

# Proof of the Herglotz Representation Theorem

## 1 Preliminary Results

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  denote the unit disc and  $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$  the right half-plane. We consider the Borel  $\sigma$ -algebra on  $\mathbb{C}$ .

We follow a proof that can be found, e.g., in [1, Section 3].

Let  $C(\partial\mathbb{D})$  be the space of continuous, real-valued functions on  $\partial\mathbb{D}$ , equipped with the supremum norm  $\|\cdot\|_\infty$ , and let  $C(\partial\mathbb{D})^*$  be its dual space.

We first need some technical lemmas that establish the compactness and convergence properties needed for the main theorem.

**Lemma 1** (Compactness in the dual ball). *The set*

$$K = \{\Lambda \in C(\partial\mathbb{D})^* : |\Lambda(f)| \leq 1 \text{ for all } f \in C(\partial\mathbb{D}) \text{ with } \|f\|_\infty \leq 1\}$$

*is a compact metric space with respect to the induced weak\*-topology of  $C(\partial\mathbb{D})^*$ .*

*Proof.* By the Banach-Alaoglu theorem,  $K$  is weak\*-compact as the closed unit ball in the dual space.

Since  $C(\partial\mathbb{D})$  is a separable vector space (the unit circle is a compact metric space) and  $K$  is weak\*-compact, the space  $K$  is metrizable in the weak\*-topology. This follows because  $C(\partial\mathbb{D})^*$  has a countable family of continuous functions that separates points, given by evaluation at a countable dense subset of  $C(\partial\mathbb{D})$ .  $\square$

Let  $p: \mathbb{D} \rightarrow \mathbb{H}$  be analytic with  $p(0) = 1$ . Let  $u = \operatorname{Re} p$  denote its real part.

Consider a sequence  $(r_n)_{n \in \mathbb{N}}$  in  $(0, 1)$  such that  $r_n \rightarrow 1$  as  $n \rightarrow \infty$  (for example,  $r_n = \frac{n+1}{n+2}$ ). For every  $n \in \mathbb{N}$ , define  $u_n: \mathbb{D} \rightarrow \mathbb{R}$  by

$$u_n(z) = u(r_n z), \quad z \in \mathbb{D}.$$

For every  $n \in \mathbb{N}$ , define the linear functional  $\Lambda_n \in C(\partial\mathbb{D})^*$  by

$$\Lambda_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) u_n(e^{it}) dt, \quad f \in C(\partial\mathbb{D}).$$

**Lemma 2** (Boundedness of the functionals). *For every  $n \in \mathbb{N}$ , we have  $|\Lambda_n(f)| \leq 1$  for all  $f \in C(\partial\mathbb{D})$  with  $\|f\|_\infty \leq 1$ .*

*Proof.* Let  $n \in \mathbb{N}$  and  $f \in C(\partial\mathbb{D})$  with  $\|f\|_\infty \leq 1$ . Since  $u_n(\zeta) > 0$  for all  $\zeta \in \partial\mathbb{D}$  (as  $u_n$  is the real part of a function mapping to the right half-plane), we have

$$|\Lambda_n(f)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})| u_n(e^{it}) dt \leq \frac{1}{2\pi} \int_0^{2\pi} u_n(e^{it}) dt.$$

Since  $u_n$  is harmonic on  $\mathbb{D}$  and continuous on  $\overline{\mathbb{D}}$ , by the mean value property for harmonic functions,

$$\frac{1}{2\pi} \int_0^{2\pi} u_n(e^{it}) dt = u_n(0) = u(0) = \operatorname{Re} p(0) = 1.$$

This completes the proof.  $\square$

**Lemma 3** (Existence of a convergent subsequence). *There exists a subsequence  $(\Lambda_{n_k})_{k \in \mathbb{N}}$  of  $(\Lambda_n)_{n \in \mathbb{N}}$  and a functional  $\Lambda \in C(\partial\mathbb{D})^*$  such that*

$$\lim_{k \rightarrow \infty} \Lambda_{n_k}(f) = \Lambda(f), \quad \text{for all } f \in C(\partial\mathbb{D}).$$

*Proof.* By Lemma 2, the sequence  $(\Lambda_n)_{n \in \mathbb{N}}$  lies in  $K$ . By Lemma 1, there exist  $\Lambda \in K$  and a subsequence  $(\Lambda_{n_k})_{k \in \mathbb{N}}$  of  $(\Lambda_n)_{n \in \mathbb{N}}$  that is weak\*-convergent to  $\Lambda$ .  $\square$

## 2 The Representing Measure

**Lemma 4** (Existence of the probability measure). *There exists a probability measure  $\mu$  on  $\partial\mathbb{D}$  and a subsequence  $(r_{n_k})_{k \in \mathbb{N}}$  such that*

$$\lim_{k \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) u(r_{n_k} e^{it}) dt = \int_{\partial\mathbb{D}} f(\zeta) d\mu(\zeta)$$

for all  $f \in C(\partial\mathbb{D})$  with  $f(\zeta) \geq 0$  for all  $\zeta \in \partial\mathbb{D}$ .

*Proof.* Let  $\Lambda \in C(\partial\mathbb{D})^*$  and  $(r_{n_k})_{k \in \mathbb{N}}$  be given by Lemma 3. Since  $u(z) > 0$  for all  $z \in \mathbb{D}$  (as  $p$  maps to the right half-plane), in view of Lemma 3,  $\Lambda$  is a positive functional on  $C(\partial\mathbb{D})$ .

By the Riesz-Markov-Kakutani representation theorem, there exists a measure  $\mu$  on  $\partial\mathbb{D}$  such that

$$\Lambda(f) = \int_{\partial\mathbb{D}} f(\zeta) d\mu(\zeta)$$

for all  $f \in C(\partial\mathbb{D})$  with  $f(\zeta) \geq 0$  for all  $\zeta \in \partial\mathbb{D}$ .

For  $f \equiv 1$ , by the mean value property, we have

$$\mu(\partial\mathbb{D}) = \lim_{k \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} u(r_{n_k} e^{it}) dt = 1.$$

Therefore,  $\mu$  is a probability measure.  $\square$

**Lemma 5** (Properties of the Herglotz kernel integral). *Let  $\mu$  be a probability measure on  $\partial\mathbb{D}$  and define*

$$q(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta), \quad z \in \mathbb{D}.$$

*Then  $q$  is analytic on  $\mathbb{D}$ , takes values in  $\mathbb{H}$ , and satisfies  $q(0) = 1$ .*

*Proof.* The function  $z \mapsto \frac{\zeta + z}{\zeta - z}$  is analytic for  $z \in \mathbb{D}$  and  $\zeta \in \partial\mathbb{D}$  (since  $|z| < 1 = |\zeta|$ ). By dominated convergence and Morera's theorem,  $q$  is analytic on  $\mathbb{D}$ .

For the real part, note that

$$\operatorname{Re} \frac{\zeta + z}{\zeta - z} = \frac{1 - |z|^2}{|\zeta - z|^2} > 0$$

for  $z \in \mathbb{D}$  and  $\zeta \in \partial\mathbb{D}$ . Therefore,  $q$  maps to  $\mathbb{H}$ .

Finally,  $q(0) = \int_{\partial\mathbb{D}} \frac{\zeta}{\zeta} d\mu(\zeta) = \mu(\partial\mathbb{D}) = 1$ .  $\square$

### 3 Key Technical Lemmas

**Lemma 6** (Poisson-type integral formula). *For every  $r \in (0, 1)$ ,*

$$u(rz) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \cdot u(re^{it}) dt, \quad \text{for all } z \in \mathbb{D}.$$

*Proof.* Fix  $r \in (0, 1)$  and  $z \in \mathbb{D}$ . The function  $p(r \cdot)$  is analytic on  $\frac{1}{r}\mathbb{D}$ , and since  $\frac{1}{r} > 1$ , by the Cauchy integral formula,

$$p(rz) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{p(r\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{p(re^{it})e^{it}}{e^{it} - z} dt.$$

Since  $|\frac{1}{z}| > 1$ , the function  $\zeta \mapsto \frac{p(r\zeta)}{\frac{1}{z} - \zeta}$  is analytic on a disc centered at 0 with radius larger than 1. By the Cauchy-Goursat theorem for a circle,

$$0 = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{p(r\zeta)}{\frac{1}{z} - \zeta} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{p(re^{it})\bar{z}}{e^{-it} - \bar{z}} dt.$$

Adding these two integral formulas yields

$$p(rz) = \frac{1}{2\pi} \int_0^{2\pi} p(re^{it}) \cdot \frac{1 - |z|^2}{|e^{it} - z|^2} dt = \frac{1}{2\pi} \int_0^{2\pi} p(re^{it}) \cdot \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} dt.$$

Taking the real part of both sides completes the proof.  $\square$

**Lemma 7** (Uniqueness via real parts). *Let  $p, q: \mathbb{D} \rightarrow \mathbb{H}$  be analytic with  $\operatorname{Re} p = \operatorname{Re} q$  and  $p(0) = q(0)$ . Then  $p = q$ .*

*Proof.* The function  $p - q$  is analytic on  $\mathbb{D}$  and  $\operatorname{Re}(p - q) \equiv 0$  on  $\mathbb{D}$ . By the Cauchy-Riemann equations,  $(p - q)' \equiv 0$ . Therefore,  $p - q$  is constant on  $\mathbb{D}$ , and since  $p(0) = q(0)$ , we conclude  $p = q$ .  $\square$

### 4 Main Theorem

**Theorem 8** (Herglotz representation theorem). *Let  $p: \mathbb{D} \rightarrow \mathbb{H}$  be analytic with  $p(0) = 1$ . Then there exists a unique probability measure  $\mu$  on  $\mathbb{C}$  with support on  $\partial\mathbb{D}$  such that*

$$p(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta), \quad \text{for all } z \in \mathbb{D}.$$

*Proof. Existence.* Let  $\mu$  be given by Lemma 4. Let  $z \in \mathbb{D}$ . Using Lemmas 4 to 6, we have

$$\begin{aligned} \int_{\partial\mathbb{D}} \operatorname{Re} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) &= \lim_{k \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \cdot u(r_{n_k} e^{it}) dt \\ &= \lim_{k \rightarrow \infty} u(r_{n_k} z) = u(z). \end{aligned}$$

Let  $q$  be as in Lemma 5. Then  $\operatorname{Re} q = \operatorname{Re} p = u$  and  $q(0) = p(0) = 1$ . By Lemma 7,  $p = q$ , so  $p$  has the desired representation.

**Uniqueness.** Assume there exist probability measures  $\mu_1, \mu_2$  on  $\mathbb{C}$  with support on  $\partial\mathbb{D}$  such that

$$p(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_1(\zeta) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_2(\zeta), \quad z \in \mathbb{D}.$$

Let  $\eta = \mu_2 - \mu_1$ . Then  $\eta$  is a signed finite measure on  $\mathbb{C}$  with support on  $\partial\mathbb{D}$ . Moreover, for every  $j \in \{1, 2\}$ , using the geometric series expansion (valid for  $|z| < 1$ ),

$$\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_j(\zeta) = \int_{\partial\mathbb{D}} \left(1 + 2 \sum_{n=1}^{\infty} \zeta^{-n} z^n\right) d\mu_j(\zeta) = \mu_j(\partial\mathbb{D}) + 2 \sum_{n=1}^{\infty} z^n \int_{\partial\mathbb{D}} \zeta^{-n} d\mu_j(\zeta).$$

Therefore,

$$\int_{\partial\mathbb{D}} \zeta^{-n} d\mu_1(\zeta) = \int_{\partial\mathbb{D}} \zeta^{-n} d\mu_2(\zeta), \quad \text{for all } n \in \mathbb{N}.$$

This implies

$$\int_{\partial\mathbb{D}} f(\zeta) d\mu_1(\zeta) = \int_{\partial\mathbb{D}} f(\zeta) d\mu_2(\zeta)$$

for every polynomial  $f$  on  $\mathbb{C}$ . Since the polynomials (or equivalently, the Laurent polynomials  $\{\zeta^n : n \in \mathbb{Z}\}$ ) are dense in  $C(\partial\mathbb{D})$  by the Stone-Weierstrass theorem, and both  $\mu_1$  and  $\mu_2$  are probability measures, we conclude  $\mu_1 = \mu_2$ .  $\square$

**Remark 9.** The Herglotz representation theorem is fundamental in complex analysis and harmonic analysis. It provides a correspondence between analytic functions with positive real part and probability measures on the circle, analogous to how the Poisson integral formula represents harmonic functions in terms of their boundary values.

**Remark 10.** The kernel  $\frac{\zeta+z}{\zeta-z}$  appearing in the theorem is known as the *Herglotz kernel* or *Riesz-Herglotz kernel*. Its real part,

$$\operatorname{Re} \frac{\zeta + z}{\zeta - z} = \frac{1 - |z|^2}{|\zeta - z|^2},$$

is the Poisson kernel for the unit disc.

## References

- [1] John B. Garnett, *Bounded Analytic Functions*, Saunders College Publishing/Harcourt Brace, 1981.
- [2] Walter Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.
- [3] Walter Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, 1991.