

Proof of the Poisson Integral Formula

1 Main Results

Theorem 1 (Poisson Integral Formula). *Let $u : \mathbb{C} \rightarrow \mathbb{R}$ be harmonic on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on the closed unit disc $\overline{\mathbb{D}}$. Then for all $z \in \mathbb{D}$,*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} u(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left(\frac{e^{it} + z}{e^{it} - z} \right) u(e^{it}) dt$$

The kernel

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2} = \operatorname{Re} \left(\frac{e^{it} + z}{e^{it} - z} \right)$$

is called the *Poisson kernel* for the unit disc.

2 Key Lemmas and Intermediate Results

Let E be (e.g.) a complex Banach space.

Lemma 2 (Cauchy Formula on Scaled Discs). *Let $f : \mathbb{C} \rightarrow E$ be analytic on \mathbb{D} , and let $z \in \mathbb{D}$, $r \in (0, 1)$. Then*

$$f(rz) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z} f(re^{it}) dt.$$

This follows from the standard Cauchy integral formula applied to the circle of radius r .

Lemma 3 (Goursat Vanishing Integral). *Under the same hypotheses as lemma 2, we have*

$$\int_0^{2\pi} \frac{\bar{z}}{\overline{e^{it}} - \bar{z}} f(re^{it}) dt = 0.$$

This is a consequence of the Cauchy–Goursat theorem.

Lemma 4 (Real Part of Herglotz Kernel). *Let $x, w \in \mathbb{C}$. For $|x| = 1$ and $w \in \mathbb{D}$,*

$$\operatorname{Re} \left(\frac{x + w}{x - w} \right) = \frac{1 - |w|^2}{|x - w|^2}.$$

This identity connects the Herglotz kernel to the Poisson kernel and is verified by direct algebraic computation.

3 Poisson Formula for Analytic Functions

Proposition 5 (Poisson Formula for Analytic Functions on Scaled Disc). *Let $f : \mathbb{C} \rightarrow E$ be analytic on \mathbb{D} , and let $z \in \mathbb{D}$, $r \in (0, 1)$. Then*

$$f(rz) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} f(re^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left(\frac{e^{it} + z}{e^{it} - z} \right) f(re^{it}) dt.$$

This follows by combining lemmas 2 to 4.

4 Proof of the Poisson Integral Formula

The proof of theorem 1 proceeds in several steps:

- (i) **Harmonic extension:** Since u is harmonic on \mathbb{D} , there exists an analytic function $f : \mathbb{D} \rightarrow \mathbb{C}$ such that $u = \operatorname{Re} f$ on \mathbb{D} .
- (ii) **Scaled formula:** For $r \in (0, 1)$ and $z \in \mathbb{D}$, apply proposition 5 to obtain

$$u(rz) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} u(re^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left(\frac{e^{it} + z}{e^{it} - z} \right) u(re^{it}) dt.$$

- (iii) **Approximation sequence:** Consider a sequence $r_n \rightarrow 1$ with $r_n \in (0, 1)$ and the integral representation of $u(r_n \cdot)$. For $r_n \rightarrow 1$ we can apply the dominated convergence theorem as the integrand is uniformly bounded (using continuity of u on $\overline{\mathbb{D}}$ and the fact that the Poisson kernel is bounded for z in a compact subset of \mathbb{D}).

The continuity of u on $\overline{\mathbb{D}}$ ensures that $u(r_n e^{it}) \rightarrow u(e^{it})$ as $r_n \rightarrow 1$, yielding the desired formula.

Remark 6. (The Poisson Formula for Analytic Functions) If $f : \mathbb{C} \rightarrow E$ is analytic on \mathbb{D} and continuous on $\overline{\mathbb{D}}$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} f(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left(\frac{e^{it} + z}{e^{it} - z} \right) f(e^{it}) dt.$$

5 A Counterexample

We present an example showing that a harmonic function on \mathbb{D} can be extended continuously to $\overline{\mathbb{D}}$, while this is not possible for the associated holomorphic function f .

Example 7 (Failure of Poisson Formula without Continuity). Consider the function defined by the power series

$$f(z) = -i \sum_{n=2}^{\infty} \frac{z^n}{n \log n}.$$

This series has radius of convergence $R = 1$ because

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{\log(n+1)}{\log n} = 1.$$

Thus, f is holomorphic on the open unit disc \mathbb{D} .

Analysis of the Real Part: Let $z = e^{i\theta}$ be a point on the boundary $\partial\mathbb{D}$. The real part of $f(z)$ on the boundary is

$$\operatorname{Re} f(e^{i\theta}) = \operatorname{Re} \left(-i \sum_{n=2}^{\infty} \frac{\cos(n\theta) + i \sin(n\theta)}{n \log n} \right) = \sum_{n=2}^{\infty} \frac{\sin(n\theta)}{n \log n}.$$

This is a trigonometric sine series with coefficients $b_n = \frac{1}{n \log n}$. The coefficients satisfy:

1. They are monotonically decreasing to zero: $b_n \searrow 0$.
2. The sequence $nb_n = \frac{1}{\log n}$ tends to 0 as $n \rightarrow \infty$.

By the Chaundy–Jolliffe Theorem (or standard results on uniform convergence of sine series), if (b_n) is monotonic decreasing and $nb_n \rightarrow 0$, then the series $\sum b_n \sin(n\theta)$ converges uniformly on $\theta \in [0, 2\pi]$.

Since the series converges uniformly to a continuous function on the circle $\partial\mathbb{D}$, and $\operatorname{Re} f$ is the harmonic extension of this boundary function to the interior, $\operatorname{Re} f$ is continuous on the closed unit disc $\overline{\mathbb{D}}$.

Analysis of the Imaginary Part: The imaginary part of $f(z)$ on the boundary would correspond to the series

$$\operatorname{Im} f(e^{i\theta}) = \operatorname{Im} \left(-i \sum_{n=2}^{\infty} \frac{\cos(n\theta) + i \sin(n\theta)}{n \log n} \right) = - \sum_{n=2}^{\infty} \frac{\cos(n\theta)}{n \log n}.$$

Consider the behavior at $\theta = 0$ (i.e., $z = 1$). The series becomes

$$- \sum_{n=2}^{\infty} \frac{1}{n \log n}.$$

By the integral test,

$$\int_2^{\infty} \frac{dx}{x \log x} = [\log(\log x)]_2^{\infty} = \infty.$$

Thus, the series diverges to $-\infty$ at $z = 1$.

Conclusion: Inside the disc, as $r \rightarrow 1^-$ along the real axis, the value of

$$\operatorname{Im} f(r) = - \sum_{n=2}^{\infty} \frac{r^n}{n \log n}$$

tends to $-\infty$. The function f is unbounded near $z = 1$, so it cannot be extended to a continuous function on the closed unit disc $\overline{\mathbb{D}}$.

References

- [1] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.