$$\mathcal{S}_i = \left\{ \varphi \in S_\lambda \mid \chi(S \setminus Z_\varphi) = -i \right\}.$$