S&DS 351: Stochastic Processes - Homework 3

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February 7, 2025

Problem 1

(10 points) Is it possible for a transient state to be periodic? If so, construct an example of such a Markov chain; otherwise, give a mathematical proof why not.

Note: I (fortunately) solved this after proving problem 3, so for a more thorough proof on how this example is transient, please see Problem 3.

Yes, it is possible for a transient state to be periodic. Consider a 1-dimensional asymmetric random walk on $\mathbb{Z}$:

$$X_n = X_{n-1} + Z_n, \quad \text{where} \quad \mathbb{P}(Z_n = +1) = p \quad \text{and} \quad \mathbb{P}(Z_n = -1) = 1 - p,$$

for some $p \in (0, 1)$ with $p \neq \frac{1}{2}$. Starting at state 0, state 0 is transient (see Problem 3).

Define the period as $d_i = \gcd\{n : P^n(i, i) > 0\}$, where P is the transition matrix.
In the random walk, the walk must trivially take as many $+1$ steps as -1 steps to reach the initial state. Thus one can only return to state x starting from x in an even number of steps. Note that this holds for all integers. Hence for each integer x,

$$(P^n)(x, x) > 0 \implies n \text{ is even.}$$

$$(P^n)(x, x) = 0 \implies n \text{ is odd.}$$

Therefore, the greatest common divisor of all such n is 2, and every state $x \in \mathbb{Z}$ has period 2.

Problem 2

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix P. Let $k \geq 1$ be an integer.

(a) (5 points) Prove that $Y_n = X_{kn}$ is also a Markov chain. Find its transition matrix.

Since $Y_n = X_{kn}$, the conditional probability for Y_{n+1} can be defined as

$$\mathbb{P}\left(X_{k(n+1)} = y_{n+1} \mid X_{kn} = y_n, X_{k(n-1)} = y_{n-1}, \ldots, X_0 = y_0\right)$$

Because $\{X_n\}$ is a Markov chain, it satisfies the Markov property,

$$P\big(X_{m+1} = x_{m+1} \,\big|\, X_m = x_m, \ldots, X_0 = x_0\big) = P\big(X_{m+1} = x_{m+1} \,\big|\, X_m = x_m\big).$$

Applying for the k-steps from time kn to time $k(n+1)$,

$$\mathbb{P}\left(X_{k(n+1)} = y_{n+1} \mid X_{kn} = y_n, X_{k(n-1)} = y_{n-1}, \ldots, X_0 = y_0\right) = \mathbb{P}\left(X_{k(n+1)} = y_{n+1} \mid X_{kn} = y_n\right)$$

Therefore,

$$\mathbb{P}\left(Y_{n+1} = y_{n+1} \mid Y_n = y_n, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0\right) = \mathbb{P}\left(Y_{n+1} = y_{n+1} \mid Y_n = y_n\right)$$

Thus, $\{Y_n\}$ satisfies the Markov property and is a Markov chain.

Solving for the transition matrix, note that for any state $i, j \in S$,

$$\mathbb{P}(Y_{n+1} = j \mid Y_n = i) = \mathbb{P}(X_{k(n+1)} = j \mid X_{kn} = i)$$

In the Markov chain $\{X_n\}$,

$$\mathbb{P}(X_{k(n+1)} = j \mid X_{kn} = i) = (P^k)_{ij}$$

Thus, the one-step transition probability for $\{Y_n\}$ is

$$\mathbb{P}(Y_{n+1} = j \mid Y_n = i) = (P^k)_{ij}.$$

Therefore, the transition matrix for $\{Y_n\}$ is P^k.

(b) (10 points) Suppose that the original chain $\{X_n\}$ is irreducible. Is $\{Y_n\}$ irreducible? If so, prove it; if not, provide a counterexample.

Consider a Markov chain, with state space $S = \{X_0, X_1\}$. Define its transition matrix as

$$P_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since X_0 and X_1 communicate, this chain is trivially irreducible.

Suppose $k = 2$. The transition matrix of $Y_m = X_{2n}$ is

$$P_Y = P_X^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

X_0 and X_1 do not communicate in the new chain, as X_0 and X_1 are closed subsets on S. Therefore, Y_n is not irreducible.

Note that this can be generalized to any Markov chain $X_0, X_1, ..., X_k$, where $\mathbb{P}_X(i, j) = 1$ for $j = (i + 1) \bmod k$ and $\mathbb{P}_X(i, j) = 0$ for all other i, for all j. This chain is trivially irreducible, and yields a transition matrix P^k which is a $k \times k$ identity matrix. As such, the chain Y_n is not irreducible, as the states $X_0, X_1, ..., X_k$ are closed subsets on S.

(c) (10 points) Suppose that the original chain $\{X_n\}$ is aperiodic. Is $\{Y_n\}$ aperiodic? If so, prove it; if not, provide a counterexample.

- A Markov chain with transition matrix P is called *aperiodic* if every state i has period 1. Concretely, the period of a state i is defined as

$$d(i) = \gcd\Big\{ n \geq 1 : (P^n)_{ii} > 0 \Big\}.$$

  If this gcd equals 1 for every i, then the chain is aperiodic.

- In the chain $\{Y_n\}$ defined by $Y_n := X_{kn}$, its transition probabilities arise from the k-step transition probabilities of the original chain. In particular,

$$\mathbb{P}(Y_{m+1} = j \mid Y_m = i) = \mathbb{P}(X_{k(m+1)} = j \mid X_{km} = i) = \left(P^k\right)_{ij}.$$

  Hence the transition matrix of $\{Y_n\}$ is exactly P^k, and the m-step transition probabilities in $\{Y_n\}$ are given by $(P^k)^m = P^{km}$.

- Therefore, the period of state i as a state of the chain $\{Y_n\}$ is

$$d_Y(i) = \gcd\Big\{ m \geq 1 : \left(P^{km}\right)_{ii} > 0 \Big\}.$$

  We must show that $d_Y(i) = 1$ whenever $d(i) = 1$ in the original chain.

Key idea: Aperiodicity implies eventually-positive returns in all large times. A standard characterization of aperiodicity is that for each state i, there exists some integer N (possibly depending on i) such that

$$(P^n)_{ii} \;>\; 0 \quad \text{for all } n \geq N.$$

(Equivalently, the set $\{\, n : (P^n)_{ii} > 0\}$ is *co-finite*, i.e. it contains all sufficiently large n.) We use this fact to see what happens when we look only at k-step transitions, i.e. the set

$$\Big\{ m : \big(P^{km}\big)_{ii} > 0 \Big\}.$$

If $(P^n)_{ii} > 0$ for all $n \geq N$, then in particular $(P^{km})_{ii} > 0$ whenever $km \geq N$. Hence for all integers

$$m \;\geq\; \lceil N/k \rceil,$$

we have $(P^{km})_{ii} > 0$. Thus

$$\Big\{ m : \big(P^{km}\big)_{ii} > 0 \Big\}$$

contains *all* sufficiently large integers m. **Period calculation for the chain $\{Y_n\}$.** For the chain $\{Y_n\}$, the period of i is

$$d_Y(i) \;=\; \gcd\Big\{ m \geq 1 : \big(P^{km}\big)_{ii} > 0 \Big\}.$$

Since this set includes *all* sufficiently large integers m, its greatest common divisor must be 1. Indeed, any infinite set of consecutive integers has gcd equal to 1. Formally:

- Because $\{X_n\}$ is aperiodic at i, there is an N such that $(P^n)_{ii} > 0$ for every $n \geq N$.
- Consequently, for every integer $m \geq \lceil N/k \rceil$, we have

  $$(P^{km})_{ii} > 0.$$

- Hence $\{m : (P^{km})_{ii} > 0\}$ contains the tail set $\{\lceil N/k \rceil, \lceil N/k \rceil + 1, \lceil N/k \rceil + 2, \dots\}$ of *all* sufficiently large integers.
- The greatest common divisor of any infinite set of consecutive integers is 1.
- Therefore $d_Y(i) = 1$.

Since $d_Y(i) = 1$ for every state i, the subsampled chain $\{Y_n\}$ is indeed aperiodic.

(d) (10 points) Suppose that the original chain $\{X_n\}$ is transient. Is $\{Y_n\}$ transient? If so, prove it; if not, provide a counterexample.

(e) (15 points) Suppose that the original chain $\{X_n\}$ is recurrent. Is $\{Y_n\}$ recurrent? If so, prove it; if not, provide a counterexample.

(f) (5 points) Suppose that the original chain X_n is irreducible and that it has period d. What is the period of each state i in the new Markov chain Y_n for $k = d$?

Since the original chain is irreducible with period d, for each state i,

$$(P_X^d)_{ii} \;>\; 0 \quad \text{and} \quad (P_X^{d-b})_{ii} \;=\; 0, \quad \forall b = 1, 2, \dots, d-1.$$

Therefore, for any multiple of d, $a \in \mathbb{Z}$ such that $a \geq 1$,

$$(P_X^d)_{ii}^a > 0$$

As such, all returns to i can occur only at multiples of d steps.

Define the transition matrix for Y_n where $k = d$ as

$$\mathbb{P}(Y_{m+1} = j \,\big|\, Y_m = i) \;=\; (P_X^k)_{ij} \;=\; (P_X^d)_{ij}$$

When $j = i$,

$$(P_Y^1)_{ii} \;=\; (P_X^d)_{ii} \;>\; 0$$

As such, state i can reach itself in a single step in chain Y, forming a self-loop in Y.

Defining the period of i in Y,

$$d_Y(i) \;=\; \gcd\big\{\, m \;\geq\; 1 : (P^{d\,m})_{ii} > 0 \big\}.$$

Note that $(P^d)_{ii} > 0$, so at $m = 1$, $(P^{d\cdot 1})_{ii} = (P^d)_{ii} > 0$. Since the chain is irreducible, raising P^d to a higher power trivially cannot change its positivity:

$$(P^{d\,2})_{ii} = \big(P^d\big)^2_{ii} > 0, \quad (P^{d\,3})_{ii} > 0, \;\; \ldots$$

$(P^{dn})_{ii} > 0$ holds for all $n \geq 1$, so the set of possible m is $\{1, 2, 3, \ldots\}$. Thus,

$$\big\{\, m : (P^{d\,m})_{ii} > 0 \big\} \;=\; \{1, 2, 3, \ldots\},$$

with a trivial gcd of 1. Note that since the d period was a state property in the original chain, $\{X_n\}$, it holds for all states in the old chain, and thus all states in the new chain.

As such, the period of each state in the new chain $\{Y_n\}$ is 1. In other words, the chain $\{Y_n\}$ is aperiodic.

Problem 3

(Asymmetric random walk, 15 points) Consider the *asymmetric* random walk on $\mathbb{Z}$, that is, $X_n = X_{n-1} + Z_n$, where $Z_1, Z_2, \ldots$ are iid and $\mathbb{P}(Z_n = +1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$, with $p \in [0, 1]$ and $p \neq \frac{1}{2}$. Show that the state 0 is a transient state.

In Lecture 7 we saw/will see that when $p = \frac{1}{2}$ this is not true anymore and the state 0 is recurrent. Can you explain intuitively why this is the case?

Hint: You may want to use Stirling's formula that $\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{2\pi n}} = 1$.

Starting from $X_0 = 0$, the random walk is at state 0 again at $t = n$ only when it has taken an equal number of $+1$ steps as -1 steps. As such, n must be even.

Suppose $n = 2k$, and k is the number of Z_i that are $+1$,

$$\mathbb{P}(X_{2k} = 0 \,|\, X_0 = 0) \;=\; \binom{2k}{k}\, p^k\, (1-p)^k$$

$$\text{Note that } \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 0 \;\; \text{if } n \text{ is odd}$$

Hence the series of return probabilities at 0 is

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \,|\, X_0 = 0) \;=\; 1 \;+\; \sum_{k=1}^{\infty} \binom{2k}{k}\, p^k\, (1-p)^k,$$

accounting for the initial state of 0.
Using Stirling's approximation,

$$n! \;\sim\; \left(\tfrac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as } n \to \infty,$$

applying to this case,

$$\binom{2k}{k} \;=\; \frac{(2k)!}{k!\,k!} \;\approx\; \frac{\sqrt{4\pi k}\,\left(\frac{2k}{e}\right)^{2k}}{2\pi k\,\left(\frac{k}{e}\right)^k \left(\frac{k}{e}\right)^k} \;=\; \frac{4^k}{\sqrt{\pi k}}$$

Therefore,

$$\binom{2k}{k}\, p^k\, (1-p)^k \;\approx\; \frac{4^k}{\sqrt{\pi k}}\, \big[p(1-p)\big]^k \;=\; \frac{\big[4\,p(1-p)\big]^k}{\sqrt{\pi k}}.$$

If $p \neq \frac{1}{2}$, then $4\,p(1-p) < 1$ (If $f(x) = x(1-x)$, then $f'(x) = -x + 1 - x = -2x + 1$. Solving for the max when $f'(x) = 0, x = \frac{1}{2}$).

Note, that as $k \to \infty$, $\big[4\,p(1-p)\big]^k$ decays exponentially. Therefore,

$$\binom{2k}{k} p^k\,(1-p)^k \;=\; O\Big(\big[4\,p(1-p)\big]^k\Big) \quad \text{and} \quad \sum_{k=1}^{\infty} \binom{2k}{k} p^k\,(1-p)^k \;<\; \infty.$$

Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n = 0 \,\big|\, X_0 = 0) \;=\; 1 \;+\; \sum_{k=1}^{\infty} \binom{2k}{k} p^k\,(1-p)^k \;<\; \infty.$$

which defines a transient state.

However, when $p = \frac{1}{2}$,

$$\binom{2k}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k \;\approx\; \frac{\big[4 \cdot 0.5(1-0.5)\big]^k}{\sqrt{\pi k}} \;=\; \frac{1}{\sqrt{\pi k}},$$

so

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \;=\; \infty.$$

which defines a recurrent state when $p = \frac{1}{2}$.

Exercise 1.8

Consider a Markov chain on the integers with

$$P(i, i+1) = 0.4 \text{ and } P(i, i-1) = 0.6 \text{ for } i > 0,$$
$$P(i, i+1) = 0.6 \text{ and } P(i, i-1) = 0.4 \text{ for } i < 0,$$
$$P(0, 1) = P(0, -1) = \frac{1}{2}.$$

This is a chain with infinitely many states, but it has a sort of probabilistic "restoring force" that always pushes back toward 0. Find the stationary distribution.

Denote the stationary distribution by $\{\pi_i\}_{i \in \mathbb{Z}}$

$$\sum_{j \in \mathbb{Z}} \pi_j P(j, i) \;=\; \pi_i \quad \text{for all } i \in \mathbb{Z},$$

and $\sum_{i \in \mathbb{Z}} \pi_i = 1$. As this is a two-sided birth-death chain,

$$\pi_i\, P(i, i+1) \;=\; \pi_{i+1}\, P(i+1, i).$$

For $i \geq 1$:

$$\pi_i \times 0.4 \;=\; \pi_{i+1} \times 0.6 \quad \Longrightarrow \quad \frac{\pi_{i+1}}{\pi_i} = \frac{0.4}{0.6} = \frac{2}{3}.$$

For $i \leq -1$:

$$\pi_{i-1} \times 0.6 \;=\; \pi_i \times 0.4 \quad \Longrightarrow \quad \frac{\pi_{i-1}}{\pi_i} = \frac{0.4}{0.6} = \frac{2}{3}.$$

When $i = 0$,

$$\pi_0 \times 0.5 \;=\; \pi_1 \times 0.6 \quad \Longrightarrow \quad \frac{\pi_1}{\pi_0} = \frac{0.5}{0.6} = \frac{5}{6}.$$
$$\pi_{-1} \times 0.6 = \pi_0 \times 0.5 \quad \Longrightarrow \quad \frac{\pi_{-1}}{\pi_0} = \frac{0.5}{0.6} = \frac{5}{6},$$

Hence,

$$\pi_1 = \frac{5}{6}\,\pi_0, \quad \pi_{-1} = \frac{5}{6}\,\pi_0.$$

As the probabilities of a jump remain the same, generalizing for all $i \geq 1$,

$$\pi_{i+1} = \frac{2}{3}\pi_i \quad \implies \quad \pi_i = \left(\frac{2}{3}\right)^{i-1} \pi_1 \quad \text{for } i \geq 1.$$

$$\pi_i = \left(\frac{2}{3}\right)^{i-1} \cdot \frac{5}{6}\pi_0 \quad \forall i \geq 1.$$

Generalizing for all $i \leq -1$,

$$\pi_{i-1} = \frac{2}{3}\pi_i \quad \implies \quad \pi_i = \frac{2}{3}^{-i-1} \pi_{-1} \quad \forall i \leq -1,$$

$$\pi_i = \left(\frac{2}{3}\right)^{-i-1} \cdot \frac{5}{6}\pi_0 \quad \forall i \leq -1.$$

Combining the two cases,

$$\pi_i = \frac{5}{6}\left(\frac{2}{3}\right)^{|i|-1} \pi_0, \quad \forall i \neq 0.$$

Solving for π_0, first recall that,

$$\sum_{i=-\infty}^{\infty} \pi_i = 1.$$

Hence,

$$\pi_0 + \sum_{i \neq 0} \frac{5}{6}\left(\tfrac{2}{3}\right)^{|i|-1} \pi_0 = 1.$$

$$\pi_0\left[1 + \frac{5}{6}\sum_{i \neq 0} \left(\tfrac{2}{3}\right)^{|i|-1}\right] = 1.$$

Exploiting the symmetry of the chain,

$$\sum_{i \neq 0} \left(\tfrac{2}{3}\right)^{|i|-1} = 2\sum_{j=1}^{\infty} \left(\tfrac{2}{3}\right)^{j-1} = 2\sum_{k=0}^{\infty} \left(\tfrac{2}{3}\right)^{k} = 2 \cdot \frac{1}{1-\frac{2}{3}} = 2 \cdot 3 = 6.$$

Thus

$$\pi_0\left[1 + \frac{5}{6} \cdot 6\right] = 6\,\pi_0 = 1 \quad \implies \quad \pi_0 = \frac{1}{6}.$$

Therefore,

$$\pi_i = \begin{cases} \dfrac{1}{6}, & i = 0 \\ \dfrac{5}{36}\left(\tfrac{2}{3}\right)^{|i|-1} & i \neq 0 \end{cases}$$

Exercise 1.16

Show that if an irreducible Markov chain has a state i such that $P(i,i) > 0$, then the chain is aperiodic. Also show by example that this sufficient condition is not necessary.

Let $\{X_n\}$ be an irreducible Markov chain on a countable state space S. Suppose there is a state $i \in S$ such that

$$P(i,i) > 0.$$

The period of state i is defined as

$$d(i) = \gcd\{n \geq 1 : P^n(i,i) > 0\},$$

Since $P(i,i) = P^1(i,i) > 0$, there is a positive probability of returning to i in exactly 1 step. As such, $1 \in \{n : P^n(i,i) > 0\}$, so any common divisor of all n must trivially divide 1.

Therefore,

$$d(i) \;=\; \gcd\{n \geq 1 : P^n(i,i) > 0\} \;=\; 1.$$

Thus, i is an aperiodic state.

By irreducibility, for any other state $j \in S$, $\exists m, k \in \mathbb{Z}$ such that $m, k \geq 1$, $P^m(j,i) > 0$, and $P^k(i,j) > 0$. Therefore as proven in lecture, for any $n \geq 1$,

$$P^{m+n+k}(j,j) \;\geq\; P^m(j,i)\, P^n(i,i)\, P^k(i,j).$$

Since $P^n(i,i) > 0$ for all $n \geq 1$, for arbitrarily many n, the probability $P^{m+n+k}(j,j)$ is strictly positive. As such,

$$d(j) \;=\; \gcd\{n \geq 1 : P^n(j,j) > 0\} \;=\; 1.$$

Thus, every state in an irreducible chain with a self-loop, i.e. $P(i,i) > 0$, is aperiodic.

To show that this sufficient condition is not necessary, consider a chain on three states $\{1, 2, 3\}$ with transition matrix:

$$\begin{aligned}
P(1,1) &= 0, \quad & P(1,2) &= 1, \quad & P(1,3) &= 0,\\
P(2,1) &= \tfrac{1}{2}, \quad & P(2,2) &= 0, \quad & P(2,3) &= \tfrac{1}{2},\\
P(3,1) &= \tfrac{1}{2}, \quad & P(3,2) &= \tfrac{1}{2}, \quad & P(3,3) &= 0.
\end{aligned}$$

Note that the chain is irreducible, but does not contain any self-loops, i.e. $P(i,i) = 0$ for $i = 1, 2, 3$.

$$\begin{gathered}
P^2(1,1) = P(1,2)P(2,1) = (1)\left(\tfrac{1}{2}\right) = \tfrac{1}{2} > 0,\\
P^3(1,1) = P(1,2)P(2,3)P(3,1) = 1 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} > 0.
\end{gathered}$$

Therefore, state 1 can return to itself in both 2 steps and 3 steps, which comprises a subset of the set of all possible return times back to state 1. The gcd of 2 and 3 is 1, or a period of 1. Note that $P^2(2,2) > 0$, $P^2(3,3) > 0$, and $P^3(2,2) > 0$, $P^3(3,3) > 0$, so every state also has period 1, making the entire chain aperiodic. Thus, $P(i,i) > 0$ for some i is not a necessary condition for a chain to be aperiodic.