Solutions to Terence Tao's *Analysis I*

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Introduction

It has been a while since I have done any formal mathematics, so take these solutions with a large heaping of salt. If there is intuition to be developed in a problem, I try to develop it before diving in.

Chapter 2: The natural numbers

The natural numbers satisfy the Peano axioms:

Axiom 1 *0 is a natural number.*

Axiom 2 *If n is a natural number, then n++ is also a natural number.*

Axiom 3 *0 is not the successor of any natural number; i.e. we have n++ $\neq 0$ for every natural number n.*

Axiom 4 *Different natural numbers must have different successors; i.e., if n and m are natural numbers, and $n \neq m$, then n++ $\neq m$++.*

Axiom 5 *Principle of mathematical induction). Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n$++$)$ is also true. Then $P(n)$ is true for every natural number n.*

Addition

For reference, addition is defined inductively as $0 + m = m$ and $(n$++$) + m = (n + m)$++.

2.2.1. (Addition is associative)
We can use induction on a to prove $a + (b + c) = (a + b) + c$.

The base case is

$$\begin{aligned}
0 + (b + c) &= (b + c) && \text{(by the definition of addition)} \\
&= b + c \\
&= (0 + b) + c && \text{(again using the definition)}
\end{aligned}$$

Now we assume the statement holds for a, and we seek to prove it for $a{+}{+}$.

$$\begin{aligned}
(a{+}{+}) + (b + c) &= (a + (b + c)){+}{+} && \text{(definition)} \\
&= ((a + b) + c){+}{+} && \text{(inductive step)} \\
&= (a + b){+}{+} + c && \text{(definition)} \\
&= ((a{+}{+}) + b) + c && \text{(definition)}
\end{aligned}$$

2.2.2. (Positive numbers have only one successor)
We will prove this by induction on a. The base case is $a = 1$ as the first positive number, and since $0{+}{+} = 1$, the base case is proved.

For the inductive step, assume the statement holds for a, that is that there exists exactly one b such that $b{+}{+} = a$. If we increment both sides, then $(b{+}{+}){+}{+} = a{+}{+}$. By the inductive step, we know $b{+}{+} = a$ and from the problem statement, we know that a is positive which completes the proof.

2.2.3. (Basic properties of natural numbers)
I will prove transitivity and the final two.

a) To prove transitivity, we can expand the $\geq$ definitions as $a \geq b \implies a = b + n$ for some natural number n, and $b \geq c \implies b = c + m$ for some natural number m. Thus

$$\begin{aligned}
a &= b + n \\
&= (c + m) + n && \text{(expand)} \\
&= c + (m + n) && \text{(by associativity)}
\end{aligned}$$

Since $m + n$ is a natural number, by definition of $\geq$, we have $a \geq c$.

b) We want to prove that $a < b$ if and only if $a{+}{+} \leq b$.

($\Leftarrow$) If $a{+}{+} \leq b$, then $b = (a{+}{+}) + n$ for some natural number n. By definition of addition, $b = (a + n){+}{+}$. By commutativity, $b = a + (n{+}{+})$. Since $n{+}{+}$ is a positive number, we have that $b \geq a$ and $b \neq a$[1] which shows $b > a$.

($\Rightarrow$) If $a \leq b$, then $b = a + n$, and $b \neq a$. This means that $n \neq 0$, and by the previous exercise,

[1] Otherwise we get a contradiction that $n{+}{+} = 0$ by cancellation.

we know that $n = m$++ for some natural number m. Replacing in the equation for b, we have

$$\begin{aligned} b &= a + (m\texttt{++}) \\ &= (m\texttt{++}) + a && \text{(commutativity)} \\ &= (a + m)\texttt{++} && \text{(addition and commutativity)} \\ &= (a\texttt{++}) + m && \text{addition} \end{aligned}$$

hence $b \geq a$++ by definition of $\geq$.

c) We want to show that $a < b$ if and only if $b = a + d$ for some positive number d.

The proof for this is part of the proof for **b)**, although the forward direction can be shown more easily by contradiction.