

S&DS 351: Stochastic Processes - Homework 3

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Problem 1

(10 points) Is it possible for a transient state to be periodic? If so, construct an example of such a Markov chain; otherwise, give a mathematical proof why not.

Note: I (fortunately) solved this after proving problem 3, so for a more thorough proof on how this example is transient, please see Problem 3.

Yes, it is possible for a transient state to be periodic. Consider a 1-dimensional asymmetric random walk on \mathbb{Z} :

$$X_n = X_{n-1} + Z_n, \quad \text{where } \mathbb{P}(Z_n = +1) = p \quad \text{and} \quad \mathbb{P}(Z_n = -1) = 1 - p,$$

for some $p \in (0, 1)$ with $p \neq \frac{1}{2}$. Starting at state 0, state 0 is transient (see Problem 3).

Define the period as $d_i = \gcd\{n : P^n(i, i) > 0\}$, where P is the transition matrix.

In the random walk, the walk must trivially take as many $+1$ steps as -1 steps to reach the initial state. Thus one can only return to state x starting from x in an even number of steps. Note that this holds for all integers. Hence for each integer x ,

$$(P^n)(x, x) > 0 \implies n \text{ is even.}$$

$$(P^n)(x, x) = 0 \implies n \text{ is odd.}$$

Therefore, the greatest common divisor of all such n is 2, and every state $x \in \mathbb{Z}$ has period 2.

Problem 2

Let X_0, X_1, \dots be a Markov chain with transition matrix P . Let $k \geq 1$ be an integer.

- (a) (5 points) Prove that $Y_n = X_{kn}$ is also a Markov chain. Find its transition matrix.

Since $Y_n = X_{kn}$, the conditional probability for Y_{n+1} can be defined as

$$\mathbb{P}(X_{k(n+1)} = y_{n+1} \mid X_{kn} = y_n, X_{k(n-1)} = y_{n-1}, \dots, X_0 = y_0)$$

Because $\{X_n\}$ is a Markov chain, it satisfies the Markov property,

$$P(X_{m+1} = x_{m+1} \mid X_m = x_m, \dots, X_0 = x_0) = P(X_{m+1} = x_{m+1} \mid X_m = x_m).$$

Applying for the k -steps from time kn to time $k(n+1)$,

$$\mathbb{P}(X_{k(n+1)} = y_{n+1} \mid X_{kn} = y_n, X_{k(n-1)} = y_{n-1}, \dots, X_0 = y_0) = \mathbb{P}(X_{k(n+1)} = y_{n+1} \mid X_{kn} = y_n)$$

Therefore,

$$\mathbb{P}(Y_{n+1} = y_{n+1} \mid Y_n = y_n, Y_{n-1} = y_{n-1}, \dots, Y_0 = y_0) = \mathbb{P}(Y_{n+1} = y_{n+1} \mid Y_n = y_n)$$

Thus, $\{Y_n\}$ satisfies the Markov property and is a Markov chain.

Solving for the transition matrix, note that for any state $i, j \in S$,

$$\mathbb{P}(Y_{n+1} = j \mid Y_n = i) = \mathbb{P}(X_{k(n+1)} = j \mid X_{kn} = i)$$

In the Markov chain $\{X_n\}$,

$$\mathbb{P}(X_{k(n+1)} = j \mid X_{kn} = i) = (P^k)_{ij}$$

Thus, the one-step transition probability for $\{Y_n\}$ is

$$\mathbb{P}(Y_{n+1} = j \mid Y_n = i) = (P^k)_{ij}.$$

Therefore, the transition matrix for $\{Y_n\}$ is P^k .

- (b) (10 points) Suppose that the original chain $\{X_n\}$ is irreducible. Is $\{Y_n\}$ irreducible? If so, prove it; if not, provide a counterexample.

Consider a Markov chain, with state space $S = \{X_0, X_1\}$. Define its transition matrix as

$$P_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since X_0 and X_1 communicate, this chain is trivially irreducible.

Suppose $k = 2$. The transition matrix of $Y_m = X_{2m}$ is

$$P_Y = P_X^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

X_0 and X_1 do not communicate in the new chain, as X_0 and X_1 are closed subsets on S . Therefore, Y_n is not irreducible.

Note that this can be generalized to any Markov chain X_0, X_1, \dots, X_k , where $\mathbb{P}_X(i, j) = 1$ for $j = (i + 1) \bmod k$ and $\mathbb{P}_X(i, j) = 0$ for all other i , for all j . This chain is trivially irreducible, and yields a transition matrix P^k which is a $k \times k$ identity matrix. The chain Y_n is not irreducible, as the states X_0, X_1, \dots, X_k are closed subsets on S .

- (c) (10 points) Suppose that the original chain $\{X_n\}$ is aperiodic. Is $\{Y_n\}$ aperiodic? If so, prove it; if not, provide a counterexample.

- A Markov chain with transition matrix P is called *aperiodic* if every state i has period 1. Concretely, the period of a state i is defined as

$$d(i) = \gcd\{n \geq 1 : (P^n)_{ii} > 0\}.$$

If this gcd equals 1 for every i , then the chain is aperiodic.

- In the chain $\{Y_n\}$ defined by $Y_n := X_{kn}$, its transition probabilities arise from the k -step transition probabilities of the original chain. In particular,

$$\mathbb{P}(Y_{m+1} = j \mid Y_m = i) = \mathbb{P}(X_{k(m+1)} = j \mid X_{km} = i) = (P^k)_{ij}.$$

Hence the transition matrix of $\{Y_n\}$ is exactly P^k , and the m -step transition probabilities in $\{Y_n\}$ are given by $(P^k)^m = P^{km}$.

- Therefore, the period of state i as a state of the chain $\{Y_n\}$ is

$$d_Y(i) = \gcd\{m \geq 1 : (P^{km})_{ii} > 0\}.$$

We must show that $d_Y(i) = 1$ whenever $d(i) = 1$ in the original chain.

Key idea: Aperiodicity implies eventually-positive returns in all large times. A standard characterization of aperiodicity is that for each state i , there exists some integer N (possibly depending on i) such that

$$(P^n)_{ii} > 0 \quad \text{for all } n \geq N.$$

(Equivalently, the set $\{n : (P^n)_{ii} > 0\}$ is *co-finite*, i.e. it contains all sufficiently large n .) We use this fact to see what happens when we look only at k -step transitions, i.e. the set

$$\{m : (P^{km})_{ii} > 0\}.$$

If $(P^n)_{ii} > 0$ for all $n \geq N$, then in particular $(P^{km})_{ii} > 0$ whenever $km \geq N$. Hence for all integers

$$m \geq \lceil N/k \rceil,$$

we have $(P^{km})_{ii} > 0$. Thus

$$\{m : (P^{km})_{ii} > 0\}$$

contains *all* sufficiently large integers m . **Period calculation for the chain $\{Y_n\}$.** For the chain $\{Y_n\}$, the period of i is

$$d_Y(i) = \gcd\{m \geq 1 : (P^{km})_{ii} > 0\}.$$

Since this set includes *all* sufficiently large integers m , its greatest common divisor must be 1. Indeed, any infinite set of consecutive integers has gcd equal to 1. Formally:

- Because $\{X_n\}$ is aperiodic at i , there is an N such that $(P^n)_{ii} > 0$ for every $n \geq N$.
- Consequently, for every integer $m \geq \lceil N/k \rceil$, we have

$$(P^{km})_{ii} > 0.$$

- Hence $\{m : (P^{km})_{ii} > 0\}$ contains the tail set $\{\lceil N/k \rceil, \lceil N/k \rceil + 1, \lceil N/k \rceil + 2, \dots\}$ of *all* sufficiently large integers.
- The greatest common divisor of any infinite set of consecutive integers is 1.
- Therefore $d_Y(i) = 1$.

Since $d_Y(i) = 1$ for every state i , the subsampled chain $\{Y_n\}$ is indeed aperiodic.

- (d) (10 points) Suppose that the original chain $\{X_n\}$ is transient. Is $\{Y_n\}$ transient? If so, prove it; if not, provide a counterexample.

We first recall the definition of transience for a Markov chain: A Markov chain $\{X_n\}$ on a state space S is *transient* if there exists at least one state $i \in S$ such that

$$\mathbb{P}_i(\text{the chain ever returns to } i) < 1.$$

Equivalently, a state i is called *transient* if

$$\mathbb{P}_i(\exists n \geq 1 : X_n = i) < 1.$$

We want to show that if X_n is transient, then the new process $Y_n = X_{kn}$ is also transient. Consider a state $i \in S$ and suppose we start from $X_0 = i$. By transience of $\{X_n\}$, we have

$$\mathbb{P}_i(\exists m \geq 1 : X_m = i) < 1.$$

Define the event

$$A = \{\exists m \geq 1 : X_m = i\}$$

and the event

$$B = \{\exists n \geq 1 : Y_n = i\} = \{\exists n \geq 1 : X_{kn} = i\}.$$

Note that $B \subseteq A$ since if $X_{kn} = i$ for some n , it is in particular true that $X_m = i$ for $m = kn$. Hence,

$$\mathbb{P}_i(B) \leq \mathbb{P}_i(A).$$

Since A has probability strictly less than 1 (by the transience of X_n), it follows that

$$\mathbb{P}_i(\exists n \geq 1 : Y_n = i) = \mathbb{P}_i(B) < 1.$$

But this is precisely the statement that from state i , the chain $\{Y_n\}$ returns to i with probability strictly less than 1. Therefore, i is transient for the chain $\{Y_n\}$ as well.

Because this argument applies to every state $i \in S$ that is transient in $\{X_n\}$, it shows that no state i can become “recurrent” under the k -step sampling. Indeed, the probability that $\{Y_n\}$ returns to i is bounded above by the probability that $\{X_n\}$ returns to i , and the latter is less than 1 for transient states.

Since transience is preserved under k -step sampling (because the event of returning at multiples of k steps is a subset of returning at any step), we conclude that if the original Markov chain $\{X_n\}$ is transient, then the sampled chain $\{Y_n\} = \{X_{kn}\}$ is also transient.

An intuitive way to see why $\{Y_n\}$ cannot suddenly become recurrent if $\{X_n\}$ is transient is:

- If the original chain rarely returns to a state i (with probability < 1), then restricting our view to every k -th step only makes returns even less likely, not more.
 - Formally, the event of “returning to i at a multiple of k ” is a strict subset of the event “returning to i at least once at any time.”
- (e) (15 points) Suppose that the original chain $\{X_n\}$ is recurrent. Is $\{Y_n\}$ recurrent? If so, prove it; if not, provide a counterexample.

Let $\{X_n\}_{n \geq 0}$ be a Markov chain on a state space \mathcal{S} with transition matrix P , and assume that $\{X_n\}$ is recurrent. Fix a state $i \in \mathcal{S}$ that is recurrent, meaning that

$$\mathbb{P}_i(\{n \geq 1 : X_n = i\} \text{ is infinite}) = 1.$$

For a fixed integer $k \geq 1$, define the k -skeleton (or subsequence) chain $\{Y_n\}_{n \geq 0}$ by

$$Y_n = X_{kn}, \quad n = 0, 1, 2, \dots$$

We wish to show that the state i is also recurrent for $\{Y_n\}$; that is,

$$\mathbb{P}_i(\{n \geq 1 : Y_n = i\} \text{ is infinite}) = 1.$$

Since i is recurrent for $\{X_n\}$, the set

$$A = \{n \geq 1 : X_n = i\}$$

is almost surely infinite. Now, note that every positive integer belongs to one of the k residue classes modulo k , i.e.,

$$\mathbb{N} = \bigcup_{r=0}^{k-1} \{n \in \mathbb{N} : n \equiv r \pmod{k}\}.$$

Thus, we can partition the return times as

$$A = \bigcup_{r=0}^{k-1} A_r, \quad \text{where } A_r = \{n \in A : n \equiv r \pmod{k}\}.$$

Because A is infinite almost surely, by the pigeonhole principle at least one of the sets A_r must be infinite (almost surely). That is, there exists an $r_0 \in \{0, 1, \dots, k-1\}$ such that

$$\mathbb{P}_i(A_{r_0} \text{ is infinite}) = 1.$$

There are now two cases to consider: **Case 1.** If $r_0 = 0$, then infinitely many returns to i occur at times that are multiples of k . In other words,

$$A_0 = \{n \geq 1 : X_n = i \text{ and } n \equiv 0 \pmod{k}\}$$

is infinite almost surely. But note that

$$\{n \geq 1 : Y_n = i\} = \{n \geq 1 : X_{kn} = i\}$$

which is exactly A_0 . Hence, i is visited infinitely often by the chain $\{Y_n\}$ and is therefore recurrent.

Case 2. If $r_0 \neq 0$, define a time-shifted chain

$$\tilde{Y}_n = X_{kn+r_0}, \quad n = 0, 1, 2, \dots$$

Since \tilde{Y}_n is merely a shifted version of the k -skeleton chain, it is also a Markov chain (with the same state space and transition matrix P^k) and has the same recurrence properties (recurrence is invariant under a finite time shift). The fact that A_{r_0} is infinite means that

$$\{n \geq 0 : \tilde{Y}_n = i\} = \{n \geq 0 : X_{kn+r_0} = i\}$$

is infinite almost surely. Therefore, i is recurrent for the chain $\{\tilde{Y}_n\}$. But since \tilde{Y}_n and Y_n differ only by a fixed time shift, it follows that the recurrence properties are the same for both chains. (In more detail, the state space of $\{Y_n\}$ decomposes into cyclic classes corresponding to the residue classes modulo k , and recurrence is a class property.) Hence, i is recurrent for $\{Y_n\}$ as well. In

either case, we conclude that if i is recurrent for $\{X_n\}$, then it is also recurrent for the chain $\{Y_n\}$ defined by $Y_n = X_{kn}$. Since the choice of $i \in \mathcal{S}$ was arbitrary (or, more precisely, since recurrence is a class property), we deduce that the entire chain $\{Y_n\}$ is recurrent.

- (f) (5 points) Suppose that the original chain X_n is irreducible and that it has period d . What is the period of each state i in the new Markov chain Y_n for $k = d$?

Since the original chain is irreducible with period d , for each state i ,

$$(P_X^d)_{ii} > 0 \quad \text{and} \quad (P_X^{d-b})_{ii} = 0, \quad \forall b = 1, 2, \dots, d-1.$$

Therefore, for any multiple of d , $a \in \mathbb{Z}$ such that $a \geq 1$,

$$(P_X^d)_{ii}^a > 0$$

As such, all returns to i can occur only at multiples of d steps.

Define the transition matrix for Y_n where $k = d$ as

$$\mathbb{P}(Y_{m+1} = j \mid Y_m = i) = (P_X^k)_{ij} = (P_X^d)_{ij}$$

When $j = i$,

$$(P_Y^1)_{ii} = (P_X^d)_{ii} > 0$$

As such, state i can reach itself in a single step in chain Y , forming a self-loop in Y .

Defining the period of i in Y ,

$$d_Y(i) = \gcd \{ m \geq 1 : (P^{dm})_{ii} > 0 \}.$$

Note that $(P^d)_{ii} > 0$, so at $m = 1$, $(P^{d \cdot 1})_{ii} = (P^d)_{ii} > 0$. Since the chain is irreducible, raising P^d to a higher power trivially cannot change its positivity:

$$(P^{d^2})_{ii} = (P^d)_{ii}^2 > 0, \quad (P^{d^3})_{ii} > 0, \quad \dots$$

$(P^{dn})_{ii} > 0$ holds for all $n \geq 1$, so the set of possible m is $\{1, 2, 3, \dots\}$. Thus,

$$\{ m : (P^{dm})_{ii} > 0 \} = \{1, 2, 3, \dots\},$$

with a trivial gcd of 1. Note that since the d period was a state property in the original chain, $\{X_n\}$, it holds for all states in the old chain, and thus all states in the new chain.

As such, the period of each state in the new chain $\{Y_n\}$ is 1. In other words, the chain $\{Y_n\}$ is aperiodic.

Problem 3

(Asymmetric random walk, 15 points) Consider the *asymmetric* random walk on \mathbb{Z} , that is, $X_n = X_{n-1} + Z_n$, where Z_1, Z_2, \dots are iid and $\mathbb{P}(Z_n = +1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$, with $p \in [0, 1]$ and $p \neq \frac{1}{2}$. Show that the state 0 is a transient state.

In Lecture 7 we saw/will see that when $p = \frac{1}{2}$ this is not true anymore and the state 0 is recurrent. Can you explain intuitively why this is the case?

Hint: You may want to use Stirling's formula that $\lim_{n \rightarrow \infty} \frac{n!}{(n/e)^n \sqrt{2\pi n}} = 1$.

Starting from $X_0 = 0$, the random walk is at state 0 again at $t = n$ only when it has taken an equal number of $+1$ steps as -1 steps. As such, n must be even.

Suppose $n = 2k$, and k is the number of Z_i that are $+1$,

$$\mathbb{P}(X_{2k} = 0 \mid X_0 = 0) = \binom{2k}{k} p^k (1-p)^k$$

Note that $\mathbb{P}(X_n = 0 \mid X_0 = 0) = 0$ if n is odd

Hence the series of return probabilities at 0 is

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \mid X_0 = 0) = 1 + \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k,$$

accounting for the initial state of 0.

Using Stirling's approximation,

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as } n \rightarrow \infty,$$

applying to this case,

$$\binom{2k}{k} = \frac{(2k)!}{k! k!} \approx \frac{\sqrt{4\pi k} \left(\frac{2k}{e}\right)^{2k}}{2\pi k \left(\frac{k}{e}\right)^k \left(\frac{k}{e}\right)^k} = \frac{4^k}{\sqrt{\pi k}}$$

Therefore,

$$\binom{2k}{k} p^k (1-p)^k \approx \frac{4^k}{\sqrt{\pi k}} [p(1-p)]^k = \frac{[4p(1-p)]^k}{\sqrt{\pi k}}.$$

If $p \neq \frac{1}{2}$, then $4p(1-p) < 1$ (If $f(x) = x(1-x)$, then $f'(x) = -x + 1 - x = -2x + 1$. Solving for the max when $f'(x) = 0, x = \frac{1}{2}$).

Note, that as $k \rightarrow \infty$, $[4p(1-p)]^k$ decays exponentially. Therefore,

$$\binom{2k}{k} p^k (1-p)^k = O([4p(1-p)]^k) \quad \text{and} \quad \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k < \infty.$$

Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \mid X_0 = 0) = 1 + \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k < \infty.$$

which defines a transient state.

However, when $p = \frac{1}{2}$,

$$\binom{2k}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k \approx \frac{[4 \cdot 0.5(1-0.5)]^k}{\sqrt{\pi k}} = \frac{1}{\sqrt{\pi k}},$$

so

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty.$$

which defines a recurrent state when $p = \frac{1}{2}$.

Exercise 1.8

Consider a Markov chain on the integers with

$$P(i, i+1) = 0.4 \text{ and } P(i, i-1) = 0.6 \text{ for } i > 0,$$

$$P(i, i+1) = 0.6 \text{ and } P(i, i-1) = 0.4 \text{ for } i < 0,$$

$$P(0, 1) = P(0, -1) = \frac{1}{2}.$$

This is a chain with infinitely many states, but it has a sort of probabilistic “restoring force” that always pushes back toward 0. Find the stationary distribution.

Denote the stationary distribution by $\{\pi_i\}_{i \in \mathbb{Z}}$

$$\sum_{j \in \mathbb{Z}} \pi_j P(j, i) = \pi_i \quad \text{for all } i \in \mathbb{Z},$$

and $\sum_{i \in \mathbb{Z}} \pi_i = 1$. As this is a two-sided birth-death chain,

$$\pi_i P(i, i+1) = \pi_{i+1} P(i+1, i).$$

For $i \geq 1$:

$$\pi_i \times 0.4 = \pi_{i+1} \times 0.6 \implies \frac{\pi_{i+1}}{\pi_i} = \frac{0.4}{0.6} = \frac{2}{3}.$$

For $i \leq -1$:

$$\pi_{i-1} \times 0.6 = \pi_i \times 0.4 \implies \frac{\pi_{i-1}}{\pi_i} = \frac{0.4}{0.6} = \frac{2}{3}.$$

When $i = 0$,

$$\pi_0 \times 0.5 = \pi_1 \times 0.6 \implies \frac{\pi_1}{\pi_0} = \frac{0.5}{0.6} = \frac{5}{6}.$$

$$\pi_{-1} \times 0.6 = \pi_0 \times 0.5 \implies \frac{\pi_{-1}}{\pi_0} = \frac{0.5}{0.6} = \frac{5}{6},$$

Hence,

$$\pi_1 = \frac{5}{6} \pi_0, \quad \pi_{-1} = \frac{5}{6} \pi_0.$$

As the probabilities of a jump remain the same, generalizing for all $i \geq 1$,

$$\pi_{i+1} = \frac{2}{3} \pi_i \implies \pi_i = \left(\frac{2}{3}\right)^{i-1} \pi_1 \quad \text{for } i \geq 1.$$

$$\pi_i = \left(\frac{2}{3}\right)^{i-1} \cdot \frac{5}{6} \pi_0 \quad \forall i \geq 1.$$

Generalizing for all $i \leq -1$,

$$\pi_{i-1} = \frac{2}{3} \pi_i \implies \pi_i = \frac{2^{-i-1}}{3} \pi_{-1} \quad \forall i \leq -1,$$

$$\pi_i = \left(\frac{2}{3}\right)^{-i-1} \cdot \frac{5}{6} \pi_0 \quad \forall i \leq -1.$$

Combining the two cases,

$$\pi_i = \frac{5}{6} \left(\frac{2}{3}\right)^{|i|-1} \pi_0, \quad \forall i \neq 0.$$

Solving for π_0 , first recall that,

$$\sum_{i=-\infty}^{\infty} \pi_i = 1.$$

Hence,

$$\pi_0 + \sum_{i \neq 0} \frac{5}{6} \left(\frac{2}{3}\right)^{|i|-1} \pi_0 = 1.$$

$$\pi_0 \left[1 + \frac{5}{6} \sum_{i \neq 0} \left(\frac{2}{3}\right)^{|i|-1} \right] = 1.$$

Exploiting the symmetry of the chain,

$$\sum_{i \neq 0} \left(\frac{2}{3}\right)^{|i|-1} = 2 \sum_{j=1}^{\infty} \left(\frac{2}{3}\right)^{j-1} = 2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = 2 \cdot \frac{1}{1 - \frac{2}{3}} = 2 \cdot 3 = 6.$$

Thus

$$\pi_0 \left[1 + \frac{5}{6} \cdot 6 \right] = 6 \pi_0 = 1 \implies \pi_0 = \frac{1}{6}.$$

Therefore,

$$\pi_i = \begin{cases} \frac{1}{6}, & i = 0 \\ \frac{5}{36} \left(\frac{2}{3}\right)^{|i|-1} & i \neq 0 \end{cases}$$

Exercise 1.16

Show that if an irreducible Markov chain has a state i such that $P(i, i) > 0$, then the chain is aperiodic. Also show by example that this sufficient condition is not necessary.

Let $\{X_n\}$ be an irreducible Markov chain on a countable state space S . Suppose there is a state $i \in S$ such that

$$P(i, i) > 0.$$

The period of state i is defined as

$$d(i) = \gcd\{n \geq 1 : P^n(i, i) > 0\},$$

Since $P(i, i) = P^1(i, i) > 0$, there is a positive probability of returning to i in exactly 1 step. As such, $1 \in \{n : P^n(i, i) > 0\}$, so any common divisor of all n must trivially divide 1.

Therefore,

$$d(i) = \gcd\{n \geq 1 : P^n(i, i) > 0\} = 1.$$

Thus, i is an aperiodic state.

By irreducibility, for any other state $j \in S$, $\exists m, k \in \mathbb{Z}$ such that $m, k \geq 1$, $P^m(j, i) > 0$, and $P^k(i, j) > 0$. Therefore as proven in lecture, for any $n \geq 1$,

$$P^{m+n+k}(j, j) \geq P^m(j, i) P^n(i, i) P^k(i, j).$$

Since $P^n(i, i) > 0$ for all $n \geq 1$, for arbitrarily many n , the probability $P^{m+n+k}(j, j)$ is strictly positive. As such,

$$d(j) = \gcd\{n \geq 1 : P^n(j, j) > 0\} = 1.$$

Thus, every state in an irreducible chain with a self-loop, i.e. $P(i, i) > 0$, is aperiodic.

To show that this sufficient condition is not necessary, consider a chain on three states $\{1, 2, 3\}$ with transition matrix:

$$\begin{aligned} P(1, 1) &= 0, & P(1, 2) &= 1, & P(1, 3) &= 0, \\ P(2, 1) &= \frac{1}{2}, & P(2, 2) &= 0, & P(2, 3) &= \frac{1}{2}, \\ P(3, 1) &= \frac{1}{2}, & P(3, 2) &= \frac{1}{2}, & P(3, 3) &= 0. \end{aligned}$$

Note that the chain is irreducible, but does not contain any self-loops, i.e. $P(i, i) = 0$ for $i = 1, 2, 3$.

$$P^2(1, 1) = P(1, 2)P(2, 1) = (1) \left(\frac{1}{2}\right) = \frac{1}{2} > 0,$$

$$P^3(1, 1) = P(1, 2)P(2, 3)P(3, 1) = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} > 0.$$

Therefore, state 1 can return to itself in both 2 steps and 3 steps, which comprises a subset of the set of all possible return times back to state 1. The gcd of 2 and 3 is 1, or a period of 1. Note that $P^2(2, 2) > 0$, $P^2(3, 3) > 0$, and $P^3(2, 2) > 0$, $P^3(3, 3) > 0$, so every state also has period 1, making the entire chain aperiodic. Thus, $P(i, i) > 0$ for some i is not a necessary condition for a chain to be aperiodic.