S&DS 351: Stochastic Processes - Homework 3

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Problem 1

(10 points) Is it possible for a transient state to be periodic? If so, construct an example of such a Markov chain; otherwise, give a mathematical proof why not.

Note: I (fortunately) solved this after proving problem 3, so for a more thorough proof on how this example is transient, please see Problem 3.

Yes, it is possible for a transient state to be periodic. Consider a 1-dimensional asymmetric random walk on $\mathbb{Z}$:

$$X_n = X_{n-1} + Z_n, \quad \text{where} \quad \mathbb{P}(Z_n = +1) = p \quad \text{and} \quad \mathbb{P}(Z_n = -1) = 1 - p,$$

for some $p \in (0, 1)$ with $p \neq \frac{1}{2}$. Starting at state 0, state 0 is transient (see Problem 3).

Define the period as $d_i = \gcd\{n : P^n(i, i) > 0\}$, where P is the transition matrix.
In the random walk, the walk must trivially take as many $+1$ steps as -1 steps to reach the initial state. Thus one can only return to state x starting from x in an even number of steps. Hence for each integer x,

$$(P^n)(x, x) > 0 \implies n \text{ is even.}$$
$$(P^n)(x, x) = 0 \implies n \text{ is odd.}$$

Therefore, the greatest common divisor of all such n is 2, and every state $x \in \mathbb{Z}$ has period 2.

Problem 2

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix P. Let $k \geq 1$ be an integer.

1. (5 points) Prove that $Y_n = X_{kn}$ is also a Markov chain. Find its transition matrix.

2. (10 points) Suppose that the original chain $\{X_n\}$ is irreducible. Is $\{Y_n\}$ irreducible? If so, prove it; if not, provide a counterexample.

3. (10 points) Suppose that the original chain $\{X_n\}$ is aperiodic. Is $\{Y_n\}$ aperiodic? If so, prove it; if not, provide a counterexample.

4. (10 points) Suppose that the original chain $\{X_n\}$ is transient. Is $\{Y_n\}$ transient? If so, prove it; if not, provide a counterexample.

5. (15 points) Suppose that the original chain $\{X_n\}$ is recurrent. Is $\{Y_n\}$ recurrent? If so, prove it; if not, provide a counterexample.

6. (5 points) Suppose that the original chain X_n is irreducible and that it has period d. What is the period of each state i in the new Markov chain Y_n for $k = d$?

Problem 3

(Asymmetric random walk, 15 points) Consider the *asymmetric* random walk on $\mathbb{Z}$, that is, $X_n = X_{n-1} + Z_n$, where $Z_1, Z_2, \ldots$ are iid and $\mathbb{P}(Z_n = +1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$, with $p \in [0, 1]$ and $p \neq \frac{1}{2}$. Show that the state 0 is a transient state.

In Lecture 7 we saw/will see that when $p = \frac{1}{2}$ this is not true anymore and the state 0 is recurrent. Can you explain intuitively why this is the case?

Hint: You may want to use Stirling's formula that $\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{2\pi n}} = 1$.

Starting from $X_0 = 0$, the random walk is at state 0 again at $t = n$ only when it has taken an equal number of $+1$ steps as -1 steps. As such, n must be even.

Suppose $n = 2k$, and k is the number of Z_i that are $+1$,

$$\mathbb{P}(X_{2k} = 0 \,|\, X_0 = 0) \;=\; \binom{2k}{k} p^k (1-p)^k$$

$$\text{Note that } \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 0 \;\text{ if } n \text{ is odd}$$

Hence the series of return probabilities at 0 is

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 1 \,+\, \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k,$$

accounting for the initial state of 0.
Using Stirling's approximation,

$$n! \;\sim\; \left(\tfrac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as } n \to \infty,$$

applying to this case,

$$\binom{2k}{k} \;=\; \frac{(2k)!}{k!\,k!} \;\approx\; \frac{\sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k}}{2\pi k\left(\frac{k}{e}\right)^k\left(\frac{k}{e}\right)^k} \;=\; \frac{4^k}{\sqrt{\pi k}}$$

Therefore,

$$\binom{2k}{k} p^k (1-p)^k \;\approx\; \frac{4^k}{\sqrt{\pi k}}\,\big[p(1-p)\big]^k \;=\; \frac{\big[4\,p(1-p)\big]^k}{\sqrt{\pi k}}.$$

If $p \neq \frac{1}{2}$, then $4\,p(1-p) < 1$ (If $f(x) = x(1-x)$, then $f'(x) = -x + 1 - x = -2x + 1$. Solving for the max when $f'(x) = 0, x = \frac{1}{2}$).

Note, that as $k \to \infty$, $\big[4\,p(1-p)\big]^k$ decays exponentially. Therefore,

$$\binom{2k}{k} p^k (1-p)^k \;=\; O\Big(\big[4\,p(1-p)\big]^k\Big) \quad \text{and} \quad \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k \;<\; \infty.$$

Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 1 \,+\, \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k \;<\; \infty.$$

which defines a transient state.

However, when $p = \frac{1}{2}$,

$$\binom{2k}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^k \;\approx\; \frac{\big[4\cdot 0.5(1-0.5)\big]^k}{\sqrt{\pi k}} \;=\; \frac{1}{\sqrt{\pi k}},$$

so

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty.$$

which defines a recurrent state when $p = \frac{1}{2}$.

Exercise 1.8

Consider a Markov chain on the integers with

$$P(i, i+1) = 0.4 \text{ and } P(i, i-1) = 0.6 \text{ for } i > 0,$$

$$P(i, i+1) = 0.6 \text{ and } P(i, i-1) = 0.4 \text{ for } i < 0,$$

$$P(0, 1) = P(0, -1) = \frac{1}{2}.$$

This is a chain with infinitely many states, but it has a sort of probabilistic "restoring force" that always pushes back toward 0. Find the stationary distribution.

Exercise 1.10

On page **13** we argued that a limiting distribution must be stationary. This argument was clear in the case of a finite state space. For you fans of mathematical analysis, what happens in the case of a countably infinite state space? Can you still make the limiting argument work?