1) Time to collision

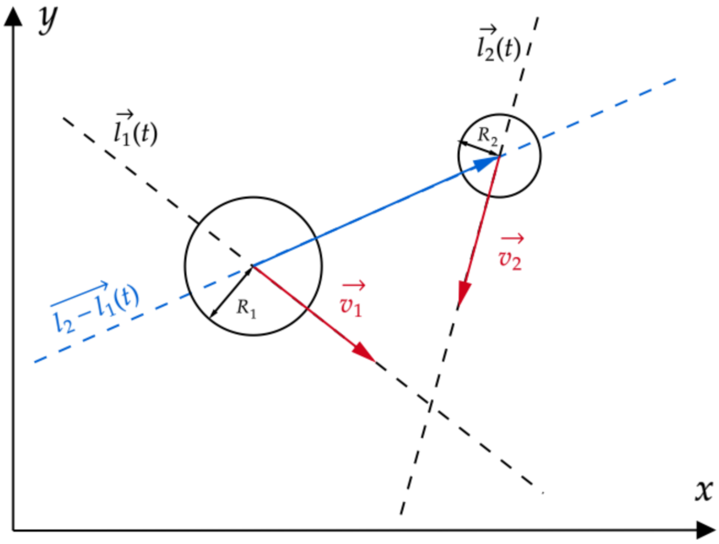

Figure 1: Two particles moving with constant speed on a collision trajectory

Parameterization of each trajectory as:

$$\vec{l}_1(t) = \vec{v}_1 \cdot t + \vec{r}_1 \tag{1}$$
$$\vec{l}_2(t) = \vec{v}_2 \cdot t + \vec{r}_2 \tag{2}$$

yields the distance between the two particles as:

$$d(t) = ||\vec{l}_2(t) - \vec{l}_1(t)|| = ||(\vec{v}_2 - \vec{v}_1)t + (\vec{r}_2 - \vec{r}_1)|| \tag{3}$$

Therefore the time of a collision can be found as the solution to:

$$d(t) = R_1 + R_2 \leftrightarrow d(t) - (R_1 + R_2) = 0 \tag{4}$$

which, in turn, corresponds to solving the 2 degree polynomial:

$$d(t)^2 - (R_1 + R_2)^2 = 0. \tag{5}$$

with:

$$d(t)^2 = \left|\left|\vec{l}_2(t) - \vec{l}_1(t)\right|\right|^2 \tag{6}$$
$$= \left|\left|\vec{v}_2 - \vec{v}_1\right|\right|^2 t^2 + 2\langle \vec{v}_2 - \vec{v}_1 \,|\, \vec{r}_2 - \vec{r}_1 \rangle t + \left|\left|\vec{r}_2 - \vec{r}_1\right|\right|^2 \tag{7}$$

the solution to (5), therefore is given by the standard formula, and yields the following:

$$t_1, t_2 = \frac{-2\langle \vec{v}_2 - \vec{v}_1 \,|\, \vec{r}_2 - \vec{r}_1 \rangle \pm \sqrt{4\langle \vec{v}_2 - \vec{v}_1 \,|\, \vec{r}_2 - \vec{r}_1 \rangle^2 - 4\left|\left|\vec{v}_2 - \vec{v}_1\right|\right|^2 \left(\left|\left|\vec{r}_2 - \vec{r}_1\right|\right|^2 - (R_1 + R_2)^2\right)}}{2\left|\left|\vec{v}_2 - \vec{v}_1\right|\right|^2}$$

2) Updating velocity

Neglecting gravity, conservation of energy and momentum in 1D:

$$\frac{1}{2}m_1 v_{1,i}^2 + \frac{1}{2}m_2 v_{2,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2$$
$$m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}$$

yields:

$$v_{1,f} = \frac{m_1 v_{1,i} - m2(v_{1,i} - 2v_{2,i})}{m_1 + m_2} \quad \text{and} \quad v_{2,f} = \frac{m_1(2v_{1,i} - v_{2,i}) + m_2 v_{2,i}}{m_1 + m_2} \tag{8}$$

On the surface, this doesn't seem to provide any viable solution to expressing the post-collision velocity in 2D. However, expressing their velocities in the basis of a line parallel to their contact-point, and one orthogonal to their contact-point:

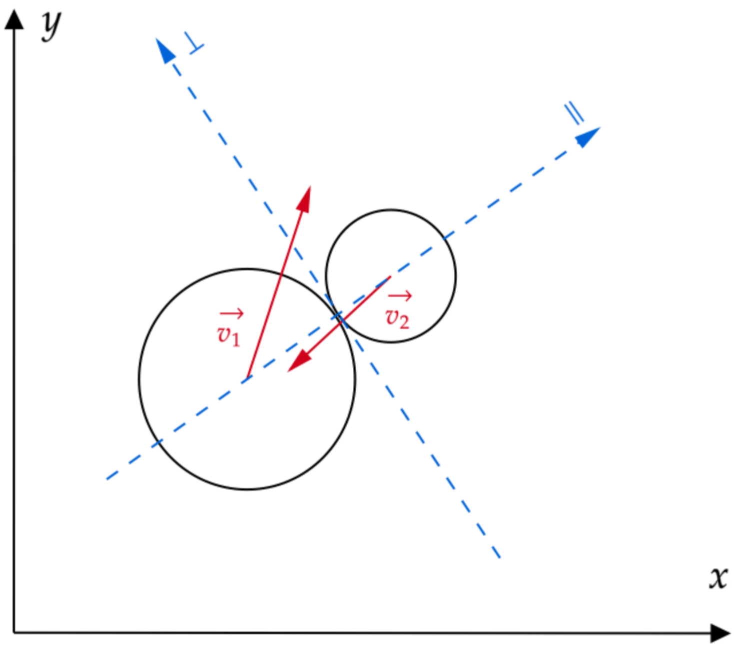

Figure 2: Two particles moving with constant speed on a collision trajectory

s.t.:

$$\vec{v}_i = v_{i,\parallel}\hat{e}_\parallel + v_{i,\perp}\hat{e}_\perp \tag{9}$$

gives the advantage the there is only an exchange of forces in direction of the contact, i.e. $\frac{d\vec{p}_\perp}{dt} = 0$, and therefore that only the component of the velocity in this direction changes.
In this case, an unit-vector in the $\parallel$ direction is given as:

$$\hat{e}_\parallel = \frac{\vec{r}_2 - \vec{r}_1}{||\vec{r}_2 - \vec{r}_1||} \tag{10}$$

and, the unit-vector in the orthogonal direction is found by simple rotation:

$$\hat{e}_\perp = \theta\left(\frac{\pi}{2}\right)\hat{e}_\parallel = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hat{e}_\parallel \tag{11}$$

finally the velocity component in each direction can be found by means of projection:

$$v_{i,\parallel} = \langle \vec{v}_i \,|\, \hat{e}_\parallel \rangle \hat{e}_\parallel \quad \text{and} \quad v_{i,\perp} = \langle \vec{v}_i \,|\, \hat{e}_\perp \rangle \hat{e}_\perp \tag{12}$$

Applying (8) on the parallel component finally gives:

$$\vec{v}_{1,f} = v_{1,i,\parallel}\hat{e}_\parallel + v_{1,f,\perp}\hat{e}_\perp = \vec{v}_{1,i} - \frac{2m_2}{m_1 + m_2} \frac{\langle \vec{v}_{1,i} - \vec{v}_{2,i} \,|\, \vec{r}_{1,i} - \vec{r}_{2,i} \rangle}{||\vec{r}_{1,i} - \vec{r}_{2,i}||^2} (\vec{r}_{1,i} - \vec{r}_{2,i}) \tag{13}$$

$$\vec{v}_{2,f} = v_{2,i,\parallel}\hat{e}_\parallel + v_{2,f,\perp}\hat{e}_\perp = \vec{v}_{2,i} - \frac{2m_1}{m_1 + m_2} \frac{\langle \vec{v}_{2,i} - \vec{v}_{1,i} \,|\, \vec{r}_{2,i} - \vec{r}_{1,i} \rangle}{||\vec{r}_{2,i} - \vec{r}_{1,i}||^2} (\vec{r}_{2,i} - \vec{r}_{1,i}) \tag{14}$$