S&DS 351: Stochastic Processes - Homework 9

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Problem 1 (10 points) Calculate using the definitions (6.2) and (6.3) in Chang's notes, the drift and variance function of the geometric Brownian motion $X(t) = \exp(\mu t + \sigma W(t))$, $t \geq 0$ where $W(t)$ is a standard Brownian motion, $\mu \in \mathbb{R}$ and $\sigma > 0$.

$$(6.2) \quad \mathbb{E}[X(t+h) - X(t) \mid X(t) = x] = \mu(x)h + o(h)$$

$$(6.3) \quad \mathrm{Var}[X(t+h) - X(t) \mid X(t) = x] = \sigma^2(x)h + o(h) \text{ as } h \downarrow 0.$$

Recall the definitions (6.2) and (6.3) above. Let $W(t)$ be a standard Brownian motion, $\mu \in \mathbb{R}$ and $\sigma > 0$, and the process be the geometric Brownian motion

$$X(t) = \exp\big(\mu t + \sigma W(t)\big), \quad \text{s.t. } t \geq 0$$

Consider the scenario where we fix $t \geq 0$ and condition on $X(t) = x$. Since $W(t+h) - W(t) \sim \mathcal{N}(0, h)$ and is independent of $\mathcal{F}_t$,

$$X(t+h) = x \exp\big(\mu h + \sigma\left(W(t+h) - W(t)\right)\big).$$

Define $\Delta W = W(t+h) - W(t)$. In order to compute drift, we can first compute

$$\mathbb{E}[X(t+h) - X(t) \mid X(t) = x] = x\Big(\mathbb{E}\big[e^{\mu h + \sigma \Delta W}\big] - 1\Big)$$

Using the moment generating function of a normal variable,

$$\mathbb{E}\big[e^{\sigma \Delta W}\big] = e^{\frac{1}{2}\sigma^2 h}$$

As such,

$$\mathbb{E}\big[e^{\mu h + \sigma \Delta W}\big] = e^{\mu h + \frac{1}{2}\sigma^2 h} = 1 + \big(\mu + \tfrac{1}{2}\sigma^2\big)h + o(h)$$

Therefore,

$$\mathbb{E}[X(t+h) - X(t) \mid X(t) = x] = x\Big(\big(\mu + \tfrac{1}{2}\sigma^2\big)h + o(h)\Big)$$

Using (6.2) yeilds the drift

$$\mu(x) = \big(\mu + \tfrac{1}{2}\sigma^2\big)x$$

Calculating the variance function, we can write $\Delta X = X(t+h) - X(t) = x\big(e^{\mu h + \sigma \Delta W} - 1\big)$. Then

$$\mathrm{Var}[\Delta X \mid X(t) = x] = x^2\, \mathrm{Var}\big[e^{\mu h + \sigma \Delta W}\big]$$

Computing the second moment yields

$$\mathbb{E}\big[e^{2(\mu h + \sigma \Delta W)}\big] = e^{2\mu h + 2\sigma^2 h} = 1 + (2\mu + 2\sigma^2)h + o(h)$$

Therefore,

$$\mathrm{Var}\big[e^{\mu h + \sigma \Delta W}\big] = e^{2\mu h + 2\sigma^2 h} - e^{2\mu h + \sigma^2 h} = e^{2\mu h}\big(e^{2\sigma^2 h} - e^{\sigma^2 h}\big) = \sigma^2 h + o(h)$$

As such,

$$\mathrm{Var}[X(t+h) - X(t) \mid X(t) = x] = x^2\big(\sigma^2 h + o(h)\big)$$

Using (6.3) yields the variance function,

$$\sigma^2(x) = \sigma^2 x^2$$

Problem 2 (15 points) Prove that the standard Gaussian density given by $p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ where $t \geq 0, x \in \mathbb{R}$ satisfies the "heat" equation:

$$\frac{\partial}{\partial t} p(t,x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p(t,x),$$

for all $t \geq 0, x \in \mathbb{R}$.

When proving that the standard Gaussian density satisfies the "heat" equation, we can need to validate its smoothness, time derivative, spatial derivatives, equality, and behavior at $t = 0$.
First, confirming smoothness, see that for every fixed $t > 0$, $x \mapsto p(t,x)$ is the product of a smooth function and the rapidly decreasing Gaussian. As such, it is C^∞ in x. Likewise, for every fixed $x \in \mathbb{R}$, $t \mapsto p(t,x)$ is C^∞ on $(0,\infty)$ since it is a composition of smooth functions of t with powers and exponentials. Therefore, all derivatives appearing below exist and can be obtained by term–wise differentiation.
Next, solving for the time derivative, see that

$$p(t,x) = \big(2\pi t\big)^{-1/2} \exp\Big(-\frac{x^2}{2t}\Big)$$

Differentiating with respect to t,

$$\frac{\partial}{\partial t} p(t,x) = -\frac{1}{2}\,(2\pi)^{-1/2} t^{-3/2} \exp\Big(-\frac{x^2}{2t}\Big) + \big(2\pi t\big)^{-1/2} \exp\Big(-\frac{x^2}{2t}\Big)\frac{x^2}{2t^2}$$

From the standard Gaussian density, see that

$$\frac{\partial}{\partial t} p(t,x) = p(t,x)\Big(-\frac{1}{2t} + \frac{x^2}{2t^2}\Big)$$

Now solving for the spatial derivatives, see that the first derivative is

$$\frac{\partial}{\partial x} p(t,x) = \big(2\pi t\big)^{-1/2} \exp\Big(-\frac{x^2}{2t}\Big)\Big(-\frac{x}{t}\Big) = -\frac{x}{t}\,p(t,x)$$

and the second derivative is

$$\frac{\partial^2}{\partial x^2} p(t,x) = -\frac{1}{t}\,p(t,x) + \frac{x^2}{t^2}\,p(t,x) = p(t,x)\Big(-\frac{1}{t} + \frac{x^2}{t^2}\Big)$$

Multiplying the second derivative by $1/2$,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} p(t,x) = p(t,x)\Big(-\frac{1}{2t} + \frac{x^2}{2t^2}\Big)$$

Comparing the time derivative and second spacial derivates,

$$\frac{\partial}{\partial t} p(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} p(t,x), \quad \forall t > 0, x \in \mathbb{R}$$

Also note that although $p(t,x)$ is singular at $t = 0$, for each x, $p(t,x) \to 0$ as $t \downarrow 0$, while the total mass $\int_{\mathbb{R}} p(t,x)\,dx = 1$ for every $t > 0$. As such, $p(t,\cdot)$ converges to the Dirac delta of 0. As such, the standard Gaussian density $p(t,x)$ is C^∞ on $(0,\infty) \times \mathbb{R}$ and satisfies the heat equation.

Problem 3 (20 points) In class we unfortunately had to skip section 6.7 on the quadratic variation of the standard Brownian motion. This section is quite important because it explains why one may expect the "rule" $(d(W(t))^2 = dt$. Read the section and solve Exercise (6.28).

(6.28) Let $X(t) = \mu t + \sigma W(t)$ be a (μ, σ^2)-Brownian motion. Show that, with probability 1, the quadratic variation of X on $[0,t]$ is $\sigma^2 t$.

First, let's decompose the increments. For $X(t) = \mu t + \sigma W(t)$, see that

$$\Delta_i^{(n)} X = X(t_{i+1}^{(n)}) - X(t_i^{(n)}) = \mu\, \Delta_i^{(n)} t + \sigma\, \Delta_i^{(n)} W$$

where $\Delta_i^{(n)} t = t_{i+1}^{(n)} - t_i^{(n)}$ and $\Delta_i^{(n)} W = W(t_{i+1}^{(n)}) - W(t_i^{(n)}) \sim \mathcal{N}\big(0, \Delta_i^{(n)} t\big)$, independent for distinct i. Expanding the quadratic sum, we can write the partial quadratic variant as

$$Q_n \;=\; \sum_i \big(\Delta_i^{(n)} X\big)^2 = \mu^2 \sum_i \big(\Delta_i^{(n)} t\big)^2 + 2\mu\sigma \sum_i \Delta_i^{(n)} t\, \Delta_i^{(n)} W + \sigma^2 \sum_i \big(\Delta_i^{(n)} W\big)^2$$

Now, let us consider each of the three terms in the sum.
First the first term, see that since $\sum_i \big(\Delta_i^{(n)} t\big) = t$,

$$\sum_i \big(\Delta_i^{(n)} t\big)^2 \;\le\; |P_n| \sum_i \Delta_i^{(n)} t \;=\; |P_n|\, t \xrightarrow[n\to\infty]{} 0$$

Hence $\mu^2 \sum (\Delta_i^{(n)} t)^2$ deterministically approaches 0.
Now consider the middle term. For simplicity, denote $M_n = \sum_i \Delta_i^{(n)} t\, \Delta_i^{(n)} W$. Since $\mathbb{E}[\Delta_i^{(n)} W] = 0$ and the increments are independent,

$$\mathbb{E}[M_n] = 0 \quad \text{and} \quad \mathbb{E}[M_n^2] = \sum_i \big(\Delta_i^{(n)} t\big)^2\, \mathrm{Var}\big[\Delta_i^{(n)} W\big] = \sum_i \big(\Delta_i^{(n)} t\big)^3$$

Similarly to the first term, see that $\sum_i (\Delta_i^{(n)} t)^3 \le |P_n| \sum_i (\Delta_i^{(n)} t)^2 \le |P_n|^2 t \to 0$. As such, $\mathbb{E}[M_n^2] \to 0$. Thus, M_n approaches 0 as $n \to \infty$ in L^2. Consequently $2\mu\sigma M_n \to 0$ in probability.
Finally considering the last term, recall that

$$[W]_t \;=\; \lim_{n\to\infty} \sum_i \big(\Delta_i^{(n)} W\big)^2 = t \quad \text{w.p. } 1$$

Multiplying by σ^2,

$$\sigma^2 \sum_i \big(\Delta_i^{(n)} W\big)^2 \xrightarrow[n\to\infty]{\sigma}{}^2\; t$$

Since the first term is deterministic and the second approaches 0, the entire sum converges w.p. 1 to $\sigma^2 t$. Hence, for any sequence of partitions whose mesh tends to 0,

$$[X]_t = \lim_{n\to\infty} \sum_i \big(X(t_{i+1}^{(n)}) - X(t_i^{(n)})\big)^2 = \sigma^2 t \quad \text{w.p. } 1.$$

Problem 4 (10 points) Prove that for all diffusions $X(t), t \ge 0$ with differentials $dX(t)$ and deterministic differentiable function $f(t), t \ge 0$ (hence, with differential $df(t) = f'(t)dt$) it holds

$$d(f(t)X(t)) = X(t)df(t) + f(t)dX(t).$$

Let $f : [0,\infty) \to \mathbb{R}$ be continuously differentiable, s.t.

$$df(t) = f'(t)\, dt$$

Let X be any continuous diffusion, i.e. a continuous semimartingale. In differential form,

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t)$$

where W is a standard Brownian motion and μ, σ are adapted processes.
Using Ito's product rule, see that for two continuous semimartingales U, V,

$$d(UV) = U\, dV + V\, dU + d[U, V]$$

with $[U, V]$ denoting the quadratic covariation.
First see that a deterministic, finite–variation process such as f satisfies $[f, X] = 0$ since its increments are $O(dt)$, while those of X are $O(dt^{1/2})$. As such, the product is $O(dt^{3/2})$. As such, the quadratic

covariation term vanishes.
When $U = X$ and $V = f$ and using $[f, X] = 0$,

$$d(f(t)X(t)) = X(t)\,df(t) + f(t)\,dX(t)$$

Thus whenever one factor is deterministic and C^1, the stochastic differential behaves exactly like the ordinary differential.

$$d\big(f(t)X(t)\big) = X(t)\,df(t) + f(t)\,dX(t)$$

Problem 5 Let $W(t), t \geq 0$ be a standard Brownian motion. For any $N > 0$ consider the sum $Z_N = N^{-1}\sum_{k=0}^{N-1} W(k/N)$.

a) (10 points) Prove that Z_N is a Gaussian random variable and find its mean and variance.

Since a standard Brownian motion $W(t)$ is a centred Gaussian process, note that every finite vector

$$\big(W(0), W(1/N), \dots, W((N-1)/N)\big)$$

is jointly Gaussian. A linear combination of jointly Gaussian random variables is again Gaussian, and

$$Z_N = \frac{1}{N}\sum_{k=0}^{N-1} W(k/N)$$

is such a linear combination. Thus, Z_N is Gaussian.
Computing its mean, see that $\forall k$, $\mathbb{E}[W(k/N)] = 0$. So,

$$\mathbb{E}[Z_N] = \frac{1}{N}\sum_{k=0}^{N-1} 0 = 0$$

Computing the variance,

$$\mathrm{Var}(Z_N) = \mathbb{E}[Z_N^2] = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\mathbb{E}\big[W(i/N)\,W(j/N)\big]$$

Note that ror Brownian motion $\mathbb{E}[W(s)W(t)] = \min\{s, t\}$.

$$\mathbb{E}[W(i/N)W(j/N)] = \frac{\min\{i, j\}}{N}$$

Therefore,

$$\mathrm{Var}(Z_N) = \frac{1}{N^3}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\min\{i, j\}$$

Evaluating the double sum,

$$\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\min\{i, j\} = \frac{N(N-1)(2N-1)}{6}$$

$$\mathrm{Var}(Z_N) = \frac{(N-1)(2N-1)}{6N^2}$$

Thus,

$$Z_N \sim \mathcal{N}\Big(0, \frac{(N-1)(2N-1)}{6N^2}\Big)$$

Note that as $N \to \infty$, variance converges to $1/3$.

b) (10 points) Recall that for any continuous function $f : \mathbb{R} \to \mathbb{R}$, the Riemann sums it holds

$$\lim_{N\to\infty} N^{-1}\sum_{k=0}^{N-1} f(k/N) = \int_0^1 f(s)ds.$$

Prove that $\int_0^1 W(t)dt$ follows a normal distribution and find its mean and variance.

(Hint: You may use without proof that if a sequence of normal random variables $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2), n \in \mathbb{N}$ converges almost surely to a random variable X then $X \sim \mathcal{N}(\lim_n \mu_n, \lim_n \sigma_n^2)$.

Recall that Brownian paths are continuous w.p. 1. Define the left-endpoint Riemann sums

$$R_N = \frac{1}{N}\sum_{k=0}^{N-1} W(k/N) = Z_N \qquad (N \geq 1)$$

Since $W(\cdot)$ is continuous on $[0,1]$ w.p. 1, the Riemann sums of W with mesh $1/N$ converge w.p. 1 to the Riemann integral,

$$\lim_{N\to\infty} R_N = \int_0^1 W(t)\,dt \quad \text{w.p. } 1$$

Next, in order to identify the limiting distribution, see that for every N, part (a) shows that $R_N = Z_N$ is Gaussian with mean 0 and variance $(N-1)(2N-1)/(6N^2)$. Therefore we have a sequence of centred normal random variables

$$R_N \sim \mathcal{N}\Big(0, \frac{(N-1)(2N-1)}{6N^2}\Big), \qquad N \geq 1$$

converging w.p. 1 to $X = \displaystyle\int_0^1 W(t)\,dt$. Recall that the variance term

$$\lim_{N\to\infty} \frac{(N-1)(2N-1)}{6N^2} = \frac{1}{3}$$

From the given hint, an almost–sure limit of normal variables remains normal, with mean and variance being the limits of the corresponding sequences. Consequently

$$\int_0^1 W(t)\,dt \;\sim\; \mathcal{N}\Big(0, \frac{1}{3}\Big)$$

Problem 6 Consider the OU process $X(t), t \geq 0$ satisfying $dX(t) = -X(t)dt + \sqrt{2}dW(t), X(0) = 0$.

a) (15 points) Prove that

$$d(e^t X(t)) = \sqrt{2}e^t dW(t).$$

In integral notation, this means for any $t \geq 0$,

$$X(t) = \sqrt{2}e^{-t}\int_0^t e^s dW(s).$$

Denote by $Y(t) = e^t X(t)$. As such, $Y(0) = X(0) = 0$. Since $t \mapsto e^t$ is C^1, Itô's product rule (which is a special case of Itô's formula taking $f(t,x) = e^t x$) yields

$$dY(t) = e^t X(t)\,dt + e^t\,dX(t) + d\big[e^t, X\big]_t$$

Because e^t is of finite variation, its quadratic covariation with the semimartingale X vanishes (as shown previously)

$$d\big[e^t, X\big]_t = 0$$

Substituting $dX(t) = -X(t)\,dt + \sqrt{2}\,dW(t)$,

$$dY(t) = e^t X(t)\,dt + e^t\big[-X(t)\,dt + \sqrt{2}\,dW(t)\big] = \sqrt{2}\,e^t\,dW(t)$$

Integrating from 0 to $t\,(>0)$ and using $Y(0) = 0$,

$$Y(t) - Y(0) = \sqrt{2}\int_0^t e^s\,dW(s) \implies e^t X(t) = \sqrt{2}\int_0^t e^s\,dW(s)$$

Dividing by e^t yields

$$X(t) = \sqrt{2}\,e^{-t}\int_0^t e^s\,dW(s)$$

In order to verify that the representation solves the SDE, define

$$\widetilde{X}(t) = \sqrt{2}\, e^{-t} \int_0^t e^s \, dW(s), \qquad t \geq 0$$

Validating the conditions, see that for:
(i) adaptedness and continuity, the stochastic integral is $(\mathcal{F}_t)$-adapted with continuous paths, hence so is $\widetilde{X}$.
(ii) the initial condition, Itô's isometry can be used to show

$$\widetilde{X}(0) = \sqrt{2} \int_0^0 e^s \, dW(s) = 0$$

(iii) dynamics, Itô's formula can be applied to $e^t \widetilde{X}(t)$, reproducing

$$d\big(e^t \widetilde{X}(t)\big) = \sqrt{2}\, e^t \, dW(t)$$

$$d\widetilde{X}(t) = -\widetilde{X}(t)\, dt + \sqrt{2}\, dW(t)$$

(iv) uniqueness, the linear SDE admits a unique strong solution, so $\widetilde{X} \equiv X$ w.p. 1.
As such,

$$d\big(e^t X(t)\big) = \sqrt{2}\, e^t \, dW(t), \qquad X(t) = \sqrt{2}\, e^{-t} \int_0^t e^s \, dW(s)$$

b) (10 points) For any $N > 0$ consider the sum

$$Z_N = \sum_{k=0}^{N-1} e^{k/N} (W(((k+1)/N)) - W(tk/N)).$$

Prove that Z_N is Gaussian and find its mean and a formula for its variance. Where does the mean and variance converge as $N \to +\infty$?

Proving gaussianity: See that the Brownian increments

$$\Delta W_k = W\big((k+1)/N\big) - W(k/N), \qquad k = 0, \ldots, N-1$$

are independent centred Gaussian variables with variance $1/N$. Since Z_N is a finite linear combination of jointly Gaussian variables, Z_N itself is Gaussian.
Calculating the mean: For each k, $\mathbb{E}[\Delta W_k] = 0$. As such,

$$\mathbb{E}[Z_N] = 0$$

Calculating variance: See that the independence of the increments yields

$$\mathrm{Var}(Z_N) = \sum_{k=0}^{N-1} e^{2k/N}\, \mathrm{Var}(\Delta W_k) = \frac{1}{N} \sum_{k=0}^{N-1} e^{2k/N}$$

Therefore,

$$\mathrm{Var}(Z_N) = \frac{1}{N} \sum_{k=0}^{N-1} e^{2k/N}$$

Trivially, the mean converges to 0 as $N \to \infty$.
Regarding the variance, see that the variance is a Riemann sum,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2k/N} = \int_0^1 e^{2s} \, ds = \frac{e^2 - 1}{2}$$

Thus,

$$Z_N \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{N}\Big(0, \frac{e^2-1}{2}\Big)$$

c) (Bonus, 5 points) Based on the intuition from Problem 5 (b), can you guess the distribution of $\int_0^t e^{-s} dW(s)$? What about the distribution of $X(t)$ as $t \to +\infty$?

Because the stochastic integral of a deterministic function against Brownian motion is Gaussian with mean 0 and variance equal to the L^2-norm of that function,

$$\int_0^t e^{-s}\, dW(s) \;\sim\; \mathcal{N}\Big(0, \int_0^t e^{-2s}\, ds\Big) = \mathcal{N}\Big(0, \tfrac{1}{2}(1 - e^{-2t})\Big).$$

Using the representation from part (a),

$$X(t) = \sqrt{2}\, e^{-t} \int_0^t e^{s}\, dW(s) \;\sim\; \mathcal{N}\Big(0, 2e^{-2t} \int_0^t e^{2s}\, ds\Big) = \mathcal{N}\big(0, 1 - e^{-2t}\big).$$

Thus, as $t \to \infty$, the variance approaches 1.

$$X(t) \xrightarrow[t\to\infty]{\mathrm{d}} \mathcal{N}(0,1)$$

Note that this limiting distribution $N(0,1)$ is precisely the stationary law of the Ornstein–Uhlenbeck process.