$$\left(\partial_{\tilde{t}} + \dot{r}\partial_r\right)S_0 = 0 ,$$