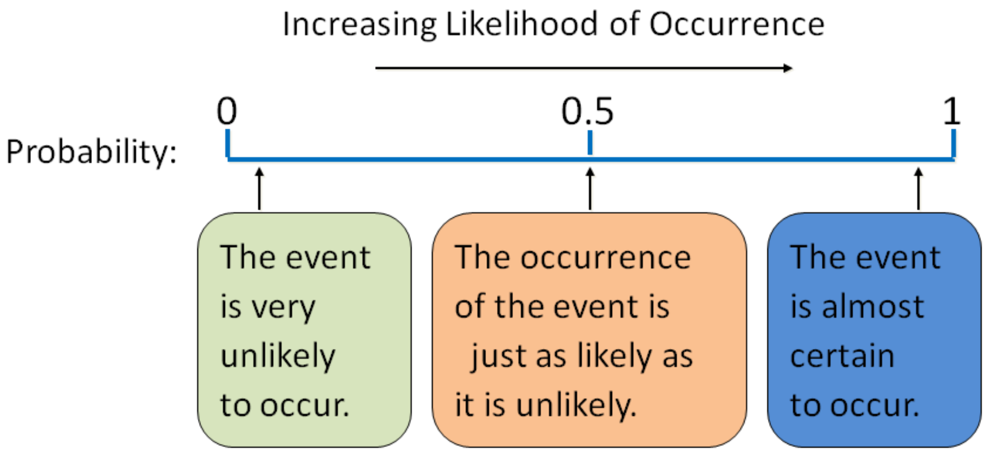
Questions

- What is the probability that Smith Oil is profitable?
- What is the probability that Smith Oil or David Mining is profitable?
- What is the probability that Smith Oil and David Mining are profitable?
- What is the probability that David Mining is profitable, given that Smith Oil is profitable?

Probability (Classical definition)

- **The ratio of the number of favourable outcomes to the total number of outcomes**

$$P(A) = \frac{\text{Number of outcomes leading to event } A}{\text{Total number of possible outcomes}}$$



$0 \leq \text{Probability} \leq 1$

Sample space & event sample

Experiment	Sample Space (possible outcomes)
Toss 2 Coins, Note Faces	HH, HT, TH, TT
Play a Football Game	Win, Lose, Tie
Inspect a Part, Note Quality	Defective, OK

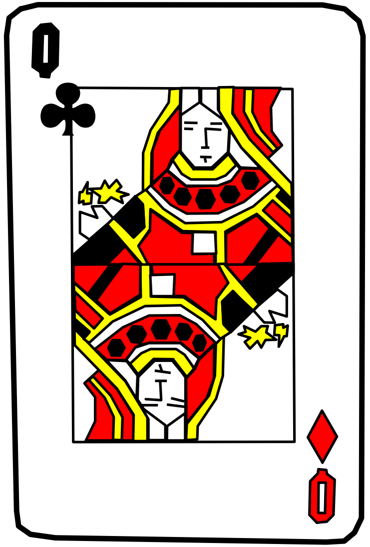
- Experiment: Toss 2 Coins. Note Faces.
- Sample Space: HH, HT, TH, TT

Events	Outcomes in event
1 Head & 1 Tail	HT, TH
Heads on 1st Coin	HH, HT
At Least 1 Head	HH, HT, TH
Heads on Both	HH

Null event

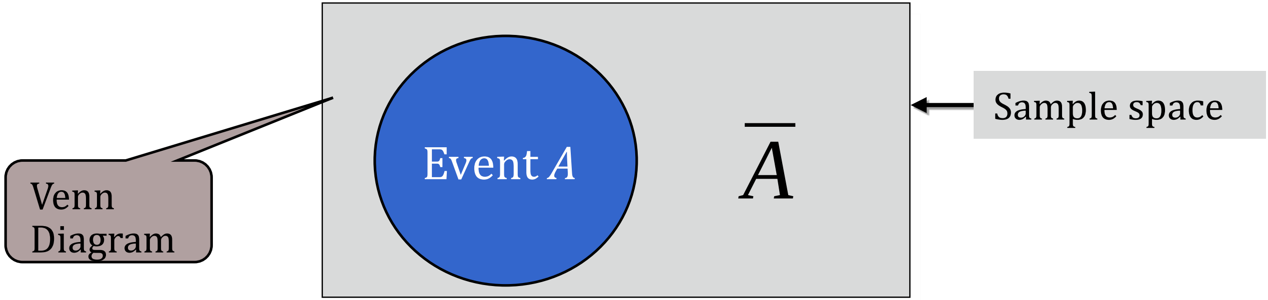
- **Null Event, denoted with Φ**
  - ➢ Club & Diamond on Card Draw

Null Event



Complement of an Event

The complement of event A is defined to be the event consisting of all sample points that are not in A. The complement of A is denoted $\overline{A}$



Example: The probability that a MSc student can pass his exam is p. The probability that he will fail is 1-p

P(success)=1- P(failure)

Profitable and non-profitable???

Compound events

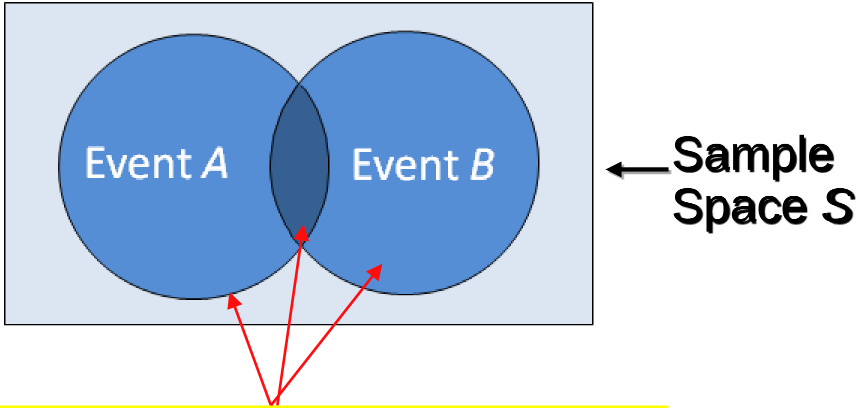
- **Union**
  - Outcomes in either Event A ***or*** Event B ***or*** Both
  - ‘OR’ Statement
  - ∪ Symbol (i.e., A **∪** B, A or B)

- **Intersection**
  - Outcomes in Both Events A ***and*** B
  - ‘AND’ Statement
  - ∩ Symbol (i.e., A **∩** B, A & B)

Union of Two Events

The union of events A and B is the set containing all sample points that are in A or B or both.

The union of events A and B is denoted by $A \cup B$, *A or B*

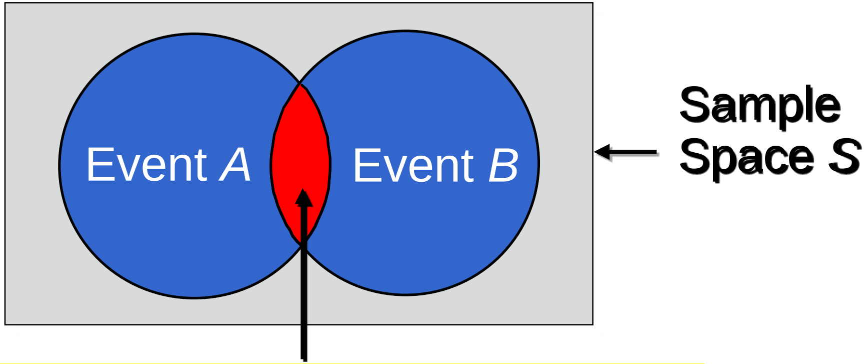


The entire blue area, including light blue and dark blue, is $A \cup B$

Intersection of Two Events

The intersection of events A and B is the set of all sample points that are in both A and B.

The intersection of events A and B is denoted by A & B, $A \cap B$



The red area is $A \cap B$

Sets in Python

business_analytics={'machine learning','big data','stats with python'} #modules of business analytics

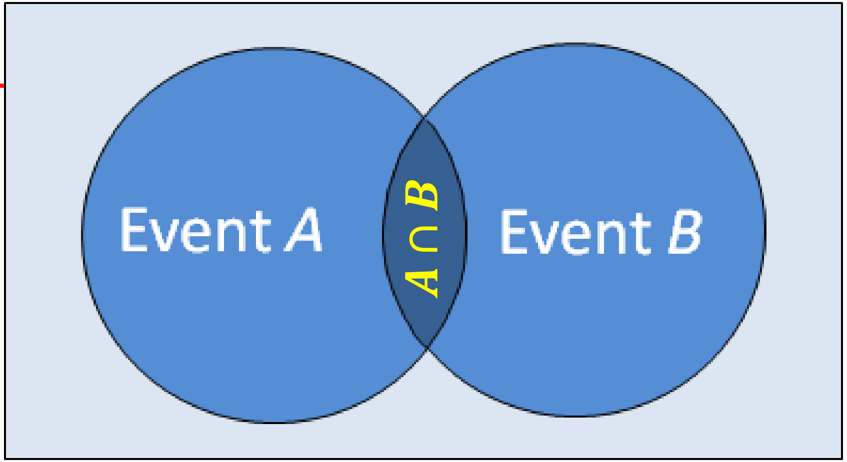
supply_chain={'machine learning','supply chain'} #modules of supply chain

common_modules=business_analytics.intersection(supply_chain) #modules shared by both

all_modules=business_analytics.union(supply_chain) # all modules

```
>>> business_analytics={'machine learning','big data','stats with python'}
>>> supply_chain={'machine learning','supply chain'}
>>> common_modules=business_analytics.intersection(supply_chain)
>>> all_modules=business_analytics.union(supply_chain)
>>> common_modules
{'machine learning'}
>>> all_modules
{'big data', 'supply chain', 'machine learning', 'stats with python'}
>>>
```

Addition Law



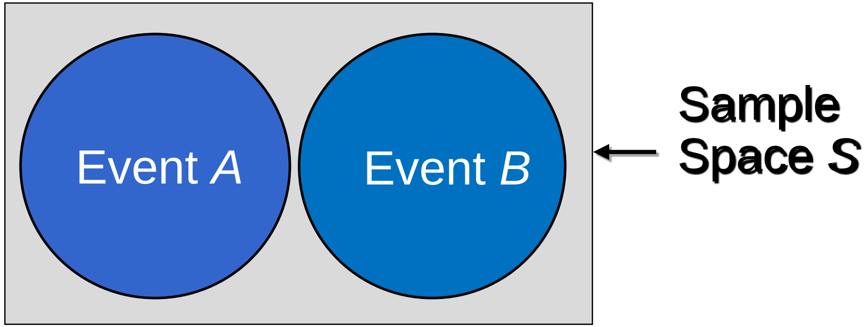
The addition law provides a way to compute the probability of event *A*, or *B*, or both *A* and *B* occurring.

The law is written as:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

Mutually Exclusive Events

Two events are mutually exclusive if, when one event occurs, the other cannot occur.



If events *A* and *B* are mutually exclusive, $P(A \cap B) = 0$.

The addition law for mutually exclusive events is:

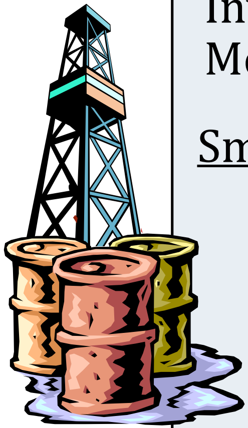
$$P(A \text{ or } B) = P(A) + P(B)$$

there's no need to include "- $P(A \cap B)$"

Examples, multiplication rule, conditional probability

Example: Mick’s Investments

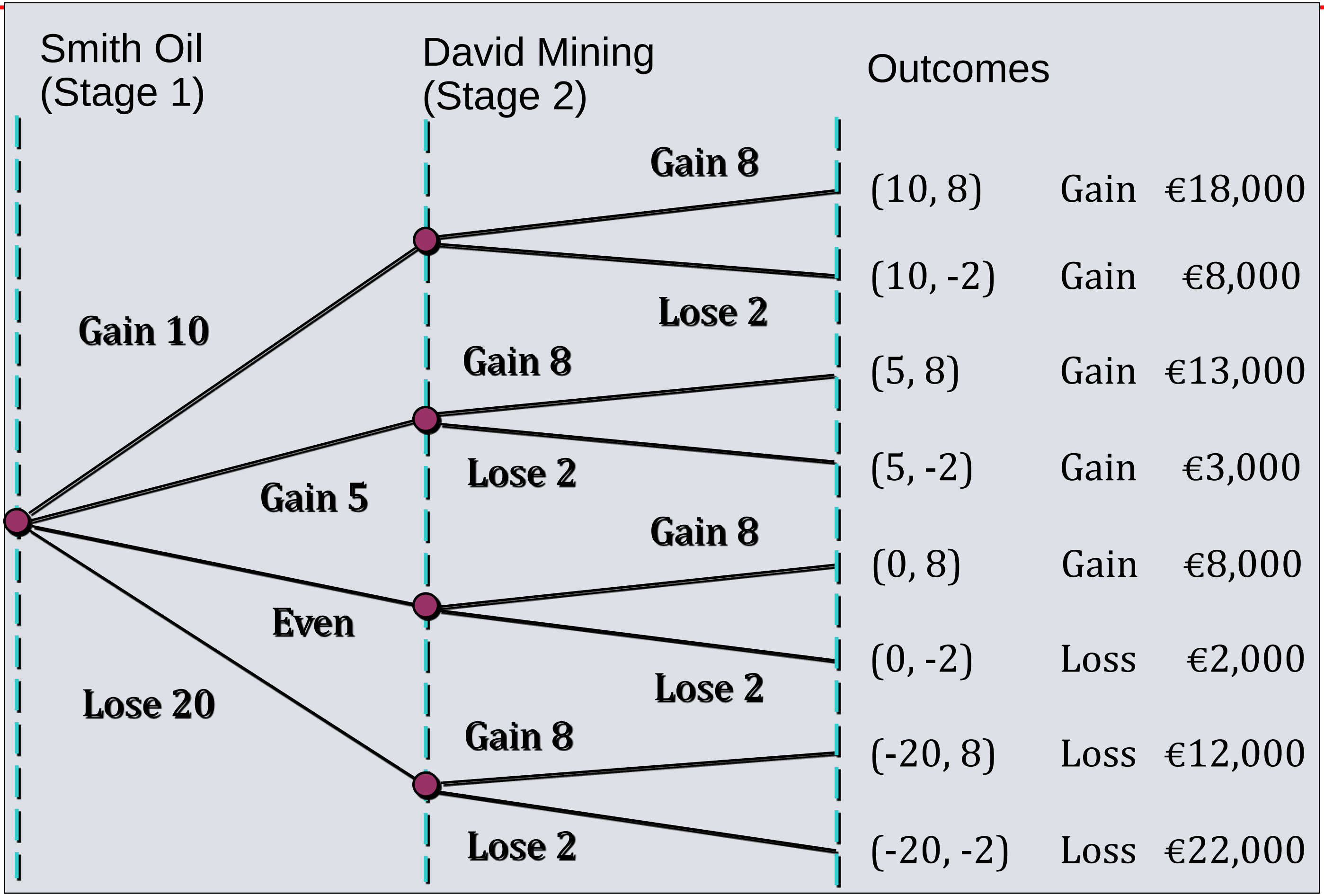
Mick has invested in two stocks, Smith Oil and David Mining. He estimated that the possible outcomes of these investments three months from now were as follows.



Investment Gain or Loss in 3 Months (in £1,000)

Smith Oil	David Mining
10	8
5	-2
0	
-20	

A Counting Rule for Multiple-Step Experiments

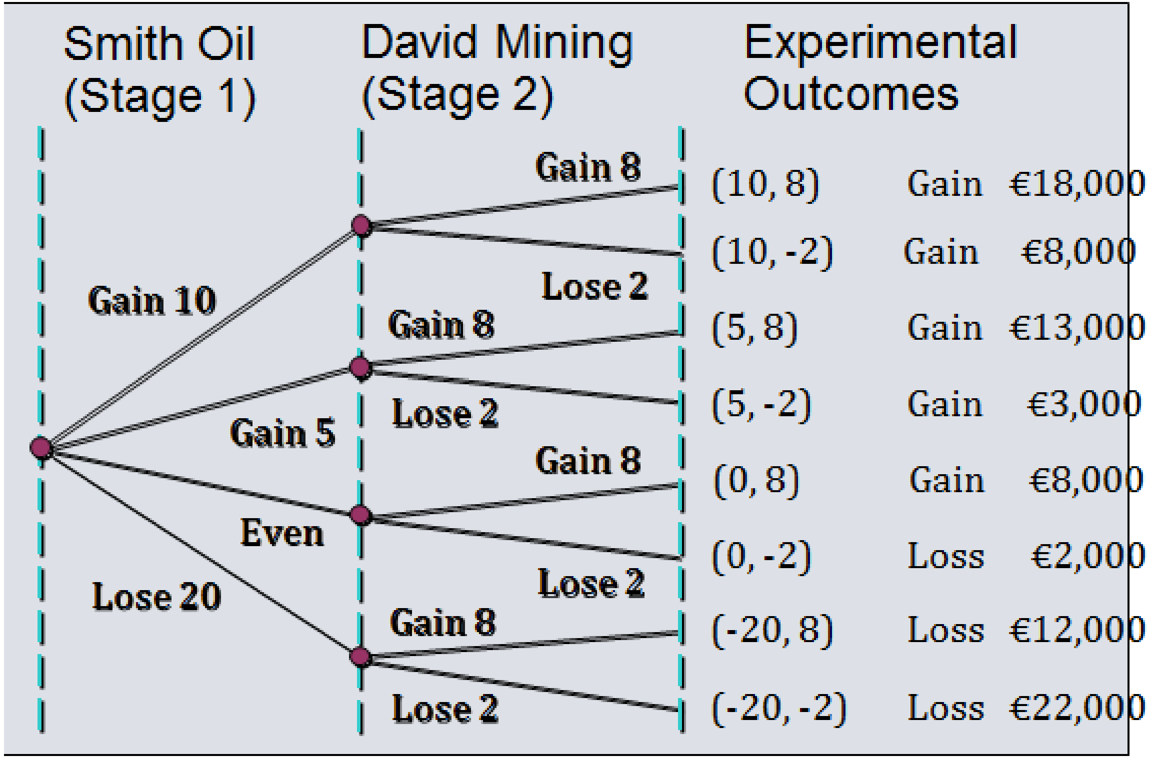


A Counting Rule for Multiple-Step Experiments

Mick's investments can be viewed as a two-step experiment. It involves two stocks, each with a set of experimental outcomes.

- Smith Oil: $n_1 = 4$
- David Mining: $n_2 = 2$
- Total Number of Experimental Outcomes: $n_1 \times n_2 = 4 \times 2 = 8$

Tree diagram



An example

Applying the subjective method, an analyst made the following probability assignments.

Exper. Outcome	Net Gain or Loss	Probability
(10, 8)	£18,000 Gain	.20
(10, -2)	£8,000 Gain	.08
(5, 8)	£13,000 Gain	.16
(5, -2)	£3,000 Gain	.26
(0, 8)	£8,000 Gain	.10
(0, -2)	£2,000 Loss	.12
(-20, 8)	£12,000 Loss	.02
(-20, -2)	£22,000 Loss	.06

Events and their Probabilities

Exper. Outcome	Net Gain or Loss	Probability
(10, 8)	€18,000 Gain	.20
(10, -2)	€8,000 Gain	.08
(5, 8)	€13,000 Gain	.16
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(-20, -2)	€22,000 Loss	.06

Event S = Smith Oil Profitable

$$S = \{(10, 8), (10, -2), (5, 8), (5, -2)\}$$

$$P(S) = P(10, 8) + P(10, -2) + P(5, 8) + P(5, -2)$$

$$= .20 + .08 + .16 + .26$$

$$= .70$$

Event D = David Mining Profitable

$$D = \{(10, 8), (5, 8), (0, 8), (-20, 8)\}$$

$$P(D) = P(10, 8) + P(5, 8) + P(0, 8) + P(-20, 8)$$

$$= .20 + .16 + .10 + .02$$

$$= .48$$

Probability of the Union of Two Events

Event S = Smith Oil Profitable

Event D = David Mining Profitable

S or D = Smith Oil Profitable
or David Mining Profitable

S or D = {(10, 8), (10, -2), (5, 8), (5, -2), (0, 8), (-20, 8)}

$$P(S \text{ or } D) = P(10, 8) + P(10, -2) + P(5, 8) + P(5, -2) + P(0, 8) + P(-20, 8)$$
$$= .20 + .08 + .16 + .26 + .10 + .02$$
$$= 0.82$$

Direct calculation

Use the formula

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

Event S = Smith Oil Profitable

Event D = David Mining Profitable

S or D = Smith Oil Profitable
or David Mining Profitable

We know: $P(S) = .70$, $P(D) = .48$, $P(S \cap D) = .36$

Thus: $P(S \text{ or } D) = P(S) + P(D) - P(S \cap D)$
$$= .70 + .48 - .36$$
$$= .82$$

(This result is the same as that obtained earlier using the definition of the probability of an event.)

Multiplication Law

The multiplication law provides a way to compute the probability of the intersection of two events.

The law is written as: $P(A \cap B) = P(B)P(A|B)$

$$P(A \cap B) = P(A)P(B|A)$$

Conditional probability

The probability of an event given that another event has occurred is called a conditional probability.

The conditional probability of *A given B* is denoted by $P(A|B)$.

The conditional probability can be obtained from the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Independent Events

If the probability of event *A* is not changed by the existence of event *B*, we would say that events *A* and *B* are <u>independent</u>.

Two events *A* and *B* are independent if:

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

The multiplication law for Independent events becomes

$$P(A \cap B) = P(A)P(B)$$

Intersection of Two Events

Exper. Outcome	Net Gain or Loss	Probability
(10, 8)	€18,000 Gain	.20
(10, -2)	€8,000 Gain	.08
(5, 8)	€13,000 Gain	.16
(5, -2)	€3,000 Gain	.26
(0, 8)	€8,000 Gain	.10
(0, -2)	€2,000 Loss	.12
(-20, 8)	€12,000 Loss	.02
(-20, -2)	€22,000 Loss	.06

Event S = Smith Oil Profitable

Event D = David Mining Profitable

$S \cap D$ = Smith Oil Profitable and David Mining Profitable

$S \cap D = \{(10, 8), (5, 8)\}$

$$P(S \cap D) = P(10, 8) + P(5, 8)$$
$$= .20 + .16$$
$$= .36$$

We are unable to use the following formula yet as we do not know the conditional probability at this moment

The law is written as: $P(A \cap B) = P(B)P(A|B)$

$P(A \cap B) = P(A)P(B|A)$

To decide if events are independent

Event S = Smith Oil Profitable

Event D = David Mining Profitable

Are events S and D independent?

Does $P(S \cap D) = P(S)P(D)$?

We know: $P(S \cap D) = .36$, $P(S) = .70$, $P(D) = .48$

But: $P(S)P(D) = (.70)(.48) = .34$, not .36

Hence: S and D are <u>not</u> independent.

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Event S = Smith Oil Profitable

Event D = David Mining Profitable

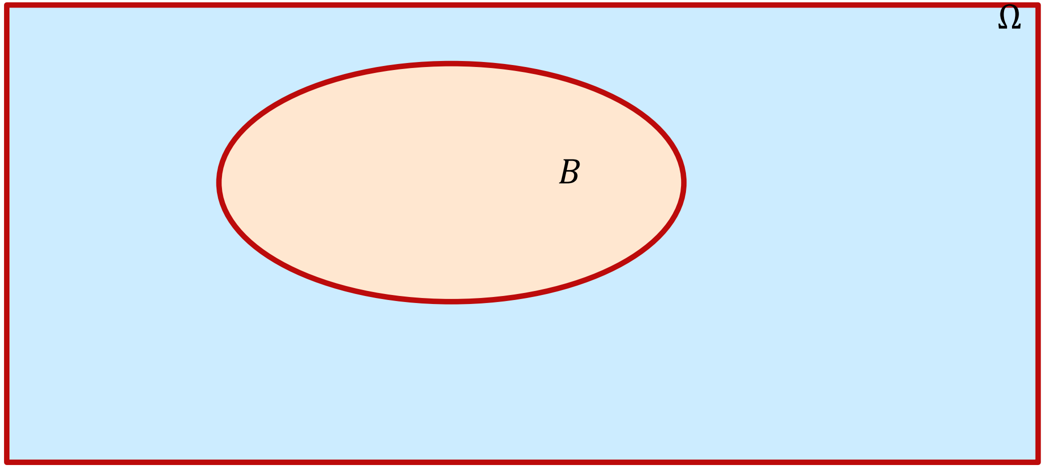
$P(D|S)$ = David Mining Profitable given Smith Oil Profitable

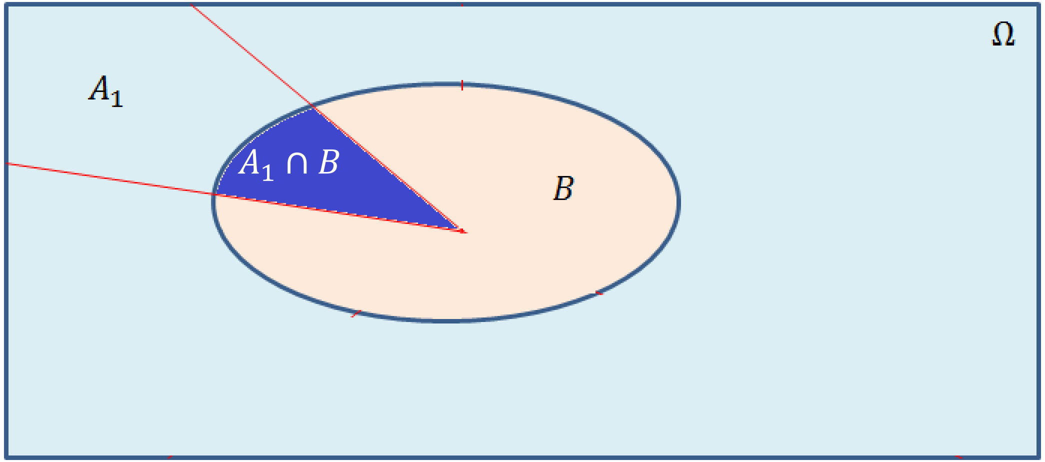
We know: $P(S \cap D) = .36$, $P(S) = .70$

Thus: $P(D|S) = \frac{P(S \cap D)}{P(S)} = \frac{0.36}{0.70} = 0.5143$

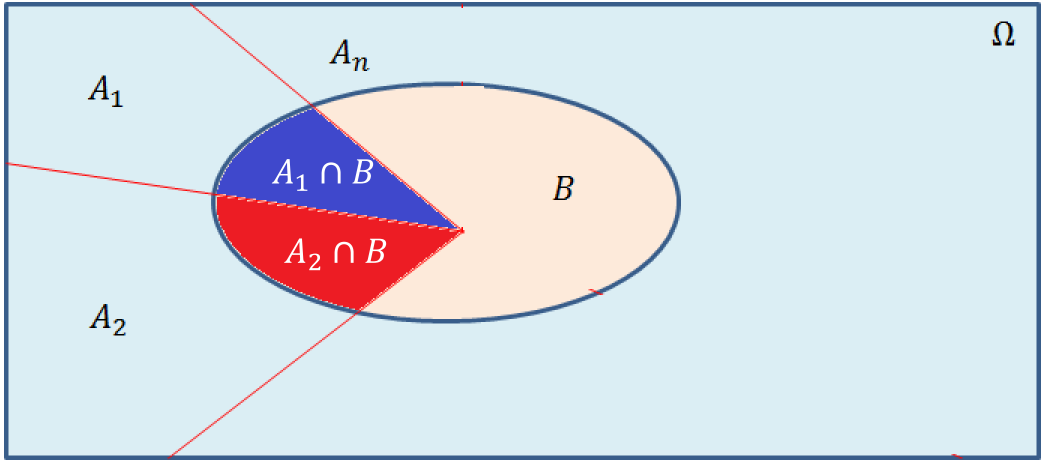
The theorem of total probability, Bayes theorem

The Theorem of Total Probability

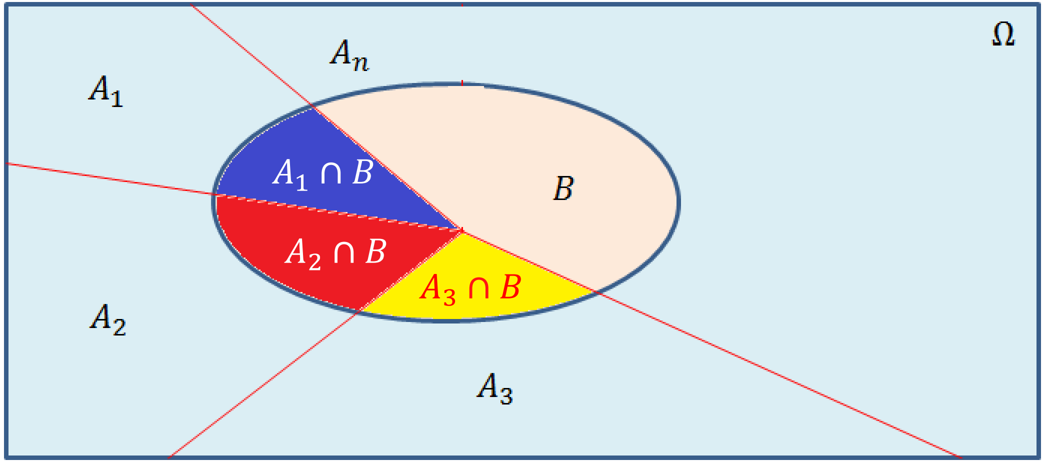


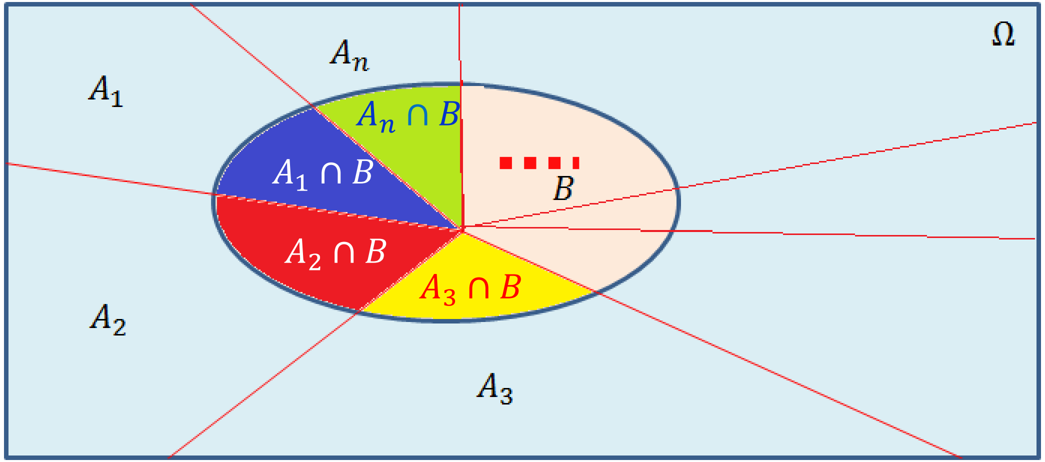


Ω
A_1
$A_1 \cap B$
B

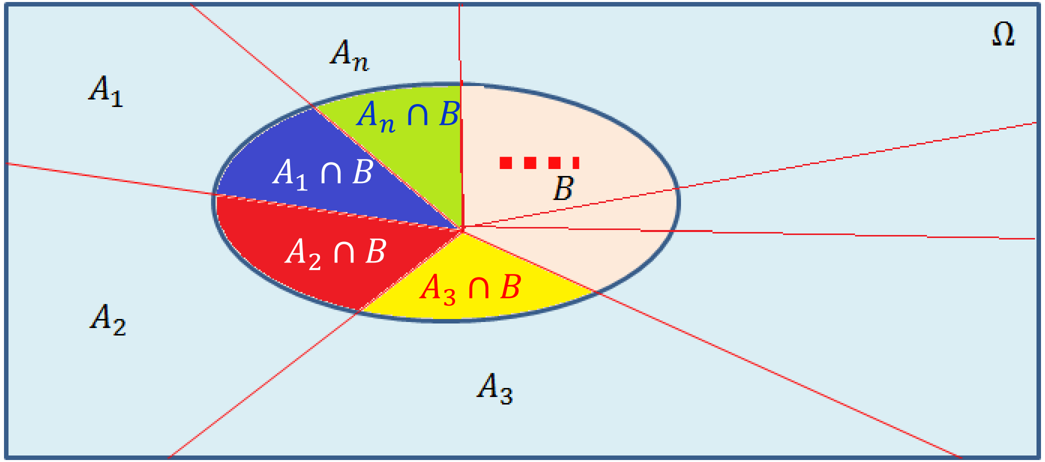


A_n
Ω
A_1
$A_1 \cap B$
B
$A_2 \cap B$
A_2


Ω
A_n
A_1
$A_1 \cap B$
B
$A_2 \cap B$
$A_3 \cap B$
A_2
A_3


Ω
A_1
A_n
$A_n \cap B$
$A_1 \cap B$
B
$A_2 \cap B$
$A_3 \cap B$
A_2
A_3

Law of total probability



- $(A_i \cap B) \cap (A_j \cap B) = \emptyset, where\ i \neq j$

- $B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$

Law of total probability:

- Then $\mathrm{P}(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$
  $= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)$

Bayes' Theorem

- Then $P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$
  $= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)$

- To find the posterior probability, $P(A_i|B)$, that event A_i will occur given that event B has occurred, we apply
  Bayes' theorem.

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}$$

- $P(A_i)$: Prior probability
- $P(B|A_i)$: Likelihood
- $P(A_i|B)$: Posterior probability

Calculating the posterior distribution can be difficult. Some methods such as MCMC (Markov chain Monte Carlo) and Variational Inference can be applied

Example 2: Application of Bayes' Theorem

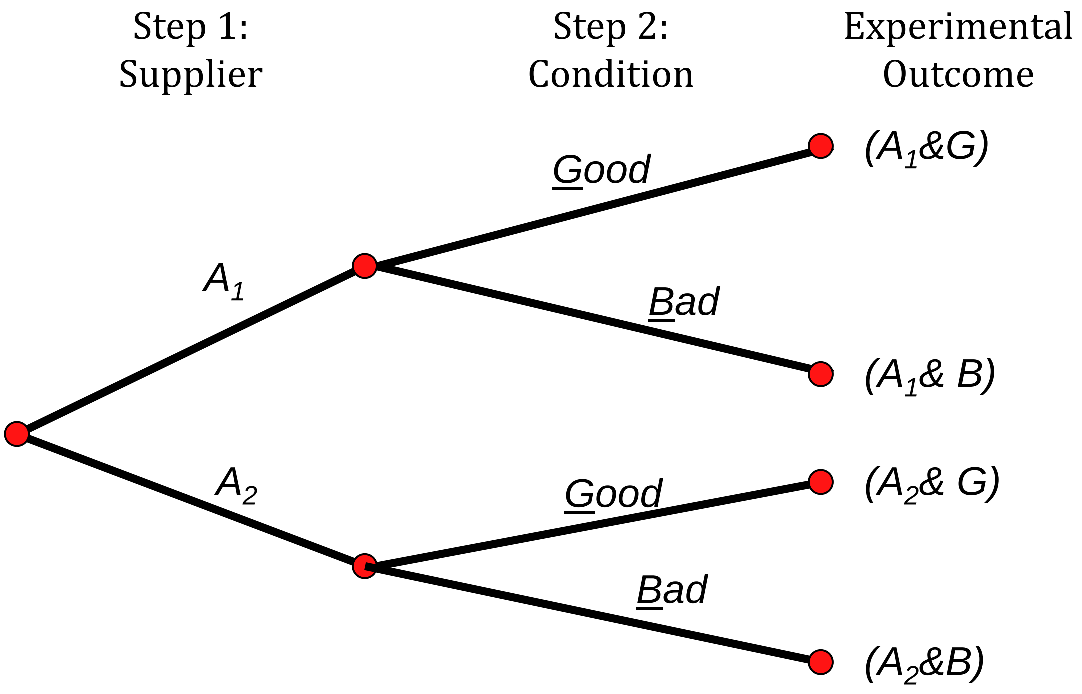
- Consider a manufacturing firm that receives shipment of parts from two suppliers.
- Let
  - A1: event that a part is received from supplier 1; and
  - A2: event that a part is received from supplier 2
- Suppose that
  - 65% of our parts are supplied by supplier 1 and
  - 35% of our parts are supplied by supplier 2.
  - That is:
    - $P(A_1) = 0.65$ and
    - $P(A_2) = 0.35$.
- **The above gives prior probabilities**

Quality levels differ between suppliers

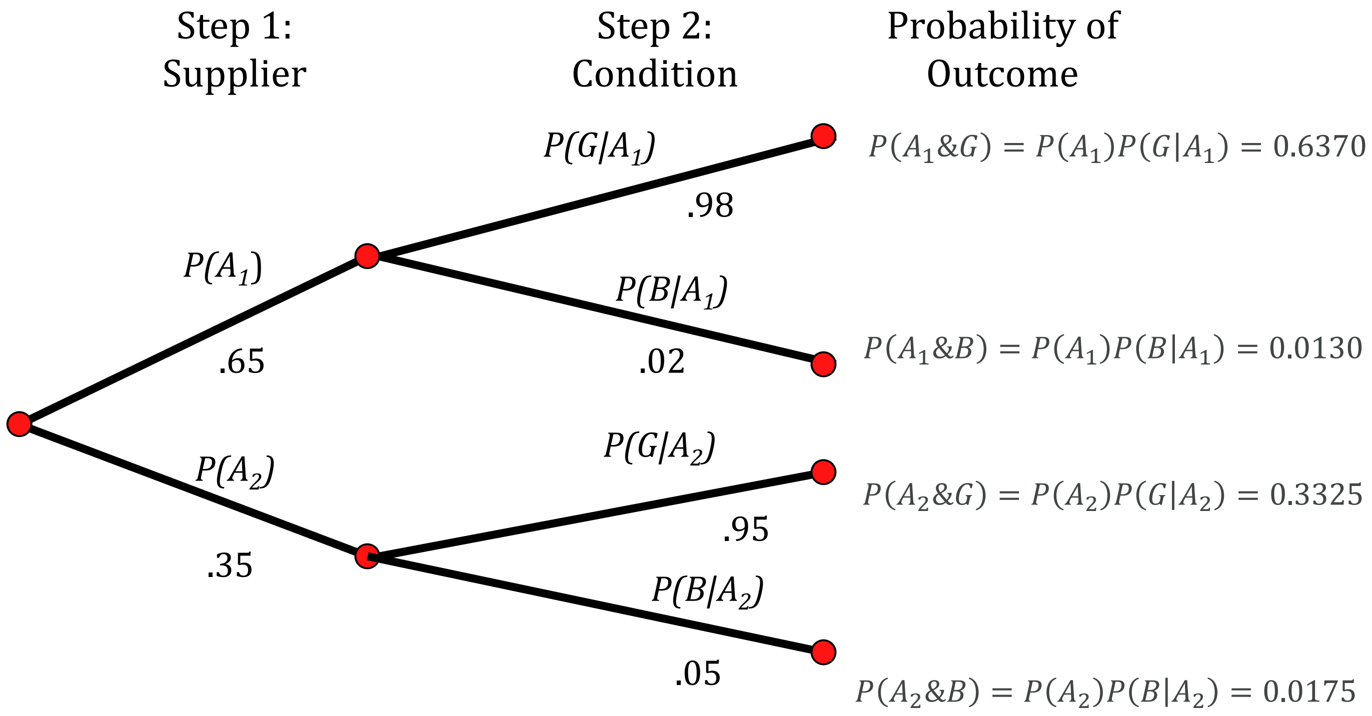
	Percentage Good Parts	Percentage Bad Parts
Supplier 1	98	2
Supplier 2	95	5

- Let
  - G: a part is good, and
  - B: a part is bad
- According to the above table, we have
  - $P(G|A_1)$ =0.98 and $P(B|A_1)$ =0.02
  - $P(G|A_2)$ =0.95 and $P(B|A_2)$ = 0.05
- **The above gives new information, or likelihood**

Tree Diagram for Two-Supplier Example



Probability Tree for the Two-Supplier Example



Application of Bayes' theorem

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$= \frac{(.65)(.02)}{(.65)(.02) + (.35)(.05)} = \frac{.0130}{.0305} = .4262$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$= \frac{(.35)(.05)}{(.65)(.02) + (.35)(.05)} = \frac{.0175}{.0305} = .5738$$

The above obtains the posterior probabilities, which updates our knowledge

Summary I

- Sample space
- Events
  - Null events
  - Complementary events
  - Mutually exclusive events
  - Independent events
- Compound events
  - Union
  - Intersection
- Conditional probability
- Bayes' theorem

Summary II

- OR
  - $P(A \cup B)=P(A)+P(B)- P(A \cap B)$

- AND
  - $P(A \cap B)=P(A)P(B|A)$
  - $P(A \cap B)=P(B)P(A|B)$

- Exclusive
  - If A and B are mutually exclusive, then $P(A \cap B)=0$

- Independent
  - If A is not independent of B, then
    $P(A \cap B) = P(A)P(B|A)$
  - If A is independent of B, then
    $P(A \cap B)=P(A)P(B)$