Solutions to Terence Tao's *Analysis I*

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Contents

It has been a while since I have done any formal mathematics, so take these solutions with a large heaping of salt. If there is intuition to be developed in a problem, I try to develop it before diving in.

2 The natural numbers

2.1 Peano axioms

The natural numbers satisfy the Peano axioms:

Axiom 1 0 *is a natural number.*

Axiom 2 *If n is a natural number, then $n{+}{+}$ is also a natural number.*

Axiom 3 0 *is not the successor of any natural number; i.e. we have $n{+}{+} \neq 0$ for every natural number n.*

Axiom 4 *Different natural numbers must have different successors; i.e., if n and m are natural numbers, and $n \neq m$, then $n{+}{+} \neq m{+}{+}$.*

Axiom 5 *Principle of mathematical induction). Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.*

2.2 Addition exercises

For reference, addition is defined inductively as $0 + m = m$ and $(n{+}{+}) + m = (n + m){+}{+}$.

2.2.1. (Addition is associative)

We can use induction on a to prove $a + (b + c) = (a + b) + c$.

The base case is

$$\begin{aligned} 0 + (b + c) &= (b + c) && \text{(by the definition of addition)} \\ &= b + c \\ &= (0 + b) + c && \text{(again using the definition)} \end{aligned}$$

Now we assume the statement holds for a, and we seek to prove it for $a{+}{+}$.

$$\begin{aligned} (a{+}{+}) + (b + c) &= (a + (b + c)){+}{+} && \text{(definition)} \\ &= ((a + b) + c){+}{+} && \text{(inductive step)} \\ &= (a + b){+}{+} + c && \text{(definition)} \\ &= ((a{+}{+}) + b) + c && \text{(definition)} \end{aligned}$$

2.2.2. (Positive numbers have only one successor)

We will prove this by induction on a. The base case is $a = 1$ as the first positive number, and since $0{+}{+} = 1$, the base case is proved.

For the inductive step, assume the statement holds for a, that is that there exists exactly one b such that $b{+}{+} = a$. If we increment both sides, then $(b{+}{+}){+}{+} = a{+}{+}$. By the inductive step, we know $b{+}{+} = a$ and from the problem statement, we know that a is positive which completes the proof.

2.2.3. (Basic properties of natural numbers)

I will prove transitivity and the final two.

a) To prove transitivity, we can expand the $\geq$ definitions as $a \geq b \implies a = b + n$ for some natural number n, and $b \geq c \implies b = c + m$ for some natural number m. Thus

$$\begin{aligned} a &= b + n \\ &= (c + m) + n && \text{(expand)} \\ &= c + (m + n) && \text{(by associativity)} \end{aligned}$$

Since $m + n$ is a natural number, by definition of $\geq$, we have $a \geq c$.

b) We want to prove that $a < b$ if and only if $a{+}{+} \leq b$.

($\Leftarrow$) If $a{+}{+} \leq b$, then $b = (a{+}{+}) + n$ for some natural number n. By definition of addition, $b = (a+n){+}{+}$. By commutativity, $b = a + (n{+}{+})$. Since $n{+}{+}$ is a positive number, we have that $b \geq a$ and $b \neq a$[1] which shows $b > a$.

($\Rightarrow$) If $a \leq b$, then $b = a + n$, and $b \neq a$. This means that $n \neq 0$, and by the previous exercise, we know that $n = m{+}{+}$ for some natural number m. Replacing in the equation for b, we have

$$\begin{aligned} b &= a + (m{+}{+}) & \\ &= (m{+}{+}) + a & \text{(commutativity)} \\ &= (a+m){+}{+} & \text{(addition and commutativity)} \\ &= (a{+}{+}) + m & \text{addition} \end{aligned}$$

hence $b \geq a{+}{+}$ by definition of $\geq$.

c) We want to show that $a < b$ if and only if $b = a + d$ for some positive number d.

The proof for this is part of the proof for **b)**, although the forward direction can be shown more easily by contradiction.

[1] Otherwise we get a contradiction that $n{+}{+} = 0$ by cancellation.