S&DS 351: Stochastic Processes - Homework 2

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Problem 1.1 (15 points)

Suppose that we are given a Markov Chain $X_0, X_1, \ldots$. Show that for any $1 \leq k \leq n-1$, and states $i, j, j_1, j_2, \ldots, j_{n-k}$:

$$P(X_n = i | X_{n-k} = j, X_{n-k-1} = j_1, X_{n-k-2} = j_2, \ldots, X_0 = j_{n-k}) = P(X_n = i | X_{n-k} = j).$$

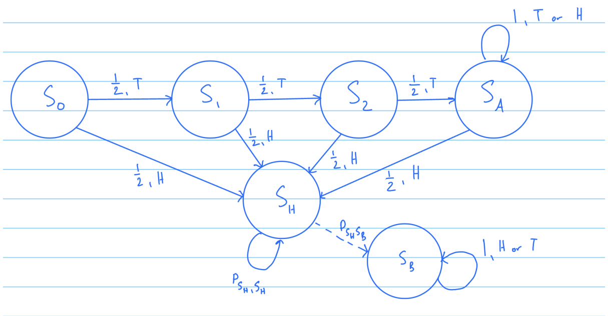
Prove this result and show your steps.

Suppose

Problem 1.2 (10 points)

Consider the following coin-tossing game. A single fair coin is flipped sequentially. Alice wins if `TTT` comes up first and Bob wins if `HTT` comes up. What is the probability that Alice wins?

Let S be a set of possible states. Define S_0 as the starting state with no flips or no partial pattern of TTT or HTT. Define S_1 as the state where there has been exactly one tail in a row, i.e. the most recent flip was T, without any prior heads. Define S_2 as the state where the last two flips were consecutive tails, without any prior heads. Define S_A as the state where the last three flips were consecutive tails, i.e. the absorbing state where Alice has won. Define S_B as the state where the last three flips were consecutive heads, i.e. the absorbing state where Bob has won. Finally, define S_H to be the state where at least one heads has been seen.
This yields the following Markov Chain:



This yields the transition matrix,

$$P = \begin{bmatrix} P_{S_0,S_0} & P_{S_0,S_1} & P_{S_0,S_2} & P_{S_0,S_A} & P_{S_0,S_B} & P_{S_0,S_H} \\ P_{S_1,S_0} & P_{S_1,S_1} & P_{S_1,S_2} & P_{S_1,S_A} & P_{S_1,S_B} & P_{S_1,S_H} \\ P_{S_2,S_0} & P_{S_2,S_1} & P_{S_2,S_2} & P_{S_2,S_A} & P_{S_2,S_B} & P_{S_2,S_H} \\ P_{S_A,S_0} & P_{S_A,S_1} & P_{S_A,S_2} & P_{S_A,S_A} & P_{S_A,S_B} & P_{S_A,S_H} \\ P_{S_B,S_0} & P_{S_B,S_1} & P_{S_B,S_2} & P_{S_B,S_A} & P_{S_B,S_B} & P_{S_B,S_H} \\ P_{S_H,S_0} & P_{S_H,S_1} & P_{S_H,S_2} & P_{S_H,S_A} & P_{S_H,S_B} & P_{S_H,S_H} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & P_{S_H,S_B} \neq 0 & P_{S_H,S_H} \neq 0 \end{bmatrix}$$

Note that after S_H has been reached, Alice can no longer win, as HTT will always preceed TTT, i.e. $\lim_{n\to\infty} \pi_{S_H} = (0,0,0,0,1,0)$, or Bob will eventually always win after S_H is reached. Since S_A is the only absorbing state where Alice can win, the probabilty of reaching S_A is given by,

$$p_0 = \mathbb{P}(S_A|S_0), p_1 = \mathbb{P}(S_A|S_1), p_2 = \mathbb{P}(S_A|S_2)$$

Using the probability transition matrix and that any transition to S_H prohibits Alice from winning,

$$p_2 = 1 \cdot P_{S_2,S_A} + 0 \cdot P_{S_2,S_H} = \frac{1}{2}$$

$$p_1 = P_{S_1,S_2} \cdot P_{S_2,S_A} + 0 \cdot P_{S_1,S_H} = \frac{1}{4}$$

$$p_0 = P_{S_0,S_1} \cdot P_{S_1,S_2} \cdot P_{S_2,S_A} + 0 \cdot P_{S_0,S_H} = \frac{1}{8}$$

Thus, the probability that Alice wins is $\frac{1}{8}$.

Problem 1.3 (Moran model)

In population genetics, the Moran model is a simple Markov chain-based model to describe evolutionary competition. Consider a population of N individuals, divided into two types called A and B. The population evolves according to the following mechanism: At each time $n \geq 0$, two individuals are selected from the current population by independent random sampling *with replacement*. The first individual gives birth to a copy of itself, which joins the population together with its parent. Then, the second individual dies and is removed from the population. The result of these two steps is the population at time $n+1$. The random samplings at different times are all independent and uniform. Note that by design, the size of the total population stays at N.

(a) (5 points) Let X_n denote the number of individuals of type A at time n. Show that $\{X_n : n \geq 0\}$ is a Markov chain. Identify the state space and find the transition probabilities $P = (P_{ij})$.

(b) (5 points) For $N = 3$, write down the transition matrix P explicitly. Find the stationary distribution(s), i.e., the distribution(s) π with $\pi = P\pi$. Is it unique?

Problem 1.4

Let M be a positive integer. Let $Y_0, Y_1, \dots$ be iid and uniformly distributed on $\{1, \dots, M\}$. Let $X_n = \min\{Y_0, \dots, Y_n\}$.

(a) (5 points) Show that $\{X_n : n \geq 0\}$ is a Markov chain.

(b) (5 points) Identify the state space and find the probability transition matrix P.

(c) (5 points) Find the stationary distribution π. Is it unique? Provide an intuitive explanation for your conclusion.

(d) (5 points) Let $Y_0 = 1$. Prove that the marginal distribution of X_n converges to the stationary distribution, i.e., if $\pi_n(i) = P(X_n = i), i = 1, \dots, M$ then for all $i = 1, \dots, M$:

$$\lim_{n\to\infty} \pi_n(i) = \pi(i).$$

Problems from Chang

Exercise 1.1

Let $X_0, X_1, \ldots$ be a Markov chain, and let A and B be subsets of the state space.

(a) Is it true that $P\{X_2 \in B \mid X_1 = x_1, X_0 \in A\} = P\{X_2 \in B \mid X_1 = x_1\}$? Give a proof or counterexample.

(b) Is it true that $P\{X_2 \in B \mid X_1 \in A, X_0 = x_0\} = P\{X_2 \in B \mid X_1 \in A\}$? Give a proof or counterexample.

Exercise 1.3

Let $\{X_n\}$ be a finite-state Markov chain and let A be a subset of the state space. Suppose we want to determine the expected time until the chain enters the set A, starting from an arbitrary initial state. That is, letting $\tau_A = \inf\{n \geq 0 : X_n \in A\}$ denote the first time to hit A [defined to be 0 if $X_0 \in A$], we want to determine $\mathbb{E}_i(\tau_A)$. Show that

$$\mathbb{E}_i(\tau_A) = 1 + \sum_k P(i,k)\mathbb{E}_k(\tau_A)$$

for $i \notin A$.

Exercise 1.4

You are tossing a coin repeatedly. Which pattern would you expect to see faster: `HH` or `HT`? For example, if you get the sequence `TTHHHTH`..., then you see `HH` at the 4th toss and `HT` at the 6th. Letting N_1 and N_2 denote the times required to see `HH` and `HT`, respectively, can you guess intuitively whether $\mathbb{E}(N_1)$ is smaller than, the same as, or larger than $\mathbb{E}(N_2)$? Go ahead, make a guess [and my day]. Why don't you also simulate some to see how the answer looks; I recommed a computer, buf if you like tossing real coins, enjoy yourself by all means. Finally, you can use the reasoning of Exercise [1.3] to solve the problem and evaluate $\mathbb{E}(N_i)$. A hint is to set up a Markov chain having the 4 states `HH`, `HT`, `TH`, and `TT`.

Exercise 1.6

[A moving average process] Moving average models are used frequently in time series analysis, economics and engineering. For these models, one assumes that there is an underlying, unobserved process $\ldots, Y_{-1}, Y_0, Y_1, \ldots$ of *iid* random variables. A ***moving average process*** takes an average (possibly a weighted average) of these *iid* random variables in a "sliding window." For example, suppose that at time n we simply take the average of the Y_n and Y_{n-1}, defining $X_n = \frac{1}{2}(Y_n + Y_{n-1})$. Our goal is to show that the process $X_0, X_1, \ldots$ defined in this way is not Markov. As a simple example, suppose that the distribution of the *iid* Y random variables is $\mathbb{P}\{Y_i = 1\} = 1/2 = \mathbb{P}\{Y_i = -1\}$.

(a) Show that $X_0, X_1, \ldots$ is not a Markov chain.

(b) Show that $X_0, X_1, \ldots$ is not an rth order Markov chain for any finite r.