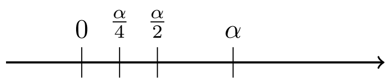
Infinitesimal Calculus

Definition: An **infinitesimal** is a quantity that is closer to zero than any standard real number but is not zero itself.

Theorem: If α is an infinitesimal, then $\alpha \notin \mathbb{R}$.

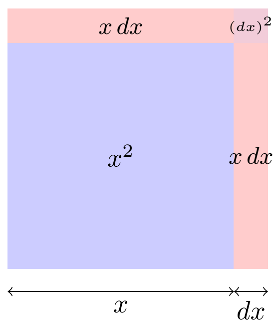
Proof: Let α be an infinitesimal. Clearly there are smaller positive real numbers than α.



Therefore, infinitesimals do not exist.

Geometric Approach to the Derivative of x^2

When we increase the side length of a square from x to $x + dx$, the change in area can be seen geometrically:



The total change in area dA is:

$$dA = 2x\,dx + (dx)^2$$

Remember that $A = x^2$

$$\begin{aligned}
dA &= 2x\,dx + (dx)^2 \\
dx^2 &= 2x\,dx + (dx)^2 \\
\frac{dx^2}{dx} &= \frac{2x\,dx + (dx)^2}{dx} \\
\frac{d}{dx}(x^2) &= 2x + dx \quad \text{(dx is basically 0)} \\
\frac{d}{dx}(x^2) &= 2x
\end{aligned}$$

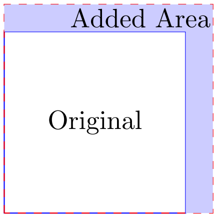
Where:

- $2x\,dx$ represents the two red rectangles (each with area $x\,dx$)
- $(dx)^2$ represents the tiny purple square at the corner

Since dx is infinitesimal, $(dx)^2$ is "even more infinitesimal" and can be ignored:

$$\frac{dA}{dx} \approx 2x$$

Why This Works Geometrically



The derivative represents the rate of change of area with respect to side length:

- As $dx \to 0$, the purple corner square becomes negligible
- Added area $\approx$ two rectangular strips of width dx and length x
- Thus, instantaneous rate of change $= 2x$

This geometric approach shows why the derivative of x^2 is $2x$ in a visual, intuitive way that matches our understanding of area.

Alternate Geometric Approach to the Derivative of $\sqrt{x}$ via a Circular Sector

Let

$$r = \sqrt{x} \quad \Longrightarrow \quad x = r^2.$$

Now, consider a circular sector with radius r and a fixed central angle $\theta = 2$ radians. (Note that using $\theta = 2$ makes the area of the sector equal to

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{2}{2\pi}\pi r^2 = r^2 = x.$$

) Now, if we change the radius by an infinitesimal amount

$$dr = d\sqrt{x},$$

the corresponding change in area is approximately the area of a thin annular (ring-like) sector. This small area is given by the product of the "arc length" of the sector and dr. Since the arc length of a sector is

$$s = r\,\theta = 2r,$$

we have

$$dA \approx s\,dr = 2r\,dr.$$

But because $A = x$, also $dA = dx$ and therefore

$$dx \approx 2r\,d\sqrt{x}.$$

Dividing both sides by dx (and taking the infinitesimal limit) gives

$$\frac{d\sqrt{x}}{dx} \approx \frac{1}{2r} = \frac{1}{2\sqrt{x}},$$

which is exactly the derivative of $\sqrt{x}$.