

$$\begin{aligned}
\liminf_{z \rightarrow \infty} \frac{f_S(z)}{f_B(z)} &\geq \liminf_{z \rightarrow \infty} \frac{f_S(az)}{f_B(z)} + 1 = \liminf_{z \rightarrow \infty} \frac{f_B(az)}{f_B(z)} \frac{f_S(az)}{f_B(az)} + 1 \\
&= a^{\rho/(1+\rho)} \liminf_{z \rightarrow \infty} \frac{f_S(z)}{f_B(z)} + 1,
\end{aligned}$$