S&DS 351: Stochastic Processes - Homework 8

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April 18, 2025

Chang Problems:

[5.8] *The strong Markov property* is an extension of the restarting property of Proposition **5.5** from fixed times c to random *stopping times* γ: For a stopping time γ, the process x defined by $X(t) = W(\gamma + t) - W(\gamma)$ is a Brownian motion, independent of the path of W up to time γ. Explain the role of the stopping time requirement by explaining how the restarting property can fail for a random time that isn't a stopping time. For example, let $M = \max\{B_t : 0 \le t \le 1\}$ and let $\beta = \inf\{t : B_t = M\}$; this is the first time at which B achieves its maximum height over the time interval $[0, 1]$. Clearly β is not a stopping time, since we must look at the whole path $\{B_t : 0 \le t \le 1\}$ to determine when the maximum is attained. Argue that the restarted process $X(t) = W(\beta + t) - W(\beta)$ is not a standard Brownian motion.

Because $B_{\beta+t} \le B_\beta$ for every $0 \le t \le 1-\beta$, we get $X(1-\beta) = B_1 - B_\beta \le 0$ almost surely, contradicting the symmetry of a $N(0, 1-\beta)$ law and hence proving that X cannot be a standard Brownian motion, which shows why the strong Markov property demands β to be a stopping time.

[5.9] [Ornstein-Uhlenbeck process] Define a process X by

$$X(t) = e^{-t} W(e^{2t})$$

for $t \ge 0$, where W is a standard Brownian motion. X is called an *Ornstein-Uhlenbeck process.*

(a) Find the covariance function of X.

The process X is obtained from a standard Brownian motion W by the deterministic space–time change

$$X(t) = e^{-t} W\big(e^{2t}\big), \qquad t \ge 0.$$

Because W is Gaussian with mean 0, X is also Gaussian with mean 0, so its second-order behaviour is completely described by its covariance function. Fix $s, t \ge 0$ and—without loss of generality—assume $s \le t$; then

$$\mathbb{E}\big[X(s)X(t)\big] = \mathbb{E}\Big[e^{-s} W(e^{2s})\, e^{-t} W(e^{2t})\Big] = e^{-(s+t)}\, \mathbb{E}\Big[W(e^{2s})\, W(e^{2t})\Big].$$

Brownian motion has the covariance $\mathbb{E}[W(u)W(v)] = \min\{u, v\}$, so

$$\mathbb{E}\big[X(s)X(t)\big] = e^{-(s+t)} \min\big\{e^{2s}, e^{2t}\big\} = e^{-(s+t)} e^{2s} = e^{-(t-s)}.$$

By symmetry in (s, t) this extends to all $s, t \ge 0$ and gives

$$\boxed{\operatorname{Cov}\big(X(s), X(t)\big) = e^{-|t-s|}, \qquad s, t \ge 0.}$$

Hence X is a *stationary* centered Gaussian process with exponentially decaying covariance, the hallmark of the Ornstein–Uhlenbeck family.

(b) Evaluate the functions μ and σ^2, defined by

$$\mu(x, t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[X(t+h) - X(t) \mid X(t) = x]$$

$$\sigma^2(x, t) = \lim_{h \downarrow 0} \frac{1}{h} \operatorname{Var}[X(t+h) - X(t) \mid X(t) = x].$$

To identify the "infinitesimal" drift and diffusion of X, expand $X(t+h)$ around t. Write

$$\Delta_h := W\big(e^{2(t+h)}\big) - W(e^{2t}), \qquad \text{so that} \quad X(t+h) = e^{-(t+h)}\big[W(e^{2t}) + \Delta_h\big].$$

Conditional distribution of Δ_h. Since W has independent increments, Δ_h is independent of $W(e^{2t})$ and is Gaussian with mean 0 and variance

$$\operatorname{Var}(\Delta_h) = e^{2(t+h)} - e^{2t} = e^{2t}\big(e^{2h} - 1\big).$$

Conditioning on $X(t) = x$. The event $\{X(t) = x\}$ pins down the value of $W(e^{2t})$:

$$X(t) = x \implies e^{-t}W(e^{2t}) = x \implies W(e^{2t}) = e^t x.$$

Therefore, under this conditioning,

$$\mathbb{E}[\Delta_h \,|\, X(t) = x] = 0, \quad \operatorname{Var}[\Delta_h \,|\, X(t) = x] = e^{2t}\big(e^{2h} - 1\big).$$

First conditional moment.

$$\begin{aligned}
\mathbb{E}\big[X(t+h) - X(t) \,\big|\, X(t) = x\big] &= \mathbb{E}\Big[e^{-(t+h)}W(e^{2t}) - e^{-t}W(e^{2t}) + e^{-(t+h)}\Delta_h \,\Big|\, X(t) = x\Big] \\
&= \big(e^{-(t+h)} - e^{-t}\big)\, e^t x + e^{-(t+h)}\mathbb{E}[\Delta_h \,|\, X(t) = x] \\
&= (e^{-h} - 1)x.
\end{aligned}$$

Hence

$$\mu(x,t) = \lim_{h\downarrow 0} \frac{1}{h}\, \mathbb{E}\big[X(t+h) - X(t) \,|\, X(t) = x\big] = \lim_{h\downarrow 0} \frac{e^{-h} - 1}{h}\, x = -x.$$

Second conditional moment.

$$\begin{aligned}
\operatorname{Var}\big[X(t+h) - X(t) \,\big|\, X(t) = x\big] &= \operatorname{Var}\Big[e^{-(t+h)}\Delta_h\Big] \\
&= e^{-2(t+h)}\operatorname{Var}[\Delta_h \,|\, X(t) = x] \\
&= e^{-2h}\big(e^{2h} - 1\big) \\
&= 2h + o(h) \quad (h \downarrow 0).
\end{aligned}$$

Consequently

$$\sigma^2(x,t) = \lim_{h\downarrow 0} \frac{1}{h}\operatorname{Var}\big[X(t+h) - X(t) \,|\, X(t) = x\big] = 2.$$

Interpretation. The limits $\mu(x,t) = -x$ and $\sigma^2(x,t) = 2$ coincide with the drift and twice the diffusion coefficient in the stochastic differential equation

$$dX_t = -X_t\, dt + \sqrt{2}\, dW_t,$$

whose unique stationary solution is precisely the Ornstein–Uhlenbeck process we constructed by the time–space transform of Brownian motion.

[5.10] Let W be a standard Brownian motion.

(i) Defining $\tau_b = \inf\{t : W(t) = b\}$ for $b > 0$ as above, show that τ_b has probability density function

$$f_{\tau_b}(t) = \frac{b}{\sqrt{2\pi}} t^{-3/2} e^{-b^2/(2t)}$$

for $t > 0$.

(i) Density of the hitting time τ_b.
For $b > 0$ let

$$\tau_b = \inf\{t > 0 : W(t) = b\}.$$

By the reflection principle

$$P\{\tau_b \le t\} = P\Big\{\sup_{0\le s\le t} W(s) \ge b\Big\} = 2P\{W(t) \ge b\}.$$

Because $W(t) \sim \mathcal{N}(0,t)$ we have

$$P\{W(t) \geq b\} = 1 - \Phi\Big(\frac{b}{\sqrt{t}}\Big),$$

where Φ is the standard normal distribution function and $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ its density. Hence

$$F_{\tau_b}(t) = 2\Big[1 - \Phi\Big(\frac{b}{\sqrt{t}}\Big)\Big] \qquad (t > 0).$$

Differentiate to obtain the density:

$$f_{\tau_b}(t) = 2\varphi\Big(\frac{b}{\sqrt{t}}\Big)\Big(-\frac{b}{2}t^{-3/2}\Big)(-1) = \frac{b}{\sqrt{2\pi}}\, t^{-3/2} \exp\Big(-\frac{b^2}{2t}\Big), \qquad t > 0.$$

Thus τ_b has the inverse–Gaussian density claimed.

(ii) Show that for $0 < t_0 < t_1$,

$$P\{W(t) = 0 \text{ for some } t \in (t_0, t_1)\} = \frac{2}{\pi} \tan^{-1}\left(\sqrt{\frac{t_1}{t_0} - 1}\right) = \frac{2}{\pi} \cos^{-1}\left(\sqrt{\frac{t_0}{t_1}}\right).$$

[Hint: The last equality is simple trigonometry. For the previous equality, condition on the value of $W(t_0)$, use part (i), and Fubini (or perhaps integration by parts).]

Fix $0 < t_0 < t_1$ and write $\Delta = t_1 - t_0$. By the Markov property, conditioning on $W(t_0) = x$ and letting

$$\tau_{|x|} = \inf\{s > 0 : W_x(s) = 0\} \quad (W_x(0) = x),$$

we obtain

$$P\{W(t) = 0 \text{ for some } t \in (t_0, t_1)\} = \int_{\mathbb{R}} P\{\tau_{|x|} \leq \Delta\} \frac{e^{-x^2/(2t_0)}}{\sqrt{2\pi t_0}}\, dx.$$

Because $\tau_{|x|}$ has the density from part (i) with $b = |x|$,

$$P\{\tau_{|x|} \leq \Delta\} = \int_0^{\Delta} \frac{|x|}{\sqrt{2\pi}} s^{-3/2} \exp\Big(-\frac{x^2}{2s}\Big) ds.$$

Insert this and use symmetry to restrict to $x > 0$:

$$P = 2\int_0^{\infty} \frac{e^{-x^2/(2t_0)}}{\sqrt{2\pi t_0}} \int_0^{\Delta} \frac{x}{\sqrt{2\pi}} s^{-3/2} e^{-x^2/(2s)}\, ds\, dx.$$

Fubini's theorem allows us to swap the integrals:

$$P = \frac{2}{2\pi\sqrt{t_0}} \int_0^{\Delta} s^{-3/2} \int_0^{\infty} x \exp\Big(-x^2\Big(\frac{1}{2t_0} + \frac{1}{2s}\Big)\Big)\, dx\, ds.$$

For $\alpha > 0$, $\int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2\alpha}$; here

$$\alpha = \frac{1}{2t_0} + \frac{1}{2s} = \frac{s + t_0}{2st_0}, \quad \frac{1}{2\alpha} = \frac{st_0}{s + t_0}.$$

Therefore

$$P = \frac{1}{2\pi\sqrt{t_0}} \int_0^{\Delta} \frac{2t_0\, s^{-1/2}}{s + t_0}\, ds = \frac{\sqrt{t_0}}{\pi} \int_0^{\Delta} \frac{s^{-1/2}}{s + t_0}\, ds.$$

Set $s = t_0 u^2$ $(u \geq 0)$; then $ds = 2t_0 u\, du$ and the upper limit becomes

$$u_{\max} = \sqrt{\frac{\Delta}{t_0}} = \sqrt{\frac{t_1}{t_0} - 1}.$$

Substituting gives

$$P = \frac{\sqrt{t_0}}{\pi} \int_0^{u_{\max}} \frac{1}{\sqrt{t_0}u} \frac{2t_0 u}{t_0(1 + u^2)}\, du = \frac{2}{\pi} \int_0^{u_{\max}} \frac{du}{1 + u^2} = \frac{2}{\pi} \tan^{-1}(u_{\max}).$$

Finally

$$u_{\max} = \sqrt{\frac{t_1}{t_0} - 1}, \quad \text{so} \quad P = \frac{2}{\pi}\tan^{-1}\Big(\sqrt{\frac{t_1}{t_0} - 1}\Big) = \frac{2}{\pi}\cos^{-1}\Big(\sqrt{\frac{t_0}{t_1}}\Big),$$

the last equality being the elementary identity

$$\tan^{-1}\big(\sqrt{z-1}\big) = \cos^{-1}\big(z^{-1/2}\big) \qquad (z > 1).$$

[5.13] Let $(X(t), Y(t))$ be a two-dimensional standard Brownian motion; that is, let $\{X(t)\}$ and $\{Y(t)\}$ be standard Brownian motion processes that are independent of each other. Let $b > 0$, and define $\tau = \inf\{t : X(t) = b\}$. Find the probability density function of $Y(\tau)$. That is, find the probability density of the height at which the two-dimensional Brownian motion first hits the vertical line $x = b$.

[Hint: The answer is a Cauchy distribution.]

Let

$$\tau = \inf\{t > 0 : X(t) = b\}, \qquad b > 0,$$

so that $(X(\tau), Y(\tau)) = (b, Y(\tau))$ is the first point where the planar Brownian motion hits the vertical line $x = b$. Because X and Y are independent one–dimensional Brownian motions started at 0, the law of $Y(\tau)$ can be obtained in three steps. **1. Density of the hitting time τ.** From the reflection principle (see Problem 5.10 (i))

$$f_\tau(t) \;=\; \frac{b}{\sqrt{2\pi}}\, t^{-3/2}\, e^{-b^2/(2t)}, \qquad t > 0.$$

2. Conditional law of $Y(\tau)$ given $\tau = t$. For fixed t the increment $Y(t)$ is independent of X and satisfies

$$Y(\tau) \mid \{\tau = t\} \;\sim\; \mathcal{N}\big(0, t\big), \quad \text{i.e.} \quad g_t(y) \;=\; \frac{1}{\sqrt{2\pi t}}\, e^{-y^2/(2t)}, \qquad y \in \mathbb{R}.$$

3. Unconditional density of $Y(\tau)$. Using the law–of–total–probability and Fubini,

$$\begin{aligned} f_{Y(\tau)}(y) &= \int_0^\infty g_t(y)\, f_\tau(t)\, dt \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi t}}\, e^{-y^2/(2t)}\, \frac{b}{\sqrt{2\pi}}\, t^{-3/2}\, e^{-b^2/(2t)}\, dt \\ &= \frac{b}{2\pi}\int_0^\infty t^{-2}\, \exp\Big(-\frac{b^2+y^2}{2t}\Big)\, dt. \end{aligned}$$

Evaluate the integral by the substitution $u = (b^2+y^2)/(2t)$, so that $t = (b^2+y^2)/(2u)$ and $dt = -\frac{b^2+y^2}{2u^2}\, du$:

$$\begin{aligned} \int_0^\infty t^{-2} e^{-(b^2+y^2)/(2t)}\, dt &= \int_\infty^0 \Big(\frac{2u}{b^2+y^2}\Big)^2 e^{-u}\, \Big(-\frac{b^2+y^2}{2u^2}\Big)\, du \\ &= \frac{1}{\frac{1}{2}(b^2+y^2)}\int_0^\infty e^{-u}\, du = \frac{2}{b^2+y^2}. \end{aligned}$$

Substituting back,

$$f_{Y(\tau)}(y) = \frac{b}{2\pi}\,\frac{2}{b^2+y^2} = \frac{b}{\pi\big(b^2+y^2\big)}, \qquad y \in \mathbb{R}.$$

4. Identification with the Cauchy distribution. The density

$$y \;\longmapsto\; \frac{b}{\pi\big(b^2+y^2\big)}$$

is the centered Cauchy density with scale parameter b. Hence

$$Y(\tau) \;\sim\; \text{Cauchy}(0, b),$$

confirming the hint and completing the proof.

[5.15] Let $0 < s < t < u$.

(a) Show that $\mathbb{E}(W_sW_t \mid W_u) = \frac{s}{t}\mathbb{E}(W_t^2 \mid W_u)$.

Fix $0 < s < t < u$ and recall that $\{W_r\}_{r\geq 0}$ is a centred Gaussian process with independent increments. Because W_u is non–degenerate, any finite-dimensional vector built from the path is jointly Gaussian, so conditional expectations are obtained by linear regression. A convenient way to organise the calculation is to decompose the *Brownian bridge*

$$W_r = \frac{r}{u}\,W_u + B_r, \qquad 0 \leq r \leq u,$$

where $\{B_r\}_{0\leq r\leq u}$ is a (mean–0) Gaussian bridge independent of W_u with covariance

$$\mathbb{E}[B_rB_{r'}] = \frac{r(u-r')}{u}, \qquad r \leq r'.$$

Step 1: Conditional second moment of W_t.

$$W_t^2 = \Big(\frac{t}{u}\,W_u\Big)^2 + 2\frac{t}{u}\,W_uB_t + B_t^2.$$

Taking conditional expectation given W_u (noting that B_t is independent of W_u and has mean 0):

$$\mathbb{E}(W_t^2 \mid W_u) = \frac{t^2}{u^2}\,W_u^2 + \mathbb{E}(B_t^2) = \frac{t^2}{u^2}\,W_u^2 + \frac{t(u-t)}{u}.$$

Step 2: Conditional mixed moment $\mathbb{E}(W_sW_t \mid W_u)$. Using the same decomposition,

$$\begin{aligned} W_sW_t &= \Big(\frac{s}{u}W_u + B_s\Big)\Big(\frac{t}{u}W_u + B_t\Big) \\ &= \frac{st}{u^2}W_u^2 + \frac{s}{u}W_uB_t + \frac{t}{u}W_uB_s + B_sB_t. \end{aligned}$$

Conditioning on W_u kills the linear terms in B_s, B_t and replaces B_sB_t by its covariance:

$$\mathbb{E}(W_sW_t \mid W_u) = \frac{st}{u^2}W_u^2 + \mathbb{E}(B_sB_t) = \frac{st}{u^2}W_u^2 + \frac{s(u-t)}{u}.$$

Step 3: Relation asserted in part (a). Multiply the result of Step 1 by $\frac{s}{t}$:

$$\frac{s}{t}\,\mathbb{E}(W_t^2 \mid W_u) = \frac{s}{t}\Big(\frac{t^2}{u^2}W_u^2 + \frac{t(u-t)}{u}\Big) = \frac{st}{u^2}W_u^2 + \frac{s(u-t)}{u} = \mathbb{E}(W_sW_t \mid W_u),$$

which establishes

$$\boxed{\mathbb{E}(W_sW_t \mid W_u) = \frac{s}{t}\,\mathbb{E}(W_t^2 \mid W_u).}$$

(b) Find $\mathbb{E}(W_t^2 \mid W_u)$ [you know $\mathrm{Var}(W_t \mid W_u)$ and $\mathbb{E}(W_t \mid W_u)$!] and use this to show that

$$\mathrm{Cov}(W_s, W_t \mid W_u) = \frac{s(u-t)}{u}.$$

We already have

$$\boxed{\mathbb{E}(W_t^2 \mid W_u) = \frac{t(u-t)}{u} + \frac{t^2}{u^2}W_u^2.}$$

For the conditional covariance,

$$\mathrm{Cov}(W_s, W_t \mid W_u) = \mathbb{E}(W_sW_t \mid W_u) - \mathbb{E}(W_s \mid W_u)\,\mathbb{E}(W_t \mid W_u).$$

Since $\{W_r\}$ is a martingale,

$$\mathbb{E}(W_r \mid W_u) = \frac{r}{u}\,W_u \qquad (0 \leq r \leq u),$$

so that

$$\mathbb{E}(W_s \mid W_u)\,\mathbb{E}(W_t \mid W_u) = \frac{st}{u^2} W_u^2.$$

Subtracting this from the expression in Step 2 yields

$$\boxed{\operatorname{Cov}(W_s, W_t \mid W_u) = \frac{s(u-t)}{u}.}$$

[5.17] Verify that the definitions (**5.13**) and (**5.14**) give Brownian bridges.

$$(5.13) \quad X(t) = W(t) - tW(1) \quad \text{for } 0 \le t \le 1.$$

$$(5.14) \quad Y(t) = (1-t)W\left(\frac{t}{1-t}\right) \quad \text{for } 0 \le t < 1, \quad Y(1) = 0$$

Solution. Definition of a Brownian bridge on $[0,1]$.

A centred, continuous Gaussian process $\{B(t)\}_{0\le t\le 1}$ is called a *Brownian bridge* provided

$$B(0) = B(1) = 0, \qquad \operatorname{Cov}\big(B(s), B(t)\big) = \min\{s,t\} - st,\ 0 \le s,t \le 1.$$

The covariance formula is often written, when $s \le t$, as $s(1-t)$.

$$\textbf{(5.13)}\ X(t) = W(t) - tW(1),\ 0 \le t \le 1.$$

Claim : X is a Brownian bridge.

(1) End points. $X(0) = W(0) = 0,\ \ X(1) = W(1) - W(1) = 0.$

(2) Gaussianity. $X(t)$ is a fixed linear combination of $\{W(r)\}_{0\le r\le 1}$, so every finite-dimensional distribution is multivariate normal.

(3) Mean. $\mathbb{E}[X(t)] = \mathbb{E}[W(t)] - t\mathbb{E}[W(1)] = 0.$

(4) Covariance. For $0 \le s \le t \le 1$,

$$\begin{aligned}\operatorname{Cov}(X(s), X(t)) &= \mathbb{E}[(W(s) - sW(1))(W(t) - tW(1))] \\ &= \min\{s,t\} - ts - s(1) + st \\ &= s(1-t).\end{aligned}$$

This is $\min\{s,t\} - st$ in general.

(5) Continuity. X inherits almost–sure continuity from W.

All axioms being satisfied, X is a Brownian bridge.

$$\textbf{(5.14)}\ Y(t) = \begin{cases} (1-t)\,W(\dfrac{t}{1-t}), & 0 \le t < 1, \\ 0, & t = 1. \end{cases}$$

Claim : Y is a Brownian bridge.

Let $\theta(t) = \dfrac{t}{1-t}$, so that $\theta : [0,1) \to [0,\infty)$ is strictly increasing.

(1) End points. $Y(0) = (1-0)W(0) = 0, \qquad Y(1) = 0$ by definition.

(2) Gaussianity. For $t < 1$, $Y(t)$ is a scalar multiple of $W(\theta(t))$; any finite vector $(Y(t_1), \ldots, Y(t_k))$ is therefore a linear image of $(W(\theta(t_1)), \ldots, W(\theta(t_k)))$, hence Gaussian.

(3) Mean. $\mathbb{E}[Y(t)] = 0.$

(4) Covariance. Fix $0 \le s \le t < 1$. $\theta(s) \le \theta(t) \implies \min\{\theta(s), \theta(t)\} = \theta(s)$.

$$\begin{aligned}\operatorname{Cov}(Y(s), Y(t)) &= (1-s)(1-t)\,\mathbb{E}\Big[W(\theta(s))\,W(\theta(t))\Big]\\ &= (1-s)(1-t)\,\theta(s)\\ &= (1-s)(1-t)\,\frac{s}{1-s} = s(1-t).\end{aligned}$$

Thus $\operatorname{Cov}(Y(s), Y(t)) = \min\{s, t\} - st$ for all $s, t \le 1$.

(5) Continuity and the value at $t = 1$.

First compute the variance: $\operatorname{Var}[Y(t)] = t(1-t) \xrightarrow[t\to 1^-]{} 0$. Hence $Y(t) \to 0$ in L^2 and therefore in probability as $t \to 1^-$. Because W admits a continuous modification, one may choose that modification and verify that $t \mapsto Y(t)$ is almost surely continuous on $[0, 1)$ and converges to 0 at $t = 1$; redefining $Y(1) = 0$ yields an a.s. continuous version on $[0, 1]$.

$$\boxed{X \text{ and } Y \text{ both satisfy the defining properties of a Brownian bridge on } [0,1].}$$

Consequently, (5.13) and (5.14) indeed "manufacture" Brownian bridges from a single standard Brownian motion.

Problem 1. (15 points) Let $W(t), t \ge 0$ be a standard Brownian motion. Prove that it is a Gaussian process, i.e., for all $n \in \mathbb{N}, t_1, \ldots, t_n \ge 0$ and $a_1, \ldots, a_n \in \mathbb{R}$, the distribution of $\sum_{i=1}^n a_i W(t_i)$ is Gaussian.

We recall the usual definition of a *standard Brownian motion* $\{W(t)\}_{t\ge 0}$:

$$W(0) = 0, \qquad \text{for } 0 \le s < t\,,\ W(t) - W(s) \sim N(0, t-s), \qquad \{W(t) - W(s)\}_{0\le s<t} \text{ are independent.}$$

In what follows let

$$n \in \mathbb{N}, \qquad t_1, \ldots, t_n \ge 0, \qquad a_1, \ldots, a_n \in \mathbb{R},$$

and set

$$S := \sum_{i=1}^n a_i W(t_i).$$

Our task is to prove that S is (univariate) Gaussian. **Step 1. Reduction to strictly increasing times.** If some of the t_i are equal we can merge coefficients; if they are merely unordered we may relabel indices so that

$$0 \le t_{(1)} < t_{(2)} < \cdots < t_{(m)}, \qquad b_j := \sum_{i:\, t_i = t_{(j)}} a_i, \quad 1 \le j \le m \le n,$$

and write $S = \sum_{j=1}^m b_j W(t_{(j)})$. Hence *without loss of generality we assume* $0 < t_1 < \cdots < t_n$. **Step 2. Express S in terms of increments.** Define the independent Gaussian increments

$$\Delta_1 := W(t_1) - W(0) = W(t_1), \qquad \Delta_k := W(t_k) - W(t_{k-1}),\ 2 \le k \le n.$$

Then S can be rewritten as

$$S = \sum_{i=1}^n a_i\Big(\Delta_1 + \cdots + \Delta_i\Big) = \sum_{k=1}^n \Big(\sum_{i=k}^n a_i\Big)\Delta_k =: \sum_{k=1}^n c_k \Delta_k,$$

where we set $c_k := \sum_{i=k}^n a_i$. **Step 3. Use closure of the Gaussian family under linear combinations.** Each increment Δ_k is Gaussian:

$$\Delta_k \sim N\big(0, t_k - t_{k-1}\big) \quad (\text{put } t_0 := 0),$$

and the vector $(\Delta_1, \dots, \Delta_n)$ has *independent* components by definition of Brownian motion. Because independent Gaussian variables are also jointly Gaussian, any *deterministic* linear combination of them is again Gaussian. Concretely,

$$S = \sum_{k=1}^{n} c_k \Delta_k$$

is a sum of independent $N(0, \sigma_k^2)$ variables multiplied by deterministic scalars c_k, hence

$$S \sim N\Big(0,\ \sum_{k=1}^{n} c_k^2 (t_k - t_{k-1})\Big),$$

i.e. S is Gaussian. **Step 4. Characteristic–function verification (optional but instructive).** For completeness, compute the characteristic function of S:

$$\begin{aligned}\varphi_S(\lambda) = \mathbb{E}\, e^{i\lambda S} &= \prod_{k=1}^{n} \mathbb{E} \exp\big(i\lambda c_k \Delta_k\big) \\ &= \prod_{k=1}^{n} \exp\Big(-\tfrac{1}{2}\lambda^2 c_k^2 (t_k - t_{k-1})\Big) = \exp\Big(-\tfrac{1}{2}\lambda^2 \sum_{k=1}^{n} c_k^2 (t_k - t_{k-1})\Big),\end{aligned}$$

the characteristic function of a centred normal distribution, confirming the previous step. **Conclusion.** For arbitrary n, times $t_1, \dots, t_n \geq 0$ and coefficients $a_1, \dots, a_n \in \mathbb{R}$, the linear form $S = \sum_{i=1}^{n} a_i W(t_i)$ is Gaussian. Therefore the finite–dimensional distributions of $\{W(t)\}$ are multivariate normal, so W is indeed a *Gaussian process.*