S&DS 351: Stochastic Processes - Homework 8

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April 18, 2025

Chang Problems:

[5.8] *The strong Markov property* is an extension of the restarting property of Proposition **5.5** from fixed times c to random *stopping times* γ: For a stopping time γ, the process x defined by $X(t) = W(\gamma + t) - W(\gamma)$ is a Brownian motion, independent of the path of W up to time γ. Explain the role of the stopping time requirement by explaining how the restarting property can fail for a random time that isn't a stopping time. For example, let $M = \max\{B_t : 0 \le t \le 1\}$ and let $\beta = \inf\{t : B_t = M\}$; this is the first time at which B achieves its maximum height over the time interval $[0, 1]$. Clearly β is not a stopping time, since we must look at the whole path $\{B_t : 0 \le t \le 1\}$ to determine when the maximum is attained. Argue that the restarted process $X(t) = W(\beta + t) - W(\beta)$ is not a standard Brownian motion.

[5.9] **[Ornstein-Uhlenbeck process]** Define a process X by

$$X(t) = e^{-t} W(e^{2t})$$

for $t \ge 0$, where W is a standard Brownian motion. X is called an *Ornstein-Uhlenbeck process.*

(a) Find the covariance function of X.

(b) Evaluate the functions μ and σ^2, defined by

$$\mu(x, t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[X(t + h) - X(t) \mid X(t) = x]$$

$$\sigma^2(x, t) = \lim_{h \downarrow 0} \frac{1}{h} \mathrm{Var}[X(t + h) - X(t) \mid X(t) = x].$$

[5.10] Let W be a standard Brownian motion.

(i) Defining $\tau_b = \inf\{t : W(t) = b\}$ for $b > 0$ as above, show that τ_b has probability density function

$$f_{\tau_b}(t) = \frac{b}{\sqrt{2\pi}} t^{-3/2} e^{-b^2/(2t)}$$

for $t > 0$.

(ii) Show that for $0 < t_0 < t_1$,

$$P\{W(t) = 0 \text{ for some } t \in (t_0, t_1)\} = \frac{2}{\pi} \tan^{-1}\left(\sqrt{\frac{t_1}{t_0} - 1}\right) = \frac{2}{\pi} \cos^{-1}\left(\sqrt{\frac{t_0}{t_1}}\right).$$

[Hint: The last equality is simple trigonometry. For the previous equality, condition on the value of $W(t_0)$, use part (i), and Fubini (or perhaps integration by parts).]

J. Chang, February 2, 2007

[5.13] Let $(X(t), Y(t))$ be a two-dimensional standard Brownian motion; that is, let $\{X(t)\}$ and $\{Y(t)\}$ be standard Brownian motion processes that are independent of each other. Let $b > 0$, and define $\tau = \inf\{t : X(t) = b\}$. Find the probability density function of $Y(\tau)$. That is, find the probability density of the height at which the two-dimensional Brownian motion first hits the vertical line $x = b$.

[Hint: The answer is a Cauchy distribution.]

[5.15] Let $0 < s < t < u$.

(a) Show that $\mathbb{E}(W_s W_t \mid W_u) = \frac{s}{t}\mathbb{E}(W_t^2 \mid W_u)$.

(b) Find $\mathbb{E}(W_t^2 \mid W_u)$ [you know $\mathrm{Var}(W_t \mid W_u)$ and $\mathbb{E}(W_t \mid W_u)$!] and use this to show that

$$\mathrm{Cov}(W_s, W_t \mid W_u) = \frac{s(u-t)}{u}.$$

[5.17] Verify that the definitions (**5.13**) and (**5.14**) give Brownian bridges.

$$(5.13) \quad X(t) = W(t) - tW(1) \quad \text{for } 0 \le t \le 1.$$

$$(5.14) \quad Y(t) = (1-t)W\left(\frac{t}{1-t}\right) \quad \text{for } 0 \le t < 1, \quad Y(1) = 0$$

Problem 1. (15 points) Let $W(t), t \ge 0$ be a standard Brownian motion. Prove that it is a Gaussian process, i.e., for all $n \in \mathbb{N}, t_1, \ldots, t_n \ge 0$ and $a_1, \ldots, a_n \in \mathbb{R}$, the distribution of $\sum_{i=1}^n a_i W(t_i)$ is Gaussian.