S&DS 351: Stochastic Processes - Homework 3

Bryan SebaRaj

Professor Ilias Zadik

February 7, 2025

Problem 1

(10 points) Is it possible for a transient state to be periodic? If so, construct an example of such a Markov chain; otherwise, give a mathematical proof why not.

Note: I (fortunately) solved this after proving problem 3, so for a more thorough proof on how this example is transient, please see Problem 3.

Yes, it is possible for a transient state to be periodic. Consider a 1-dimensional asymmetric random walk on $\mathbb{Z}$:

$$X_n = X_{n-1} + Z_n, \quad \text{where} \quad \mathbb{P}(Z_n = +1) = p \quad \text{and} \quad \mathbb{P}(Z_n = -1) = 1 - p,$$

for some $p \in (0, 1)$ with $p \neq \frac{1}{2}$. Starting at state 0, state 0 is transient (see Problem 3).

Define the period as $d_i = \gcd\{n : P^n(i, i) > 0\}$, where P is the transition matrix.
In the random walk, the walk must trivially take as many $+1$ steps as -1 steps to reach the initial state. Thus one can only return to state x starting from x in an even number of steps. Note that this holds for all integers. Hence for each integer x,

$$(P^n)(x, x) > 0 \implies n \text{ is even.}$$

$$(P^n)(x, x) = 0 \implies n \text{ is odd.}$$

Therefore, the greatest common divisor of all such n is 2, and every state $x \in \mathbb{Z}$ has period 2.

Problem 2

Let $X_0, X_1, \dots$ be a Markov chain with transition matrix P. Let $k \geq 1$ be an integer.

1. (5 points) Prove that $Y_n = X_{kn}$ is also a Markov chain. Find its transition matrix.

   By the definition $Y_n = X_{kn}$,

   $$P\big(Y_{n+1} = j \,\big|\, Y_0 = i_0, \dots, Y_n = i_n\big) = P\big(X_{k(n+1)} = j \,\big|\, X_0 = i_0, X_k = i_1, \dots, X_{kn} = i_n\big).$$

   Since $\{X_n\}$ is a Markov chain, it satisfies the Markov property:

   $$P\big(X_{m+1} = x_{m+1} \,\big|\, X_0 = x_0, \dots, X_m = x_m\big) = P\big(X_{m+1} = x_{m+1} \,\big|\, X_m = x_m\big).$$

   Applying this property step-by-step for the transitions from time kn to time $k(n+1)$ (i.e., k steps), we obtain:

   $$P\big(X_{k(n+1)} = j \,\big|\, X_{kn} = i_n\big) = \big(P^k\big)(i_n, j).$$

   Because all intermediate states between X_{kn} and $X_{k(n+1)}$ do not affect this probability beyond the conditioning on X_{kn}, we conclude:

   $$P\big(X_{k(n+1)} = j \,\big|\, X_0 = i_0, \dots, X_{kn} = i_n\big) = P\big(X_{k(n+1)} = j \,\big|\, X_{kn} = i_n\big) = \big(P^k\big)(i_n, j).$$

Hence,

$$P\big(Y_{n+1} = j \,\big|\, Y_0 = i_0, \ldots, Y_n = i_n\big) = P\big(Y_{n+1} = j \,\big|\, Y_n = i_n\big),$$

proving that $\{Y_n\}$ is a Markov chain.

Finally, by the computation above, the one-step transition probability for Y_n from state i_n to state j is exactly $\big(P^k\big)(i_n, j)$. Therefore, the transition matrix of $\{Y_n\}$ is P^k.

2. (10 points) Suppose that the original chain $\{X_n\}$ is irreducible. Is $\{Y_n\}$ irreducible? If so, prove it; if not, provide a counterexample.

3. (10 points) Suppose that the original chain $\{X_n\}$ is aperiodic. Is $\{Y_n\}$ aperiodic? If so, prove it; if not, provide a counterexample.

4. (10 points) Suppose that the original chain $\{X_n\}$ is transient. Is $\{Y_n\}$ transient? If so, prove it; if not, provide a counterexample.

5. (15 points) Suppose that the original chain $\{X_n\}$ is recurrent. Is $\{Y_n\}$ recurrent? If so, prove it; if not, provide a counterexample.

6. (5 points) Suppose that the original chain X_n is irreducible and that it has period d. What is the period of each state i in the new Markov chain Y_n for $k = d$?

Problem 3

(Asymmetric random walk, 15 points) Consider the *asymmetric* random walk on $\mathbb{Z}$, that is, $X_n = X_{n-1} + Z_n$, where $Z_1, Z_2, \ldots$ are iid and $\mathbb{P}(Z_n = +1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$, with $p \in [0, 1]$ and $p \neq \frac{1}{2}$. Show that the state 0 is a transient state.

In Lecture 7 we saw/will see that when $p = \frac{1}{2}$ this is not true anymore and the state 0 is recurrent. Can you explain intuitively why this is the case?

Hint: You may want to use Stirling's formula that $\lim_{n\to\infty} \frac{n!}{(n/e)^n \sqrt{2\pi n}} = 1$.

Starting from $X_0 = 0$, the random walk is at state 0 again at $t = n$ only when it has taken an equal number of $+1$ steps as -1 steps. As such, n must be even.

Suppose $n = 2k$, and k is the number of Z_i that are $+1$,

$$\mathbb{P}(X_{2k} = 0 \,|\, X_0 = 0) \;=\; \binom{2k}{k}\, p^k\, (1-p)^k$$

$$\text{Note that } \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 0 \;\text{ if } n \text{ is odd}$$

Hence the series of return probabilities at 0 is

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 1 \;+\; \sum_{k=1}^{\infty} \binom{2k}{k}\, p^k\, (1-p)^k,$$

accounting for the initial state of 0.
Using Stirling's approximation,

$$n! \;\sim\; \left(\tfrac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as } n \to \infty,$$

applying to this case,

$$\binom{2k}{k} \;=\; \frac{(2k)!}{k!\,k!} \;\approx\; \frac{\sqrt{4\pi k}\,\left(\frac{2k}{e}\right)^{2k}}{2\pi k\,\left(\frac{k}{e}\right)^k \left(\frac{k}{e}\right)^k} \;=\; \frac{4^k}{\sqrt{\pi k}}$$

Therefore,

$$\binom{2k}{k} p^k (1-p)^k \approx \frac{4^k}{\sqrt{\pi k}} \big[p(1-p)\big]^k = \frac{\big[4\,p(1-p)\big]^k}{\sqrt{\pi k}}.$$

If $p \neq \frac{1}{2}$, then $4\,p(1-p) < 1$ (If $f(x) = x(1-x)$, then $f'(x) = -x + 1 - x = -2x + 1$. Solving for the max when $f'(x) = 0, x = \frac{1}{2}$).

Note, that as $k \to \infty$, $\big[4\,p(1-p)\big]^k$ decays exponentially. Therefore,

$$\binom{2k}{k} p^k (1-p)^k = O\Big(\big[4\,p(1-p)\big]^k\Big) \quad \text{and} \quad \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k < \infty.$$

Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0) = 1 + \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k < \infty.$$

which defines a transient state.

However, when $p = \frac{1}{2}$,

$$\binom{2k}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k \approx \frac{\big[4 \cdot 0.5(1-0.5)\big]^k}{\sqrt{\pi k}} = \frac{1}{\sqrt{\pi k}},$$

so

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty.$$

which defines a recurrent state when $p = \frac{1}{2}$.

Exercise 1.8

Consider a Markov chain on the integers with

$$P(i, i+1) = 0.4 \text{ and } P(i, i-1) = 0.6 \text{ for } i > 0,$$

$$P(i, i+1) = 0.6 \text{ and } P(i, i-1) = 0.4 \text{ for } i < 0,$$

$$P(0, 1) = P(0, -1) = \frac{1}{2}.$$

This is a chain with infinitely many states, but it has a sort of probabilistic "restoring force" that always pushes back toward 0. Find the stationary distribution.

Denote the stationary distribution by $\{\pi_i\}_{i \in \mathbb{Z}}$

$$\sum_{j \in \mathbb{Z}} \pi_j P(j, i) = \pi_i \quad \text{for all } i \in \mathbb{Z},$$

and $\sum_{i \in \mathbb{Z}} \pi_i = 1$. As this is a two-sided birth-death chain,

$$\pi_i \, P(i, i+1) = \pi_{i+1} \, P(i+1, i).$$

For $i \geq 1$:

$$\pi_i \times 0.4 = \pi_{i+1} \times 0.6 \implies \frac{\pi_{i+1}}{\pi_i} = \frac{0.4}{0.6} = \frac{2}{3}.$$

For $i \leq -1$:

$$\pi_{i-1} \times 0.6 = \pi_i \times 0.4 \implies \frac{\pi_{i-1}}{\pi_i} = \frac{0.4}{0.6} = \frac{2}{3}.$$

When $i = 0$,

$$\pi_0 \times 0.5 = \pi_1 \times 0.6 \implies \frac{\pi_1}{\pi_0} = \frac{0.5}{0.6} = \frac{5}{6}.$$

$$\pi_{-1} \times 0.6 = \pi_0 \times 0.5 \quad \Longrightarrow \quad \frac{\pi_{-1}}{\pi_0} = \frac{0.5}{0.6} = \frac{5}{6},$$

Hence,

$$\pi_1 = \frac{5}{6}\,\pi_0, \quad \pi_{-1} = \frac{5}{6}\,\pi_0.$$

As the probabilities of a jump remain the same, generalizing for all $i \geq 1$,

$$\pi_{i+1} = \frac{2}{3}\pi_i \quad \Longrightarrow \quad \pi_i = \left(\frac{2}{3}\right)^{i-1} \pi_1 \quad \text{for } i \geq 1.$$

$$\pi_i \;=\; \left(\frac{2}{3}\right)^{i-1} \cdot \frac{5}{6}\,\pi_0 \quad \forall i \geq 1.$$

Generalizing for all $i \leq -1$,

$$\pi_{i-1} = \frac{2}{3}\pi_i \quad \Longrightarrow \quad \pi_i = \frac{2}{3}^{-i-1} \pi_{-1} \quad \forall i \leq -1,$$

$$\pi_i \;=\; \left(\frac{2}{3}\right)^{-i-1} \cdot \frac{5}{6}\,\pi_0 \quad \forall i \leq -1.$$

Combining the two cases,

$$\pi_i \;=\; \frac{5}{6}\left(\frac{2}{3}\right)^{|i|-1} \pi_0, \quad \forall i \neq 0.$$

Solving for π_0, first recall that,

$$\sum_{i=-\infty}^{\infty} \pi_i \;=\; 1.$$

Hence,

$$\pi_0 \;+\; \sum_{i\neq 0} \frac{5}{6}\left(\tfrac{2}{3}\right)^{|i|-1} \pi_0 \;=\; 1.$$

$$\pi_0\Big[1 \;+\; \frac{5}{6}\sum_{i\neq 0}\left(\tfrac{2}{3}\right)^{|i|-1}\Big] \;=\; 1.$$

Exploiting the symmetry of the chain,

$$\sum_{i\neq 0}\left(\tfrac{2}{3}\right)^{|i|-1} \;=\; 2\sum_{j=1}^{\infty}\left(\tfrac{2}{3}\right)^{j-1} \;=\; 2\sum_{k=0}^{\infty}\left(\tfrac{2}{3}\right)^{k} \;=\; 2\cdot\frac{1}{1-\frac{2}{3}} \;=\; 2\cdot 3 \;=\; 6.$$

Thus

$$\pi_0\Big[1 + \frac{5}{6}\cdot 6\Big] \;=\; 6\,\pi_0 \;=\; 1 \quad \Longrightarrow \quad \pi_0 \;=\; \frac{1}{6}.$$

Therefore,

$$\pi_i = \begin{cases} \dfrac{1}{6}, & i = 0 \\ \dfrac{5}{36}\left(\tfrac{2}{3}\right)^{|i|-1} & i \neq 0 \end{cases}$$

Exercise 1.16

Show that if an irreducible Markov chain has a state i such that $P(i,i) > 0$, then the chain is aperiodic. Also show by example that this sufficient condition is not necessary.

Let $\{X_n\}$ be an irreducible Markov chain on a countable state space S. Suppose there is a state $i \in S$ such that

$$P(i,i) > 0.$$

The period of state i is defined as

$$d(i) \;=\; \gcd\{n \geq 1 : P^n(i,i) > 0\},$$

Since $P(i,i) = P^1(i,i) > 0$, there is a positive probability of returning to i in exactly 1 step. As such, $1 \in \{n : P^n(i,i) > 0\}$, so any common divisor of all n must trivially divide 1.

Therefore,

$$d(i) \;=\; \gcd\{n \geq 1 : P^n(i,i) > 0\} \;=\; 1.$$

Thus, i is an aperiodic state.

By irreducibility, for any other state $j \in S$, $\exists m, k \in \mathbb{Z}$ such that $m, k \geq 1$, $P^m(j,i) > 0$, and $P^k(i,j) > 0$. Therefore as proven in lecture, for any $n \geq 1$,

$$P^{m+n+k}(j,j) \;\geq\; P^m(j,i)\, P^n(i,i)\, P^k(i,j).$$

Since $P^n(i,i) > 0$ for all $n \geq 1$, for arbitrarily many n, the probability $P^{m+n+k}(j,j)$ is strictly positive. As such,

$$d(j) \;=\; \gcd\{n \geq 1 : P^n(j,j) > 0\} \;=\; 1.$$

Thus, every state in an irreducible chain with a self-loop, i.e. $P(i,i) > 0$, is aperiodic.

To show that this sufficient condition is not necessary, consider a chain on three states $\{1,2,3\}$ with transition matrix:

$$\begin{aligned} &P(1,1) = 0, \quad P(1,2) = 1, \quad P(1,3) = 0, \\ &P(2,1) = \tfrac{1}{2}, \quad P(2,2) = 0, \quad P(2,3) = \tfrac{1}{2}, \\ &P(3,1) = \tfrac{1}{2}, \quad P(3,2) = \tfrac{1}{2}, \quad P(3,3) = 0. \end{aligned}$$

Note that the chain is irreducible, but does not contain any self-loops, i.e. $P(i,i) = 0$ for $i = 1,2,3$.

$$\begin{aligned} P^2(1,1) &= P(1,2)P(2,1) = (1)\left(\tfrac{1}{2}\right) = \tfrac{1}{2} > 0, \\ P^3(1,1) &= P(1,2)P(2,3)P(3,1) = 1 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} > 0. \end{aligned}$$

Therefore, state 1 can return to itself in both 2 steps and 3 steps, which comprises a subset of the set of all possible return times back to state 1. The gcd of 2 and 3 is 1, or a period of 1. Note that $P^2(2,2) > 0$, $P^2(3,3) > 0$, and $P^3(2,2) > 0$, $P^3(3,3) > 0$, so every state also has period 1, making the entire chain aperiodic. Thus, $P(i,i) > 0$ for some i is not a necessary condition for a chain to be aperiodic.