$$\liminf_{z\to\infty}\frac{f_S(z)}{f_B(z)} \geq \liminf_{z\to\infty}\frac{f_S(az)}{f_B(z)} + 1 = \liminf_{z\to\infty}\frac{f_B(az)}{f_B(z)}\frac{f_S(az)}{f_B(az)} + 1$$

$$= a^{\rho/(1+\rho)} \liminf_{z\to\infty}\frac{f_S(z)}{f_B(z)} + 1,$$