S&DS 351: Stochastic Processes - Homework 4

Bryan SebaRaj

Professor Ilias Zadik

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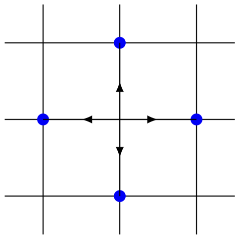
1. (Nearest-neighbor walk in two dimensions) Let $\{X_n : n \geq 0\}$ denote the nearest-neighbor walk on $\mathbb{Z}^2$: given $X_n = (a, b)$, the next position X_{n+1} is equally likely to be one of **four** possibilities: $(a+1, b)$, $(a-1, b)$, $(a, b+1)$, or $(a, b-1)$, as shown in the following picture.



(a) (8 points) Let u_n denote the probability of returning to the origin in n steps if the chain starts from the origin. For integer $m \geq 0$, show that $u_{2m+1} = 0$ and

$$u_{2m} = \frac{1}{4^{2m}} \sum_{k=0}^{m} \frac{(2m)!}{\{k!(m-k)!\}^2}$$

(Hint: the walk either moves horizontally or vertically. If $2k$ steps are horizontal, then $2(m-k)$ steps are vertical.)

(b) (5 points) Use the fact that $\sum_{k=0}^{m} \binom{m}{k}^2 = \binom{2m}{m}$ to arrive at the simple expression

$$u_{2m} = \frac{1}{4^{2m}} \binom{2m}{m}^2.$$

(c) (10 points) As we did in class, use Stirling's approximation $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ as $n \to \infty$ to conclude that $\sum_{n \geq 0} u_n = \infty$. (Hint: recall the fact mentioned in class: $\sum_{m \geq 1} m^{-\alpha} = \infty$ if $\alpha \leq 1$ and $< \infty$ if $\alpha > 1$.)

(d) (2 points) Use the classification criterion to conclude that the chain is *recurrent*.

2. (Nearest-neighbor walk in three dimensions) Now that we have warmed up, let's consider $\{X_n : n \geq 0\}$ being the nearest-neighbor walk on $\mathbb{Z}^3$: given $X_n = (a, b, c)$, the next position X_{n+1} is equally likely to be one of **six** possibilities: $(a+1, b, c)$, $(a-1, b, c)$, $(a, b+1, c)$, $(a, b-1, c)$, $(a, b, c+1)$, or $(a, b, c-1)$.

(a) (8 points) Again, let u_n denote the probability of returning to the origin in n steps if the chain starts from the origin. For integer $m \geq 0$, show that $u_{2m+1} = 0$ and

$$u_{2m} = \frac{1}{6^{2m}} \sum_{k=0}^{m} \sum_{j=0}^{m-k} \frac{(2m)!}{\{j!k!(m-j-k)!\}^2}$$

(Hint: Mimic the reasoning from part (a) in Problem 1, this time in three dimensions.)

(b) (5 points) Simplify the above expression as

$$u_{2m} = \frac{1}{2^{2m}} \binom{2m}{m} \sum_{k=0}^{m} \sum_{j=0}^{m-k} p_{k,j}, \text{ where } p_{k,j} = \frac{1}{3^m} \binom{m}{k} \binom{m-k}{j}.$$

(c) (10 points) Show that

$$\sum_{k=0}^{m} \sum_{j=0}^{m-k} p_{k,j} = 1.$$

(Hint: how many ways to distribute m balls into 3 urns?)

(d) (Bonus, 5 points) Using Stirling's approximation, one can show that

$$\max_{k,j} p_{k,j} \leq \frac{C}{m},$$

for some positive constant C, where the maximum is over all $k, j \geq 0$ such that $k + j \leq m$.

(e) (5 points) Combine parts (b), (c), (d) and use Stirling's approximation to show that $u_{2m} \leq \frac{C'}{m^{3/2}}$ for some constant C' and $\sum_{n \geq 0} u_n < \infty$.

(f) (2 points) Use the classification criterion to conclude that the chain is *transient*.

3. Let i be a recurrent state of a Markov Chain. Assume another state j is accessible from i.

(a) (3 points) Prove that

$$\mathbb{P}(N_i = \infty | X_0 = i) = 1.$$

(b) (10 points) Prove that

$$\mathbb{P}(T_j < \infty | X_0 = i) = 1.$$

(c) (5 points) Prove that

$$\mathbb{P}(T_i < \infty | X_0 = j) = 1.$$

(d) (2 points) Is it true that j needs also to be recurrent?

4. Let j be a transient state of a Markov chain and i an arbitrary state of the Markov chain.

(a) (10 points) Prove that for some constant $C > 0$ and some $p \in (0, 1)$ it holds for all $t \geq 1$

$$\mathbb{P}(N_j \geq t | X_0 = i) = 1 \leq Cp^t.$$

(b) (5 points) Conclude that

$$\lim_{n \to \infty} P^n_{i,j} = 0.$$

Chang 1.25. Prove the following proposition:
The total variation distance $\|\lambda - \mu\|$ may also be expressed in the alternative forms:

$$\|\lambda - \mu\| = \sup_{A \subseteq S} |\lambda(A) - \mu(A)| = \frac{1}{2} \sum_{i \in S} |\lambda(i) - \mu(i)| = 1 - \sum_{i \in S} \min\{\lambda(i), \mu(i)\}.$$