Proof of the Poisson Integral Formula

1 Main Results

Theorem 1 (Poisson Integral Formula)**.** *Let $u : \mathbb{C} \to \mathbb{R}$ be harmonic on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on the closed unit disc $\overline{\mathbb{D}}$. Then for all $z \in \mathbb{D}$,*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2}\, u(e^{it})\, dt.$$

The kernel

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}$$

is called the *Poisson kernel* for the unit disc.

2 Key Lemmas and Intermediate Results

Let E be (e.g.) a complex Banach space.

Lemma 2 (Cauchy Formula on Scaled Discs)**.** *Let $f : \mathbb{C} \to E$ be analytic on $\mathbb{D}$, and let $z \in \mathbb{D}$, $r \in (0, 1)$. Then*

$$f(rz) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z}\, f(re^{it})\, dt.$$

This follows from the standard Cauchy integral formula applied to the circle of radius r.

Lemma 3 (Goursat Vanishing Integral)**.** *Under the same hypotheses as lemma 2, we have*

$$\int_0^{2\pi} \frac{\overline{z}}{\overline{e^{it}} - \overline{z}}\, f(re^{it})\, dt = 0.$$

This is a consequence of the Cauchy–Goursat theorem.

Lemma 4 (Real Part of Herglotz Kernel)**.** *Let $x, w \in \mathbb{C}$. For $|x| = 1$ and $w \in \mathbb{D}$,*

$$\mathrm{Re}\left(\frac{x + w}{x - w}\right) = \frac{1 - |w|^2}{|x - w|^2}.$$

This identity connects the Herglotz kernel to the Poisson kernel and is verified by direct algebraic computation.

3 Poisson Formula for Analytic Functions

Proposition 5 (Poisson Formula for Analytic Functions). *Let $f : \mathbb{C} \to E$ be analytic on $\mathbb{D}$, and let $z \in \mathbb{D}$, $r \in (0, 1)$. Then*

$$f(rz) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) f(re^{it})\, dt.$$

This follows by combining lemmas 2 to 4.

4 Proof of the Poisson Integral Formula

The proof of theorem 1 proceeds in several steps:

(i) **Harmonic extension:** Since u is harmonic on $\mathbb{D}$, there exists an analytic function $f : \mathbb{D} \to \mathbb{C}$ such that $u = \operatorname{Re} f$ on $\mathbb{D}$.

(ii) **Scaled formula:** For $r \in (0, 1)$ and $z \in \mathbb{D}$, apply proposition 5 to obtain

$$u(rz) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) u(re^{it})\, dt.$$

(iii) **Approximation sequence:** Consider a sequence $r_n \to 1$ with $r_n \in (0, 1)$ and the integral representation of $u(r_n \cdot)$. For $r_n \to 1$ we can apply the dominated convergence theorem as the integrand is uniformly bounded (using continuity of u on $\overline{\mathbb{D}}$ and the fact that the Poisson kernel is bounded for z in a compact subset of $\mathbb{D}$).

The continuity of u on $\overline{\mathbb{D}}$ ensures that $u(r_n e^{it}) \to u(e^{it})$ as $r_n \to 1$, yielding the desired formula.

5 A Counterexample

We present an example showing that a harmonic function on $\mathbb{D}$ can be extended continuously to $\overline{\mathbb{D}}$, while this is not possible for the associated holomorphic function f.

Example 6 (Failure of Poisson Formula without Continuity). Consider the function defined by the power series

$$f(z) = -i \sum_{n=2}^{\infty} \frac{z^n}{n \log n}.$$

This series has radius of convergence $R = 1$ because

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{\log(n+1)}{\log n} = 1.$$

Thus, f is holomorphic on the open unit disc $\mathbb{D}$.

Analysis of the Real Part: Let $z = e^{i\theta}$ be a point on the boundary $\partial\mathbb{D}$. The real part of $f(z)$ on the boundary is

$$\operatorname{Re} f(e^{i\theta}) = \operatorname{Re}\left(-i\sum_{n=2}^{\infty}\frac{\cos(n\theta) + i\sin(n\theta)}{n\log n}\right) = \sum_{n=2}^{\infty}\frac{\sin(n\theta)}{n\log n}.$$

This is a trigonometric sine series with coefficients $b_n = \frac{1}{n\log n}$. The coefficients satisfy:

1. They are monotonically decreasing to zero: $b_n \searrow 0$.
2. The sequence $nb_n = \frac{1}{\log n}$ tends to 0 as $n \to \infty$.

By the Chaundy–Jolliffe Theorem (or standard results on uniform convergence of sine series), if (b_n) is monotonic decreasing and $nb_n \to 0$, then the series $\sum b_n \sin(n\theta)$ converges uniformly on $\theta \in [0, 2\pi]$.

Since the series converges uniformly to a continuous function on the circle $\partial\mathbb{D}$, and $\operatorname{Re} f$ is the harmonic extension of this boundary function to the interior, $\operatorname{Re} f$ is continuous on the closed unit disc $\overline{\mathbb{D}}$.

Analysis of the Imaginary Part: The imaginary part of $f(z)$ on the boundary would correspond to the series

$$\operatorname{Im} f(e^{i\theta}) = \operatorname{Im}\left(-i\sum_{n=2}^{\infty}\frac{\cos(n\theta) + i\sin(n\theta)}{n\log n}\right) = -\sum_{n=2}^{\infty}\frac{\cos(n\theta)}{n\log n}.$$

Consider the behavior at $\theta = 0$ (i.e., $z = 1$). The series becomes

$$-\sum_{n=2}^{\infty}\frac{1}{n\log n}.$$

By the integral test,

$$\int_2^{\infty}\frac{dx}{x\log x} = [\log(\log x)]_2^{\infty} = \infty.$$

Thus, the series diverges to $-\infty$ at $z = 1$.

Conclusion: Inside the disc, as $r \to 1^-$ along the real axis, the value of

$$\operatorname{Im} f(r) = -\sum_{n=2}^{\infty}\frac{r^n}{n\log n}$$

tends to $-\infty$. The function f is unbounded near $z = 1$, so it cannot be extended to a continuous function on the closed unit disc $\overline{\mathbb{D}}$.

References

[1] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.