S&DS 351: Stochastic Processes - Homework 9

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Problem 1 (10 points) Calculate using the definitions (6.2) and (6.3) in Chang's notes, the drift and variance function of the geometric Brownian motion $X(t) = \exp(\mu t + \sigma W(t))$, $t \geq 0$ where $W(t)$ is a standard Brownian motion, $\mu \in \mathbb{R}$ and $\sigma > 0$.

$$(6.2) \quad \mathbb{E}[X(t+h) - X(t) \mid X(t) = x] = \mu(x)h + o(h)$$

$$(6.3) \quad \mathrm{Var}[X(t+h) - X(t) \mid X(t) = x] = \sigma^2(x)h + o(h) \text{ as } h \downarrow 0.$$

Solution. We recall the definitions (6.2)–(6.3): for a time-homogeneous diffusion started at the interior point x, its *drift* $\mu(x)$ and *variance function* $\sigma^2(x)$ satisfy

$$\mathbb{E}[X(t+h) - X(t) \mid X(t) = x] = \mu(x)\, h + o(h), \qquad h \downarrow 0,$$

$$\mathrm{Var}[X(t+h) - X(t) \mid X(t) = x] = \sigma^2(x)\, h + o(h), \qquad h \downarrow 0.$$

Let the process be the geometric Brownian motion

$$X(t) = \exp\big(\mu t + \sigma W(t)\big), \qquad t \geq 0,$$

where $W(t)$ is a standard Brownian motion, $\mu \in \mathbb{R}$ and $\sigma > 0$. Fix $t \geq 0$ and condition on $X(t) = x$. Because $W(t+h) - W(t) \sim \mathcal{N}(0, h)$ and is independent of $\mathcal{F}_t$, we may write

$$X(t+h) = x \exp\big(\mu h + \sigma\, (W(t+h) - W(t))\big).$$

Denote $\Delta W := W(t+h) - W(t)$.
Step 1: Drift. We compute

$$\mathbb{E}[X(t+h) - X(t) \mid X(t) = x] = x \Big(\mathbb{E}\big[e^{\mu h + \sigma \Delta W}\big] - 1 \Big).$$

Using the moment generating function of a normal variable,

$$\mathbb{E}\big[e^{\sigma \Delta W}\big] = e^{\frac{1}{2}\sigma^2 h},$$

so

$$\mathbb{E}\big[e^{\mu h + \sigma \Delta W}\big] = e^{\mu h + \frac{1}{2}\sigma^2 h} = 1 + \big(\mu + \tfrac{1}{2}\sigma^2\big)h + o(h).$$

Hence

$$\mathbb{E}[X(t+h) - X(t) \mid X(t) = x] = x\Big(\big(\mu + \tfrac{1}{2}\sigma^2\big)h + o(h)\Big).$$

Comparing with (6.2) yields

$$\boxed{\mu(x) = \big(\mu + \tfrac{1}{2}\sigma^2\big)x.}$$

Step 2: Variance function. Write $\Delta X := X(t+h) - X(t) = x\big(e^{\mu h + \sigma \Delta W} - 1\big)$. Then

$$\mathrm{Var}[\Delta X \mid X(t) = x] = x^2\, \mathrm{Var}\big[e^{\mu h + \sigma \Delta W}\big].$$

Compute the second moment:

$$\mathbb{E}\big[e^{2(\mu h + \sigma \Delta W)}\big] = e^{2\mu h + 2\sigma^2 h} = 1 + (2\mu + 2\sigma^2)h + o(h).$$

Therefore

$$\mathrm{Var}\big[e^{\mu h + \sigma \Delta W}\big] = e^{2\mu h + 2\sigma^2 h} - e^{2\mu h + \sigma^2 h} = e^{2\mu h}\big(e^{2\sigma^2 h} - e^{\sigma^2 h}\big) = \sigma^2 h + o(h).$$

Consequently

$$\mathrm{Var}[X(t+h) - X(t) \mid X(t) = x] = x^2\big(\sigma^2 h + o(h)\big).$$

Matching with (6.3) we obtain

$$\boxed{\sigma^2(x) = \sigma^2 x^2.}$$

Conclusion. The geometric Brownian motion has state-dependent drift and variance function

$$\mu(x) = \big(\mu + \tfrac{1}{2}\sigma^2\big)x, \qquad \sigma^2(x) = \sigma^2 x^2, \qquad x > 0.$$

These satisfy the required $o(h)$ error terms in (6.2) and (6.3), completing the proof.

Problem 2 (15 points) Prove that the standard Gaussian density given by $p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ where $t \geq 0, x \in \mathbb{R}$ satisfies the "heat" equation:

$$\frac{\partial}{\partial t} p(t,x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p(t,x),$$

for all $t \geq 0, x \in \mathbb{R}$.

Solution. We verify directly that the standard Gaussian density

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}, \qquad t > 0, \ x \in \mathbb{R},$$

solves the one–dimensional heat equation

$$\frac{\partial}{\partial t} p(t,x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p(t,x).$$

1. Smoothness. For every fixed $t > 0$ the mapping $x \mapsto p(t,x)$ is the product of a smooth function and the rapidly decreasing Gaussian, hence is C^∞ in x. Likewise, for each fixed $x \in \mathbb{R}$ the mapping $t \mapsto p(t,x)$ is C^∞ on $(0,\infty)$ because it is a composition of smooth functions of t with powers and exponentials. Thus all derivatives appearing below exist and can be obtained by term–wise differentiation.

2. The time derivative. Write $p(t,x)$ as

$$p(t,x) = (2\pi t)^{-1/2} \exp\!\Big(-\frac{x^2}{2t}\Big).$$

Differentiating with respect to t yields

$$\frac{\partial}{\partial t} p(t,x) = -\frac{1}{2} (2\pi)^{-1/2} t^{-3/2} \exp\!\Big(-\frac{x^2}{2t}\Big) + (2\pi t)^{-1/2} \exp\!\Big(-\frac{x^2}{2t}\Big) \frac{x^2}{2t^2}.$$

Recognising the common factor $p(t,x)$ we obtain

$$\frac{\partial}{\partial t} p(t,x) = p(t,x)\Big(-\frac{1}{2t} + \frac{x^2}{2t^2}\Big).$$

3. The spatial derivatives. First derivative:

$$\frac{\partial}{\partial x} p(t,x) = (2\pi t)^{-1/2} \exp\!\Big(-\frac{x^2}{2t}\Big)\Big(-\frac{x}{t}\Big) = -\frac{x}{t}\, p(t,x).$$

Second derivative:

$$\frac{\partial^2}{\partial x^2} p(t,x) = -\frac{1}{t}\, p(t,x) + \frac{x^2}{t^2}\, p(t,x) = p(t,x)\Big(-\frac{1}{t} + \frac{x^2}{t^2}\Big).$$

Multiplying by 1/2 gives

$$\frac{1}{2} \frac{\partial^2}{\partial x^2} p(t,x) = p(t,x)\Big(-\frac{1}{2t} + \frac{x^2}{2t^2}\Big).$$

4. Equality of both sides. Comparing the expressions obtained in Steps 2 and 3 we see

$$\frac{\partial}{\partial t}p(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x),$$

for every $t > 0$ and $x \in \mathbb{R}$.

5. Behaviour at $t = 0$. Although $p(t,x)$ is singular at $t = 0$, for each fixed x we have $p(t,x) \to 0$ as $t \downarrow 0$, while the total mass $\int_{\mathbb{R}} p(t,x)\,dx = 1$ for every $t > 0$; hence $p(t,\cdot)$ converges in the sense of distributions to the Dirac delta at 0, the classical initial condition for the heat kernel.

Conclusion. The standard Gaussian density $p(t,x)$ is C^∞ on $(0,\infty) \times \mathbb{R}$ and satisfies the heat equation with the delta initial condition, completing the proof.

Problem 3 (20 points) In class we unfortunately had to skip section 6.7 on the quadratic variation of the standard Brownian motion. This section is quite important because it explains why one may expect the "rule" $(d(W(t))^2 = dt$. Read the section and solve Exercise (6.28).

(6.28) Let $X(t) = \mu t + \sigma W(t)$ be a (μ, σ^2)-Brownian motion. Show that, with probability 1, the quadratic variation of X on $[0,t]$ is $\sigma^2 t$.

Solution. Fix $t > 0$ and let $\left\{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k(n)}^{(n)} = t\right\}_{n \geq 1}$ be a sequence of deterministic partitions whose mesh

$$|P_n| = \max_{0 \leq i < k(n)} \left(t_{i+1}^{(n)} - t_i^{(n)}\right) \xrightarrow[n\to\infty]{} 0.$$

For a càdlàg process X the *quadratic variation* on $[0,t]$ along $\{P_n\}$ is

$$[X]_t := \lim_{n\to\infty} \sum_{i=0}^{k(n)-1} \left(X(t_{i+1}^{(n)}) - X(t_i^{(n)})\right)^2,$$

whenever the limit exists (in probability or almost surely) and is independent of the chosen sequence of partitions.

1. Decomposition of the increments. For $X(t) = \mu t + \sigma W(t)$ we have

$$\Delta_i^{(n)} X := X(t_{i+1}^{(n)}) - X(t_i^{(n)}) = \mu\,\Delta_i^{(n)} t + \sigma\,\Delta_i^{(n)} W,$$

where $\Delta_i^{(n)} t := t_{i+1}^{(n)} - t_i^{(n)}$ and $\Delta_i^{(n)} W := W(t_{i+1}^{(n)}) - W(t_i^{(n)}) \sim \mathcal{N}\left(0, \Delta_i^{(n)} t\right)$, independent for distinct i.

2. Expanding the quadratic sum. Write the partial quadratic variation

$$Q_n := \sum_i \left(\Delta_i^{(n)} X\right)^2 = \mu^2 \sum_i \left(\Delta_i^{(n)} t\right)^2 + 2\mu\sigma \sum_i \Delta_i^{(n)} t\,\Delta_i^{(n)} W + \sigma^2 \sum_i \left(\Delta_i^{(n)} W\right)^2.$$

We treat the three terms separately.

3. The deterministic term vanishes. Because $\sum_i \left(\Delta_i^{(n)} t\right) = t$,

$$\sum_i \left(\Delta_i^{(n)} t\right)^2 \leq |P_n| \sum_i \Delta_i^{(n)} t = |P_n|\,t \xrightarrow[n\to\infty]{} 0.$$

Hence $\mu^2 \sum_i (\Delta_i^{(n)} t)^2 \to 0$ deterministically.

4. The mixed term vanishes in L^2. Set $M_n := \sum_i \Delta_i^{(n)} t\,\Delta_i^{(n)} W$. Because $\mathbb{E}[\Delta_i^{(n)} W] = 0$ and the increments are independent,

$$\mathbb{E}[M_n] = 0, \qquad \mathbb{E}[M_n^2] = \sum_i \left(\Delta_i^{(n)} t\right)^2 \mathrm{Var}\left[\Delta_i^{(n)} W\right] = \sum_i \left(\Delta_i^{(n)} t\right)^3.$$

As before $\sum_i (\Delta_i^{(n)} t)^3 \leq |P_n| \sum_i (\Delta_i^{(n)} t)^2 \leq |P_n|^2 t \to 0$, so $\mathbb{E}[M_n^2] \to 0$. Thus $M_n \xrightarrow[n\to\infty]{} 0$ in L^2, hence also in probability. Consequently $2\mu\sigma M_n \to 0$ in probability.

5. Quadratic variation of Brownian motion. It is a classical fact (proved in Section 6.7) that

$$[W]_t = \lim_{n\to\infty} \sum_i \left(\Delta_i^{(n)} W\right)^2 = t \quad \text{almost surely.}$$

Multiplying by σ^2 yields

$$\sigma^2 \sum_i \big(\Delta_i^{(n)} W\big)^2 \xrightarrow[n\to\infty]{a.s.} \sigma^2 t.$$

6. Putting the pieces together. We have shown

$$Q_n = \underbrace{\mu^2 \sum_i (\Delta_i^{(n)} t)^2}_{\longrightarrow 0} + \underbrace{2\mu\sigma M_n}_{\xrightarrow{\mathbb{P}} 0} + \underbrace{\sigma^2 \sum_i (\Delta_i^{(n)} W)^2}_{\xrightarrow{a.s.} \sigma^2 t}.$$

Since the first term is deterministic and the second vanishes in probability, we may extract a subsequence (still indexed by n) such that $M_n \to 0$ almost surely; along this subsequence the entire sum converges almost surely to $\sigma^2 t$.

7. Conclusion. Hence, for any sequence of partitions whose mesh tends to 0,

$$[X]_t = \lim_{n\to\infty} \sum_i \big(X(t_{i+1}^{(n)}) - X(t_i^{(n)})\big)^2 = \sigma^2 t \quad \text{with probability 1.}$$

Therefore the (μ, σ^2)–Brownian motion $X(t) = \mu t + \sigma W(t)$ possesses quadratic variation

$$\boxed{[X]_t = \sigma^2 t \qquad \text{a.s.}}$$

for every $t \geq 0$, completing the proof.

Problem 4 (10 points) Prove that for all diffusions $X(t), t \geq 0$ with differentials $dX(t)$ and deterministic differentiable function $f(t), t \geq 0$ (hence, with differential $df(t) = f'(t)dt$) it holds

$$d(f(t)X(t)) = X(t)df(t) + f(t)dX(t).$$

Proof. Let $f : [0, \infty) \to \mathbb{R}$ be continuously differentiable, so

$$df(t) = f'(t)\, dt.$$

Let X be any *continuous diffusion* (hence a continuous semimartingale); in differential form one may write

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t),$$

where W is a standard Brownian motion and μ, σ are adapted processes.

1. Ito product rule. For two continuous semimartingales U, V,

$$d(UV) = U\, dV + V\, dU + d[U, V],$$

with $[U, V]$ the quadratic covariation.

2. Quadratic covariation term vanishes. A *deterministic finite–variation* process such as f satisfies $[f, X] \equiv 0$ because its increments are $O(dt)$, while those of X are $O(dt^{1/2})$; the product is $O(dt^{3/2})$, negligible compared with dt. **3. Apply (3) with** $U = X$, $V = f$**.** Using $[f, X] = 0$ and (1)–(2),

$$d(f(t)X(t)) = X(t)\, df(t) + f(t)\, dX(t).$$

Substituting $df(t) = f'(t)\, dt$ if desired gives

$$d(f(t)X(t)) = X(t)f'(t)\, dt + f(t)\big[\mu(t)\, dt + \sigma(t)\, dW(t)\big].$$

4. Interpretation. Equation (4) is a direct analogue of the classical product rule. The only extra term in Ito's general product formula—$d[f, X]$—drops out because f has no stochastic (quadratic-variation) component. Thus *whenever one factor is deterministic and C^1, the stochastic differential behaves exactly like the ordinary differential.*

$$\boxed{d\big(f(t)X(t)\big) = X(t)\, df(t) + f(t)\, dX(t)}$$

□

Problem 5 Let $W(t), t \geq 0$ be a standard Brownian motion. For any $N > 0$ consider the sum $Z_N = N^{-1} \sum_{k=0}^{N-1} W(k/N)$.

a) (10 points) Prove that Z_N is a Gaussian random variable and find its mean and variance.

Solution to Problem 5 a) Z_N **is Gaussian and computation of its mean and variance.** Because a standard Brownian motion $W(t)$ is a centred Gaussian process, every finite vector

$$\big(W(0), W(1/N), \ldots, W((N-1)/N)\big)$$

is jointly Gaussian. A linear combination of jointly Gaussian random variables is again Gaussian, and

$$Z_N = \frac{1}{N} \sum_{k=0}^{N-1} W(k/N)$$

is such a linear combination, hence Z_N itself is Gaussian.
Its mean is easily found: for all k one has $\mathbb{E}[W(k/N)] = 0$, so

$$\mathbb{E}[Z_N] = \frac{1}{N} \sum_{k=0}^{N-1} 0 = 0.$$

To compute the variance, write

$$\mathrm{Var}(Z_N) = \mathbb{E}[Z_N^2] = \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \mathbb{E}\big[W(i/N)\, W(j/N)\big].$$

For Brownian motion $\mathbb{E}[W(s)W(t)] = \min\{s, t\}$, so here

$$\mathbb{E}[W(i/N)W(j/N)] = \frac{\min\{i, j\}}{N}.$$

Thus

$$\mathrm{Var}(Z_N) = \frac{1}{N^3} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \min\{i, j\}.$$

A classical double–sum evaluation gives

$$\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \min\{i, j\} = \frac{(N-1)N(2N-1)}{6},$$

so

$$\mathrm{Var}(Z_N) = \frac{(N-1)(2N-1)}{6N^2}.$$

Hence

$$Z_N \sim \mathcal{N}\Big(0, \frac{(N-1)(2N-1)}{6N^2}\Big).$$

Notice that as $N \to \infty$, this variance converges to $1/3$.

b) (10 points) Recall that for any continuous function $f : \mathbb{R} \to \mathbb{R}$, the Riemann sums it holds

$$\lim_{N \to \infty} N^{-1} \sum_{k=0}^{N-1} f(k/N) = \int_0^1 f(s)ds.$$

Prove that $\int_0^1 W(t)dt$ follows a normal distribution and find its mean and variance.

(Hint: You may use without proof that if a sequence of normal random variables $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2), n \in \mathbb{N}$ converges almost surely to a random variable X then $X \sim \mathcal{N}(\lim_n \mu_n, \lim_n \sigma_n^2)$.)

b) The random variable $\int_0^1 W(t)\,dt$ is normal with mean 0 and variance $1/3$.

Step 1 – Pathwise Riemann–sum approximation. Brownian paths are almost surely continuous. Define the (left-endpoint) Riemann sums

$$R_N := \frac{1}{N} \sum_{k=0}^{N-1} W(k/N) = Z_N \qquad (N \geq 1).$$

Because $W(\cdot)$ is continuous on $[0,1]$ with probability one, the Riemann sums of W with mesh $1/N$ converge almost surely to the Riemann integral:

$$\lim_{N\to\infty} R_N = \int_0^1 W(t)\,dt \quad \text{a.s.}$$

Step 2 – Identification of the limiting distribution. For every N, part (a) shows that $R_N = Z_N$ is Gaussian with mean 0 and variance $(N-1)(2N-1)/(6N^2)$. Therefore we have a sequence of centred normal random variables

$$R_N \sim \mathcal{N}\Big(0, \frac{(N-1)(2N-1)}{6N^2}\Big), \qquad N \geq 1,$$

converging almost surely to $X := \int_0^1 W(t)\,dt$. The variance term satisfies

$$\lim_{N\to\infty} \frac{(N-1)(2N-1)}{6N^2} = \frac{1}{3}.$$

By the stated hint, an almost–sure limit of normal variables remains normal, with mean and variance being the limits of the corresponding sequences. Consequently

$$\int_0^1 W(t)\,dt \ \sim \ \mathcal{N}\Big(0, \frac{1}{3}\Big).$$

Conclusion. We have proved that the integral of standard Brownian motion over $[0,1]$ is itself Gaussian, centred, with variance $1/3$, completing the solution of Problem 5.

Problem 6 Consider the OU process $X(t), t \geq 0$ satisfying $dX(t) = -X(t)dt + \sqrt{2}dW(t), X(0) = 0$.

a) (15 points) Prove that

$$d(e^t X(t)) = \sqrt{2}e^t dW(t).$$

In integral notation, this means for any $t \geq 0$,

$$X(t) = \sqrt{2}e^{-t} \int_0^t e^s dW(s).$$

Prove rigorously that for the Ornstein–Uhlenbeck (OU) process

$$dX(t) = -X(t)\,dt + \sqrt{2}\,dW(t), \qquad X(0) = 0,$$

one has

$$d\big(e^t X(t)\big) = \sqrt{2}\,e^t\,dW(t)$$

and therefore

$$X(t) = \sqrt{2}\,e^{-t} \int_0^t e^s\,dW(s) \qquad (t \geq 0).$$

Proof. Throughout we work on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ that satisfies the usual hypotheses and carries a standard $(\mathcal{F}_t)$-Brownian motion W. Because both drift and diffusion coefficients of the SDE are globally Lipschitz and of linear growth, there exists a *unique* strong, continuous, adapted solution X (e.g. Theorem 5.2.1 in Karatzas & Shreve). **Step 1: Applying Itô's formula to the product $e^t X(t)$.** Denote by $Y(t) := e^t X(t)$. We have $Y(0) = X(0) = 0$. Since $t \mapsto e^t$ is C^1, Itô's product rule (which is a special case of Itô's formula taking $f(t,x) = e^t x$) yields

$$dY(t) = e^t X(t)\,dt + e^t\,dX(t) + d\big[e^t, X\big]_t.$$

Because e^t is of finite variation, its quadratic covariation with the semimartingale X vanishes:

$$d[e^t, X]_t = 0.$$

Substituting $dX(t) = -X(t)\, dt + \sqrt{2}\, dW(t)$ we obtain

$$dY(t) = e^t X(t)\, dt + e^t\big[-X(t)\, dt + \sqrt{2}\, dW(t)\big] = \sqrt{2}\, e^t\, dW(t).$$

Step 2: Integrating the differential identity. Integrate from 0 to $t\,(> 0)$ and use $Y(0) = 0$:

$$Y(t) - Y(0) = \sqrt{2}\int_0^t e^s\, dW(s) \implies e^t X(t) = \sqrt{2}\int_0^t e^s\, dW(s).$$

Dividing by e^t gives the explicit representation

$$X(t) = \sqrt{2}\, e^{-t}\int_0^t e^s\, dW(s).$$

Step 3: Verifying that the representation indeed solves the SDE. Define

$$\widetilde{X}(t) := \sqrt{2}\, e^{-t}\int_0^t e^s\, dW(s), \qquad t \geq 0.$$

(i) Adaptedness and continuity. The stochastic integral is $(\mathcal{F}_t)$-adapted with continuous paths, hence so is $\widetilde{X}$.

(ii) Initial condition. Using Itô isometry,

$$\widetilde{X}(0) = \sqrt{2}\int_0^0 e^s\, dW(s) = 0.$$

(iii) Dynamics. Applying Itô's formula once more, now to $e^t \widetilde{X}(t)$, reproduces

$$d\big(e^t \widetilde{X}(t)\big) = \sqrt{2}\, e^t\, dW(t),$$

which rearranges to

$$d\widetilde{X}(t) = -\widetilde{X}(t)\, dt + \sqrt{2}\, dW(t).$$

(iv) Uniqueness. The linear SDE admits a unique strong solution, so $\widetilde{X} \equiv X$ almost surely. **Conclusion.** The integrating-factor calculation is thus fully justified:

$$d\big(e^t X(t)\big) = \sqrt{2}\, e^t\, dW(t), \qquad X(t) = \sqrt{2}\, e^{-t}\int_0^t e^s\, dW(s).$$

b) (10 points) For any $N > 0$ consider the sum

$$Z_N = \sum_{k=0}^{N-1} e^{k/N}\big(W(((k+1)/N)) - W(tk/N)\big).$$

Prove that Z_N is Gaussian and find its mean and a formula for its variance. Where does the mean and variance converge as $N \to +\infty$?

(b) The random sum Z_N Define

$$Z_N := \sum_{k=0}^{N-1} e^{k/N}\Big(W\big((k+1)/N\big) - W(k/N)\Big), \qquad N \geq 1.$$

Gaussianity. The Brownian increments

$$\Delta W_k := W\big((k+1)/N\big) - W(k/N), \qquad k = 0, \dots, N-1,$$

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are independent centred Gaussian variables with variance $1/N$. Because Z_N is a finite linear combination of jointly Gaussian variables, Z_N itself is Gaussian. *Mean.* For each k, $\mathbb{E}[\Delta W_k] = 0$, so

$$\boxed{\mathbb{E}[Z_N] = 0.}$$

Variance. Independence of the increments yields

$$\mathrm{Var}(Z_N) = \sum_{k=0}^{N-1} e^{2k/N} \mathrm{Var}(\Delta W_k) = \frac{1}{N} \sum_{k=0}^{N-1} e^{2k/N}.$$

Hence

$$\boxed{\mathrm{Var}(Z_N) = \frac{1}{N} \sum_{k=0}^{N-1} e^{2k/N}.}$$

Limit as $N \to \infty$. The variance is a Riemann sum:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2k/N} = \int_0^1 e^{2s}\, ds = \frac{e^2 - 1}{2}.$$

Thus

$$Z_N \xrightarrow[N \to \infty]{\mathrm{d}} \mathcal{N}\!\left(0, \frac{e^2 - 1}{2}\right).$$

c) (Bonus, 5 points) Based on the intuition from Problem 5 (b), can you guess the distribution of $\int_0^t e^{-s} dW(s)$? What about the distribution of $X(t)$ as $t \to +\infty$?

(c) Distribution of $\displaystyle\int_0^t e^{-s}\, dW(s)$ **and of** $X(t)$ **as** $t \to \infty$

Because the stochastic integral of a deterministic function against Brownian motion is Gaussian with mean 0 and variance equal to the L^2-norm of that function, we have

$$\int_0^t e^{-s}\, dW(s) \sim \mathcal{N}\!\left(0, \int_0^t e^{-2s}\, ds\right) = \mathcal{N}\!\left(0, \tfrac{1}{2}(1 - e^{-2t})\right).$$

Using the representation from part (a),

$$X(t) = \sqrt{2}\, e^{-t} \int_0^t e^s\, dW(s) \sim \mathcal{N}\!\left(0, 2e^{-2t} \int_0^t e^{2s}\, ds\right) = \mathcal{N}\!\left(0, 1 - e^{-2t}\right).$$

Hence, as $t \to \infty$, the variance approaches 1 and

$$\boxed{X(t) \xrightarrow[t \to \infty]{\mathrm{d}} \mathcal{N}(0, 1).}$$

This limiting distribution $N(0, 1)$ is precisely the stationary (invariant) law of the Ornstein–Uhlenbeck process.