Homework #8

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Problem 1: Prove that when all the eigenvalues of A have strictly negative real parts, then there exists constants $c, \lambda > 0$ such that

$$|| \, e^{At} \leq ce^{-\lambda t} \, ||, \quad \forall t \in \mathbb{R} \, .$$

Solution:

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Let

$$A = VJV^{-1}, \quad J = \Lambda + N, \quad \Lambda N = N\Lambda, \quad N^p = 0$$

Note that $N = 0$ if there is no Jordan form. The proof is the same/simpler.

Then,

$$e^{At} = Ve^{\Lambda t}e^{Nt}V^{-1}.$$

Define:

$$\lambda_m = \min_j |Re(\lambda_j)|$$

Then,

$$|| \, e^{At} \, || = || \, Ve^{\Lambda t}e^{Nt}V^{-1} \, || \leq || \, V \, || \cdot || \, V^{-1} \, || \cdot e^{-\lambda_m t} \cdot || \, e^{Nt} \, ||$$

Where:

$$|| \, e^{Nt} \, || = \left\| \sum_{j=0}^{p-1} \frac{N^j t^j}{j!} \right\| \leq \sum_{j=0}^{p-1} \frac{t^j}{j!} \, || \, N \, ||^j$$

Note:

$$|| \, N \, || = 1 \implies || \, e^{Nt} \, || \leq \sum_{j=0}^{p-1} \frac{t^j}{j!}$$

Back to the other thing:

$$|| \, e^{At} \, || \leq || \, V \, || \cdot || \, V^{-1} \, || \cdot e^{-\lambda_m t} \cdot \sum_{j=0}^{p-1} \frac{t^j}{j!}$$

We can ignore the V and V^{-1} terms because they are constants.

Define: $0 < \epsilon < \lambda_m$. This gives:

$$e^{-\lambda_m t} = e^{-(\lambda_m - \epsilon + \epsilon)t} = e^{-(\lambda_m - \epsilon)t} \cdot e^{-\epsilon t}$$

So,

$$\begin{aligned} e^{-\lambda_m t} \cdot \sum_{j=0}^{p-1} \frac{t^j}{j!} &= e^{-(\lambda_m - \epsilon)t} \cdot e^{-\epsilon t} \cdot \sum_{j=0}^{p-1} \frac{t^j}{j!} \\ &= e^{-(\lambda_m - \epsilon)t} \left(\sum_{j=0}^{p-1} \frac{t^j}{j! \cdot e^{\epsilon t}} \right) \end{aligned}$$

So, you can pick $\lambda = \lambda_m - \epsilon$ and c that bounds the sum and V and V^{-1} terms such that:

$$||\, e^{At} \,|| \leq c e^{-\lambda t}, \quad \forall t \in \mathbb{R}$$

Problem 2: Verify that $\left(e^{At}\right)^T = e^{A^T t}$

Solution:

Using $(A^k)^T = (A^T)^k$ and $(A+B)^T = A^T + B^T$:

$$\begin{aligned}
\left(e^{At}\right)^T &= \left(\sum_{j=0}^{\infty} \frac{A^j t^j}{j!}\right)^T \\
&= \sum_{j=0}^{\infty} \frac{(A^T)^j t^j}{j!} \\
&= e^{A^T t}
\end{aligned}$$

Problem 3: Consider the continuous-time LTI system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n$$

and suppose there exists a positive constant μ and positive-definite matrices $P, Q \in \mathbb{R}^n$ for which the Lyapunov equation

$$A^T P + PA + 2\mu P = -Q$$

holds. Show that all eigenvalues of A have real parts less than $-\mu$.

Solution:

Given this hint: Start by showing that all eigenvalues of A have real parts less than $-\mu$ if and only if all eigenvalues of $A + \mu I$ have real parts less than 0 (i.e., $A + \mu I$ is a stability matrix).

Proof of the hint:

$$Av = \lambda v$$
$$Av + \mu v = \lambda v + \mu v$$
$$(A + \mu I)v = (\lambda + \mu)v$$
$$Re(\lambda(A)) < -\mu \iff Re(\lambda(A) + \mu) < 0 \iff Re(\lambda(A + \mu I)) < 0$$

Now the actual problem:

Set up:

$$A^T P + PA + 2\mu P = -Q$$
$$(A + \mu I)^T P + P(A + \mu I) = -Q$$

Theorem: all evals of $A + \mu I$ have real parts less than 0. The above hint then gives:

$$Re(\lambda(A)) < -\mu \iff Re(\lambda(A + \mu I)) < 0$$

Problem 4: Consider the system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

with

$$A := \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Given a 1×2 matrix $f := \begin{pmatrix} f_1 & f_2 \end{pmatrix}$, compute the characteristic polynomial of $A + Bf$

(b) Select f_1 and f_2 so that the eigenvalues of $A + Bf$ are both zero.

(c) For the matrix f computed above, is the closed-loop system

$$\dot{x} = (A + Bf)x$$

stable?

Solution:

$$\begin{aligned} A + Bf &= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ f_1 & f_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 + f_1 & -2 + f_2 \end{pmatrix} \end{aligned}$$

The characteristic polynomial is

$$\begin{aligned} \lambda^2 - (\mathrm{Tr}(A + Bf))\lambda + \det(A + Bf) &= 0 \\ \lambda^2 - (-2 + f_1)\lambda + (1 - f_1) &= 0 \\ \lambda^2 - (f_1 - 2)\lambda + (1 - f_2) &= 0 \\ \lambda &= \frac{1}{2}\left((f_1 - 2) \pm \sqrt{(f_1 - 2)^2 - 4(1 - f_2)}\right) \\ &= \frac{1}{2}\left(f_1 - 2 \pm \sqrt{(f_1^2 - 4f_1 + 4) + 4f_2 - 4}\right) \\ &= \frac{1}{2}\left(f_1 - 2 \pm \sqrt{f_1^2 - 4f_1 + 4f_2}\right) \end{aligned}$$

To get the eigenvalues to be zero, we need:

$$\begin{aligned} f_1^2 - 4f_1 + 4f_2 &= 0 \\ f_1 &= 2 \end{aligned}$$

Solve for f_2:

$$\begin{aligned} 4 - 8 + 4f_2 &= 0 \\ f_2 &= 1 \end{aligned}$$

So, $f = \begin{pmatrix} 2 & 1 \end{pmatrix}$.

The closed-loop system is:

$$\begin{aligned} \dot{x} &= (A + Bf)x \\ &= \begin{pmatrix} 0 & 1 \\ -1 + f_1 & -2 + f_2 \end{pmatrix} x \\ &= \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} x \end{aligned}$$

The trace and determinant are both negative, so the system is a saddle. So it is not stable.