Friday, Aug 30

Sample Size Selection

Recall that when estimating μ with $\bar{y}$ under simple random sampling we have that

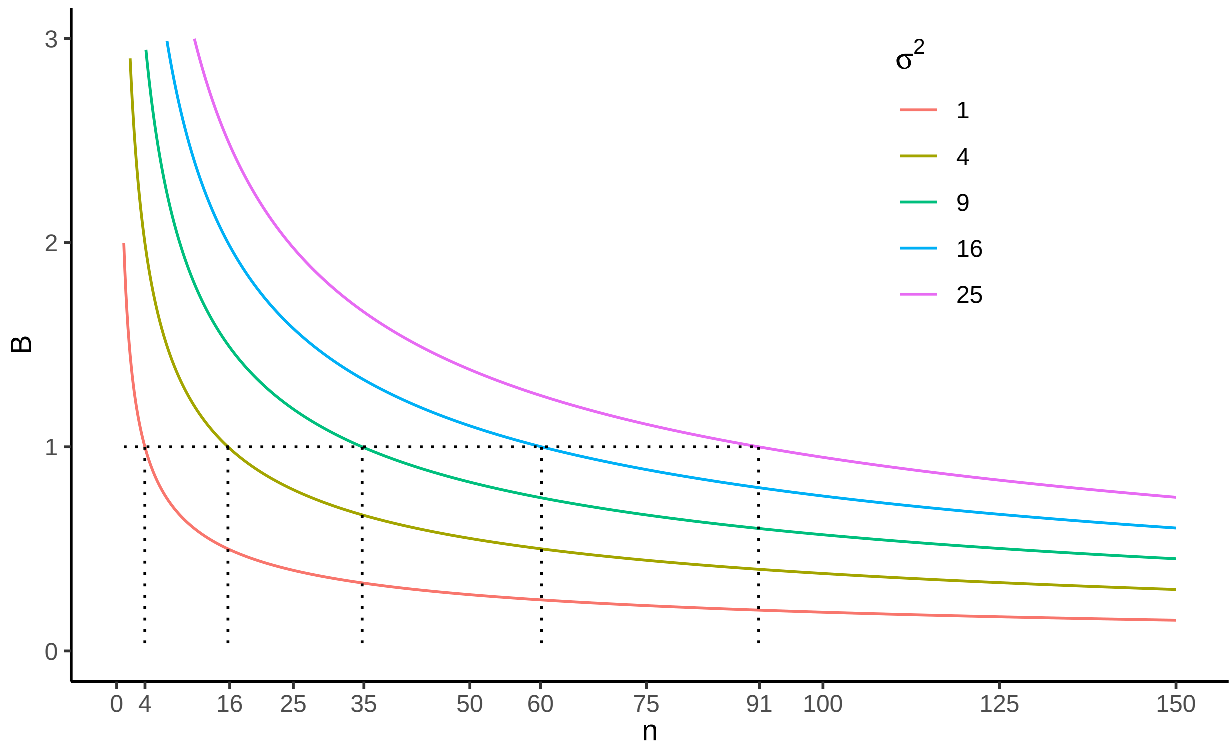
$$V(\bar{y}) = \left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}.$$

and that the bound on the error of estimation is

$$B = 2\sqrt{V(\bar{y})} = 2\sqrt{\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}.$$

The known relationship between B and n can be used to select a n for a desired B.

Example: What n would yield $B = 1$ if $N = 1000$ and $\sigma^2 = 1$, 4, 9, 16, or 25 when using a simple random sampling design?



More generally, we have the approximate relationship

$$n \approx \frac{4N\sigma^2}{B^2 N + 4\sigma^2}.$$

Example: What sample size would yield $B \approx 1$ if $N = 1000$ and $\sigma^2 = 25$?

Similarly if we are estimating τ with $\hat{\tau} = N\bar{y}$ then under simple random sampling we have that

$$B = 2\sqrt{N^2\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}} \Leftrightarrow n \approx \frac{4N\sigma^2}{B^2/N + 4\sigma^2}.$$

Example: What sample size would yield $B \approx 100$ if $N = 1000$ and $\sigma^2 = 25$?

What is the impact of B and σ^2 on n?

Specification of σ^2

How do we specify σ^2 when selecting n?

1. Expert judgment.
2. Pilot survey of the same population.
3. Previous survey of a similar population.
4. Statistical relationships involving σ^2.

Statistical Relationships Involving σ^2

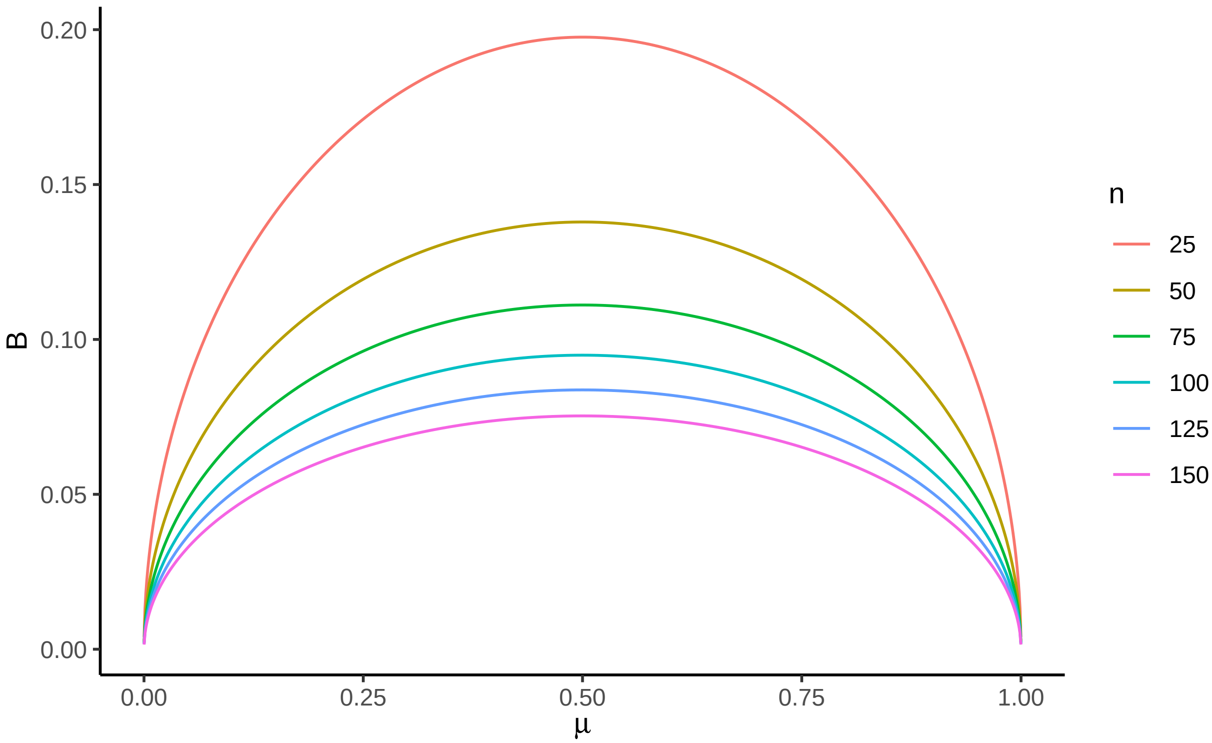
1. If the distribution of the target variable in the population is approximately normal/bell-shaped, then $\sigma \approx$ range/4 or $\sigma \approx$ range/6. The range is the distance between the largest and smallest values of the target variable (ignoring outliers).
2. If the distribution of the target variable in the population is right-skewed then often σ^2 is approximately proportional to μ, or proportional to a power of μ (so larger μ's imply larger σ^2's).
3. If the target variable is an *indicator variable* so that μ is the *proportion* of elements in the population that are in a category, then we have that

   $$\sigma^2 = \frac{N}{N-1}\mu(1-\mu).$$

   This reaches its maximum when $\mu = 0.5$ so

   $$\sigma^2 \leq 0.25N/(N-1) \approx 0.25$$

   for all μ. Thus the bound on the error of estimation is at its maximum when $\mu = 0.5$ as shown below.



This suggests a strategy for computing an *upper bound* on n by using $\sigma^2 \approx 0.25$ when the target variable is categorical if we do not have a good guess of μ.

Example: Recall the survey using a simple random sampling design to estimate the proportion of students at a university of 20000 that own an Android mobile phone. If we wanted to estimate μ (the proportion of students at the university that own an Android mobile phone) with a bound on the error of estimation of $B = 0.01$, what sample size would be needed if (a) we used a prior survey that estimated that μ is 0.4 and (b) if we wanted to use an upper bound on n?

Summary of Simple Random Sampling Notation and Formulas

The Population

We have a *population* of N units/elements. The mean (μ), total (τ), and variance (σ^2) of the target variable for these N units/elements are defined as

$$\mu = \sum_{i=1}^{N} y_i, \ \tau = \sum_{i=1}^{N} y_i, \ \sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \mu)^2.$$

The Sample

We select a *sample* of n units/elements using a *simple random sampling* design. The mean ($\bar{y}$) and variance (s^2) of the target variable for these n units/elements are defined as

$$\bar{y} = \frac{1}{n}\sum_{i \in \mathcal{S}} y_i, \ s^2 = \frac{1}{n-1}\sum_{i \in \mathcal{S}}(y_i - \bar{y})^2.$$

The Estimators

An *estimator* of μ is $\bar{y} = \frac{1}{n}\sum_{i \in \mathcal{S}} y_i$. An estimator of τ is

$$\hat{\tau} = \frac{N}{n}\sum_{i \in \mathcal{S}} y_i,$$

which can also be written as $\hat{\tau} = N\bar{y}$. We can also use s^2 as an estimator of σ^2 if necessary.

The Sampling Distributions

Based on the *sampling distribution* of an estimator we can find its mean and variance, and several other useful quantities like the standard error and the bound on the error of estimation.

Sampling Distribution of $\bar{y}$ The estimator $\bar{y}$ has a mean of μ and a variance of

$$V(\bar{y}) = \left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}.$$

The *standard error* of $\bar{y}$ is its standard deviation and so simply the square root of the variance so that

$$\sqrt{V(\bar{y})} = \sqrt{\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}.$$

The *bound on the error of estimation* of using $\bar{y}$ to estimate μ is twice the standard error of $\bar{y}$ so that

$$B = 2\sqrt{V(\bar{y})} = 2\sqrt{\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}.$$

The *confidence interval* for estimating μ using $\bar{y}$ is

$$\bar{y} \pm 2\sqrt{\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}} \Leftrightarrow \left(\bar{y} - 2\sqrt{\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}, \bar{y} + 2\sqrt{\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}\right).$$

The Sampling Distribution of $\hat{\tau}$ The estimator $\hat{\tau}$ has a mean of τ and a variance of

$$V(\hat{\tau}) = N^2\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}.$$

The *standard error* of $\hat{\tau}$ is its standard deviation and so simply the square root of the variance so that

$$\sqrt{V(\hat{\tau})} = \sqrt{N^2\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}.$$

The *bound on the error of estimation* of using $\hat{\tau}$ to estimate τ is twice the standard error of $\hat{\tau}$ so that

$$B = 2\sqrt{V(\hat{\tau})} = 2\sqrt{N^2\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}.$$

The *confidence interval* for estimating τ using $\hat{\tau}$ is

$$\hat{\tau} \pm 2\sqrt{N^2\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}} \Leftrightarrow \left(\hat{\tau} - 2\sqrt{N^2\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}, \hat{\tau} + 2\sqrt{N^2\left(1 - \frac{n}{N}\right)\frac{\sigma^2}{n}}\right).$$

Note: The estimator of τ, its standard error, and its bound on the error of estimation are N times the corresponding quantities for μ.

Note: Sometimes we use the results of a survey to compute the *estimated* variance, standard error, bound on the error of estimation, or confidence interval. This is done by replacing σ^2 with s^2 in any formula using σ^2.