S&DS 351: Stochastic Processes - Homework 3

Bryan SebaRaj

Professor Ilias Zadik

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Problem 1

(10 points) Is it possible for a transient state to be periodic? If so, construct an example of such a Markov chain; otherwise, give a mathematical proof why not.

Note: I (fortunately) solved this after proving problem 3, so for a more thorough proof on how this example is transient, please see Problem 3.

Yes, it is possible for a transient state to be periodic. Consider a 1-dimensional asymmetric random walk on $\mathbb{Z}$:

$$X_n = X_{n-1} + Z_n, \quad \text{where} \quad \mathbb{P}(Z_n = +1) = p \quad \text{and} \quad \mathbb{P}(Z_n = -1) = 1 - p,$$

for some $p \in (0, 1)$ with $p \neq \frac{1}{2}$. Starting at state 0, state 0 is transient (see Problem 3).

Define the period as $d_i = \gcd\{n : P^n(i, i) > 0\}$, where P is the transition matrix.
In the random walk, the walk must trivially take as many $+1$ steps as -1 steps to reach the initial state. Thus one can only return to state x starting from x in an even number of steps. Note that this holds for all integers. Hence for each integer x,

$$(P^n)(x, x) > 0 \implies n \text{ is even.}$$
$$(P^n)(x, x) = 0 \implies n \text{ is odd.}$$

Therefore, the greatest common divisor of all such n is 2, and every state $x \in \mathbb{Z}$ has period 2.

Problem 2

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix P. Let $k \geq 1$ be an integer.

1. (5 points) Prove that $Y_n = X_{kn}$ is also a Markov chain. Find its transition matrix.

2. (10 points) Suppose that the original chain $\{X_n\}$ is irreducible. Is $\{Y_n\}$ irreducible? If so, prove it; if not, provide a counterexample.

3. (10 points) Suppose that the original chain $\{X_n\}$ is aperiodic. Is $\{Y_n\}$ aperiodic? If so, prove it; if not, provide a counterexample.

4. (10 points) Suppose that the original chain $\{X_n\}$ is transient. Is $\{Y_n\}$ transient? If so, prove it; if not, provide a counterexample.

5. (15 points) Suppose that the original chain $\{X_n\}$ is recurrent. Is $\{Y_n\}$ recurrent? If so, prove it; if not, provide a counterexample.

6. (5 points) Suppose that the original chain X_n is irreducible and that it has period d. What is the period of each state i in the new Markov chain Y_n for $k = d$?

Problem 3

(Asymmetric random walk, 15 points) Consider the *asymmetric* random walk on $\mathbb{Z}$, that is, $X_n = X_{n-1} + Z_n$, where $Z_1, Z_2, \dots$ are iid and $\mathbb{P}(Z_n = +1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$, with $p \in [0, 1]$ and $p \neq \frac{1}{2}$. Show that the state 0 is a transient state.

In Lecture 7 we saw/will see that when $p = \frac{1}{2}$ this is not true anymore and the state 0 is recurrent. Can you explain intuitively why this is the case?

Hint: You may want to use Stirling's formula that $\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{2\pi n}} = 1$.

Starting from $X_0 = 0$, the random walk is at state 0 again at $t = n$ only when it has taken an equal number of $+1$ steps as -1 steps. As such, n must be even.

Suppose $n = 2k$, and k is the number of Z_i that are $+1$,

$$\mathbb{P}(X_{2k} = 0 \,|\, X_0 = 0) \;=\; \binom{2k}{k} p^k (1-p)^k$$

$$\text{Note that } \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 0 \;\text{ if } n \text{ is odd}$$

Hence the series of return probabilities at 0 is

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 1 \;+\; \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k,$$

accounting for the initial state of 0.
Using Stirling's approximation,

$$n! \;\sim\; \left(\tfrac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as } n \to \infty,$$

applying to this case,

$$\binom{2k}{k} \;=\; \frac{(2k)!}{k!\,k!} \;\approx\; \frac{\sqrt{4\pi k}\,\left(\frac{2k}{e}\right)^{2k}}{2\pi k\,\left(\frac{k}{e}\right)^k \left(\frac{k}{e}\right)^k} \;=\; \frac{4^k}{\sqrt{\pi k}}$$

Therefore,

$$\binom{2k}{k} p^k (1-p)^k \;\approx\; \frac{4^k}{\sqrt{\pi k}} \big[p(1-p)\big]^k \;=\; \frac{\big[4\,p(1-p)\big]^k}{\sqrt{\pi k}}.$$

If $p \neq \frac{1}{2}$, then $4\,p(1-p) < 1$ (If $f(x) = x(1-x)$, then $f'(x) = -x + 1 - x = -2x + 1$. Solving for the max when $f'(x) = 0, x = \frac{1}{2}$).

Note, that as $k \to \infty$, $\big[4\,p(1-p)\big]^k$ decays exponentially. Therefore,

$$\binom{2k}{k} p^k (1-p)^k \;=\; O\Big(\big[4\,p(1-p)\big]^k\Big) \quad \text{and} \quad \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k \;<\; \infty.$$

Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 1 \;+\; \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k \;<\; \infty.$$

which defines a transient state.

However, when $p = \frac{1}{2}$,

$$\binom{2k}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k \;\approx\; \frac{\big[4 \cdot 0.5(1-0.5)\big]^k}{\sqrt{\pi k}} \;=\; \frac{1}{\sqrt{\pi k}},$$

so

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \;=\; \infty.$$

which defines a recurrent state when $p = \frac{1}{2}$.

Exercise 1.8

Consider a Markov chain on the integers with

$$P(i, i+1) = 0.4 \text{ and } P(i, i-1) = 0.6 \text{ for } i > 0,$$
$$P(i, i+1) = 0.6 \text{ and } P(i, i-1) = 0.4 \text{ for } i < 0,$$
$$P(0, 1) = P(0, -1) = \frac{1}{2}.$$

This is a chain with infinitely many states, but it has a sort of probabilistic "restoring force" that always pushes back toward 0. Find the stationary distribution.

From the "resisting force" there is a single stationary distribution, expected to be symmetric around 0, with a geometric decay away from 0. **Step 2. Stationarity equations and ratio method.** Denote the stationary distribution by $\{\pi_i\}_{i\in\mathbb{Z}}$, satisfying

$$\sum_{j\in\mathbb{Z}} \pi_j P(j, i) \;=\; \pi_i \quad \text{for all } i \in \mathbb{Z},$$

and $\sum_{i\in\mathbb{Z}} \pi_i = 1$. Because this is a (two-sided) birth-death type chain, one may use the standard balance equations:

$$\pi_i \, P(i, i+1) \;=\; \pi_{i+1} \, P(i+1, i).$$

Concretely, for $i \geq 1$:

$$\pi_i \times 0.4 \;=\; \pi_{i+1} \times 0.6 \quad \Longrightarrow \quad \frac{\pi_{i+1}}{\pi_i} = \frac{0.4}{0.6} = \frac{2}{3}.$$

For $i \leq -1$:

$$\pi_i \times 0.6 \;=\; \pi_{i+1} \times 0.4 \quad \Longrightarrow \quad \frac{\pi_{i+1}}{\pi_i} = \frac{0.6}{0.4} = \frac{3}{2}.$$

We also need to handle the special transitions at $i = 0$. The balance equation between 0 and 1 yields:

$$\pi_0 \times 0.5 \;=\; \pi_1 \times 0.6 \quad \Longrightarrow \quad \frac{\pi_1}{\pi_0} = \frac{0.5}{0.6} = \frac{5}{6}.$$

And for $i = -1$:

$$\pi_{-1} \times 0.6 = \pi_0 \times 0.5 \quad \Longrightarrow \quad \frac{\pi_{-1}}{\pi_0} = \frac{0.5}{0.6} = \frac{5}{6},$$

Hence,

$$\pi_1 = \frac{5}{6}\,\pi_0, \quad \pi_{-1} = \frac{5}{6}\,\pi_0.$$

This shows the symmetry $\pi_1 = \pi_{-1}$ indeed. *(a) For $i > 0$:*

$$\frac{\pi_{i+1}}{\pi_i} = \frac{2}{3} \quad \Longrightarrow \quad \pi_i = \left(\frac{2}{3}\right)^{i-1} \pi_1 \quad \text{for } i \geq 1.$$

But $\pi_1 = \frac{5}{6}\,\pi_0$, so

$$\pi_i \;=\; \left(\frac{2}{3}\right)^{i-1} \cdot \frac{5}{6}\,\pi_0 \quad \text{for } i \geq 1.$$

(b) For $i < 0$:

$$\frac{\pi_{i+1}}{\pi_i} = \frac{3}{2} \quad \Longrightarrow \quad \pi_i = \frac{2}{3}\,\pi_{i+1} \quad \text{for } i \leq -2,$$

stepping upward until $i = -1$, for which we already have $\pi_{-1} = \frac{5}{6}\pi_0$. Iterating gives

$$\pi_{-2} \;=\; \frac{2}{3}\,\pi_{-1} \;=\; \frac{2}{3}\cdot\frac{5}{6}\,\pi_0 \;=\; \frac{5}{9}\,\pi_0,$$
$$\pi_{-3} \;=\; \frac{2}{3}\,\pi_{-2} \;=\; \frac{2}{3}\cdot\frac{5}{9}\,\pi_0 \;=\; \frac{10}{27}\,\pi_0,$$

and so on. In fact, a direct pattern emerges, and for $i < 0$,

$$\pi_i \;=\; \frac{5}{6}\left(\frac{2}{3}\right)^{|i|-1}\pi_0.$$

Step 3. A unified formula. To summarize, set $|0| - 1 = -1$ in the exponents interpreted carefully, or write piecewise. A concise way is:

$$\pi_0 = \pi_0, \quad \pi_i \;=\; \frac{5}{6}\left(\frac{2}{3}\right)^{|i|-1}\pi_0 \quad \text{for } i \neq 0.$$

We still must determine the constant π_0 by requiring

$$\sum_{i=-\infty}^{\infty} \pi_i \;=\; 1.$$

Hence

$$\pi_0 \;+\; \sum_{i\neq 0} \frac{5}{6}\left(\tfrac{2}{3}\right)^{|i|-1}\pi_0 \;=\; 1.$$

Factor out π_0:

$$\pi_0\Big[1 \;+\; \frac{5}{6}\sum_{i\neq 0}\left(\tfrac{2}{3}\right)^{|i|-1}\Big] \;=\; 1.$$

Next we split the sum at $i > 0$ and $i < 0$, noticing the symmetry:

$$\sum_{i\neq 0}\left(\tfrac{2}{3}\right)^{|i|-1} \;=\; 2\sum_{j=1}^{\infty}\left(\tfrac{2}{3}\right)^{j-1} \;=\; 2\sum_{k=0}^{\infty}\left(\tfrac{2}{3}\right)^{k} \;=\; 2\cdot\frac{1}{1-\frac{2}{3}} \;=\; 2\cdot 3 \;=\; 6.$$

Thus

$$\frac{5}{6}\cdot 6 \;=\; 5,$$

and so

$$\pi_0\Big[1+5\Big] \;=\; 6\,\pi_0 \;=\; 1 \quad\Longrightarrow\quad \pi_0 \;=\; \frac{1}{6}.$$

Therefore,

$$\pi_0 = \frac{1}{6}, \quad \pi_i \;=\; \frac{5}{6}\left(\frac{2}{3}\right)^{|i|-1}\frac{1}{6} \;=\; \frac{5}{36}\left(\frac{2}{3}\right)^{|i|-1}, \quad i \neq 0.$$

Rewriting compactly,

$$\pi_i = \begin{cases} \dfrac{1}{6}, & i = 0, \\ \dfrac{5}{36}\left(\tfrac{2}{3}\right)^{|i|-1}, & i \neq 0. \end{cases}$$

Exercise 1.16

Show that if an irreducible Markov chain has a state i such that $P(i,i) > 0$, then the chain is aperiodic. Also show by example that this sufficient condition is not necessary.

Let $\{X_n\}$ be an irreducible Markov chain on a countable state space S. Suppose there is a state $i \in S$ such that

$$P(i,i) > 0.$$

The period of state i is defined as

$$d(i) \;=\; \gcd\{n \geq 1 : P^n(i,i) > 0\},$$

Since $P(i,i) = P^1(i,i) > 0$, there is a positive probability of returning to i in exactly 1 step. As such, $1 \in \{n : P^n(i,i) > 0\}$, so any common divisor of all n must trivially divide 1.

Therefore,

$$d(i) \;=\; \gcd\{n \geq 1 : P^n(i,i) > 0\} \;=\; 1.$$

Thus, i is an aperiodic state.

By irreducibility, for any other state $j \in S$, there exist integers $m, k \geq 1$ such that $P^m(j,i) > 0$ and $P^k(i,j) > 0$. Then for any $n \geq 1$,

$$P^{m+n+k}(j,j) \;\geq\; P^m(j,i)\, P^n(i,i)\, P^k(i,j).$$

Since $P^n(i,i) > 0$ for all $n \geq 1$ (as argued above, using the self-loop at i repeatedly if necessary), we conclude that for arbitrarily many n, the probability $P^{m+n+k}(j,j)$ is strictly positive. It follows that

$$d(j) \;=\; \gcd\{n \geq 1 : P^n(j,j) > 0\} \;=\; 1.$$

Hence *every* state in an irreducible chain with a self-loop ($P(i,i) > 0$ for some i) has period 1, which means the chain is aperiodic. We now show that having $P(i,i) > 0$ for some state i is *not* a necessary condition for aperiodicity by giving a simple Markov chain that is irreducible, aperiodic, and yet has no self-loops (i.e. $P(i,i) = 0$ for all i). Consider a chain on three states $\{1,2,3\}$ with transition matrix:

$$P \;=\; \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Explicitly:

$$\begin{aligned} &P(1,1) = 0, \quad &&P(1,2) = 1, \quad &&P(1,3) = 0, \\ &P(2,1) = \tfrac{1}{2}, \quad &&P(2,2) = 0, \quad &&P(2,3) = \tfrac{1}{2}, \\ &P(3,1) = \tfrac{1}{2}, \quad &&P(3,2) = \tfrac{1}{2}, \quad &&P(3,3) = 0. \end{aligned}$$

Observe that:

- The chain is *irreducible* because each state communicates with all others (e.g. from 1 you can reach 2, then 3, and back to 1, etc.).
- There are *no self-loops*: $P(i,i) = 0$ for $i = 1,2,3$.

To see that state 1 is aperiodic, note:

$$P^2(1,1) = P(1,2)P(2,1) = (1)\left(\tfrac{1}{2}\right) = \tfrac{1}{2} > 0,$$

and

$$P^3(1,1) = P(1,2)P(2,3)P(3,1) = 1 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} > 0.$$

Hence 1 can return to itself in both 2 steps and 3 steps, so the set of possible return times to 1 contains $\{2,3\}$. The greatest common divisor of 2 and 3 is 1, which implies the period of 1 is 1. By irreducibility, every other state also has period 1, making the entire chain aperiodic. This example proves that $P(i,i) > 0$ for some i is not a necessary condition for a chain to be aperiodic.