Mathematical Aside - Multivariable Functions

May 25, 2017

1 Mathematical Aside: Introduction to Level Curves

In our study of production functions, we've learned that the equation

$$q(L, K)$$

defines a multivariable function. In general, a function that takes two variables as inputs lives in a **three dimensional space**. The following demonstration is aimed at helping you build some intuition for how we can understand these.

It might seem a bit difficult to wrap your head around a function with two inputs as variables. After all, a function of the form $y = f(x)$ is already graphed in two dimensions. How could we understand a function of the from $f(x, y)$? Well, as it turns out, you have already learned about a particular kind of two variable function! Can you guess? It's a circle!

The equation of a circle is given by

$$x^2 + y^2 = k$$

where k is the radius of the circle. The simplest case is that of the unit circle for which

$$x^2 + y^2 = 1$$

1.1 The Circle in Polar Coordinates

How would you verify this claim? You can begin by considering the points (1,0), (0,1), (-1,0), (0,-1) and verify that all these points obey the above equation. But how can we deal with other points along the circle? The easiest approach by far is to make a substitution. We will introduce an angle θ in the interval $[0, 2\pi]$ and relate it to x and y via the following equations:

$$x = \cos(\theta) \tag{1}$$
$$y = \sin(\theta) \tag{2}$$

This substitution defines a new set of coordinates which we call **polar coordinates**. Now, we plug in these expression into the original equation

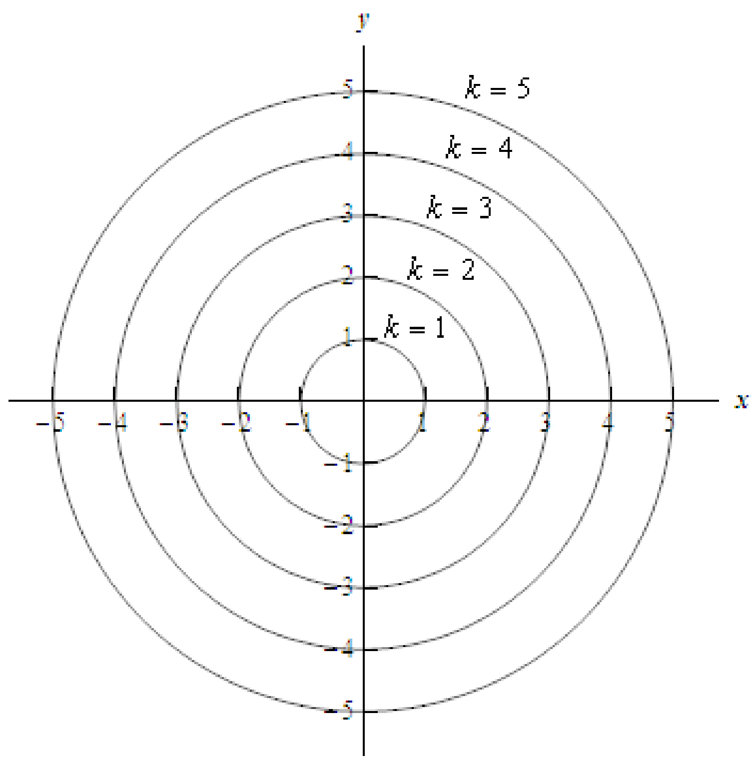
$$\begin{aligned} 1 &= x^2 + y^2 && (3) \\ &= (\cos(\theta))^2 + (\sin(\theta))^2 && (4) \\ &= 1 && (5) \end{aligned}$$

which is what we wanted to demonstrate.

From here, we can recognize that the more general case of $x^2 + y^2 = k$ follows from the coordinates

$$x = k\cos(\theta) \tag{6}$$
$$y = k\sin(\theta) \tag{7}$$

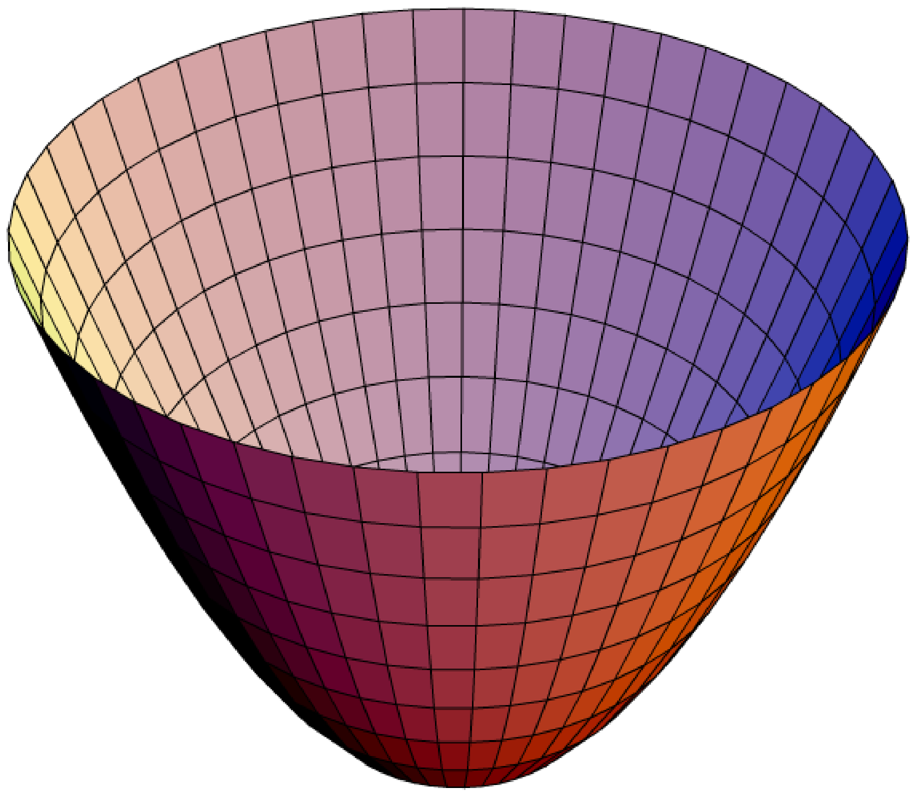
so that the equations for the unit circle corresponded to the special case of $k = 1$. The figure below has circles plotted for various levels of k.



circle level curves

In this case we have the circles lying on the xy-coordinate plane. But what if we could have a circle *above* the plane? Particulalry, what would we get if we let k describe the **height** of an axis perpendicular to the xy-plane? We would then have an expression for our multivariable function $f(x, y) = k$ defining slices of planes at different heights. The circles in the above image would be like shadows of our stacked circles. These are known as **level curves**. Finally, by letting $k \in \mathbb{R}$ define a set of real numbers, we would obtain the following three-dimensional figure:

Such a figure is known as a paraboloid.



paraboloid