S&DS 351: Stochastic Processes - Homework 8

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Chang Problems:

[5.8] *The strong Markov property* is an extension of the restarting property of Proposition **5.5** from fixed times c to random *stopping times* γ: For a stopping time γ, the process x defined by $X(t) = W(\gamma + t) - W(\gamma)$ is a Brownian motion, independent of the path of W up to time γ. Explain the role of the stopping time requirement by explaining how the restarting property can fail for a random time that isn't a stopping time. For example, let $M = \max\{B_t : 0 \leq t \leq 1\}$ and let $\beta = \inf\{t : B_t = M\}$; this is the first time at which B achieves its maximum height over the time interval $[0, 1]$. Clearly β is not a stopping time, since we must look at the whole path $\{B_t : 0 \leq t \leq 1\}$ to determine when the maximum is attained. Argue that the restarted process $X(t) = W(\beta + t) - W(\beta)$ is not a standard Brownian motion.

Since $B_{\beta+t} \leq B_\beta$, $\forall t$ s.t. $0 \leq t \leq 1 - \beta$, $X(1-\beta) = B_1 - B_\beta \leq 0$ w.p. 1; this contradicts the symmetry of a $N(0, 1-\beta)$ law and proves that X cannot be a standard Brownian motion, showing why the strong Markov property requires β to be a stopping time.

[5.9] [Ornstein-Uhlenbeck process] Define a process X by

$$X(t) = e^{-t} W(e^{2t})$$

for $t \geq 0$, where W is a standard Brownian motion. X is called an *Ornstein-Uhlenbeck process.*

(a) Find the covariance function of X.

Let the process X be obtained from a standard Brownian motion W by deterministic space–time change as given. Because W is Gaussian with mean 0, X is also Gaussian with mean 0. As such, its second-order behaviour is described by its covariance function. Suppose $s, t \geq 0$ and WLOG assume $s \leq t$. Then,

$$\mathbb{E}\big[X(s)X(t)\big] = \mathbb{E}\Big[e^{-s} W(e^{2s})\, e^{-t} W(e^{2t})\Big] = e^{-(s+t)}\, \mathbb{E}\Big[W(e^{2s})\, W(e^{2t})\Big]$$

Recall that Brownian motion has covariance $\mathbb{E}[W(u)W(v)] = \min\{u, v\}$.

$$\mathbb{E}\big[X(s)X(t)\big] = e^{-(s+t)} \min\big\{e^{2s}, e^{2t}\big\} = e^{-(s+t)} e^{2s} = e^{-(t-s)}$$

By symmetry in (s, t), this can be extended to all $s, t \geq 0$, yielding

$$\mathrm{Cov}\big(X(s), X(t)\big) = e^{-|t-s|}, \qquad s, t \geq 0$$

(b) Evaluate the functions μ and σ^2, defined by

$$\mu(x, t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[X(t+h) - X(t) \mid X(t) = x]$$

$$\sigma^2(x, t) = \lim_{h \downarrow 0} \frac{1}{h} \mathrm{Var}[X(t+h) - X(t) \mid X(t) = x].$$

Expand $X(t+h)$ around t,

$$\Delta_h = W\big(e^{2(t+h)}\big) - W(e^{2t}), \qquad \text{so that} \quad X(t+h) = e^{-(t+h)} \big[W(e^{2t}) + \Delta_h\big]$$

See that since W has independent increments, Δ_h is independent of $W(e^{2t})$ and is Gaussian with mean 0 and variance.

$$\mathrm{Var}(\Delta_h) = e^{2(t+h)} - e^{2t} = e^{2t}\big(e^{2h} - 1\big)$$

Also see that when $\{X(t) = x\}$, the value of $W(e^{2t})$ becomes

$$X(t) = e^{-t} W(e^{2t}) = x$$

$$W(e^{2t}) = e^t x$$

Therefore, in this case,

$$\mathbb{E}[\Delta_h \,|\, X(t) = x] = 0 \quad \text{and} \quad \mathrm{Var}[\Delta_h \,|\, X(t) = x] = e^{2t}\big(e^{2h} - 1\big)$$

Evaluating the first conditional moment,

$$\begin{aligned}
\mathbb{E}\big[X(t+h) - X(t) \,\big|\, X(t) = x\big] &= \mathbb{E}\Big[e^{-(t+h)} W(e^{2t}) - e^{-t} W(e^{2t}) + e^{-(t+h)} \Delta_h \,\Big|\, X(t) = x\Big] \\
&= (e^{-(t+h)} - e^{-t})\, e^t x + e^{-(t+h)} \mathbb{E}[\Delta_h \,|\, X(t) = x] \\
&= (e^{-h} - 1)x
\end{aligned}$$

Therefore,

$$\mu(x,t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}\big[X(t+h) - X(t) \,|\, X(t) = x\big] = \lim_{h \downarrow 0} \frac{e^{-h} - 1}{h} x = -x$$

Evaluating the second conditional moment,

$$\begin{aligned}
\mathrm{Var}\big[X(t+h) - X(t) \,\big|\, X(t) = x\big] &= \mathrm{Var}\Big[e^{-(t+h)} \Delta_h\Big] \\
&= e^{-2(t+h)}\, \mathrm{Var}[\Delta_h \,|\, X(t) = x] \\
&= e^{-2h}\big(e^{2h} - 1\big)
\end{aligned}$$

Therefore,

$$\sigma^2(x,t) = \lim_{h \downarrow 0} \frac{1}{h} \mathrm{Var}\big[X(t+h) - X(t) \,|\, X(t) = x\big] = 2$$

[5.10] Let W be a standard Brownian motion.

(i) Defining $\tau_b = \inf\{t : W(t) = b\}$ for $b > 0$ as above, show that τ_b has probability density function

$$f_{\tau_b}(t) = \frac{b}{\sqrt{2\pi}} t^{-3/2} e^{-b^2/(2t)}$$

for $t > 0$.

By the reflection principle, see that

$$P\{\tau_b \le t\} = P\Big\{\sup_{0 \le s \le t} W(s) \ge b\Big\} = 2P\{W(t) \ge b\}$$

Since $W(t) \sim \mathcal{N}(0, t)$,

$$P\{W(t) \ge b\} = 1 - \Phi\Big(\frac{b}{\sqrt{t}}\Big)$$

where Φ is the standard normal distribution function and $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ its density. Therefore,

$$F_{\tau_b}(t) = 2\Big[1 - \Phi\Big(\frac{b}{\sqrt{t}}\Big)\Big] \quad \text{when } t > 0$$

Differentiating to obtain the density,

$$f_{\tau_b}(t) = 2\varphi\Big(\frac{b}{\sqrt{t}}\Big)\Big(-\frac{b}{2} t^{-3/2}\Big)(-1) = \frac{b}{\sqrt{2\pi}} t^{-3/2} e^{-b^2/(2t)}, \quad \text{when } t > 0$$

Thus, τ_b has a pdf as claimed.

(ii) Show that for $0 < t_0 < t_1$,

$$P\{W(t) = 0 \text{ for some } t \in (t_0, t_1)\} = \frac{2}{\pi}\tan^{-1}\left(\sqrt{\frac{t_1}{t_0} - 1}\right) = \frac{2}{\pi}\cos^{-1}\left(\sqrt{\frac{t_0}{t_1}}\right).$$

[Hint: The last equality is simple trigonometry. For the previous equality, condition on the value of $W(t_0)$, use part (i), and Fubini (or perhaps integration by parts).]

Suppose $0 < t_0 < t_1$ and define $\Delta = t_1 - t_0$. By the Markov property, conditioning on $W(t_0) = x$ and

$$\tau_{|x|} = \inf\{s > 0 : W_x(s) = 0\} \quad \text{s.t. } \big(W_x(0) = x\big)$$

yields

$$P\{W(t) = 0, t \in (t_0, t_1)\} = \int_{\mathbb{R}} P\{\tau_{|x|} \le \Delta\} \frac{e^{-x^2/(2t_0)}}{\sqrt{2\pi t_0}}\, dx$$

Since $\tau_{|x|}$ has the density from part (i) with $b = |x|$,

$$P\{\tau_{|x|} \le \Delta\} = \int_0^{\Delta} \frac{|x|}{\sqrt{2\pi}} s^{-3/2} e^{-x^2/2s} ds$$

Restricting to $x > 0$ based on symmetry,

$$P = 2\int_0^{\infty} \frac{e^{-x^2/(2t_0)}}{\sqrt{2\pi t_0}} \int_0^{\Delta} \frac{x}{\sqrt{2\pi}} s^{-3/2} e^{-x^2/2s}\, ds\, dx$$

$$P = \frac{2}{2\pi\sqrt{t_0}} \int_0^{\Delta} s^{-3/2} \int_0^{\infty} x \exp\Big(-x^2\Big(\frac{1}{2t_0} + \frac{1}{2s}\Big)\Big)\, dx\, ds$$

For $\alpha > 0$, see that $\int_0^\infty xe^{-\alpha x^2} dx = \frac{1}{2\alpha}$. Since

$$\alpha = \frac{1}{2t_0} + \frac{1}{2s} = \frac{s + t_0}{2st_0}$$

$$\frac{1}{2\alpha} = \frac{st_0}{s + t_0}$$

Therefore,

$$P = \frac{1}{2\pi\sqrt{t_0}} \int_0^{\Delta} \frac{2t_0\, s^{-1/2}}{s + t_0}\, ds = \frac{\sqrt{t_0}}{\pi} \int_0^{\Delta} \frac{s^{-1/2}}{s + t_0}\, ds$$

Let $s = t_0 u^2$ $(u \ge 0)$. As such, $ds = 2t_0 u\, du$ and the upper limit becomes

$$u_{\max} = \sqrt{\frac{\Delta}{t_0}} = \sqrt{\frac{t_1}{t_0} - 1}$$

Substituting this into the integral,

$$P = \frac{\sqrt{t_0}}{\pi} \int_0^{u_{\max}} \frac{1}{\sqrt{t_0}u} \frac{2t_0 u}{t_0(1 + u^2)}\, du = \frac{2}{\pi} \int_0^{u_{\max}} \frac{du}{1 + u^2} = \frac{2}{\pi}\tan^{-1}(u_{\max})$$

Given that $u_{\max} = \sqrt{\frac{t_1}{t_0} - 1}$

$$P = \frac{2}{\pi}\tan^{-1}\Big(\sqrt{\frac{t_1}{t_0} - 1}\Big) = \frac{2}{\pi}\cos^{-1}\Big(\sqrt{\frac{t_0}{t_1}}\Big)$$

[5.13] Let $(X(t), Y(t))$ be a two-dimensional standard Brownian motion; that is, let $\{X(t)\}$ and $\{Y(t)\}$ be standard Brownian motion processes that are independent of each other. Let $b > 0$, and define $\tau = \inf\{t : X(t) = b\}$. Find the probability density function of $Y(\tau)$. That is, find the probability density of the height at which the two-dimensional Brownian motion first hits the vertical line $x = b$.

[Hint: The answer is a Cauchy distribution.]

Let $\tau = \inf\{t > 0 : X(t) = b$ when $b > 0$, so $(X(\tau), Y(\tau)) = (b, Y(\tau))$ is the first point where the two-dimentional Brownian motion hits $x = b$. Since X and Y are independent one–dimensional Brownian motions started at 0, $Y(\tau)$ can be obtained. From the reflection principle,

$$f_\tau(t) \;=\; \frac{b}{\sqrt{2\pi}}\, t^{-3/2}\, e^{-b^2/(2t)}, \quad \text{when } t > 0$$

For fixed t, $Y(t)$ is independent of X and satisfies

$$Y(\tau) \mid \{\tau = t\} \;\sim\; \mathcal{N}(0, t)$$

$$g_t(y) \;=\; \frac{1}{\sqrt{2\pi t}}\, e^{-y^2/(2t)}, \qquad y \in \mathbb{R}$$

Using the law–of–total–probability and Fubini,

$$\begin{aligned}
f_{Y(\tau)}(y) &= \int_0^\infty g_t(y)\, f_\tau(t)\, dt \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi t}}\, e^{-y^2/(2t)}\, \frac{b}{\sqrt{2\pi}}\, t^{-3/2}\, e^{-b^2/(2t)}\, dt \\
&= \frac{b}{2\pi} \int_0^\infty t^{-2}\, e^{-(b^2+y^2)/(2t)}\, dt
\end{aligned}$$

Evaluating the integral by substitution $u = (b^2 + y^2)/(2t)$, s.t. $t = (b^2 + y^2)/(2u)$ and $dt = -(b^2 + y^2)/(2u^2)\, du$,

$$\begin{aligned}
\int_0^\infty t^{-2} e^{-(b^2+y^2)/(2t)}\, dt &= \int_\infty^0 \Big(\frac{2u}{b^2+y^2}\Big)^2 e^{-u} \left(-\frac{b^2+y^2}{2u^2}\right) du \\
&= \frac{1}{\frac{1}{2}(b^2+y^2)} \int_0^\infty e^{-u}\, du \\
&= \frac{2}{b^2+y^2}
\end{aligned}$$

Therefore,

$$f_{Y(\tau)}(y) = \frac{b}{2\pi}\, \frac{2}{b^2+y^2} = \frac{b}{\pi\big(b^2+y^2\big)}, \qquad \forall\; y \in \mathbb{R}$$

Finally, see that the density $y \;\longmapsto\; \frac{b}{\pi\left(b^2+y^2\right)}$ is the centered Cauchy density with b. Thus,

$$Y(\tau) \;\sim\; \text{Cauchy}(0, b)$$

[5.15] Let $0 < s < t < u$.

(a) Show that $\mathbb{E}(W_s W_t \mid W_u) = \frac{s}{t}\mathbb{E}(W_t^2 \mid W_u)$.

Suppose. $0 < s < t < u$. Note that $\{W_r\}_{r \geq 0}$ is a centred Gaussian process with independent increments. Since W_u is non–degenerate, any finite vector built from the path is jointly Gaussian. As such, conditional expectations can be obtained by linear regression. Let us decompose the Brownian bridge.

$$W_r = \frac{r}{u}\, W_u + B_r, \qquad 0 \leq r \leq u$$

where $\{B_r\}_{0 \leq r \leq u}$ is a mean–0 Gaussian bridge independent of W_u with covariance

$$\mathbb{E}[B_r B_{r'}] = \frac{r(u - r')}{u}, \qquad r \leq r'$$

Calculating the conditional momemt of W_t.

$$W_t^2 = \Big(\frac{t}{u}\, W_u\Big)^2 + 2\frac{t}{u}\, W_u B_t + B_t^2$$

Note that B_t is independent of W_u and has mean 0. Taking conditional expectation given W_u yields

$$\mathbb{E}(W_t^2 \mid W_u) = \frac{t^2}{u^2} W_u^2 + \mathbb{E}(B_t^2) = \frac{t^2}{u^2} W_u^2 + \frac{t(u-t)}{u}$$

Using the same decomposition,

$$W_s W_t = \Big(\frac{s}{u} W_u + B_s\Big)\Big(\frac{t}{u} W_u + B_t\Big) = \frac{st}{u^2} W_u^2 + \frac{s}{u} W_u B_t + \frac{t}{u} W_u B_s + B_s B_t$$

Conditioning on W_u removes the linear terms in B_s, B_t and replaces $B_s B_t$ with its covariance,

$$\mathbb{E}(W_s W_t \mid W_u) = \frac{st}{u^2} W_u^2 + \mathbb{E}(B_s B_t) = \frac{st}{u^2} W_u^2 + \frac{s(u-t)}{u}$$

Therefore,

$$\frac{s}{t}\, \mathbb{E}(W_t^2 \mid W_u) = \frac{s}{t}\Big(\frac{t^2}{u^2} W_u^2 + \frac{t(u-t)}{u}\Big) = \frac{st}{u^2} W_u^2 + \frac{s(u-t)}{u} = \mathbb{E}(W_s W_t \mid W_u)$$

Thus,

$$\mathbb{E}(W_s W_t \mid W_u) = \frac{s}{t}\, \mathbb{E}(W_t^2 \mid W_u)$$

(b) Find $\mathbb{E}(W_t^2 \mid W_u)$ [you know $\mathrm{Var}(W_t \mid W_u)$ and $\mathbb{E}(W_t \mid W_u)$!] and use this to show that

$$\mathrm{Cov}(W_s, W_t \mid W_u) = \frac{s(u-t)}{u}.$$

Recall that we have already shown that

$$\mathbb{E}(W_t^2 \mid W_u) = \frac{t(u-t)}{u} + \frac{t^2}{u^2} W_u^2$$

For conditional covariance,

$$\mathrm{Cov}(W_s, W_t \mid W_u) = \mathbb{E}(W_s W_t \mid W_u) - \mathbb{E}(W_s \mid W_u)\, \mathbb{E}(W_t \mid W_u)$$

Since $\{W_r\}$ is a martingale,

$$\mathbb{E}(W_r \mid W_u) = \frac{r}{u} W_u \qquad \text{for some } 0 \le r \le u$$

As such,

$$\mathbb{E}(W_s \mid W_u)\, \mathbb{E}(W_t \mid W_u) = \frac{st}{u^2} W_u^2$$

Thus,

$$\mathrm{Cov}(W_s, W_t \mid W_u) = \frac{s}{t}\Big(\frac{t(u-t)}{u} + \frac{t^2}{u^2} W_u^2\Big) - \frac{st}{u^2} W_u^2 = \frac{s(u-t)}{u}$$

[5.17] Verify that the definitions (**5.13**) and (**5.14**) give Brownian bridges.

$$(5.13) \quad X(t) = W(t) - tW(1) \quad \text{for } 0 \le t \le 1.$$

$$(5.14) \quad Y(t) = (1-t) W\left(\frac{t}{1-t}\right) \quad \text{for } 0 \le t < 1, \quad Y(1) = 0$$

Recall that a centred, continuous Gaussian process $\{B(t)\}_{0 \le t \le 1}$ is called a Brownian bridge given that

$$B(0) = B(1) = 0, \quad \text{and} \quad \mathrm{Cov}\big(B(s), B(t)\big) = \min\{s,t\} - st, \quad \text{given } 0 \le s, t \le 1$$

When $s \le t$, covariance is $s(1-t)$. To demonstrate that (5.13) is Brownian bridge, consider the required properties.
(1) End points. $X(0) = W(0) = 0,\ \ X(1) = W(1) - W(1) = 0.$
(2) Gaussianity. $X(t)$ is a fixed linear combination of $\{W(r)\}_{0\le r\le 1}$, so every finite-dimensional distribution is multivariate normal.
(3) Mean. $\mathbb{E}[X(t)] = \mathbb{E}[W(t)] - t\mathbb{E}[W(1)] = 0.$
(4) Covariance. For $0 \le s \le t \le 1$,

$$\mathrm{Cov}(X(s), X(t)) = \mathbb{E}[(W(s) - sW(1))(W(t) - tW(1))] = \min\{s,t\} - ts - s(1) + st = s(1-t)$$

See that this is $\min\{s,t\} - st$.
(5) Continuity. X inherits almost–sure continuity from W.
Given that all axioms are satisfied, X is a Brownian bridge.

To demonstrate that (5.14) is Brownian bridge, consider the required properties.

Let $\theta(t) = \dfrac{t}{1-t}$, s.t. $\theta : [0,1) \to [0,\infty)$ is strictly increasing.
(1) End points. $Y(0) = (1-0)W(0) = 0, \qquad Y(1) = 0$ by definition.
(2) Gaussianity. For $t < 1$, $Y(t)$ is a scalar multiple of $W(\theta(t))$; any finite vector $(Y(t_1),\dots,Y(t_k))$ is thus a linear image of $(W(\theta(t_1)),\dots,W(\theta(t_k)))$, and is Gaussian.
(3) Mean. Trivially, $\mathbb{E}[Y(t)] = 0$
(4) Covariance. Fix $0 \le s \le t < 1$. $\theta(s) \le \theta(t) \implies \min\{\theta(s),\theta(t)\} = \theta(s)$

$$\mathrm{Cov}(Y(s), Y(t)) = (1-s)(1-t)\,\mathbb{E}\Big[W\big(\theta(s)\big)\,W\big(\theta(t)\big)\Big] = (1-s)(1-t)\,\theta(s) = (1-s)(1-t)\,\frac{s}{1-s} = s(1-t)$$

Therefore, $\mathrm{Cov}(Y(s), Y(t)) = \min\{s,t\} - st\ \forall s,t \le 1$. *(5) Continuity and the value at* $t = 1$. Computing the variance, $\mathrm{Var}[Y(t)] = t(1-t) \xrightarrow[t\to 1^-]{} 0$. As such, $Y(t) \to 0$ is in L^2 and therefore in probability as $t \to 1^-$. Since W admits a continuous modification, we can choose a modification and confirm $t \mapsto Y(t)$ is continuous w.p. 1 on $[0,1)$ and converges to 0 at $t = 1$. Therefore, redefining $Y(1) = 0$ yields a continuous version on $[0,1]$.
Thus, X and Y both satisfy the defining properties of a Brownian bridge on $[0,1]$.

Problem 1. (15 points) Let $W(t), t \ge 0$ be a standard Brownian motion. Prove that it is a Gaussian process, i.e., for all $n \in \mathbb{N}, t_1,\dots,t_n \ge 0$ and $a_1,\dots,a_n \in \mathbb{R}$, the distribution of $\sum_{i=1}^n a_i W(t_i)$ is Gaussian.

Recall the definition of standard Brownian motion $\{W(t)\}_{t\ge 0}$,

$$W(0) = 0, \quad \text{for } 0 \le s < t\ ,\ W(t) - W(s) \sim N(0, t-s), \quad \text{s.t. } \{W(t) - W(s)\}_{0\le s<t} \text{ are independent}$$

Let $n \in \mathbb{N}, t_1,\dots,t_n \ge 0, a_1,\dots,a_n \in \mathbb{R}$, and set $S = \sum_{i=1}^n a_i W(t_i)$. Note that if some of the t_i are equal, we can merge coefficients and if they are unordered, we can re-label indices s.t.

$$0 \le t_{(1)} < t_{(2)} < \cdots < t_{(m)}, \qquad b_j := \sum_{i:\, t_i = t_{(j)}} a_i, \quad 1 \le j \le m \le n,$$

and denote $S = \sum_{j=1}^m b_j W(t_{(j)})$. Hence WLOG $0 < t_1 < \cdots < t_n$. Defining the independent Gaussian increments as

$$\Delta_1 := W(t_1) - W(0) = W(t_1) \quad \text{and} \quad \Delta_k := W(t_k) - W(t_{k-1}),\ 2 \le k \le n$$

As such, S can be rewritten as

$$S = \sum_{i=1}^n a_i \Big(\Delta_1 + \cdots + \Delta_i\Big) = \sum_{k=1}^n \Big(\sum_{i=k}^n a_i\Big)\Delta_k = \sum_{k=1}^n c_k \Delta_k$$

where $c_k = \sum_{i=k}^n a_i$.
Note that each increment Δ_k is Gaussian.

$$\Delta_k \sim N\big(0, t_k - t_{k-1}\big) \quad \text{s.t. } t_0 = 0$$

and the vector $(\Delta_1, \ldots, \Delta_n)$ has independent components through the definition of Brownian motion. Because independent Gaussian variables are also jointly Gaussian, any deterministic linear combination of them must also be Gaussian. $S = \sum_{k=1}^{n} c_k \Delta_k$ is a sum of independent $N(0, \sigma_k^2)$ variables multiplied by deterministic scalars c_k. As such,

$$S \sim N\Big(0,\ \sum_{k=1}^{n} c_k^2 (t_k - t_{k-1})\Big)$$

or S is Gaussian.