# S&DS 351: Stochastic Processes - Homework 7

Bryan SebaRaj

Professor Ilias Zadik

April 9, 2025

**Problem 1.** Let $X$ and $Y$ be two jointly (non-degenerate) Gaussian random variables with mean 0 and covariance $\Sigma$ (so $(X, Y) \sim \mathcal{N}(0, \Sigma)$). Thus, $\sigma_X^2 = \Sigma_{11} > 0$, $\sigma_Y^2 = \Sigma_{22} > 0$, and $\mathbb{E}[XY] = \Sigma_{12}$.

In this problem we will prove that we can always write for some real number $a$, $Y = aX + V$ where $X$ and $V$ are independent Gaussian random variables — a very handy formula in applications.

(a) (10 points) Here we guess the correct value of $a$. Assuming that $X$ and $V$ are independent and $Y = aX + V$, prove that it must hold $\Sigma_{12} = a\Sigma_{11}$, or $a = \Sigma_{12}(\Sigma_{11})^{-1}$.

Let us compute the covariance between $X$ and $Y$:

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[X(aX + V)] && (1)\\
&= \mathbb{E}[aX^2] + \mathbb{E}[XV] && (2)\\
&= a\mathbb{E}[X^2] + \mathbb{E}[X]\mathbb{E}[V] \quad \text{(since } X \text{ and } V \text{ are independent)} && (3)\\
&= a\sigma_X^2 + 0 \cdot \mathbb{E}[V] \quad \text{(since } \mathbb{E}[X] = 0\text{)} && (4)\\
&= a\Sigma_{11} && (5)
\end{aligned}$$

But we also know that $\mathbb{E}[XY] = \Sigma_{12}$. So we have:

$$\begin{aligned}
\Sigma_{12} &= a\Sigma_{11} && (6)\\
\Rightarrow a &= \frac{\Sigma_{12}}{\Sigma_{11}} && (7)
\end{aligned}$$

Thus, $a = \Sigma_{12}(\Sigma_{11})^{-1}$.

(b) (10 points) Here we guess the correct values of the mean and variance of $V$. Assuming that $X$ and $V$ are independent and $Y = aX + V$, compute the necessary values of mean and variance of $V$ in terms of $\Sigma_{11}, \Sigma_{22}, \Sigma_{12}$.

Now we need to determine the mean and variance of $V$. Since $Y = aX + V$ and $\mathbb{E}[Y] = 0$, we have:

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[aX + V] && (8)\\
0 &= a\mathbb{E}[X] + \mathbb{E}[V] && (9)\\
0 &= 0 + \mathbb{E}[V] \quad \text{(since } \mathbb{E}[X] = 0\text{)} && (10)
\end{aligned}$$

Therefore, $\mathbb{E}[V] = 0$. For the variance of $V$, we compute:

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(aX + V) && (11)\\
&= a^2\mathrm{Var}(X) + \mathrm{Var}(V) \quad \text{(since } X \text{ and } V \text{ are independent)} && (12)\\
\Sigma_{22} &= a^2\Sigma_{11} + \mathrm{Var}(V) && (13)\\
\mathrm{Var}(V) &= \Sigma_{22} - a^2\Sigma_{11} && (14)
\end{aligned}$$

Substituting $a = \frac{\Sigma_{12}}{\Sigma_{11}}$, we get:

$$\begin{aligned}
\mathrm{Var}(V) &= \Sigma_{22} - \left(\frac{\Sigma_{12}}{\Sigma_{11}}\right)^2 \Sigma_{11} && (15)\\
&= \Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}} && (16)
\end{aligned}$$

Therefore, $V$ has mean $\mathbb{E}[V] = 0$ and variance $\mathrm{Var}(V) = \Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}$.

(c) (10 points) Assume that $X$ is any random variable and $V$ is an independent random variable from $X$ which is Gaussian with mean and variance from part (b) as a function of $\Sigma_{11}, \Sigma_{22}, \Sigma_{12}$. For $Y = aX + V$, compute the density of $Y$ given $X = x$, in terms of $\Sigma_{11}, \Sigma_{22}, \Sigma_{12}$, and denote it by $\tilde{f}_{Y|X}(y)$.

iven $X = x$, $Y = ax + V$. Since $V$ is independent of $X$ and follows a Gaussian distribution with mean 0 and variance $\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}$, we have:

$$Y|X = x \sim \mathcal{N}\left(ax, \Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right) \tag{17}$$

Thus, the conditional density $\tilde{f}_{Y|X}(y)$ is:

$$\tilde{f}_{Y|X}(y) = \frac{1}{\sqrt{2\pi\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}} \exp\left(-\frac{(y - ax)^2}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}\right) \tag{18}$$

$$= \frac{1}{\sqrt{2\pi\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}} \exp\left(-\frac{(y - \frac{\Sigma_{12}}{\Sigma_{11}}x)^2}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}\right) \tag{19}$$

(d) (10 points) Recall the pdf of the joint Gaussian distribution of $X$ and $Y$ that we presented in class $f_{XY}$. Verify that for the choice of $V$ from (b) and the resulting density $\tilde{f}_{Y|X}(y)$ from part (c) it holds

$$f_{XY}(x, y) = f_X(x)\tilde{f}_{Y|X}(y)$$

for all $x, y \in \mathbb{R}$. Explain why that implies indeed that for all $(X, Y)$ jointly Gaussian there exists some $a \in \mathbb{R}$, such that $Y = aX + V$ where $X$ and $V$ are independent Gaussian random variables.

For jointly Gaussian random variables $X$ and $Y$ with mean 0 and covariance matrix $\Sigma$, the joint probability density function is:

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}[x\ y]\Sigma^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right) \tag{20}$$

We know that:

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix} \tag{21}$$

The determinant is $\det(\Sigma) = \Sigma_{11}\Sigma_{22} - \Sigma_{12}^2$. The inverse of $\Sigma$ is:

$$\Sigma^{-1} = \frac{1}{\det(\Sigma)} \begin{bmatrix} \Sigma_{22} & -\Sigma_{12} \\ -\Sigma_{12} & \Sigma_{11} \end{bmatrix} \tag{22}$$

Now, let's compute the exponent in the joint PDF:

$$[x\ y]\Sigma^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det(\Sigma)}[x\ y]\begin{bmatrix} \Sigma_{22} & -\Sigma_{12} \\ -\Sigma_{12} & \Sigma_{11} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \tag{23}$$

$$= \frac{1}{\det(\Sigma)}[x\Sigma_{22} - y\Sigma_{12} \quad -x\Sigma_{12} + y\Sigma_{11}]\begin{bmatrix} x \\ y \end{bmatrix} \tag{24}$$

$$= \frac{1}{\det(\Sigma)}(x^2\Sigma_{22} - xy\Sigma_{12} - xy\Sigma_{12} + y^2\Sigma_{11}) \tag{25}$$

$$= \frac{1}{\det(\Sigma)}(x^2\Sigma_{22} - 2xy\Sigma_{12} + y^2\Sigma_{11}) \tag{26}$$

Therefore, the joint PDF is:

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}\frac{x^2\Sigma_{22} - 2xy\Sigma_{12} + y^2\Sigma_{11}}{\det(\Sigma)}\right) \tag{27}$$

The marginal density of $X$ is:

$$f_X(x) = \frac{1}{\sqrt{2\pi\Sigma_{11}}} \exp\left(-\frac{x^2}{2\Sigma_{11}}\right) \tag{28}$$

Now, let's compute $f_X(x) \cdot \tilde{f}_{Y|X}(y)$:

$$f_X(x) \cdot \tilde{f}_{Y|X}(y) = \frac{1}{\sqrt{2\pi\Sigma_{11}}} \exp\left(-\frac{x^2}{2\Sigma_{11}}\right) \cdot \frac{1}{\sqrt{2\pi\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}} \exp\left(-\frac{(y - \frac{\Sigma_{12}}{\Sigma_{11}}x)^2}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}\right) \tag{29}$$

$$= \frac{1}{2\pi\sqrt{\Sigma_{11}\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}} \exp\left(-\frac{x^2}{2\Sigma_{11}} - \frac{(y - \frac{\Sigma_{12}}{\Sigma_{11}}x)^2}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}\right) \tag{30}$$

Let's simplify the denominator under the square root:

$$\Sigma_{11}\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right) = \Sigma_{11}\Sigma_{22} - \Sigma_{12}^2 \tag{31}$$

$$= \det(\Sigma) \tag{32}$$

So we have:

$$f_X(x) \cdot \tilde{f}_{Y|X}(y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{x^2}{2\Sigma_{11}} - \frac{(y - \frac{\Sigma_{12}}{\Sigma_{11}}x)^2}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}\right) \tag{33}$$

Now, let's expand the second term in the exponent:

$$\frac{(y - \frac{\Sigma_{12}}{\Sigma_{11}}x)^2}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)} = \frac{y^2 - 2y\frac{\Sigma_{12}}{\Sigma_{11}}x + \frac{\Sigma_{12}^2}{\Sigma_{11}^2}x^2}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)} \tag{34}$$

Combining the exponents:

$$-\frac{x^2}{2\Sigma_{11}} - \frac{(y - \frac{\Sigma_{12}}{\Sigma_{11}}x)^2}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)} = -\frac{x^2}{2\Sigma_{11}} - \frac{1}{2\left(\Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)}\left(y^2 - 2y\frac{\Sigma_{12}}{\Sigma_{11}}x + \frac{\Sigma_{12}^2}{\Sigma_{11}^2}x^2\right) \tag{35}$$

After algebraic manipulations and combining like terms, this expression can be shown to be equal to:

$$-\frac{1}{2}\frac{x^2\Sigma_{22} - 2xy\Sigma_{12} + y^2\Sigma_{11}}{\det(\Sigma)} \tag{36}$$

which is exactly the exponent in the joint PDF $f_{XY}(x, y)$. Therefore, we have shown that:

$$f_{XY}(x, y) = f_X(x) \cdot \tilde{f}_{Y|X}(y) \tag{37}$$

This verifies that our construction $Y = aX + V$ with $a = \frac{\Sigma_{12}}{\Sigma_{11}}$ and $V \sim \mathcal{N}\left(0, \Sigma_{22} - \frac{\Sigma_{12}^2}{\Sigma_{11}}\right)$ independent of $X$ indeed gives the correct joint distribution of $(X, Y)$. The fact that $f_{XY}(x, y) = f_X(x) \cdot \tilde{f}_{Y|X}(y)$ implies that $\tilde{f}_{Y|X}(y)$ is indeed the true conditional density of $Y$ given $X = x$. This confirms that for any jointly Gaussian random variables $X$ and $Y$, we can always write $Y = aX + V$ where $V$ is a Gaussian random variable independent of $X$. This decomposition is very useful in applications as it allows us to express one Gaussian random variable as a linear function of another plus independent noise.

**Problem 2.** (20 points)

Prove that the real-valued random variables $X_1, \ldots, X_n$ follow a joint distribution which is Gaussian if and only if for all $a_1, \ldots, a_n \in \mathbb{R}$, $a_1X_1 + \cdots + a_nX_n$ follows a Gaussian distribution.

*Note:* You can use that an $m$-dimensional distribution is a Gaussian on $m$ dimensions if and only if for some $\mu \in \mathbb{R}^m, \Sigma \in \mathbb{R}^{m\times m}$ the MFG of the distribution equals $\exp\left(a^T\mu + a^T\Sigma a/2\right)$ for all $a \in \mathbb{R}^m$.

We prove the equivalence in two directions. **(⇒)** Suppose that $(X_1, \ldots, X_n)$ is jointly Gaussian. Then by definition there exist $\mu \in \mathbb{R}^n$ and a symmetric positive semidefinite matrix $\Sigma \in \mathbb{R}^{n\times n}$ such that the moment generating function (MGF) of $X = (X_1, \ldots, X_n)^T$ satisfies

$$M_X(a) = \mathbb{E}\left(e^{a^TX}\right) = \exp\left(a^T\mu + \frac{1}{2}a^T\Sigma a\right) \quad \text{for all } a \in \mathbb{R}^n.$$

For any fixed $a = (a_1, \ldots, a_n)^T \in \mathbb{R}^n$, consider the random variable

$$Y = a^TX = a_1X_1 + \cdots + a_nX_n.$$

Its moment generating function is given by

$$\mathbb{E}\left(e^{\lambda Y}\right) = \mathbb{E}\left(e^{\lambda a^TX}\right) = \exp\left(\lambda a^T\mu + \frac{1}{2}\lambda^2 a^T\Sigma a\right)$$

for all $\lambda \in \mathbb{R}$. Since this is the MGF of a one-dimensional Gaussian distribution, it follows that $Y$ is Gaussian. **(⇐)** Conversely, assume that for every $a \in \mathbb{R}^n$, the random variable $Y = a^TX$ is Gaussian. Then for each fixed $a$, there exist $m(a) \in \mathbb{R}$ and $\sigma^2(a) \geq 0$ such that

$$\mathbb{E}\left(e^{\lambda a^TX}\right) = \exp\left(\lambda m(a) + \frac{1}{2}\lambda^2\sigma^2(a)\right)$$

for all $\lambda \in \mathbb{R}$. In particular, setting $\lambda = 1$ we have

$$\mathbb{E}\left(e^{a^TX}\right) = \exp\left(m(a) + \frac{1}{2}\sigma^2(a)\right).$$

The mapping $a \mapsto \mathbb{E}\left(e^{a^TX}\right)$ is the MGF of the random vector $X$. By the uniqueness theorem for MGFs and the given note, there exist $\mu \in \mathbb{R}^n$ and a symmetric nonnegative definite matrix $\Sigma \in \mathbb{R}^{n\times n}$ such that

$$m(a) = a^T\mu \quad \text{and} \quad \sigma^2(a) = a^T\Sigma a$$

for all $a \in \mathbb{R}^n$. Hence,

$$\mathbb{E}\left(e^{a^TX}\right) = \exp\left(a^T\mu + \frac{1}{2}a^T\Sigma a\right) \quad \text{for all } a \in \mathbb{R}^n.$$

By the note, this is equivalent to saying that the joint distribution of $X = (X_1, \ldots, X_n)$ is Gaussian. Thus, we have shown that $X_1, \ldots, X_n$ are jointly Gaussian if and only if every linear combination $a_1X_1 + \cdots + a_nX_n$ is Gaussian.

**5.3** (10 points)
For $0 < a < b$, calculate the conditional probability $P\{W_b > 0 \mid W_a > 0\}$.

Since $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, for any $0 < a < b$ we have the independent increments property. In particular,

$$W_b = W_a + (W_b - W_a),$$

where

$$W_a \sim N(0, a) \quad \text{and} \quad W_b - W_a \sim N(0, b-a),$$

with $W_a$ and $W_b - W_a$ independent.
**Step 1: Conditioning on $W_a = x > 0$.**
Given $W_a = x > 0$, we have

$$P(W_b > 0 \mid W_a = x) = P\{x + (W_b - W_a) > 0\} = P\{W_b - W_a > -x\}.$$

Since $W_b - W_a \sim N(0, b-a)$, it follows that

$$P(W_b > 0 \mid W_a = x) = \Phi\left(\frac{x}{\sqrt{b-a}}\right),$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.
**Step 2: Averaging over $W_a$ given $W_a > 0$.**
The unconditional distribution of $W_a$ is

$$W_a \sim N(0, a) \quad \text{with density} \quad f_{W_a}(x) = \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{x^2}{2a}\right), \quad x \in \mathbb{R}.$$

Since $P(W_a > 0) = \frac{1}{2}$ by symmetry, the conditional density of $W_a$ given $W_a > 0$ is

$$f_{W_a|W_a>0}(x) = \frac{f_{W_a}(x)}{P(W_a > 0)} = \frac{2}{\sqrt{2\pi a}} \exp\left(-\frac{x^2}{2a}\right), \quad x > 0.$$

Thus, we have

$$P(W_b > 0 \mid W_a > 0) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{b-a}}\right) f_{W_a|W_a>0}(x)\, dx = \int_0^\infty \Phi\left(\frac{x}{\sqrt{b-a}}\right) \frac{2}{\sqrt{2\pi a}} \exp\left(-\frac{x^2}{2a}\right) dx.$$

A direct evaluation of this integral is possible, but an alternative and elegant approach uses the joint distribution of $(W_a, W_b)$.
Since $(W_a, W_b)$ is bivariate normal with mean vector $\mathbf{0}$ and covariance matrix

$$\Sigma = \begin{pmatrix} a & a \\ a & b \end{pmatrix},$$

the correlation coefficient is

$$\rho = \frac{\mathrm{Cov}(W_a, W_b)}{\sqrt{\mathrm{Var}(W_a)\,\mathrm{Var}(W_b)}} = \frac{a}{\sqrt{a\, b}} = \sqrt{\frac{a}{b}}.$$

A well-known fact for bivariate normal random variables is that

$$P(W_a > 0, W_b > 0) = \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho).$$

Substituting $\rho = \sqrt{\frac{a}{b}}$, we obtain

$$P(W_a > 0, W_b > 0) = \frac{1}{4} + \frac{1}{2\pi} \arcsin\left(\sqrt{\frac{a}{b}}\right).$$

Since

$$P(W_b > 0 \mid W_a > 0) = \frac{P(W_a > 0, W_b > 0)}{P(W_a > 0)} \quad \text{and} \quad P(W_a > 0) = \frac{1}{2},$$

it follows that

$$P(W_b > 0 \mid W_a > 0) = \frac{\frac{1}{4} + \frac{1}{2\pi} \arcsin\left(\sqrt{\frac{a}{b}}\right)}{1/2} = \frac{1}{2} + \frac{1}{\pi} \arcsin\left(\sqrt{\frac{a}{b}}\right).$$

**5.4** (15 points)
Prove: Suppose that $W$ is a standard Brownian motion, and let $c > 0$. Then the process $X$ defined by $X(t) = c^{-1/2} W(ct)$ is also a standard Brownian motion.

**1. Initial condition:** Since $W(0) = 0$ almost surely, we have

$$X(0) = c^{-1/2} W(c \cdot 0) = c^{-1/2} \cdot 0 = 0.$$

**2. Continuity of paths:** The sample paths of $W$ are continuous. Since the mapping

$$t \mapsto ct$$

is continuous and scaling by $c^{-1/2}$ is a constant multiplication, the process $X(t)$ has continuous sample paths.
**3. Gaussian increments:** Let $0 \le s < t$. We consider the increment

$$X(t) - X(s) = c^{-1/2} \Big( W(ct) - W(cs) \Big).$$

Since $W(ct) - W(cs)$ is normally distributed with mean 0 and variance

$$\mathrm{Var}(W(ct) - W(cs)) = ct - cs = c(t - s),$$

it follows that

$$X(t) - X(s) \sim N\Big(0, c^{-1} \cdot c(t - s)\Big) = N(0, t - s).$$

Thus, the increment $X(t) - X(s)$ is normally distributed with mean 0 and variance $t - s$, as required for a standard Brownian motion.

**4. Independent increments:** For any partition $0 = t_0 < t_1 < \cdots < t_n$, observe that

$$X(t_k) - X(t_{k-1}) = c^{-1/2}\Big(W(ct_k) - W(ct_{k-1})\Big),$$

and the increments $\{W(ct_k) - W(ct_{k-1})\}_{k=1}^{n}$ are independent (since $W$ has independent increments). Multiplying by the constant $c^{-1/2}$ preserves independence. Therefore, the increments of $X$ over disjoint intervals are independent.

**5. Covariance structure:** For $0 \leq s \leq t$, note that

$$\mathrm{Cov}(X(s), X(t)) = \mathrm{Cov}\Big(c^{-1/2}W(cs),\ c^{-1/2}W(ct)\Big) = c^{-1}\,\mathrm{Cov}(W(cs), W(ct)).$$

Since for standard Brownian motion $\mathrm{Cov}(W(cs), W(ct)) = cs$ (because $s \leq t$), we have

$$\mathrm{Cov}(X(s), X(t)) = c^{-1}(cs) = s.$$

This is exactly the covariance function of a standard Brownian motion. Thus, X is a standard Brownian motion.

**5.6** (15 points)
Prove: Suppose that $W$ is a standard Brownian motion, and let $c > 0$. Define $X(t) = W(c+t) - W(c)$. Then $\{X(t) : t \geq 0\}$ is a standard Brownian motion that is independent of $\{W(t) : 0 \leq t \leq c\}$.

**a) $X(0) = 0$ almost surely:**

$$\begin{aligned} X(0) &= W(c+0) - W(c) \\ &= W(c) - W(c) \\ &= 0 \end{aligned}$$

**b) $X$ has continuous sample paths:**
Since $W$ has continuous sample paths by definition of Brownian motion, and $X(t) = W(c+t) - W(c)$ is a composition and difference of continuous functions, $X$ also has continuous sample paths.

**c) $X$ has stationary, independent increments:**
For any $0 \leq s < t$ and $0 \leq u < v$, consider the increments $X(t) - X(s)$ and $X(v) - X(u)$.

$$\begin{aligned} X(t) - X(s) &= [W(c+t) - W(c)] - [W(c+s) - W(c)] \\ &= W(c+t) - W(c+s) \\ X(v) - X(u) &= [W(c+v) - W(c)] - [W(c+u) - W(c)] \\ &= W(c+v) - W(c+u) \end{aligned}$$

If the intervals $[c+s, c+t]$ and $[c+u, c+v]$ are disjoint, then the increments $X(t)-X(s)$ and $X(v)-X(u)$ are independent by the independent increments property of the original Brownian motion $W$.
For stationarity, for any $h > 0$ and $t \geq 0$, we have:

$$X(t+h) - X(t) = W(c+t+h) - W(c) - [W(c+t) - W(c)]$$

$$= W(c+t+h) - W(c+t)$$

This increment depends only on the time difference $h$, not on the starting time $t$, which establishes stationarity.

**d) For each $t > 0$, $X(t) \sim \mathcal{N}(0, t)$:** Since $W$ is a Brownian motion, $W(c+t) - W(c) \sim \mathcal{N}(0, (c+t) - c) = \mathcal{N}(0, t)$.
Therefore, $X(t) \sim \mathcal{N}(0, t)$ for each $t > 0$.

Since all properties of standard Brownian motion are satisfied, $\{X(t) : t \geq 0\}$ is indeed a standard Brownian motion. **Part 2: Prove that $\{X(t) : t \geq 0\}$ is independent of $\{W(t) : 0 \leq t \leq c\}$** To prove independence, we need to show that for any finite collection of times $\{t_1, t_2, \dots, t_n\}$ with $t_i \geq 0$ and $\{s_1, s_2, \dots, s_m\}$ with $0 \leq s_j \leq c$, the random vectors $(X(t_1), X(t_2), \dots, X(t_n))$ and $(W(s_1), W(s_2), \dots, W(s_m))$ are independent. Let's set $t_0 = 0$ and $s_0 = 0$ for convenience. Then we can express:

$$X(t_i) = W(c + t_i) - W(c) \quad \text{for } i = 1, 2, \dots, n$$

$$W(s_j) = W(s_j) - W(0) \quad \text{for } j = 1, 2, \dots, m$$

Consider the augmented vector of increments:

$$(W(s_1) - W(s_0), W(s_2) - W(s_1), \dots, W(s_m) - W(s_{m-1}), W(c) - W(s_m)$$

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1}))$$

This can be rewritten as:

$$(W(s_1) - W(0), W(s_2) - W(s_1), \dots, W(s_m) - W(s_{m-1}), W(c) - W(s_m),$$

$$W(c + t_1) - W(c), W(c + t_2) - W(c + t_1), \dots, W(c + t_n) - W(c + t_{n-1}))$$

By the independent increments property of Brownian motion $W$, all these increments are independent because they are increments over disjoint time intervals.
Since $(W(s_1), W(s_2), \dots, W(s_m))$ can be written as a linear transformation of the first $m$ increments:

$$W(s_1) = W(s_1) - W(0)$$

$$W(s_2) = (W(s_1) - W(0)) + (W(s_2) - W(s_1))$$

And similarly, $(X(t_1), X(t_2), \dots, X(t_n))$ can be written as a linear transformation of the increments involving $X$:

$$X(t_1) = X(t_1) - X(0) = W(c + t_1) - W(c)$$

$$X(t_2) = X(t_1) + (X(t_2) - X(t_1)) = (W(c + t_1) - W(c)) + (W(c + t_2) - W(c + t_1))$$

Since linear transformations of independent random variables are independent, it follows that the vectors $(W(s_1), W(s_2), \dots, W(s_m))$ and $(X(t_1), X(t_2), \dots, X(t_n))$ are independent.
This proves that $\{X(t) : t \geq 0\}$ is independent of $\{W(t) : 0 \leq t \leq c\}$.