S&DS 351 - Stochastic Processes - Lecture 2 Notes

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January 15, 2025

1 Admin

Total 9 HWs, see syllabus

2 Review

2.1 Limit Theorems

- For iid (independent and identically distributed) sequence of random variables $X_1, X_2, \ldots, X_n$ with mean μ and variance σ^2, the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ has mean μ and variance
- Weak Law of Large Numbers (WLLN): $\bar{X}_n = \frac{x_1 + \ldots + x_n}{n}$, n = 1,2,..., $\forall \epsilon > 0, P(|\bar{X}_n - \mu| \geq \epsilon) \to 0$ as $n \to \infty$
- Note: $E[\bar{(}X_n) = E[\frac{x_1 + \ldots + x_n}{n}]] = \frac{E[x_1] + \ldots + E[x_n]}{n} = \mu$
- $P(|\bar{X}_n - \mu| \geq \epsilon) = P[|\bar{X_m}| \geq \epsilon \leq \frac{Var(\bar{X_n})}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0 asn \to +\infty$
- Theorem (Central Limit theorem)
    - Formally, $\forall \in R, P(\frac{\bar{X_n} - \mu}{\sqrt{\sigma^2/n}} \leq t) \to P(X \leq t) = \int_{-\infty}^{t_1} \frac{1}{sqrt2\pi} e^{-\frac{s^2}{2} ds}$

3 Stochastic Processes

- Informally, a stochastic process is a collection of random variables that are indexed with time

3.1 Discrete-time:

- $X_0, X_1, X_2, \ldots$ are random variables
- there are contably many times of interests

3.2 Continuous-time:

- $X_t, t \geq 0$ are random variables

3.3 Discrete space:

- each X_n or X_t takes countably many values

3.4 Continuous space:

- each X_n or X_t takes values in $\mathbb{R}$

3.5 Simplete stoch process:

- The iid process for distr P, $X_0, X_1, X_2, \ldots$, each $X_i \sim P$ independently
- Great things: Law of large numbers
- bad things: poor model for reality (no dependency)

4 Markov Chains

- Discrete time + discrete space
- definition: a stochastic process $X_0, X_1, X_2, \ldots$ is a Markov chain if for all X_n take values in some countable set S and it satisfies the Markov Property: $\forall n, \forall i_0, i_1, \ldots, i_{n+1} \in S, P(X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = i_{n+1} | X_n = i_n)$
- Markov property simplified: $\forall n, X_{n+1}$ independent for $X_{n-1}, ..., X_0$ given X_n
- in words, the "future" X_{x+1} is independent of the "past" $X_{n-1}, ..., X_0$ given the "present" X_n
- this course makes the assumption of time-homogeneous Markov chains
  - $\forall n, i, j \in S, P(X_{n+1} = j | X_n = i) = P(X_1 = j | X_0 = i)$
- Let $N = |S|$, i.e. the cardinality of the state space, S

5 Probability Transition Matrix

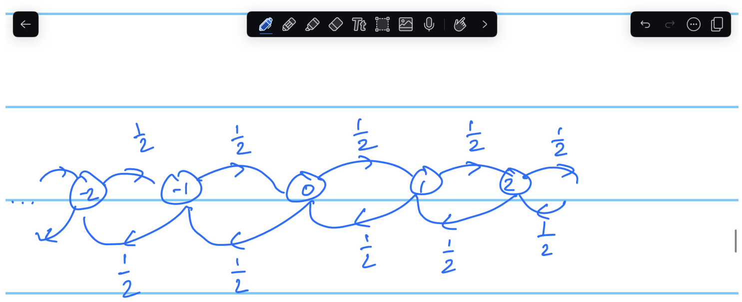
5.1 Definition

- Let a matrix $P = (P_{ij}) \in \mathbb{R}^{N \times N}$ given by $P_{ij} = P(X_1 = j | X_0 = i)$
- Properties: $P_{ij} \geq 0, \sum_{j=1}^{N} P_{ij} = 1$ (i.e. all rows sum up to 1)

- given X_0, via matrix P, I can learn "everything" about the markov chain
- given $X_0 = i_0$, for any states $i_1, i_2, \ldots, i_n \in S$, $P(X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_n = i_n | X_{n-1} = i_{n-1}) = P_{i_{n-1} i_n}$
- $P_{i_0 i_1} \mathbb{P}(X_2 = i_2 | X_1 = i_1) P(X_n = i_n, \ldots X_3 = i_3 | X_2 = i_2, X_1 = i_1, X_0 = i_0) = P_{i_0 i_1} \cdot P_{i_1 i_2} \cdot \ldots \cdot P_{i_{n-1} i_n}$

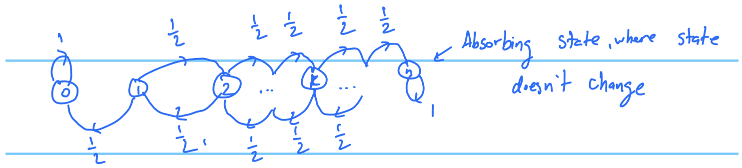
5.2 Examples

- S =Eat, Sleep, Play
-
- Random walk on $\mathbb{X}$
- $S = \mathbb{Z}$
- First, $N = |S| = \infty$
- Given $X_n = i, P(X_{n+1} = j | X_n = i) = P(X_{n+1} = j | X_n = i) = P_{ij}$



$$P_{ij} = \begin{cases} 0, & \text{o/w} \\ 1/2 & \text{if } |i-j| = 1 \end{cases}$$

- Game: Alice has k dollars an Bob flips an indepdent coin w/ prob 1/2 and for each coin: $H \to$ Alice +1 dollars and if $T \to$ Alice -1 dollar
- Alice decides to play until she either reaches n dollars or loses all her money



- To fully characterize a markov chain, we need P and we also need the initial distribution of X_0, $\pi_0(\pi_0(i) = P(X_0 = i)) \in S$
- simulate: $X_0 \sim \pi_0$, $\forall n$, if $X_n = i$, $X_{n+1} = j$ with probability P_{ij}
- This works b/c $\mathbb{P}(X_n) = i_n, \ldots, X_1 = i_1, X_0 = i_0) = \mathbb{P}(X_0 = i_0)\mathbb{P}(X_n = i_n, \ldots X_1 = i_1 | X_0 = i_0 = \pi_0(i_0) P_{i_0,i_1} \ldots P_{i_{n-1},i_n})$