On Forecasting Volatility and Option Prices: The Case of Apple stock price

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Abstract

The aim of this teaching note is to show how to price an option using the Black and Scholes (BS) formula and econometrics models of volatility. Specifically, we will use daily data of Apple stock price for the period 2005-2014 to fit a GARCH specification which will results in a marginal volatily measure that can be plugged into the BS formula in order to get the prices of different options. The unconditional and conditional variance is estimated for differents models and hence different call values are modeled and compared.

1 Introduction

An option is a financial instrument that gives the right to buy or sell an underlying asset, within a specified period of time at a specified price called the exercise or strike price. The price the buyer of the option pays in order to gain the stated right is called the option premium. An American option can be exercised at any time up to the date the option expires. A European option can be exercised only on a specified future date. The last date on which the option may be exercised is called the expiration or maturity date. Options are used as valuable tools in numerous hedging strategies as they define the price at which underlying assets can be bought or sold in the future.

The Black and Scholes formula - BS (Black and Scholes, 1973) for option pricing is a function of the parameters of the diffusion process describing the dynamics of the underlying asset price. The no-arbitrage principle states that if options are correctly priced in the market it should not be possible to make a guaranteed profit by creating portfolios of long and short positions in options and their underlying assets. By respecting this principle they derived a theoretical valuation formula for European options on common stocks.

$$C_0 = S_0 N(d_1) - X e^{-rT} N(d_2) \tag{1}$$

$$P_0 = Xe^{-rT}N(-d_2) - S_0N(-d_1)) \tag{2}$$

$$d_1 = \frac{ln(S_0/X)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln(S_0/X)+(r-\sigma^2/2)T}{\sigma\sqrt{T}} = d_1\sigma\sqrt{T}$$

Where: C_0 – current call option value
P_0– current put option value
S_0– current price of the underlying asset
X– the exercise price
e– 2.71828, the base of the natural log function
r– the continuously compounded risk free interest rate
T– time to maturity of the option in years
ln– natural logarithm function
σ– the volatility of the underlying asset's price, i.e. standard deviation of the continuously compounded rate of return
$N(d)$– the cumulative probability distribution function for the variable normally distributed with mean 0 and standard deviation of 1. The probability that a random draw from a standard normal distribution will be less than d (area under the normal curve up to d).

The Black and Scholes formula was further accommodated for different underlying assets. It is derived for a geometric Brownian motion with a constant volatility parameter σ. In order to apply the BS formula, the econometrician has to estimate the parameter σ of the diffusion process.

2 Option Pricing in the Case of Stochastic Volatility

There are two stands of literature dealing with the valuation of options when the asset return volatility is stochastic. The first approach to option pricing is with stochastic volatility in a diffusion framework. The earlier models are of univariate nature. More recently option pricing with stochastic volatility has been dealt with in a bivariate diffusion framework, in which the volatility of an asset is assumed to follow a separate stochastic process.

Hull and White (1987) studied option pricing in the case of stochastic volatility where σ^2 follows an independent geometric Brownian motion process from the stock price process. Provided that increments of the volatility process are independent of the increments of the process governing the asset price, they showed that options can be priced by averaging the BS formula over the stochastic volatility, i.e. the option price is the BS price integrated over the distribution of the mean volatility and there is no need to price the risk attached to the stochastic volatility.

Noh, Engle and Kane (1994) predict the future mean volatility using a GARCH model (Bollerslev, 1986) on the past returns and plug it into the BS formula. They conclude that the GARCH estimated volatility has more economic content than the volatility estimates (implied volatility) obtained by inverting the BS formula using observed past option prices.

The BS formula, however, ignores the stochastic nature of volatility as, for instance, it is modeled by the GARCH model. It assumes homoskedasticity when the real governing process is heteroskedastic. If we want to get prices by using the BS formula we have to plug into the formula the marginal variance of the empirical GARCH process. It must be taken into account that the BS formula under prices out-of-the money options, options on low-volatility securities, and short maturity options. Also, when there is large uncertainty on marginal measures due to near nonstationarity, plugging in the BS formula can be dangerous and give dubious results. Therefore this formula has to be used with caution regarding input data and time horizons under consideration.

The second stand of literature develops the option pricing model in the GARCH framework. During the past decade researchers have begun to study generalized autoregressive conditional heteroskedastic (GARCH) models for option pricing because of their superior performances in describing asset returns. The particular type of non-linear model that is used in finance is known as the autoregressive conditionally heteroscedastic - ARCH model (Engle, 1982). In the context of financial time series, where it is unlikely that the variance of the error terms will be constant over time, it is used as the model which describes how the variance of the error terms evolves. It is employed commonly in modeling financial time series that exhibit time-varying volatility clustering, the tendency of large changes in asset prices to follow large changes and small changes to follow small changes.

- **ARCH Model**

Under the ARCH(q) model, the conditional mean equation could take almost any form that the researcher likes, e.g.:

$$y_t = \beta_1 + \beta_2 X_{2t} + \beta_2 X_{2t} + \beta_2 X_{2t} + ... + \mu_t \tag{3}$$

$$\mu_t \sim N(0, \sigma_t^2)$$

where y_t is the variable dependent on the vector of explanatory variables x_t, whose elements may be lagged values of dependent variable y_t, while u_t is the standard error of the model and σ_t is the conditional variance of the error term.

The variance of the residuals depends on past history and we face heteroskedasticity because the variance is changing over time. One way to deal with this is to have the variance depending on the lagged period of the squared error terms.

The conditional variance of the error term, σ_t^2, depends on q lags of squared error terms:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \mu_{t-1}^2 + \alpha_2 \mu_{t-2}^2 + ... + \alpha_q \mu_{t-q}^2 \tag{4}$$

Since the ARCH model had problems deciding the value of the number of lags of the squared residual in the model q, model robustness, and conditional variance non-negativity constraint violation, it was extended through generalized ARCH, i.e. the GARCH model which is widely used in practice.

- **GARCH Model**

The GARCH model allows the conditional variance to be dependent upon its own previous lags. In the GARCH (p,q) model conditional variance is dependant upon q lags of the squared error term and p lags of the conditional variance.

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \mu_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{5}$$

In general, a GARCH(1,1) model is sufficient to capture the volatility clustering in the data and is most often used in financial theory and practice.

$$\sigma_t^2 = \alpha_0 + \alpha_1 \mu_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \qquad \alpha_1 \geq 0, \quad \beta_1 \leq 1 \quad (\alpha_1 + \beta_1) < 1 \tag{6}$$

Duan (1995) provides a general method for option pricing in the case of the GARCH processes. He uses the GARCH-M model of Engle, Lilien and Robins (1987), which combines nicely with the properties of the lognormal distribution.

The GARCH-in-mean (GARCH-M) model follows most models used in finance that assume that investors should be rewarded for taking additional risk by obtaining a higher return. One way to show this is to let the investment return be partly determined by investment risk. The GARCH-M model allows the conditional variance of asset returns to enter into the conditional mean equation. It can be presented in the following simple GARCH (1, 1)-M form:

$$y_t = \gamma + \delta \sigma_t^2 + \mu_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \mu_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

The δ can be interpreted as a risk premium. If δ is positive and statistically significant, then increased risk caused by an increase in the conditional variance leads to a rise in the mean return.

Calculating the unconditional variance from the models explained above, we can estimate the standar deviation we are looking for. In the following paragraphs we derive the unconditional variance for ARCH and GARCH representations. Finally a note on scaling daily variance is presented.

- **ARCH Unconditional Variance**

We assume a process that could be represented by an econometric model, for example:

$$y_t = \mu + \epsilon_t \tag{7}$$

with $\epsilon_t \sim \left(0, \sigma_t^2\right)$
Let the conditional variance follows an ARCH(1) model, that is:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \tag{8}$$

Using the unconditional expectation operator, we have:

$$\mathbb{E}\left(\sigma_t^2\right) = \sigma_t^2$$

$$\mathbb{E}\left(\alpha_0\right) = \alpha_0$$

$$\mathbb{E}\left(\epsilon_{t-1}^2\right) = \sigma_t^2$$

We have:

$$\sigma_t^2\left(1 - \alpha_1\right) = \alpha_0 \tag{9}$$

$$\Rightarrow \sigma_t^2 = \frac{\alpha_0}{1 - \alpha_1} \tag{10}$$

If we generalize to an ARCH(q) model, we get:

$$\begin{aligned} \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \cdots + \alpha_q \epsilon_{t-q}^2 \\ &= \alpha_0 + \sum_{k=1}^{q} \alpha_k \epsilon_{t-k}^2 \end{aligned} \tag{11}$$

where

$$\mathbb{E}\left(\epsilon_{t-1}^2\right) = \mathbb{E}\left(\epsilon_{t-2}^2\right) = \cdots = \mathbb{E}\left(\epsilon_{t-q}^2\right) = \sigma_t^2$$

then:

$$\begin{aligned} \sigma_t^2 &= \frac{\alpha_0}{1 - \alpha_1 - \alpha_2 - \cdots - \alpha_q} \\ &= \frac{\alpha_0}{1 - \sum_{k=1}^{q} \alpha_k} \end{aligned} \tag{12}$$

- **GARCH Unconditional Variance**

Suppose the same process given previously, but this time the variance also depends on its own p lags:

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \alpha_2\epsilon_{t-2}^2 + \cdots + \alpha_q\epsilon_{t-q}^2 + \beta_1\sigma_{t-1}^2 + \beta_2\sigma_{t-2}^2 + \cdots + \beta_p\sigma_{t-p}^2 \\ &= \alpha_0 + \sum_{k=1}^{q}\alpha_k\epsilon_{t-k}^2 + \sum_{l=1}^{p}\alpha_l\sigma_{t-l}^2 \qquad (13)\end{aligned}$$

The above equation give a GARCH(p,q) model. Using
$\mathbb{E}\left(\sigma_{t-1}^2\right) = \mathbb{E}\left(\sigma_{t-2}^2\right) = \cdots = \mathbb{E}\left(\sigma_{t-p}^2\right) = \sigma_t^2$

Thus, we have:

$$\begin{aligned}\sigma_t^2 &= \frac{\alpha_0}{1-\alpha_1-\alpha_2-\cdots-\alpha_q-\beta_1-\beta_2-\cdots-\beta_p} \\ &= \frac{\alpha_0}{1-\sum_{k=1}^{q}\alpha_k-\sum_{l=1}^{p}\beta_l} \qquad (14)\end{aligned}$$

- **Scaling Volatility**

Volatility calculation and scaling over different time horizons is possible only in cases when changes in the log of asset price v_t are independently and identically distributed (iid).

$$v_t = v_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim (0, \sigma^2) \qquad (15)$$

Then 1-day return is:

$$v_t - v_{t-1} = \varepsilon_t$$

with standard deviation σ. Similarly, the h-day return is:

$$v_t - v_{t-h} = \sum_{i=0}^{h-1}\varepsilon_{t-i} \qquad (16)$$

with variance $h\sigma^2$ and standard deviation $\sqrt{h\sigma^2}$. However, high-frequency financial asset returns are distinctly not iid....but it still is a good approximation.

3 Analysis of the Results

The sample includes daily data in the form of the natural logarithm (ln), excluding weekends and holidays, of the Apple stock prices, obtained for the period 12 December 1980 to 22 August 2014, comprising 8498 data points[1].

We tested for the series stationarity and found that the series is not stationary, which is supported by the estimates of the autocorrelation and partial autocorrelation functions together with the unit-root test presented below.

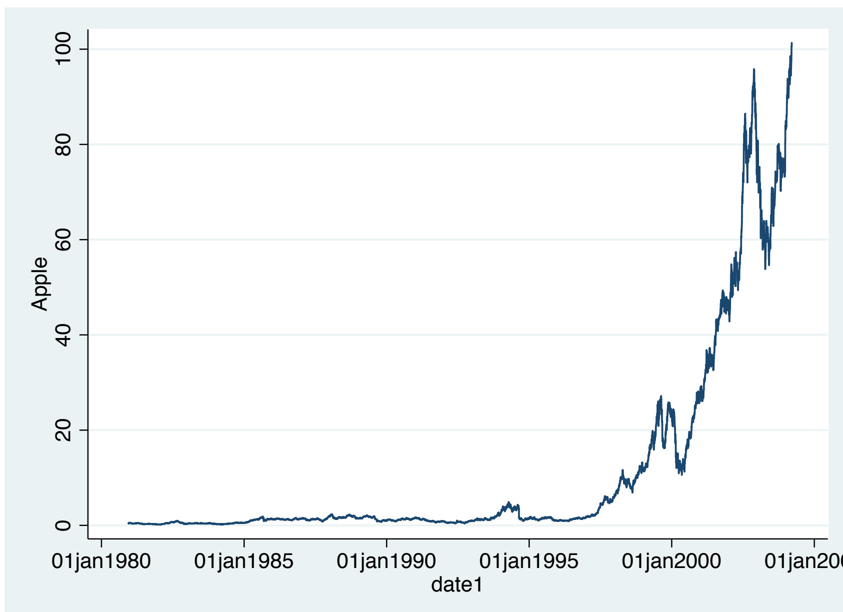


Figure 1. Daily Return of Apple (Dec 1980 – Aug 2014)

[1]The data was obtained from the yahoo finance database

Table 1. Correlogram

LAG	AC	PAC	Q	Prob>Q
1	**0.9988**	**1.0008**	**8480.2**	**0.0000**
2	**0.9976**	**-0.0179**	**16941**	**0.0000**
3	**0.9963**	**0.0059**	**25382**	**0.0000**
4	**0.9951**	**0.0204**	**33803**	**0.0000**
5	**0.9939**	**-0.0590**	**42205**	**0.0000**
6	**0.9928**	**0.0241**	**50588**	**0.0000**
7	**0.9916**	**-0.0246**	**58953**	**0.0000**
8	**0.9904**	**0.0216**	**67299**	**0.0000**
9	**0.9893**	**0.0761**	**75628**	**0.0000**
10	**0.9883**	**0.0322**	**83939**	**0.0000**

```
. dfuller apple

Dickey-Fuller test for unit root                   Number of obs   =      8497

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)              3.773            -3.430            -2.860            -2.570
------------------------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 1.0000
```

Autocorrelation coefficients are statistically significant on the significance level of 5% if they are beyond the interval (± 0.06). The correlogram showed that the ACF is long exponentially decaying and we have a unit root in the series and autocorrelation. By performing the unit root test on the series, we found that the ADF test statistic (3.773) is higher than the critical value at a 1% significance level (-3.4430), indicating that we do not reject the null hypothesis, e.g. that there is a unit root in the series. In other words, the series is not stationary.

The first step in modeling time series data is to convert the non-stationary time series to stationary one. This is important for the fact that a great deal of statistical and econometric methods are based on this assumption and can only be applied to stationary time series. Non-stationary time series are erratic and unpredictable while stationary process is mean-reverting, i.e, it fluctuates around a constant mean with constant variance. In addition, stationarity and independence of random variables are closely related because many theories that hold for independent random variables also hold for stationary time series in which independence is a required condition. The majority of these methods assume the random variables are independent (or uncorrelated); the noise is independent (or uncorrelated); and the variable and noise are independent (or uncorrelated) of each other.

In order to eliminate the unit root we found the first difference in ln(apple), i.e. exchange rate returns, ln(apple)-ln(apple(-1)), and did the test again.

Figure 2. First Difference of the Apple Stock Price (Returns)

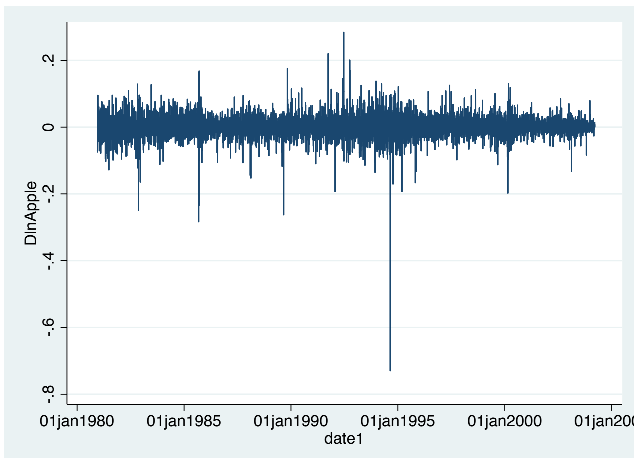


Table 2. Correlogram

```
. corrgram DlnApple
```

LAG	AC	PAC	Q	Prob>Q	-1 0 1 [Autocorrelation]	-1 0 1 [Partial Autocor]
1	**0.0140**	**0.0140**	**1.6599**	**0.1976**		
2	**−0.0246**	**−0.0248**	**6.796**	**0.0334**		
3	**−0.0285**	**−0.0278**	**13.697**	**0.0033**		
4	**0.0294**	**0.0296**	**21.052**	**0.0003**		
5	**0.0007**	**−0.0014**	**21.057**	**0.0008**		
6	**0.0131**	**0.0138**	**22.52**	**0.0010**		
7	**−0.0107**	**−0.0095**	**23.503**	**0.0014**		
8	**0.0048**	**0.0049**	**23.698**	**0.0026**		
9	**−0.0163**	**−0.0162**	**25.958**	**0.0021**		
10	**−0.0201**	**−0.0208**	**29.401**	**0.0011**		

```
. dfuller DlnApple

Dickey-Fuller test for unit root                   Number of obs   =      8496

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)            -90.910            -3.430            -2.860            -2.570
------------------------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.0000
```

We found that the ADF test (-90.910) reject the null hypothesis of unit root in the series (non stationarity) at 1% significance level (-3.430) on the return of apple stock. Thus we have shows that we do not have more than one unit root and that the first difference is stationary. Thus we concluded that we can specify the ARIMA (p,d,q) model with one unit root.

Box- Jenkins method provides a way to identify ARIMA model according to autocorrelation and partial autocorrelation graph of the series. The parameters of ARIMA consist of three components: p (autoregressive parameter), d (number of differencing), and q (moving average parameters).

There are three rules to identify ARIMA model:

- If ACF (autocorrelation graph) cut off after lag n, PACF (partial autocorrelation graph) dies down: ARIMA(0, d, n)$\Longrightarrow$identify MA(q)
- If ACF dies down, PACF cut off after lag n: ARIMA(n, d, 0)$\Longrightarrow$identify AR(p) .
- If ACF and PACF die down: mixed ARIMA model, need differencing.

In addition to Box-Jenkins method, AICc provides another way to check and identify the model. AICc is corrected Akaike Information Criterion. According to this method, the model with lowest AICc will be selected. In order to calibrate a volatility for the next three months we will use the last three month of data.

Specifically for this asset we model the following models: AIC1$\rightarrow$ arima(0, 0, 0), AIC2$\rightarrow$arima(0, 0, 1), AIC3$\rightarrow$arima(0, 0, 2) , AIC4$\rightarrow$arima(0, 0, 3), AIC5$\rightarrow$arima(0, 0, 4).

```
. estimates stats AIC1 AIC2 AIC3 AIC4 AIC5
```

Model	Obs	ll(null)	ll(model)	df	AIC	BIC
AIC1	998	.	2666.924	2	-5329.847	-5320.036
AIC2	998	.	2667.137	3	-5328.274	-5313.556
AIC3	998	.	2667.274	4	-5326.547	-5306.924
AIC4	998	.	2667.721	5	-5325.442	-5300.913
AIC5	998	.	2669.384	6	-5326.769	-5297.334

Note: N=Obs used in calculating BIC; see [R] BIC note

```
. varsoc DlnApple

   Selection-order criteria
   Sample:  29jun2001 - 18mar2004               Number of obs      =       994
```

lag	LL	LR	df	p	FPE	AIC	HQIC	SBIC
0	2654.89				.000281*	-5.33981*	-5.33794*	-5.33488*
1	2655.11	.45469	1	0.500	.000281	-5.33826	-5.33451	-5.32839
2	2655.24	.24346	1	0.622	.000282	-5.33649	-5.33087	-5.3217
3	2655.71	.94531	1	0.331	.000282	-5.33543	-5.32793	-5.3157
4	2657.1	2.7781	1	0.096	.000282	-5.33621	-5.32684	-5.31156

```
   Endogenous:  DlnApple
    Exogenous:  _cons
```

Clearly the test point to a model ARMA (0,0), which is tested below:

```
ARIMA regression

Sample:  25jun2001 - 18mar2004                  Number of obs      =       998
                                                Wald chi2(.)       =         .
Log likelihood =  2666.924                      Prob > chi2        =         .
```

DlnApple	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
DlnApple						
_cons	.0010605	.0005345	1.98	0.047	.0000129	.0021081
/sigma	.016719	.0001849	90.43	0.000	.0163567	.0170814

Note: The test of the variance against zero is one sided, and the two-sided confidence interval is truncated at zero.

The ARMA (p, q) model states that the current value of some series y depends linearly on its own previous values plus a combination of current and previous values of a white noise error term u. In the general form, the model can be written as follows:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + \theta_1 u_{t-1} + \theta_2 u_{t-2} + ... + \theta_q u_{t-q} + u_t \quad (17)$$

$$E(u_t) = 0; \quad E(u_t^2) = \sigma^2; \quad E(u_t u_s) = 0 \qquad t \neq s$$

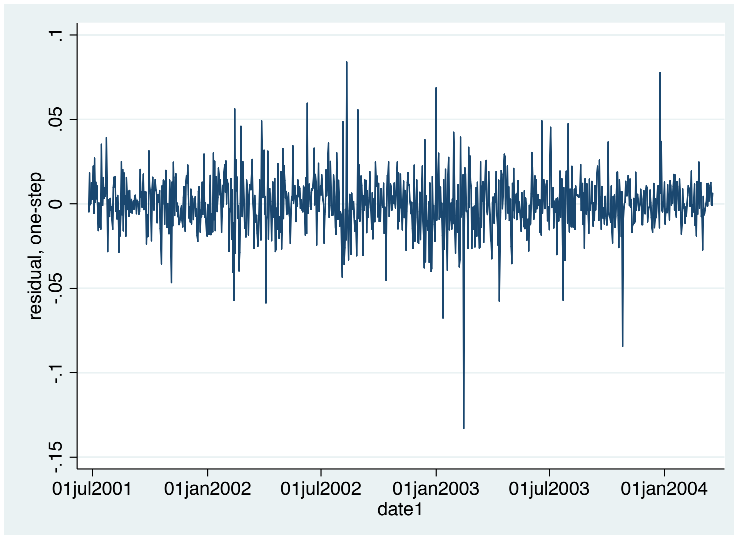
Our model for Apple returns is ARMA (0, 0):

$$y_t = 0.0010605 + u_t$$

Since series ln(apple) has 1 unit root and its first difference, apple's returns, can be estimated with the ARMA(0, 0) model, the series ln(apple) can be estimated with the ARIMA (0, 1, 0) model.

The analysis continued with a test for an ARCH effect presence in the specified model ARIMA (0,1,0). First we looked at the mean equation and saw that it is statistically significant. We further checked for the presence of an ARCH effect. In order to do that we looked at the correlogram of residuals squared that showed the presence of the ARCH effect. We saw that residuals are not normally distributed and that there is an ARCH effect, but in order to confirm this we performed the ARCH LM Test.

Figure 4. Residual ARIMA(0,1,0)



By referring to the statistic (8.191) and related probability (0.0042) we concluded that there is an ARCH effect present in the series.

```
. qui reg DlnApple L(1/4).DlnApple

. estat archlm
LM test for autoregressive conditional heteroskedasticity (ARCH)
```

lags(*p*)	chi2	df	Prob > chi2
1	**8.191**	**1**	**0.0042**

```
H0: no ARCH effects      vs.  H1: ARCH(p) disturbance
```

At this point in the analysis we tried to identify the most appropriate GARCH specification for the above ARIMA (0,1,0) model. Additionally, two other measures of volatility are estimated: historical and implied volatility.

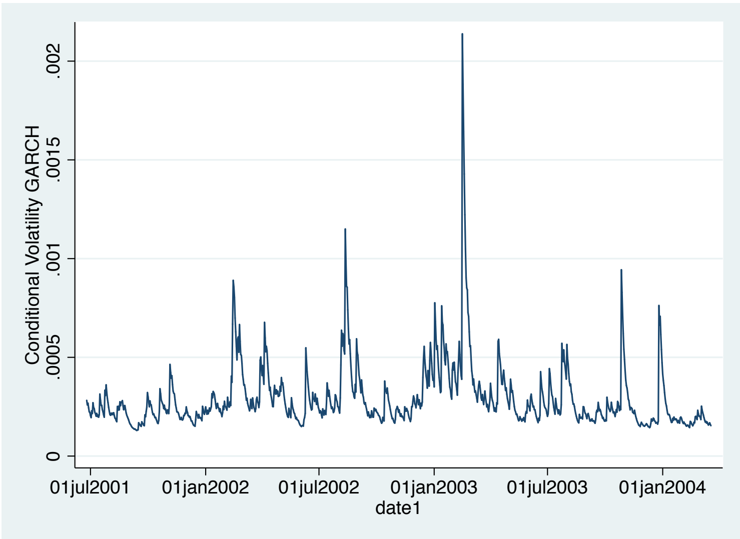
- **GARCH**

```
. arch DlnApple, arch(1) garch(1) arima(0,0,0) nolog

ARCH family regression

Sample: 25jun2001 - 18mar2004                    Number of obs   =       998
Distribution: Gaussian                           Wald chi2(.)    =         .
Log likelihood =  2692.255                       Prob > chi2     =         .
```

DlnApple	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
DlnApple						
_cons	.0017455	.0004756	3.67	0.000	.0008134	.0026776
ARCH						
arch L1.	.1002837	.0221887	4.52	0.000	.0567947	.1437728
garch L1.	.8407344	.0286212	29.37	0.000	.7846378	.8968309
_cons	.0000187	3.62e-06	5.15	0.000	.0000116	.0000257



$$y_t = 0.0017455 + u_t$$

$$\sigma_t^2 = 0.1002837 u_t^2 + 0.8407344 \sigma_{t-1}^2$$

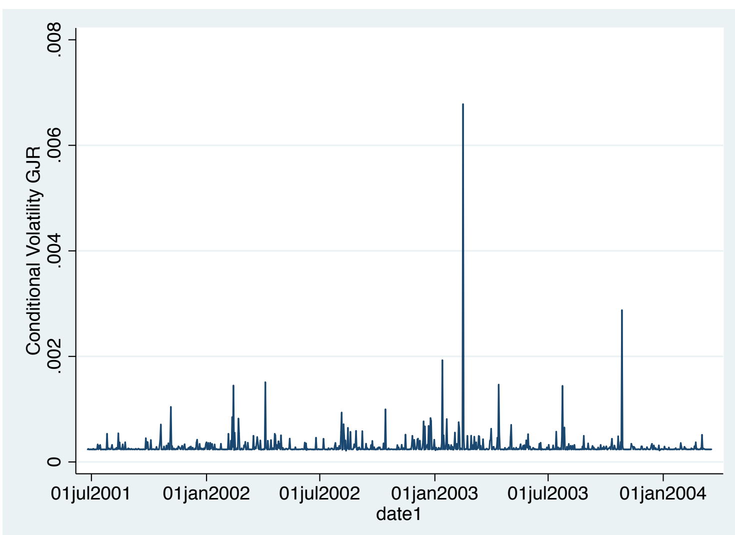
- **GARCH GJR**

```
. arch DlnApple, arch(1/1) tarch(1/1) arima(0,0,0) nolog

ARCH family regression

Sample: 25jun2001 - 18mar2004                         Number of obs   =       998
Distribution: Gaussian                                Wald chi2(.)    =         .
Log likelihood =  2683.283                            Prob > chi2     =         .
```

DlnApple	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
DlnApple						
_cons	.0010995	.000528	2.08	0.037	.0000646	.0021343
ARCH						
arch L1.	.3695543	.0700654	5.27	0.000	.2322285	.5068801
tarch L1.	-.3728632	.068846	-5.42	0.000	-.507799	-.2379275
_cons	.0002383	7.62e-06	31.27	0.000	.0002234	.0002532



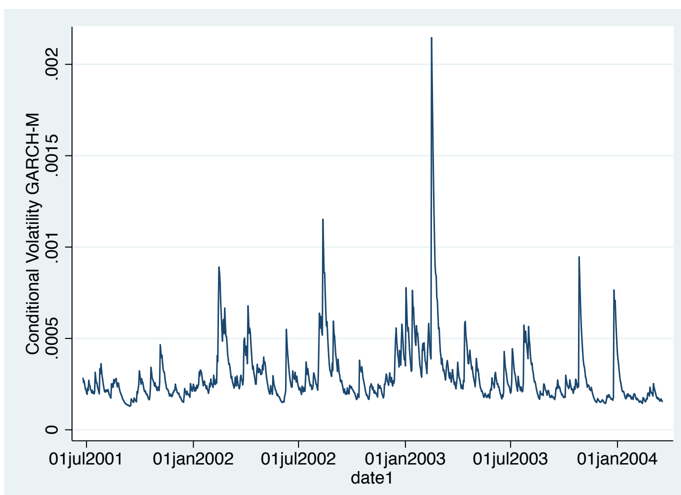
- **GARCH M**

```
. arch DlnApple, arch(1) garch(1) archm arima(0,0,0) nolog

ARCH family regression

Sample: 25jun2001 - 18mar2004                     Number of obs   =       998
Distribution: Gaussian                            Wald chi2(1)    =      0.01
Log likelihood =  2692.258                        Prob > chi2     =    0.9407
```

DlnApple	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf. Interval]	
DlnApple						
_cons	.0016587	.001315	1.26	0.207	-.0009187	.0042361
ARCHM						
sigma2	.3539658	4.759157	0.07	0.941	-8.973811	9.681742
ARCH						
arch						
L1.	.1006354	.0223762	4.50	0.000	.0567788	.144492
garch						
L1.	.8401534	.0287203	29.25	0.000	.7838627	.8964441
_cons	.0000187	3.62e-06	5.17	0.000	.0000116	.0000258



$$y_t = 0.0010995 + 0.3539658\sigma_t + u_t$$

$$\sigma_t^2 = 0.3695543u_t^2 - 0.3728632\sigma_{t-1}^2$$

- **Historical Volatility**

Historical volatility is the realized volatility of a financial instrument over a given time period. Generally, this measure is calculated by determining the average deviation from the average return of a financial instrument in the given time period. Standard deviation is the most common but not the only way to calculate historical volatility. This standar deviation was calculated for the Apple stock return in the same period considered by the GARCH models.

- **Implied Volatility**

Finally, we estimate the implied volatility of an option contract for Apple. Implied volatility is that value of the volatility of the underlying instrument which, when input in an option pricing model (such as Black–Scholes) will return a theoretical value equal to the current market price of the option. Thus we capture from yahoo finance an Apple call contract with expiration day 21-Nov-2014, see figure below.

Apple Inc. (AAPL) - NasdaqGS ★ **Follow**

101.32 ↑0.74(0.74%) Aug 22, 4:00PM EDT

After Hours : **101.45** ↑0.13 (0.13%) Aug 22, 7:59PM EDT

Options

View By Expiration: Aug 14 | Sep 14 | Oct 14 | **Nov 14** | Dec 14 | Jan 15 | Apr 15 | Jan 16

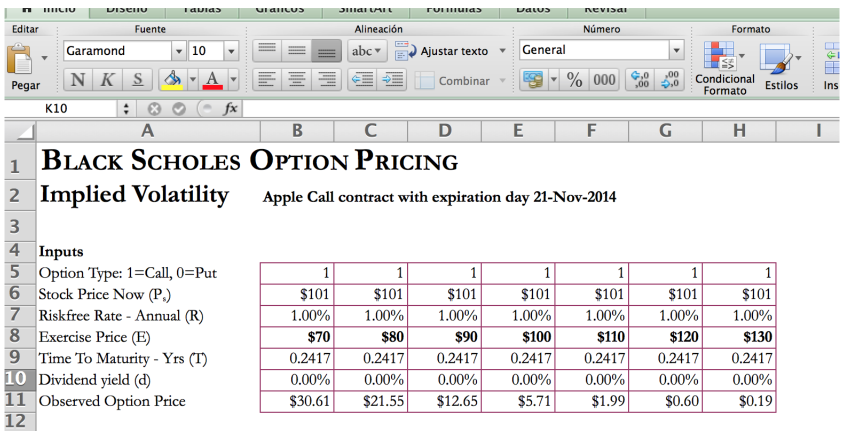
Call Options — Expire at close Saturday, November 22, 2014

Strike	Symbol	Last	Chg	Bid	Ask	Vol	Open Int
47.50	AAPL141122C00047500	53.25	0.00	52.25	55.50	121	121
50.00	AAPL141122C00050000	50.50	0.00	49.75	52.00	40	40
55.00	AAPL141122C00055000	40.80	0.00	46.20	46.75	40	40
60.00	AAPL141122C00060000	37.00	0.00	41.20	41.60	4	4
65.00	AAPL141122C00065000	36.10	0.00	36.20	36.60	5	9
70.00	AAPL141122C00070000	30.61	↑0.56	31.30	31.75	1	39
75.00	AAPL141122C00075000	25.70	0.00	26.50	26.70	60	696
80.00	AAPL141122C00080000	21.55	↑0.40	21.60	21.80	41	2,227
85.00	AAPL141122C00085000	16.95	↑0.71	16.90	17.10	14	1,003
87.50	AAPL141122C00087500	14.60	↑0.58	14.65	14.85	122	517
90.00	AAPL141122C00090000	12.65	↑0.69	12.50	12.70	805	3,790
92.50	AAPL141122C00092500	10.55	↑0.35	10.50	10.70	124	3,008
95.00	AAPL141122C00095000	8.70	↑0.45	8.70	8.85	819	10,128
97.50	AAPL141122C00097500	7.14	↑0.44	7.05	7.20	264	7,896
100.00	AAPL141122C00100000	5.71	↑0.41	5.70	5.75	2,232	17,054
105.00	AAPL141122C00105000	3.38	↑0.18	3.45	3.50	831	30,311
110.00	AAPL141122C00110000	1.99	↑0.15	1.96	2.00	1,177	15,318
115.00	AAPL141122C00115000	1.08	↑0.08	1.08	1.12	386	15,205
120.00	AAPL141122C00120000	0.60	↑0.01	0.59	0.62	1,262	15,393
125.00	AAPL141122C00125000	0.34	0.00	0.32	0.35	3	1,253
130.00	AAPL141122C00130000	0.19	↓0.01	0.19	0.20	53	1,651
135.00	AAPL141122C00135000	0.14	↑0.03	0.11	0.14	1	465

With these contracts we can use the Newton-Raphson formula for finding the root of Black-Schole equation. In this page:

http://finance.bi.no/~bernt/gcc_prog/recipes/recipes/node7.html

a program in language C is developed in order to calculate implied volatility using Newton-Raphson. Alternatively, an Excel spreadsheet can do the trick by trial an error.



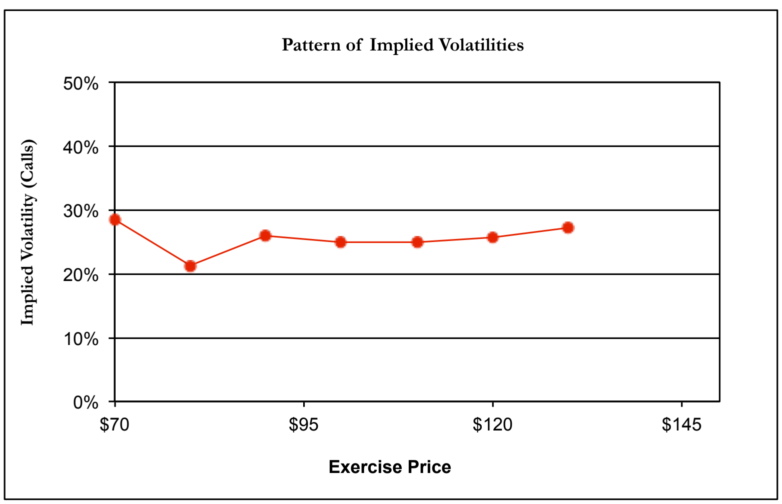
BLACK SCHOLES OPTION PRICING

Implied Volatility — **Apple Call contract with expiration day 21-Nov-2014**

Inputs

Option Type: 1=Call, 0=Put	1	1	1	1	1	1	1
Stock Price Now (P_s)	$101	$101	$101	$101	$101	$101	$101
Riskfree Rate - Annual (R)	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%
Exercise Price (E)	**$70**	**$80**	**$90**	**$100**	**$110**	**$120**	**$130**
Time To Maturity - Yrs (T)	0.2417	0.2417	0.2417	0.2417	0.2417	0.2417	0.2417
Dividend yield (d)	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Observed Option Price	$30.61	$21.55	$12.65	$5.71	$1.99	$0.60	$0.19

The results are shown below:



4 Conclussions

In this teaching note we have seen how to calculate the volatility of an underlying asset's price using different GARCH models. Besides we calculated the standard deviation of the rate of return of the underlying asset using two no econometric techniques. All the programs and data used in this paper will be upload to webcursos, so every student could calculate the results by themselves. The results presented below correspond to models considering only 3 month data.

Estimated standar deviation for Apple

GARCH (1,1)	GARCH GJR (1,1)	GARCH-M(1,1)	HISTORICAL	IMPLIED
28.1%	24.4%	28.1%	26.4%	24.97%

Finally, using these estimations we could calibrate equation (1) to valua a call 3-months option for Apple stocks. If we wait these three month we will see if GARCH models do rock!

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