Topological Data Analysis

Lecture 11

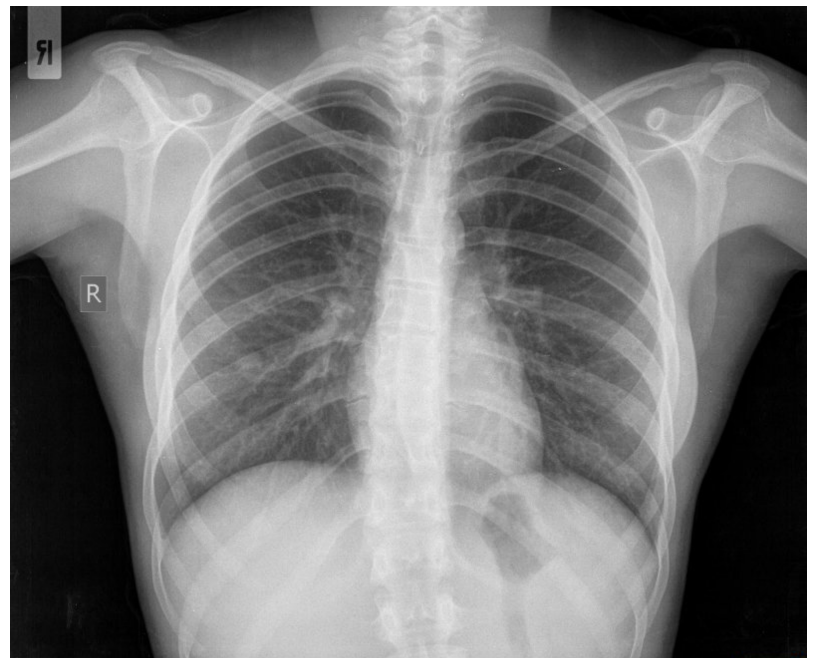
Topological Data Analysis of Digital Images

Oleg Kachan

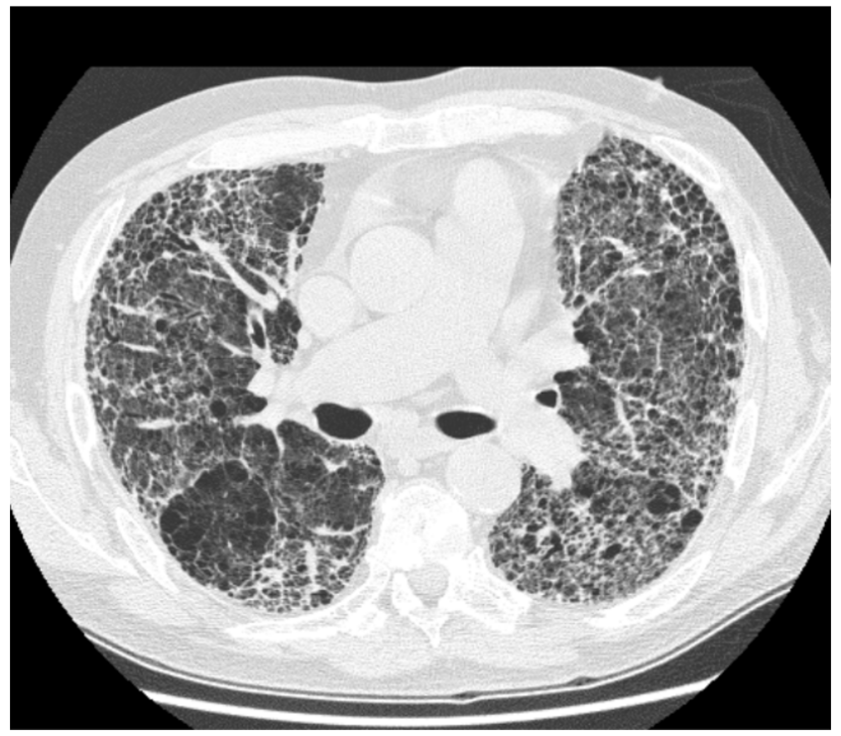
Digital Images

Medical imaging

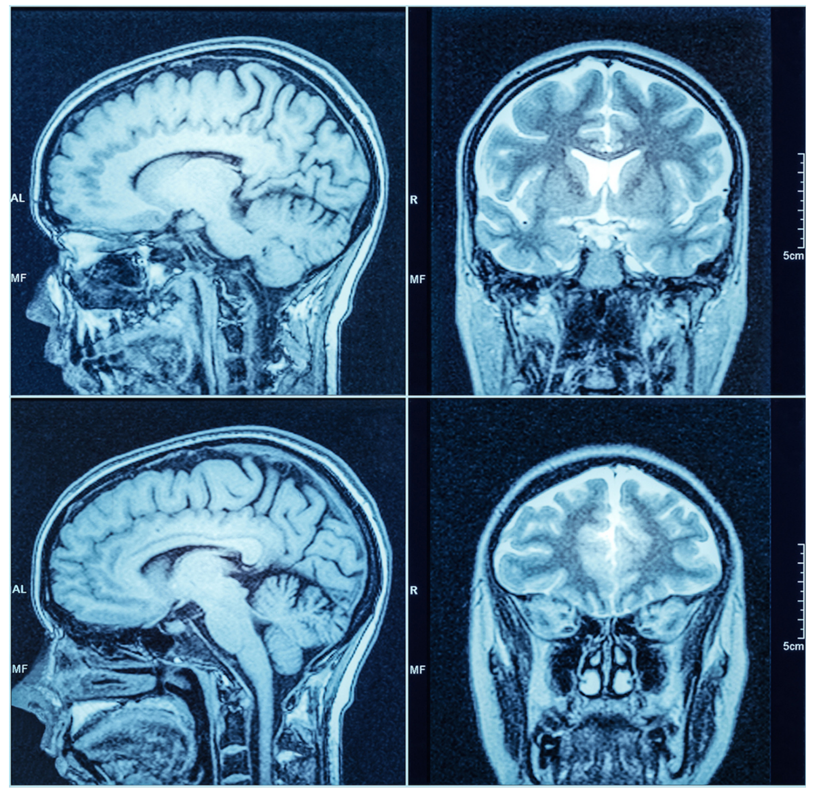
RX



CT

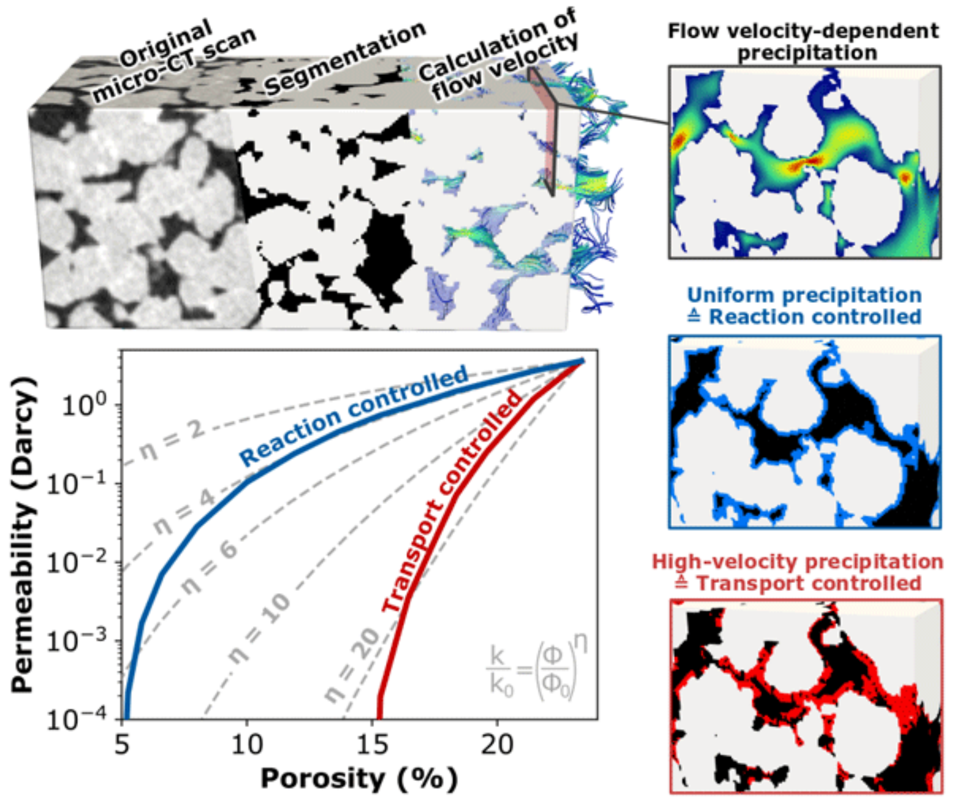


MRI

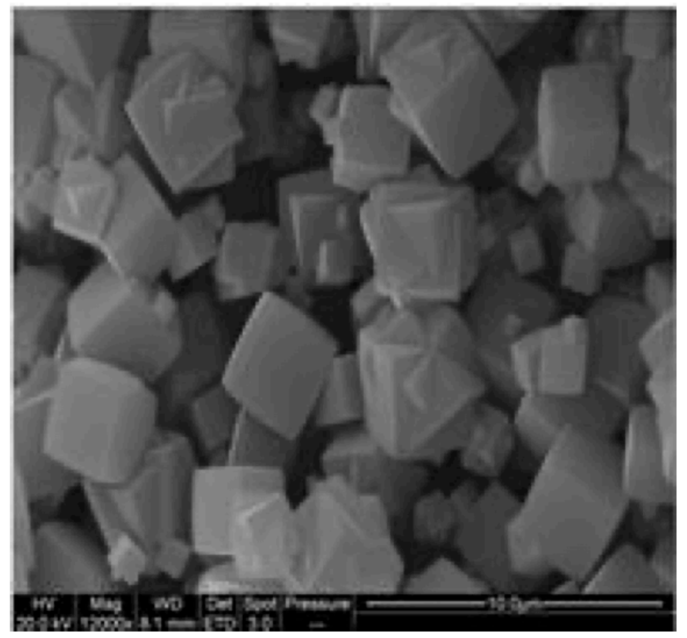


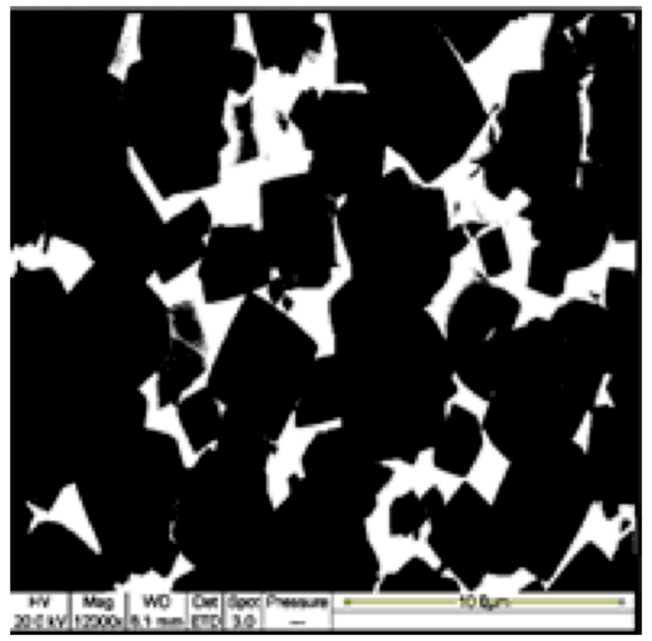
Material science

Porous media

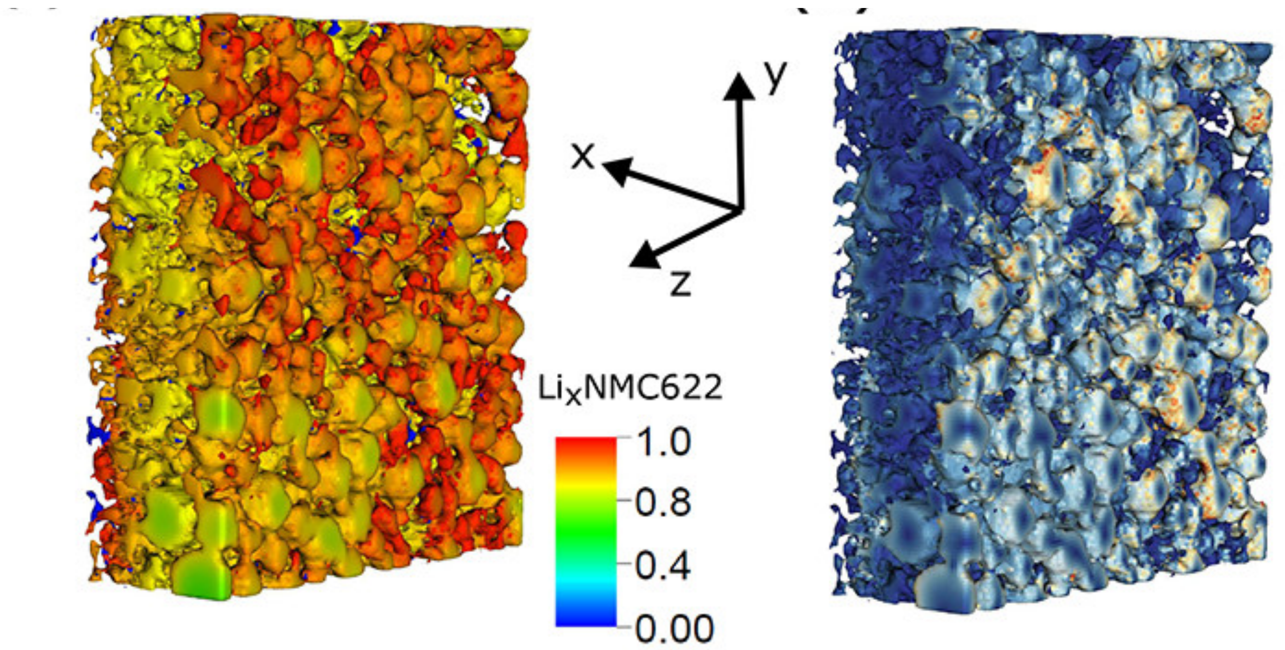


Catalysts



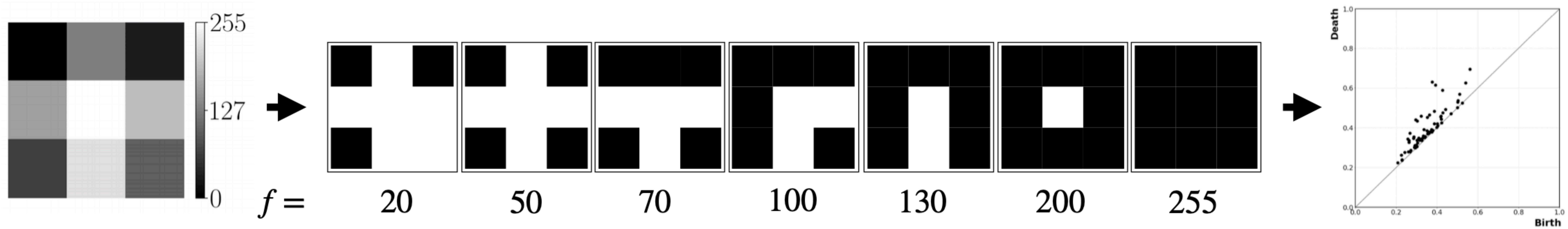


Batteries



Digital Images

Topological Data Analysis



Image/filter function

(X, f)

Filtration

$\varnothing = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_t = X$

Persistence diagram

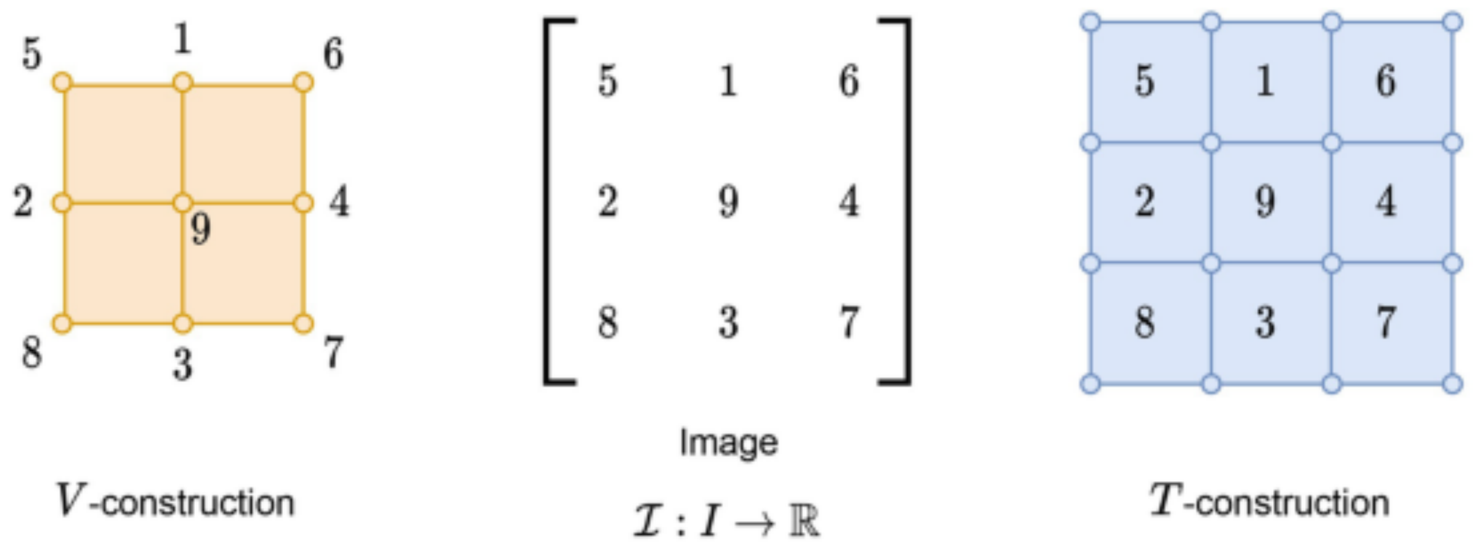
$D_f(X)$

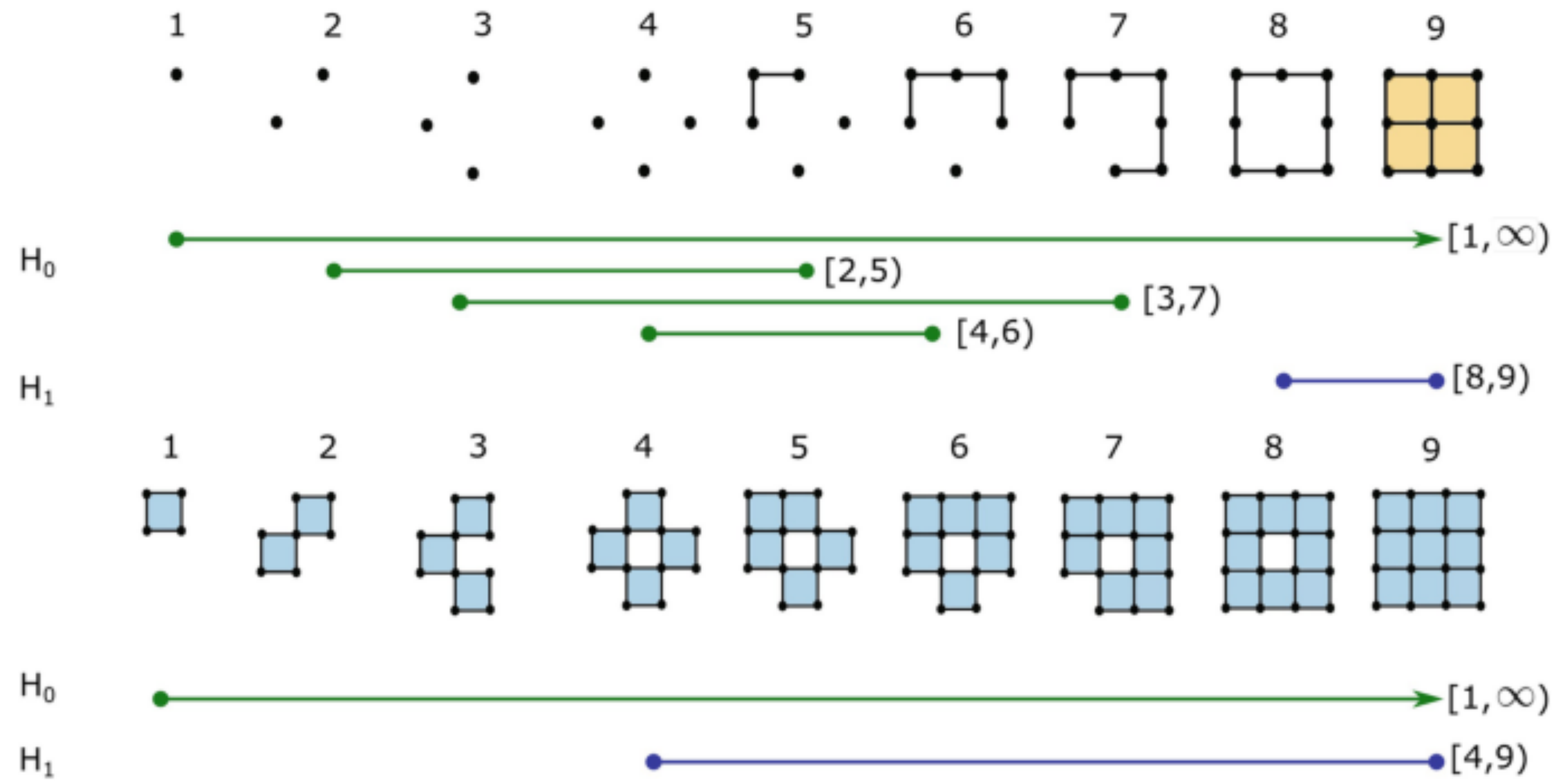
Digital Images

Cubical complex

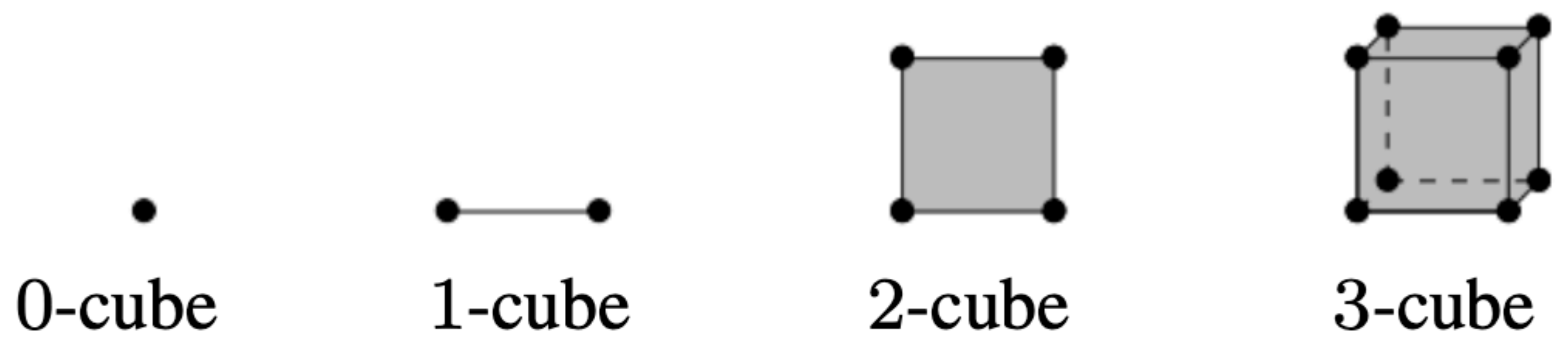
A d-dimensional image is a map $\mathscr{I} : I \subseteq \mathbb{Z}^d \to \mathbb{R}_{\geq 0}$

$\mathscr{I}(x)$ is an intensity of a voxel x.





Elementary cubes



Persistence Homology of Images

Filtration functions

Intensity

$f(x)$ is given by the intensity $\mathcal{I}(x)$ of the image.

Direction

$$f_v(x) = d(x, H(v)),$$

where v is a direction vector.

$H(v)$ is a hyperplane defined by v.

Radial

$$f_c(x) = d(x, c),$$

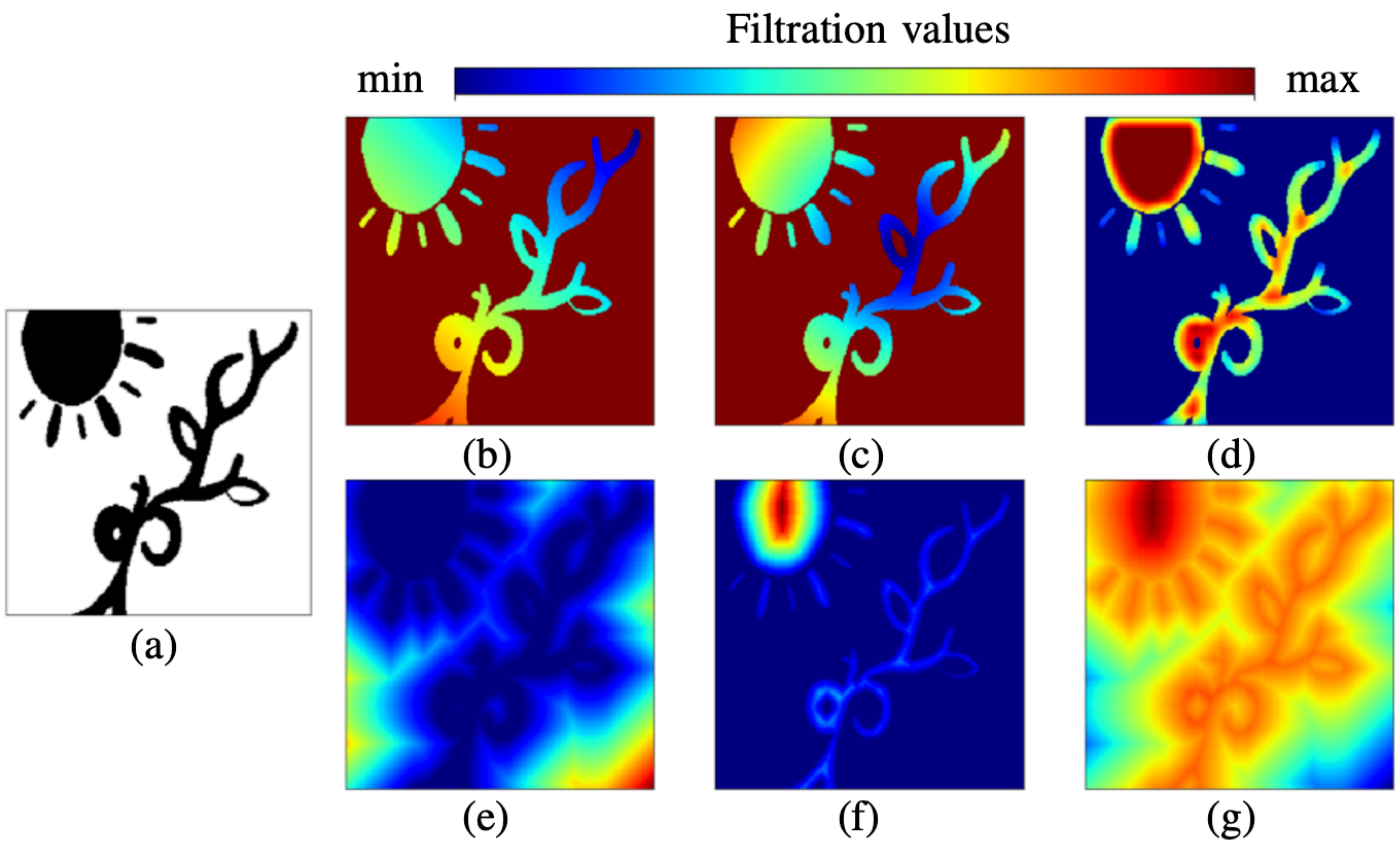
where c is a center point.

Signed Euclidean distance transform (EDT)

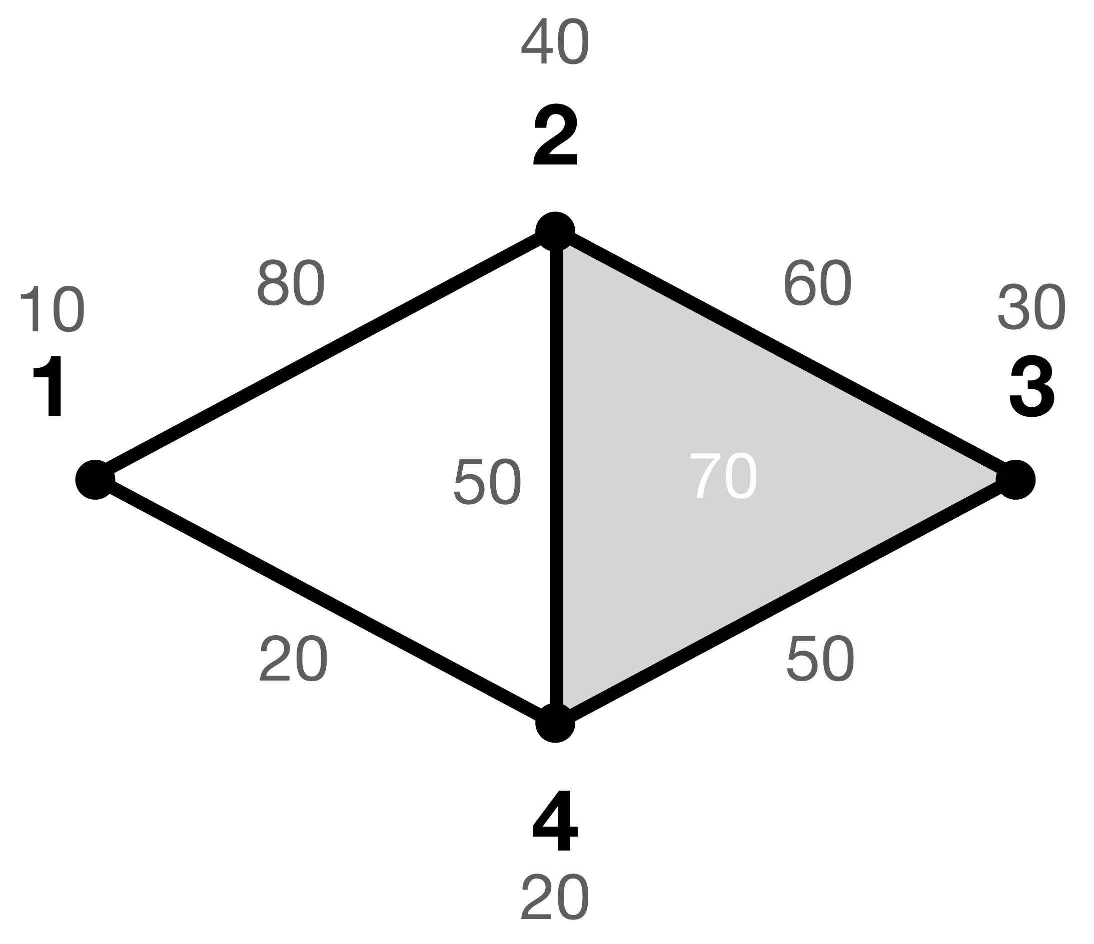
$$f(x) = \begin{cases} d(x, \partial\Omega), & x \in \Omega, \\ -d(x, \partial\Omega), & x \in \Omega^c, \end{cases}$$

where $d(x, \partial\Omega) = \inf_{y \in \partial\Omega} d(x, y)$

$\partial\Omega$ is a boundary of Ω.



Differentiability of the persistent homology



$\mathbf{R} =$

	10	20	20	30	40	50	50	60	70	80
	1	**4**	**14**	**3**	**2**	**24**	**34**	**23**	**234**	**12**
1			1							
4			1			1	1			
14										
3							1			
2						1				
24									1	
34									1	
23									1	
12										

Persistence pairing

(4, 14) 0
(2, 24) 0
(3, 34) 0
(23, 234) 1

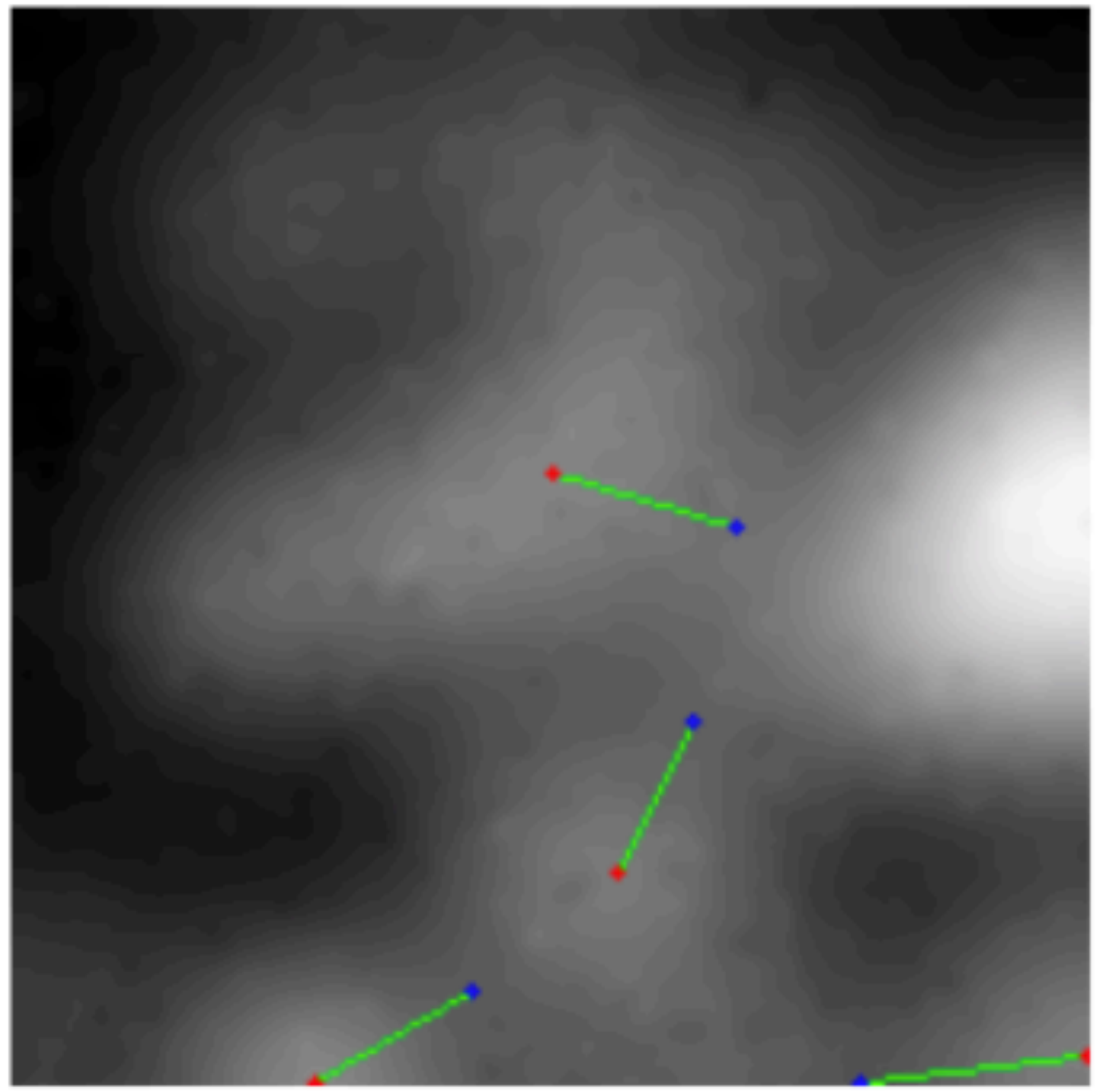
(1, ∅) 0
(12, ∅) 1

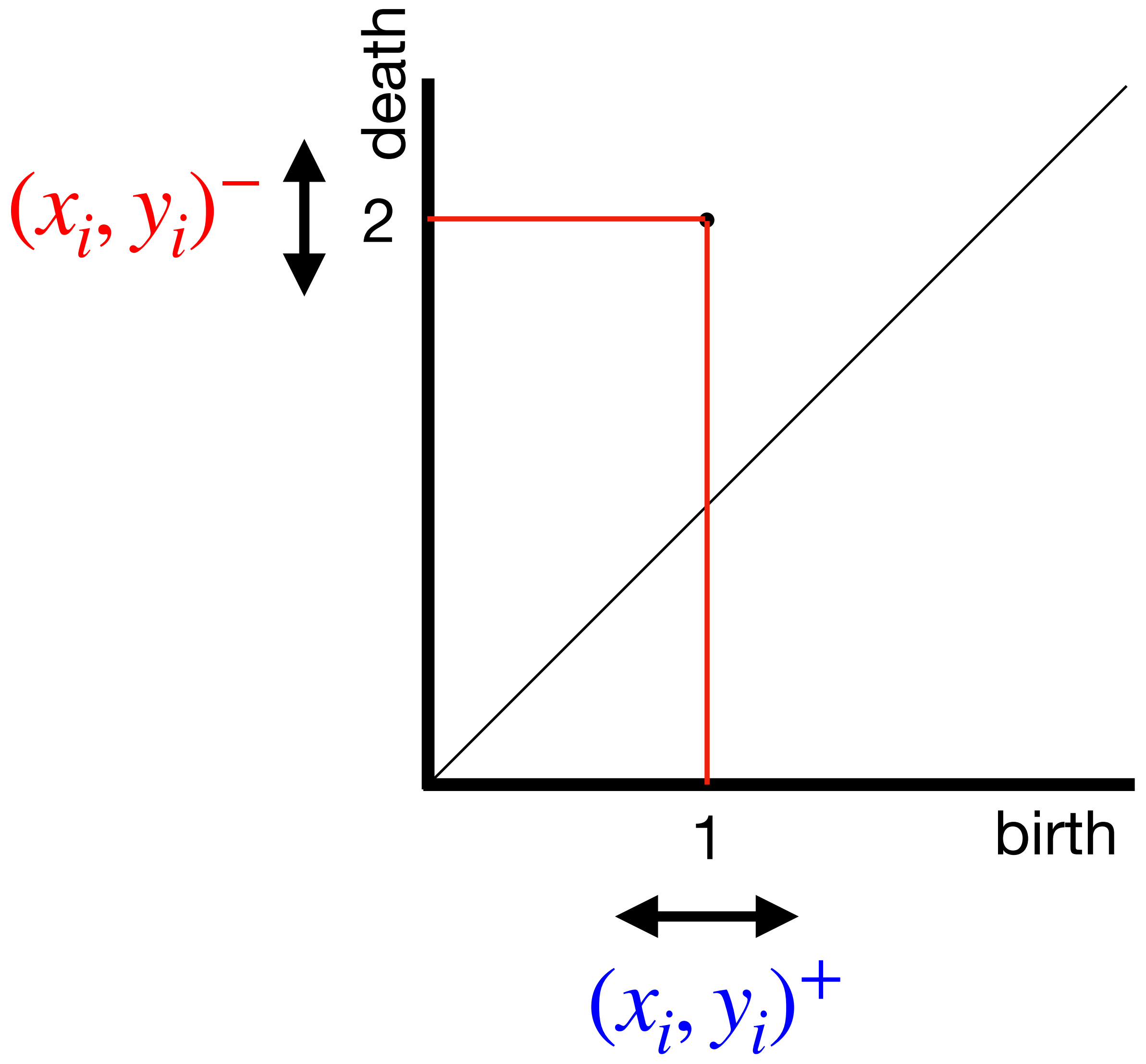
Persistence diagram

(20, 20) 0
(40, 50) 0
(30, 50) 0
(60, 70) 1

(10, ∅) 0
(80, ∅) 1

Differentiability of the persistent homology





Persistence pairing

(4, 14) 0
(2, 24) 0
(3, 34) 0
(23, 234) 1
(1, Ø) 0
(12, Ø) 1

Persistence diagram

(20, 20) 0
(40, 50) 0
(30, 50) 0
(60, 70) 1
(10, Ø) 0
(80, Ø) 1

Software

GUDHI

```
from gudhi.sklearn.cubical_persistence import CubicalPersistence

X = np.random.uniform(size=(100, 100))

pd = CubicalPersistence(newshape=[100, 100])
```

Cubical Ripser

```
import cripser

X = np.random.uniform(size=(100, 100))

pd = cripser.computePH(X, maxdim=2)
```

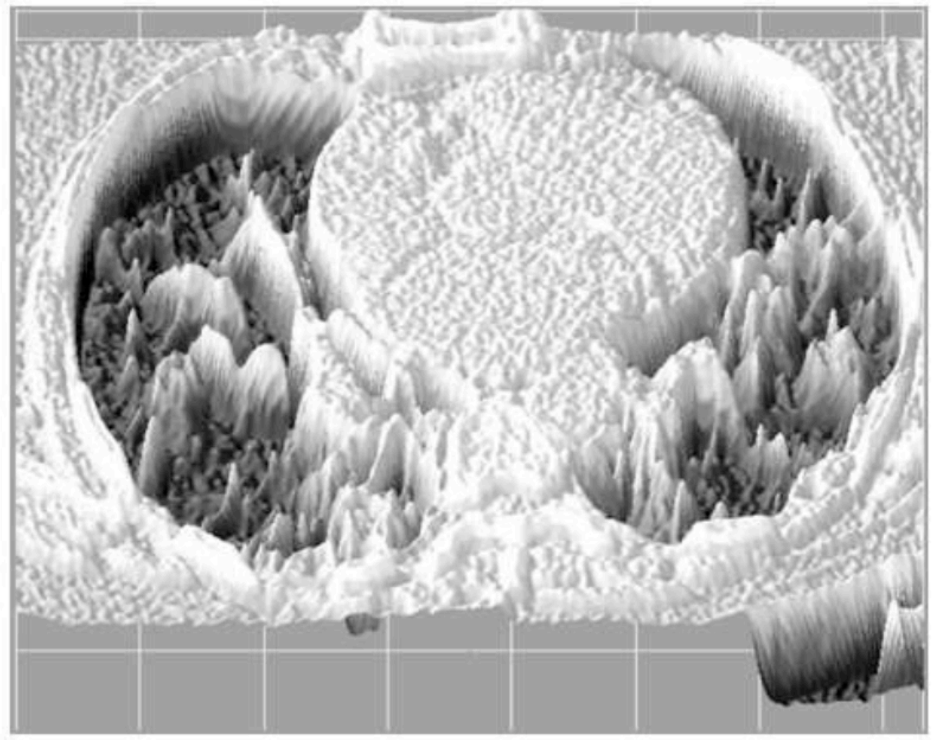
Homcloud

```
import homcloud.interface as hc

X = np.random.uniform(size=(100, 100))

hc.PDList.from_bitmap_levelset(X, "sublevel", save_to=“grayscale.pdgm")

pd0 = hc.PDList(“grayscale.pdgm").dth_diagram(0)
pd1 = hc.PDList(“grayscale.pdgm").dth_diagram(1)
```

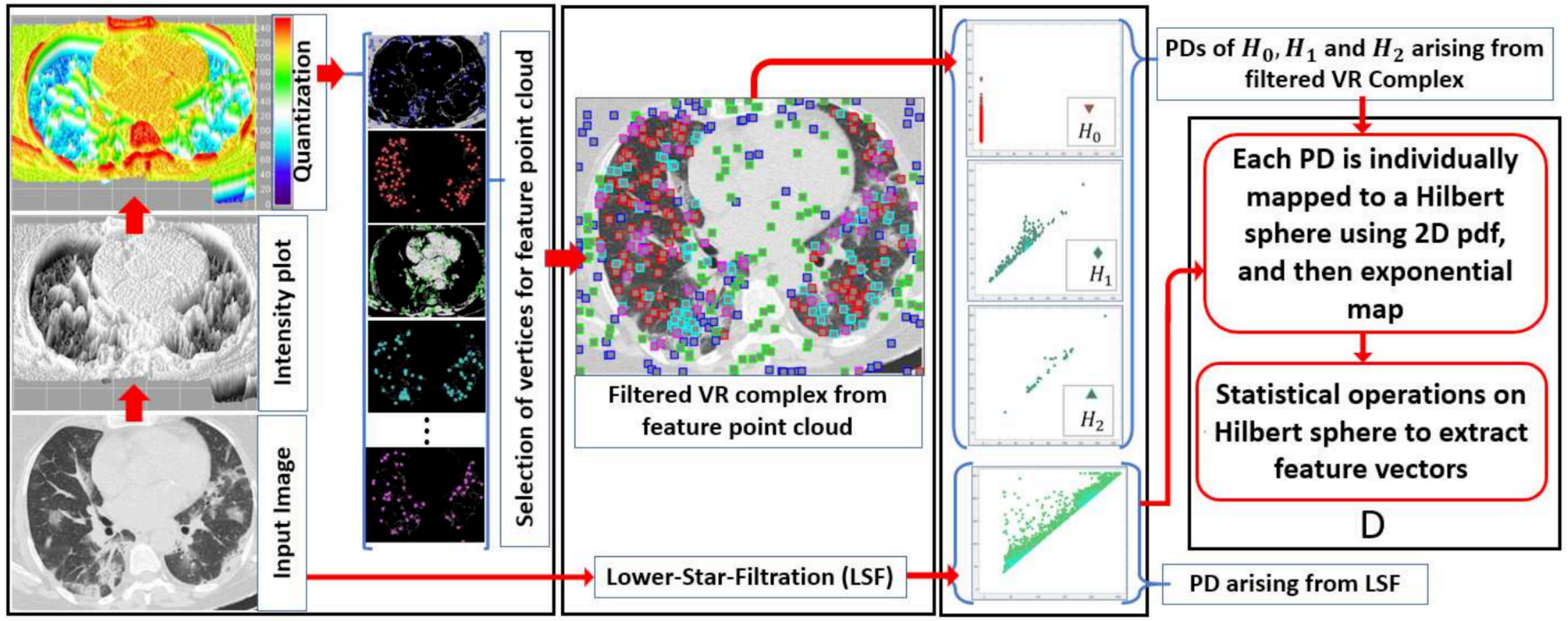
Applications

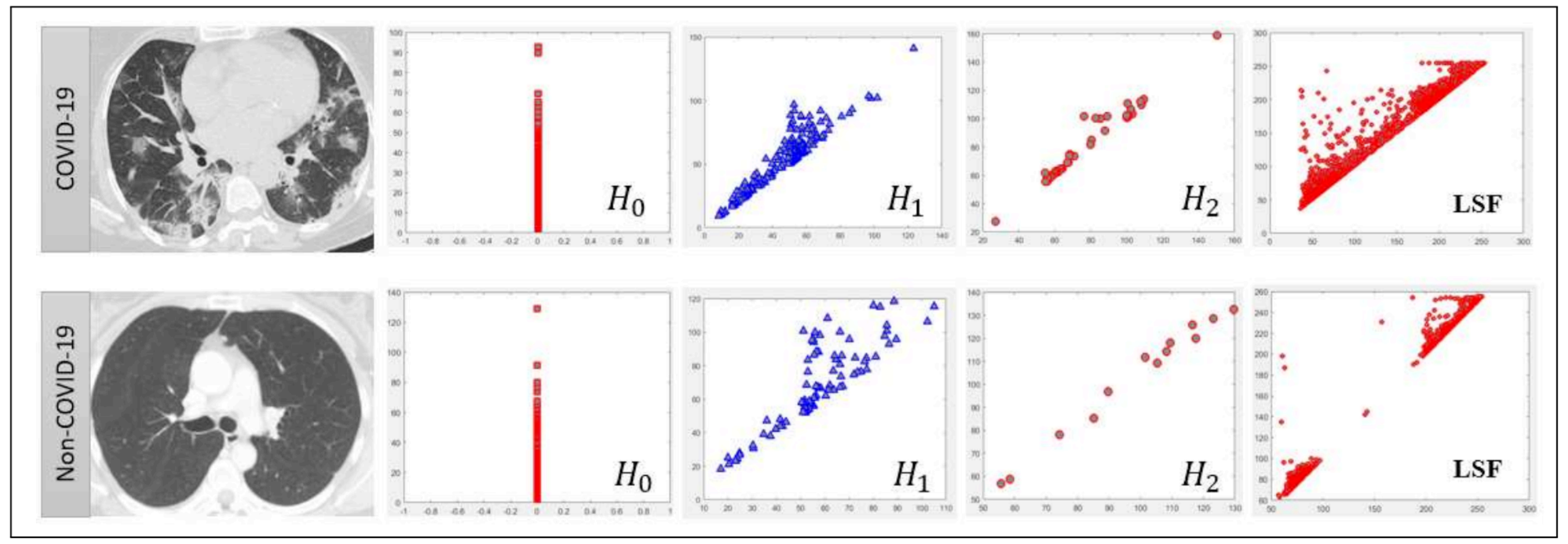
Image classification

Grayscale image



Intensity filtration (LSF)



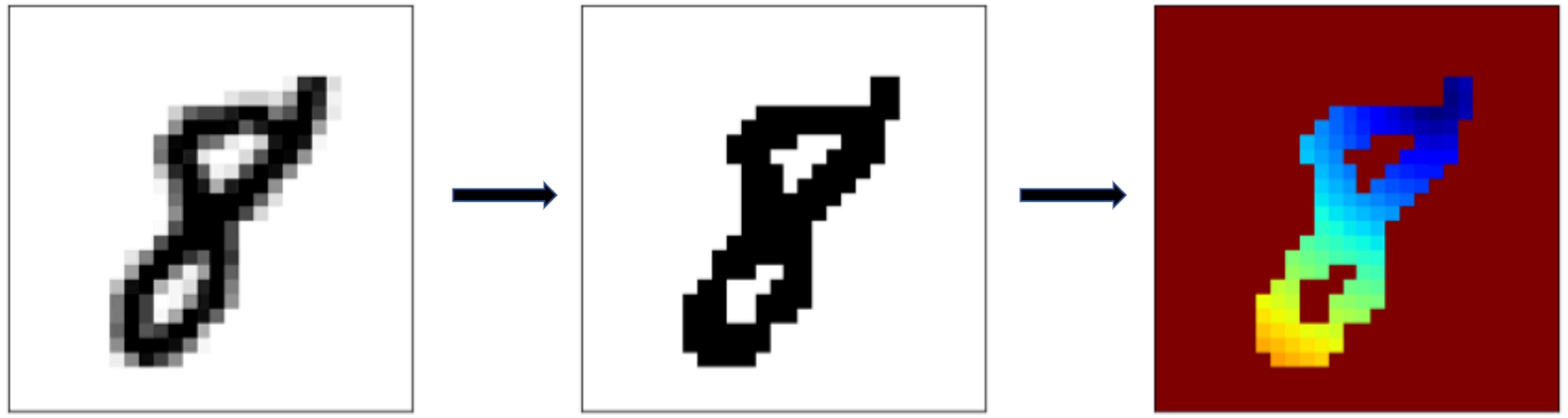


Top. Feature \ Metric	Accuracy	Precision	Recall	Specificity	F1 Score
H_0	71.6 ± 2.1	70.3 ± 3.3	74 ± 1.1	69.2 ± 3.7	72 ± 1.93
H_1	68.5 ± 0.5	67.1 ± 2.6	72.6 ± 7.7	64.4 ± 8.1	69.5 ± 32.3
H_2	64.8 ± 2.1	63.5 ± 1.8	67.7 ± 4.7	61.9 ± 2.2	65.5 ± 2.5
LSF	97.5 ± 0.6	97 ± 1	98 ± 0.6	97 ± 1	97.5 ± 0.6

Classification of COVID-19 via Homology of CT-SCAN
Iqbal et al. *(2021)*

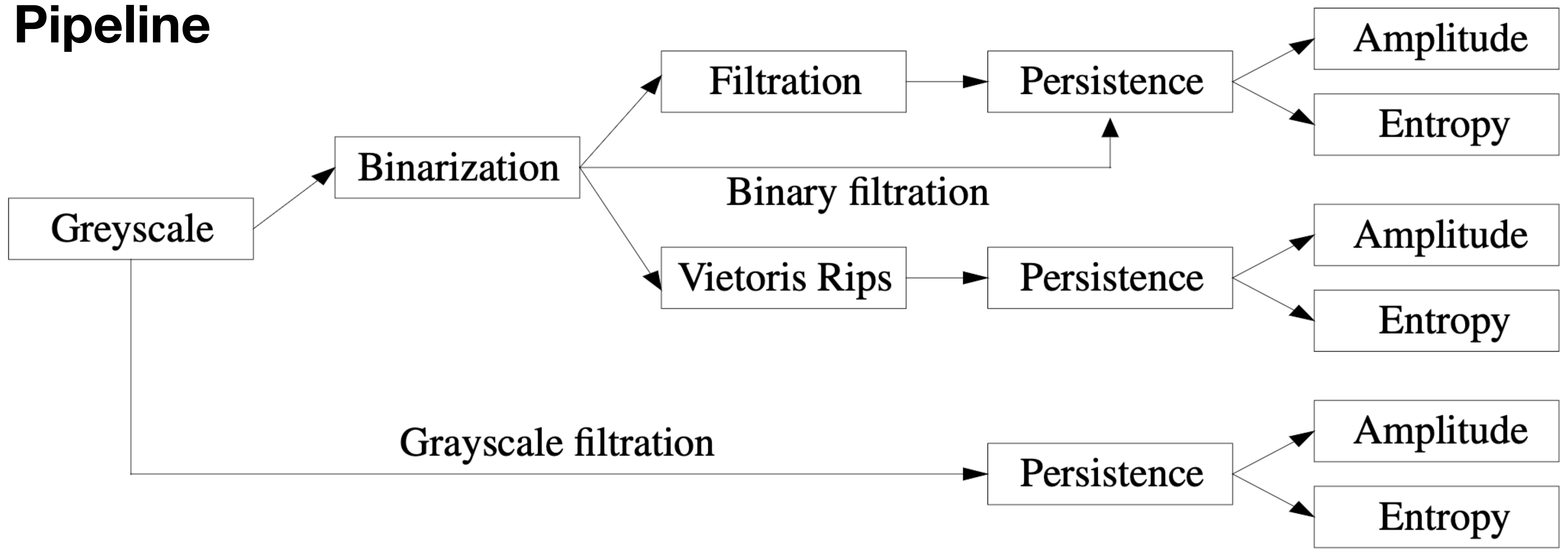
Applications

Image classification

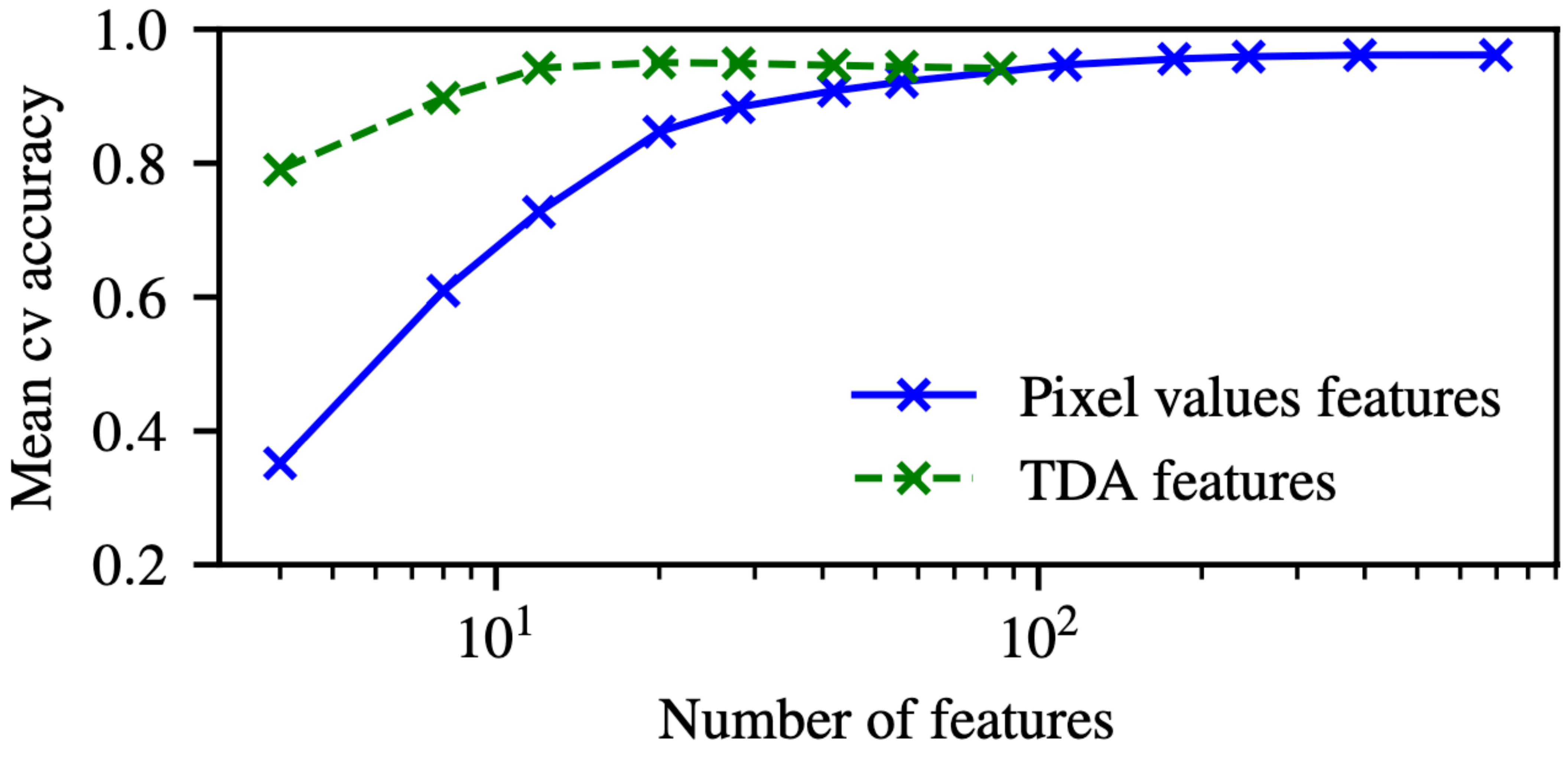


Grayscale image | **Binarized image** | **Filtered image**

Pipeline



Results



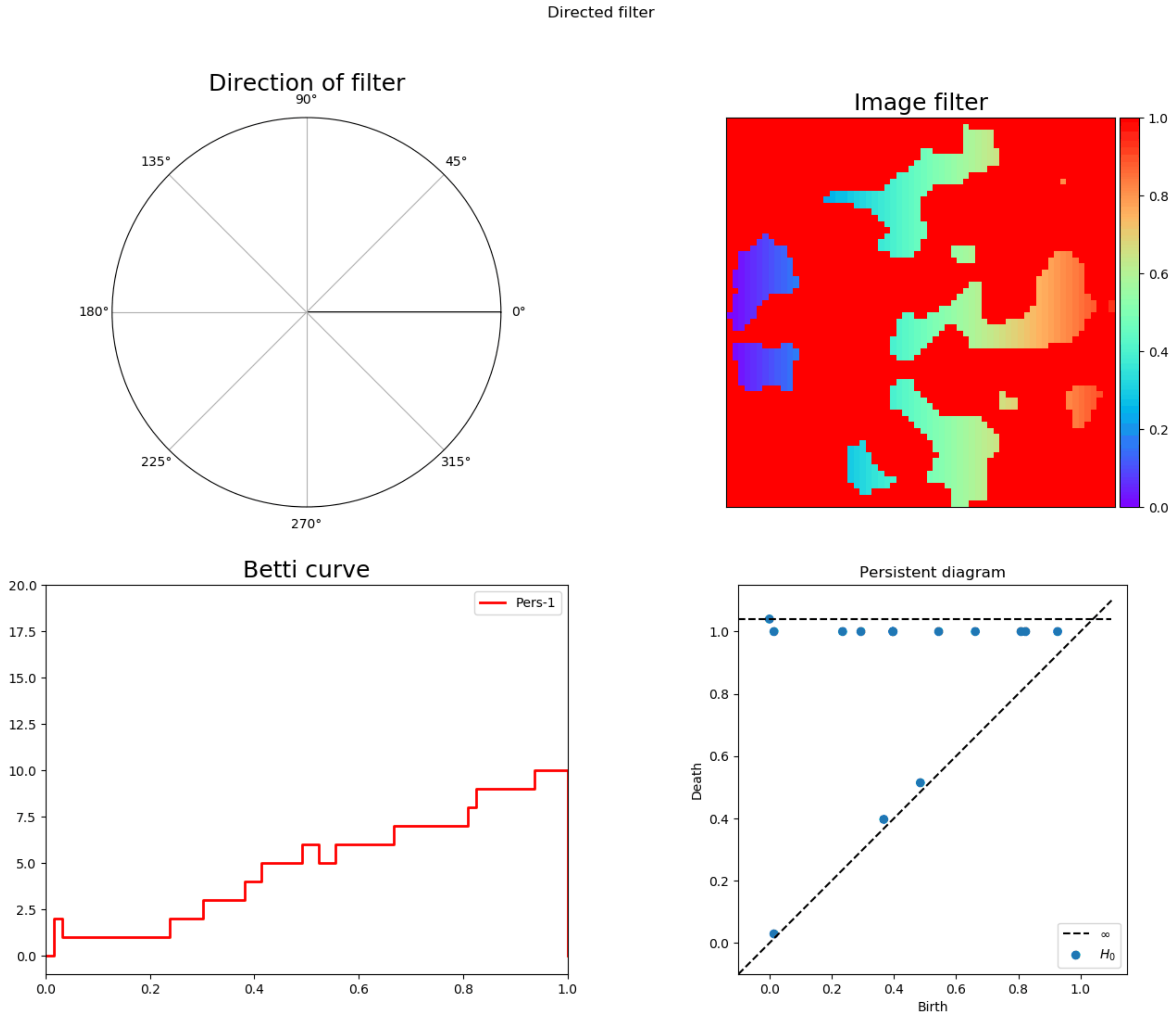
Filtrations

- direction filtration
- radial filtration
- signed EDT filtration
- density filtration

Topological Reading Lesson: Classification of MNIST Using TDA
Garin et al. *(2021)*

Applications

Image classification



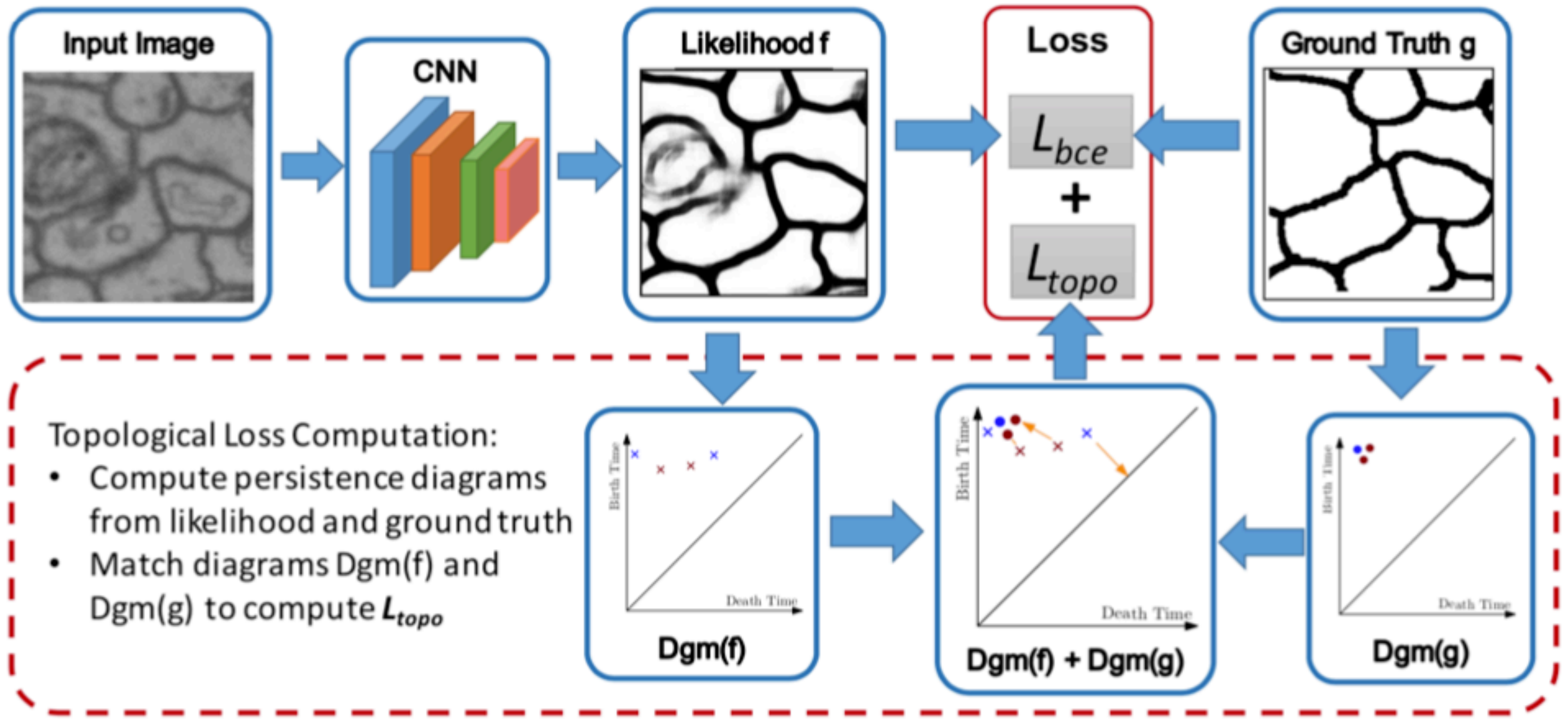
	Accuracy
Fixed filter	0.7249

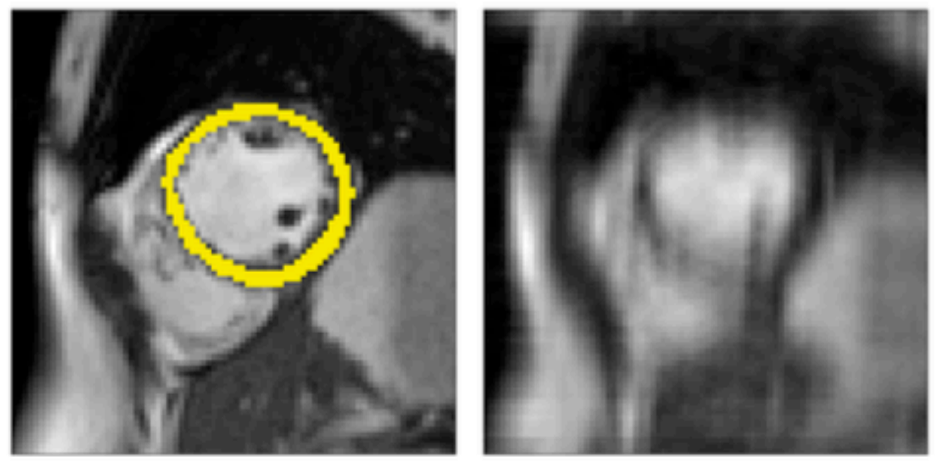
Classification accuracy

	# filters	
	4	**10**
Fixed filters	0.6631	0.7493
Fixed filters similarity matrix	0.7512	0.9220
Trainable filters similarity matrix	0.7611	0.9330

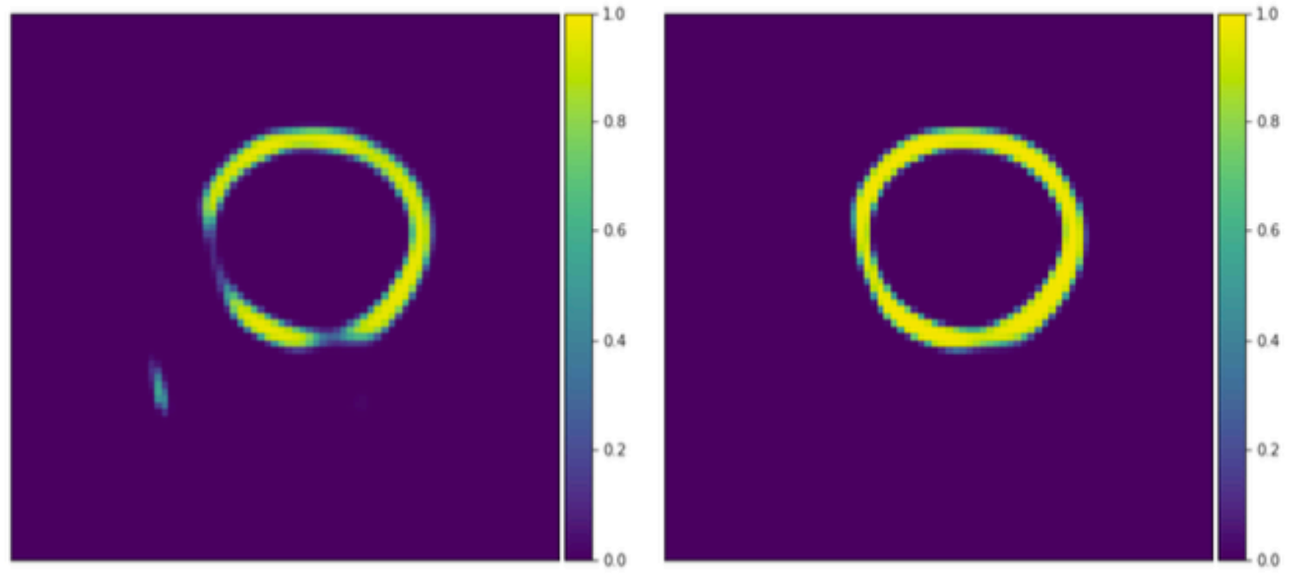
Classification accuracy

Applications
Topology-aware segmentation





(a) Left: Uncorrupted image and ground-truth segmentation. Right: Corrupted image, the input to the network.



(b) Left: The predicted segmentation from the network trained only with supervised learning. Right: The predicted segmentation from the network trained in a semi-supervised manner, incorporating the topological prior.

General form $L(g,\hat{g}) = L_{CE}(g,\hat{g}) + \lambda L_{TOPO}(g,\hat{g})$

Topological regularizer (Hu et al.)

$$L_{TOPO}(g,\hat{g}) = d_{W_2}(D(g), D(\hat{g}))$$

Topological regularizer (Clough et al.) — *"desired Betti numbers"* β_k^*

$$L_{TOPO}(g,\hat{g}) = \sum_k L_k(\beta_k^*), \quad \text{where}$$

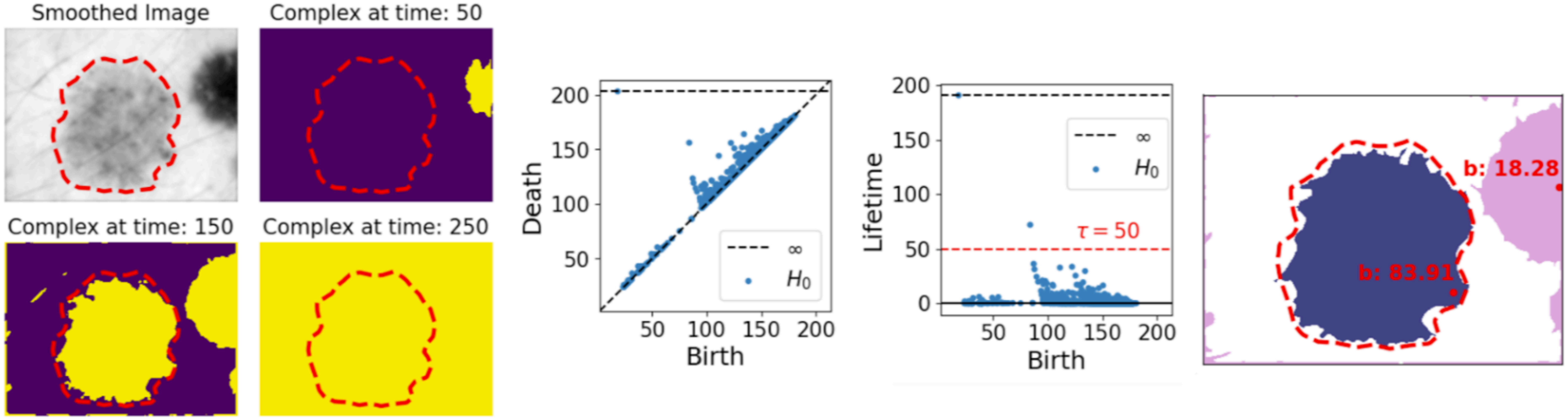
$$L_k(\beta_k^*) = \sum_{\ell=1}^{\beta_k^*} (1 - (d_{k},\ell - b_{k,\ell}))^2 + \sum_{\ell=\beta_k^*+1} (d_{k},\ell - b_{k,\ell}))^2 .$$

Topology-preserving Deep Image Segmentation
Hu et al. *NeurIPS (2019)*

A Topological Loss Function for Deep Learning-based Image Segmentation Using Persistent Homology
Clough et al. IEEE Transactions on Pattern Analysis and Machine Intelligence *(2020)*

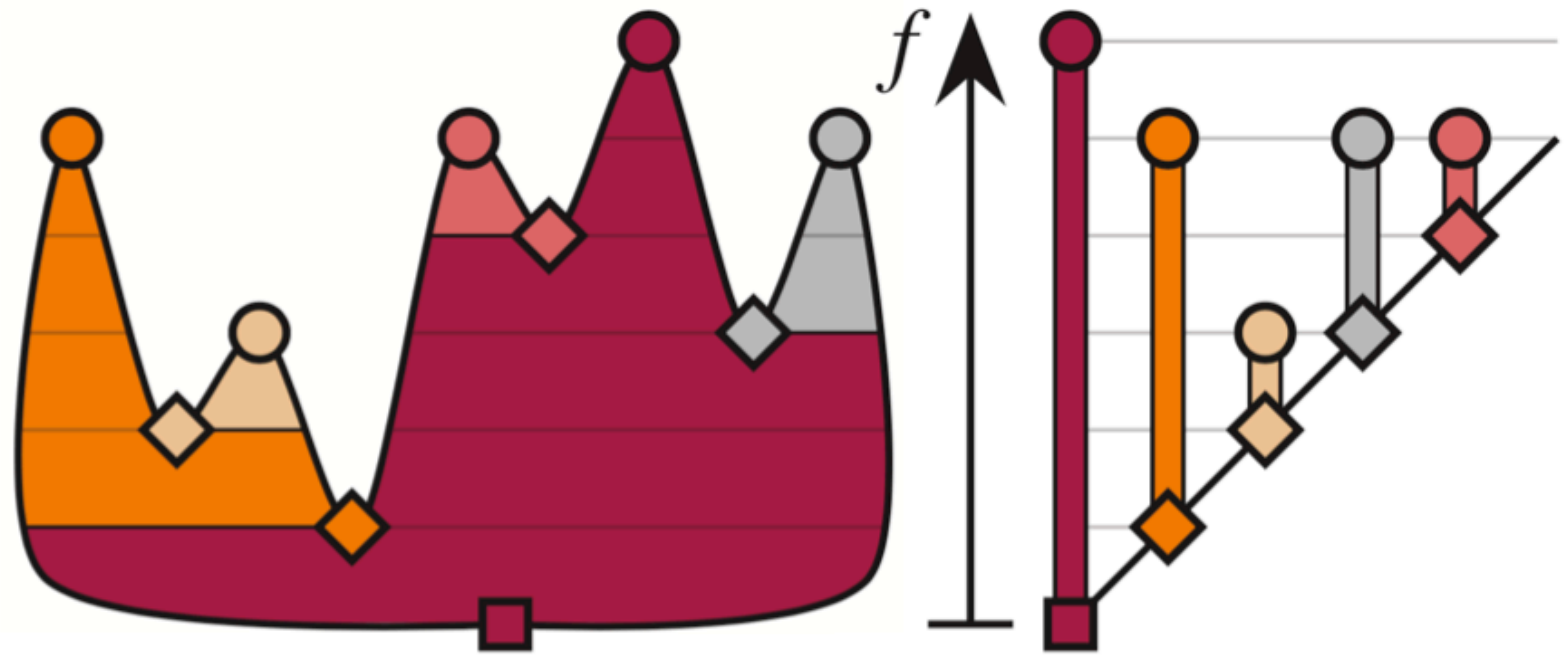
Applications

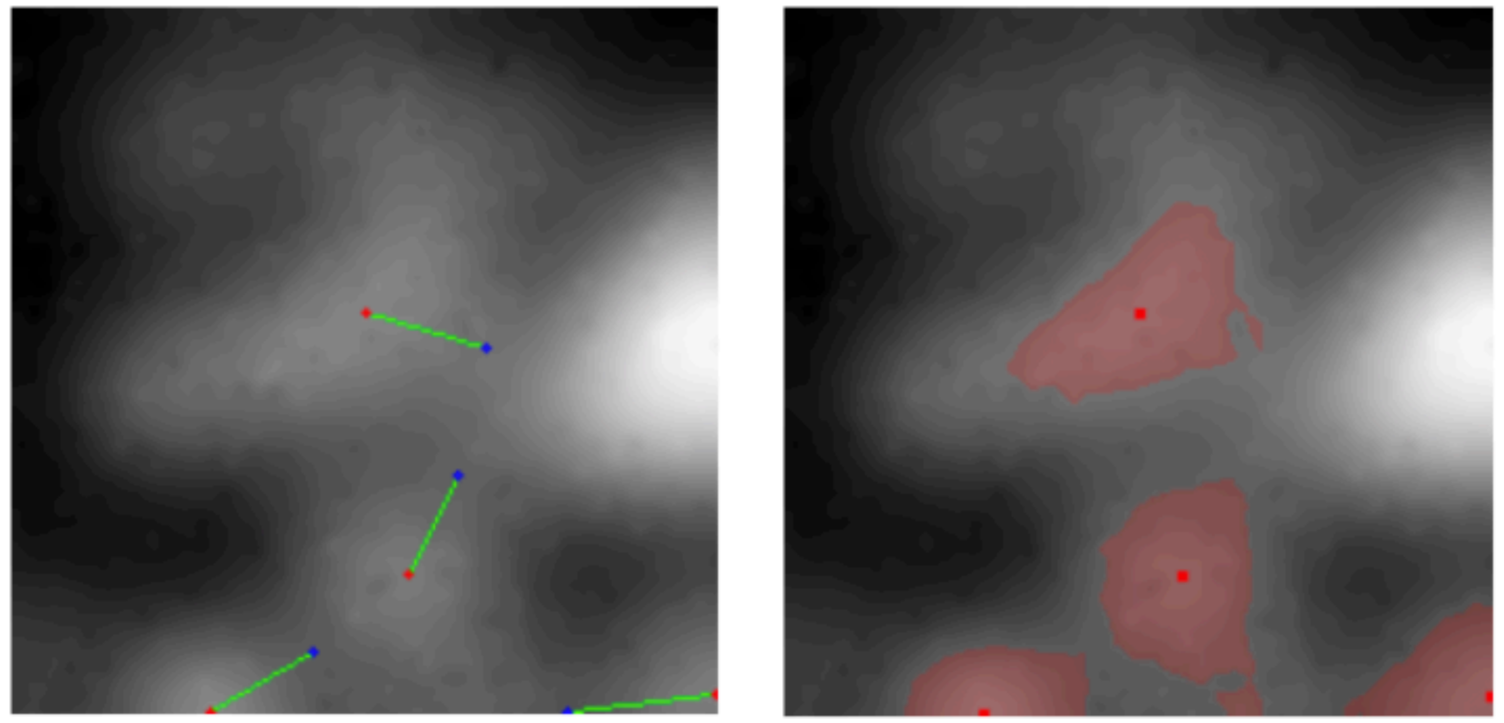
Topology-aware segmentation

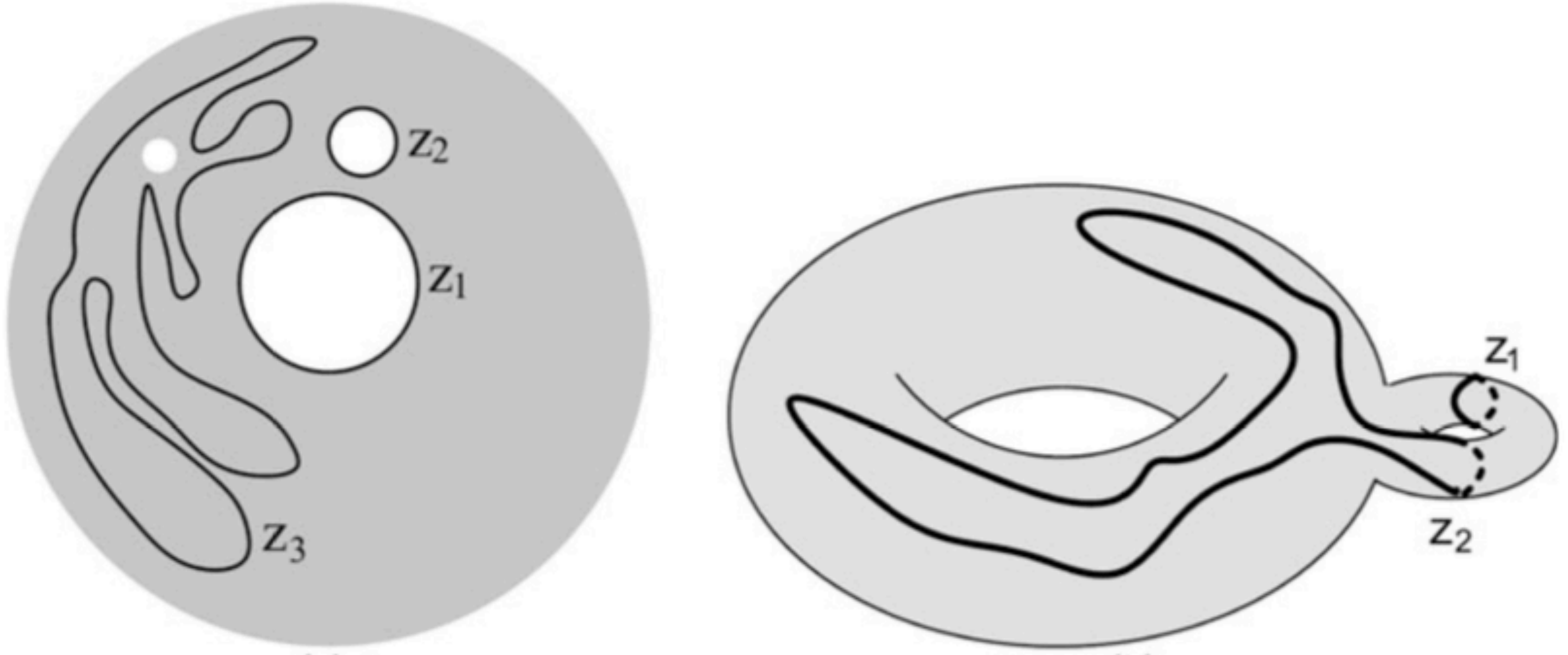


Topological Image Modification for Object Detection and Topological Image Processing of Skin Lesions
Vandaele et al. *Scientific Reports (2020)*

Applications
Topology-aware segmentation





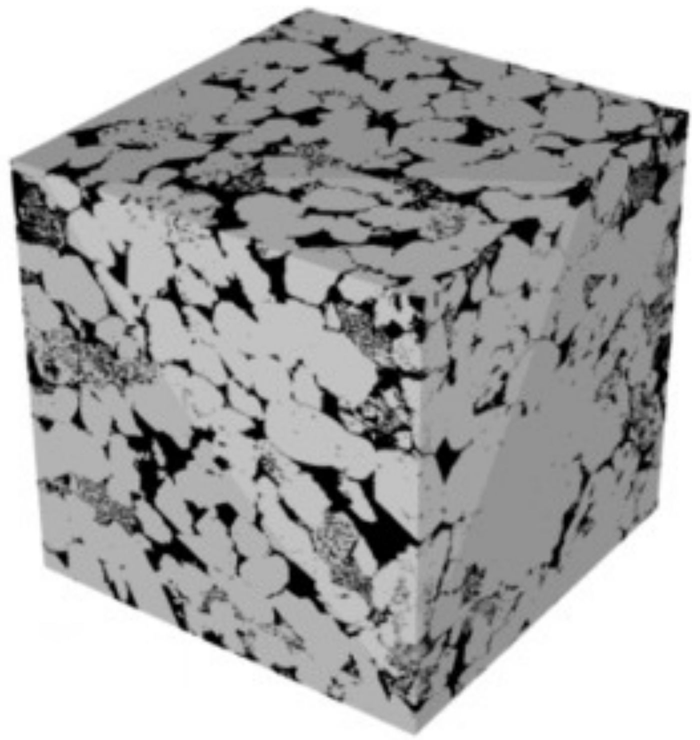


(a) (b)

Volume-optimal Cycle: Tightest Representative Cycle of a Generator in Persistent Homology
Obayashi I. *Journal on Applied Algebra and Geometry (2018)*

Applications

Permeability prediction



Porous media

Physical properties

- permeability
- conductivity
- strength
- etc.

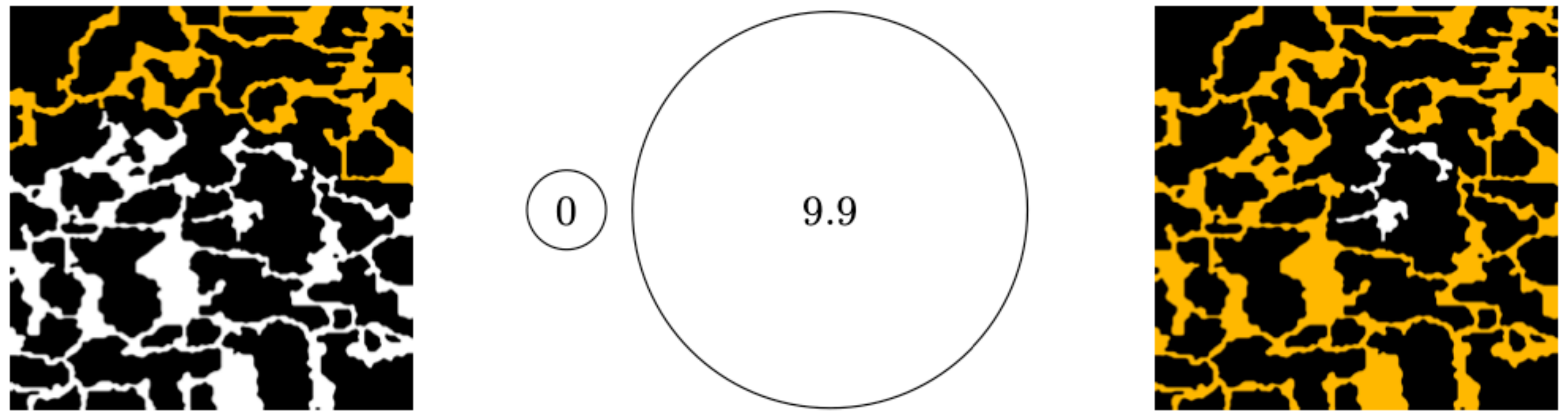


Рис. 6. Для образца А поровое пространство непроницаемо. Максимальный радиус шара проходящего через пространство пор образца Б – 9.9.

	Permeability	Filtration EDT	Filtration Vertical	Filtration Horizontal
Model [24,18]	Mean	**7.76 ± 0.00**	8.04 ± 0.00	8.28 ± 0.00
	Vertical	14.76 ± 0.00	**10.48 ± 0.00**	–
	Horizontal	12.01 ± 0.00	–	**9.45 ± 0.00**

Media	Direction	Filtration EDT	Filtration Bilateral
Bentheimer	Mean	0.527 ± 0.023	**0.412 ± 0.015**
	Top-bottom	0.613 ± 0.029	**0.411 ± 0.016**
	Left-right	0.846 ± 0.025	**0.570 ± 0.024**
	Front-back	0.699 ± 0.019	**0.575 ± 0.026**
Doddington	Mean	1.295 ± 0.037	**0.956 ± 0.042**
	Top-bottom	1.401 ± 0.057	**0.944 ± 0.042**
	Left-right	1.835 ± 0.069	**1.250 ± 0.032**
	Front-back	1.435 ± 0.053	**1.044 ± 0.049**
Ketton	Mean	1.077 ± 0.046	**0.763 ± 0.031**
	Top-bottom	1.189 ± 0.045	**1.059 ± 0.045**
	Left-right	1.459 ± 0.055	**0.889 ± 0.028**
	Front-back	1.289 ± 0.051	**0.958 ± 0.048**

Permeability prediction error, % for 2D model and 3D real data.

Bilateral Filtration for Permeability Prediction in Digital Rocks
Kachan et al. *(in preparation)*

Applications
Topology-denoising Autoencoder

$$\begin{tikzcd}
X \arrow[r, dashed, "q"] \arrow[d] \arrow[rrr, dash, bend right, "\ell_x"'] & \tilde{\mathcal{X}} \arrow[r, "f"] & \mathcal{Z} \arrow[r, "g"] & \mathcal{X}' \arrow[d] \\
D(X) \arrow[d] & & & D(X') \arrow[d] \\
d(X) \arrow[rrr, dash, bend left, "\ell_d"] & & & d(X')
\end{tikzcd}$$

Fig. 4. Topology-denoising autoencoder, variant (TDAE)