Homework #3

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Problem 1: Find the general solution of the differential equation $\vec{x}' = \vec{A}\vec{x}$ for A given by

$$A_1 = \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & -1/2 \\ 1/2 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

Draw the phase diagrams for both the original and transformed systems.

Solution:

Part 1: $A_1 = \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix}$

$\mathrm{Tr}(A)^2 = 4\det(A)$, so there is only one eigenvalue, $\lambda = \frac{1}{2}\,\mathrm{Tr}(A) = \frac{3}{2}$.

$A^T \neq \pm A$, so there is only one eigenvector.

Therefore,

$$\begin{aligned} A &= V \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1} \\ \implies \vec{y}(t) &= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \vec{y_0} \\ \implies \vec{x}(t) &= V \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} V^{-1}\vec{x}(0) \end{aligned}$$

Find the eigenvector:

$$\begin{aligned} (A_1 - \lambda I)\vec{v} &= 0 \\ \begin{pmatrix} 1/2 & 1 \\ -1/4 & -1/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \implies \frac{v_1}{2} &= -v_2 \\ \implies \vec{v} &= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \end{aligned}$$

Find $\vec{u}$:

$$(A - \frac{3}{2}I)\vec{u} = \vec{v}$$

$$\begin{pmatrix} \frac{1}{2} & 1 \\ -1/4 & -1/2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\implies \frac{1}{2}u_1 + u_2 = -2$$

$$\implies u_1 + 2u_2 = -4$$

$$u_2 = 0 \implies u_1 = -4$$

$$\implies \vec{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

$$\implies V = \begin{pmatrix} -2 & -4 \\ 1 & 0 \end{pmatrix}$$

Compute V^{-1}:

$$V^{-1} = \begin{pmatrix} 0 & 1 \\ -1/4 & -1/2 \end{pmatrix}$$

Jordan decomposition:

$$\begin{aligned} A &= V \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1} \\ &= \begin{pmatrix} -2 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 1 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1/4 & -1/2 \end{pmatrix} \end{aligned}$$

Solution for $\vec{y}$:

$$\vec{y}(t) = \begin{pmatrix} e^{\frac{3}{2}t} & te^{\frac{3}{2}t} \\ 0 & e^{\frac{3}{2}t} \end{pmatrix} \vec{y_0}$$

Solution for $\vec{x}$:

$$\vec{x}(t) = \begin{pmatrix} -2 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{3}{2}t} & te^{\frac{3}{2}t} \\ 0 & e^{\frac{3}{2}t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1/4 & -1/2 \end{pmatrix} \vec{x_0}$$

Part 2: $A_2 = \begin{pmatrix} 3 & -1/2 \\ 1/2 & 2 \end{pmatrix}$

$\mathrm{Tr}(A)^2 = 4\det(A)$, so there is only one eigenvalue, $\lambda = \frac{1}{2}\,\mathrm{Tr}(A) = \frac{5}{2}$.

$A^T \neq \pm A$, so there is only one eigenvector.

$$(A_2 - \lambda I)\vec{v} = 0$$

$$\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies v_1 - v_2 = 0$$

$$\implies \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Find $\vec{u}$:

$$(A - \frac{5}{2}I)\vec{u} = \vec{v}$$

$$\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\implies \frac{1}{2}u_1 - \frac{1}{2}u_2 = 1$$

$$\implies u_1 - u_2 = 2$$

$$u_2 = 0 \implies u_1 = 2$$

$$\implies \vec{u} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\implies V = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

Compute V^{-1}:

$$V^{-1} = \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \end{pmatrix}$$

Jordan decomposition:

$$A = V \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & 1 \\ 0 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \end{pmatrix}$$

Solution for $\vec{y}$:

$$\vec{y}(t) = \begin{pmatrix} e^{\frac{5}{2}t} & te^{\frac{5}{2}t} \\ 0 & e^{\frac{5}{2}t} \end{pmatrix} \vec{y_0}$$

Solution for $\vec{x}$:

$$\vec{x}(t) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{5}{2}t} & te^{\frac{5}{2}t} \\ 0 & e^{\frac{5}{2}t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \end{pmatrix} \vec{x_0}$$

Part 3: $A_3 = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

$\mathrm{Tr}(A)^2 = 4\det(A)$, so there is only one eigenvalue, $\lambda = \frac{1}{2}\mathrm{Tr}(A) = 2$.

$A^T \neq \pm A$, so there is only one eigenvector.

$$(A_3 - \lambda I)\vec{v} = 0$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies v_1 - v_2 = 0$$

$$\implies \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Find $\vec{u}$:

$$(A - \lambda I)\vec{u} = \vec{v}$$
$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\implies u_1 - u_2 = 1$$
$$u_2 = 0 \implies u_1 = 1$$
$$\implies \vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\implies V = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Compute V^{-1}:

$$V^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Jordan decomposition:

$$\begin{aligned} A &= V \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} V^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

Solution for $\vec{y}$:

$$\vec{y}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \vec{y_0}$$

Solution for $\vec{x}$:

$$\vec{x}(t) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \vec{x_0}$$

Phase Planes below

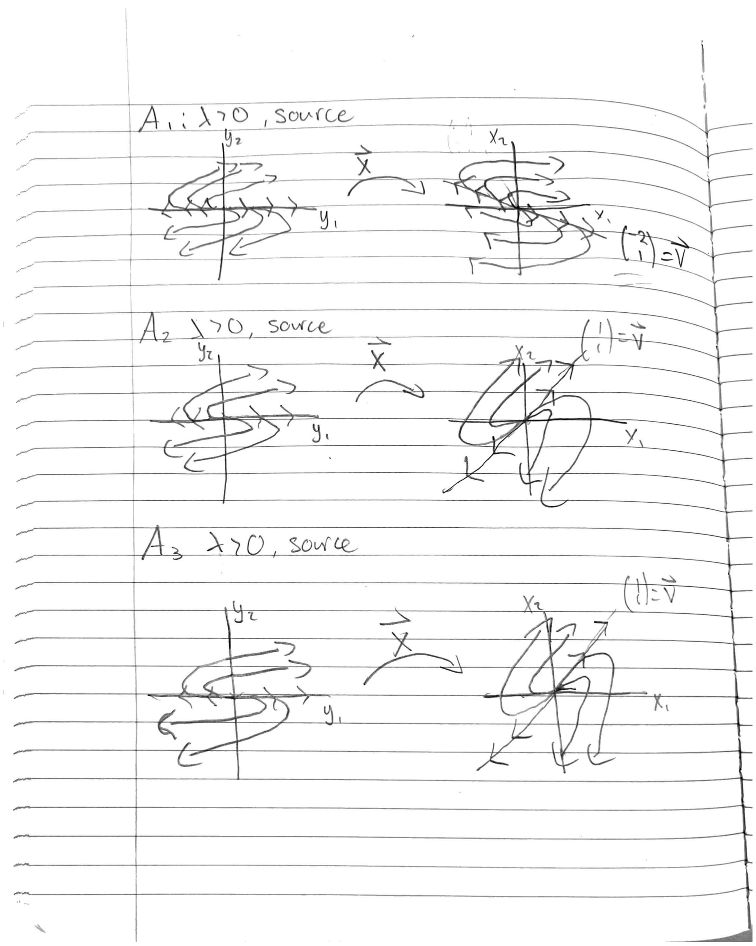

Figure 1: Phase Planes

Problem 2: Prove that for a real 2×2 matrix A, that

$$\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A^T \vec{w} \rangle$$

Solution:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then,

$$\begin{aligned}
\langle A\vec{v}, \vec{w}\rangle &= \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rangle \\
&= (av_1 + bv_2)w_1 + (cv_1 + dv_2)w_2 \\
&= av_1w_1 + bv_2w_1 + cv_1w_2 + dv_2w_2 \\
&= aw_1v_1 + cw_2v_1 + bw_1v_2 + dw_2v_2 \\
&= (aw_1 + cw_2)v_1 + (bw_1 + dw_2)v_2 \\
&= \langle \begin{pmatrix} aw_1 + cw_2 \\ bw_1 + dw_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \vec{v} \rangle \\
&= \langle A^T\vec{w}, \vec{v}\rangle
\end{aligned}$$

Problem 3: Defining $p(\lambda) = \det(\lambda I - A)$, prove that $\det(A^T - \lambda I) = p(\lambda)$, which is to say, the eigenvalues of a real matrix are the same as its transpose.

Solution:

Proof for 2×2:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then,

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \\
&= (a - \lambda)(d - \lambda) - bc \\
&= \lambda^2 - (a + d)\lambda + (ad - bc) \\
&= \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)
\end{aligned}$$

Now for $\det(A^T - \lambda I)$:

$$\begin{aligned}
\det(A^T - \lambda I) &= \det \begin{pmatrix} a - \lambda & c \\ b & d - \lambda \end{pmatrix} \\
&= (a - \lambda)(d - \lambda) - cb \\
&= \lambda^2 - (a + d)\lambda + (ad - bc) \\
&= \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) \\
&= p(\lambda)
\end{aligned}$$

Therefore, the eigenvalues of a real matrix are the same as its transpose in $\mathbb{R}^2$.

Problem 4: Prove that A^{-1} exists if and only if A^{-T} exists.

Solution:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix.

Assume A^{-1} exists.

Then,

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $\det(A) = ad - bc \neq 0$.

Consider $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\det(A^T) = ad - bc = \det(A) \neq 0$. So,

$$A^{-T} = \frac{1}{\det(A)} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

exists.

Now, assume A^{-T} exists. Then,

$$A^{-T} = \frac{1}{\det(A^T)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So, A^{-1} exists because $\det(A) = \det(A^T)$, and there is no division by zero.

Problem 5: Prove that A^{-1} does not exist if 0 is an eigenvalue of A.

Solution:

Let 0 be an eigenvalue of A. Assume, for the sake of contradiction, that A^{-1} exists.

By definition of eigenvalue and eigenvector,

$$\begin{aligned} A\vec{v} &= 0\vec{v} \quad \text{where } \vec{v} \neq 0 \\ A\vec{v} &= 0 \\ A^{-1}A\vec{v} &= A^{-1}0 \\ \vec{v} &= 0 \end{aligned}$$

This is a contradiction, so A^{-1} does not exist.

Problem 6: We say for a real 2×2 matrix A that $\vec{b} \in \operatorname{rng}(A)$ if there exists $\vec{x}$ such that

$$A\vec{x} = \vec{b}$$

Prove that $\vec{b} \in \operatorname{rng}(A)$ if and only if $\langle \vec{b}, \vec{y} \rangle = 0$ for all $\vec{y} \in \ker(A^T)$. Note, for a nontrivial 2×2 matrix, $\ker(A^T)$ is either trivial or onedimensional. Thus either A^{-T} (and thus A^{-1}) exists, or b and any $y \in \ker(A^T)$ form a basis of $\mathbb{R}^2$.

Solution:

Part 1: Assume $\vec{b} \in \operatorname{rng}(A)$.

Then, there exists $\vec{x}$ such that $A\vec{x} = \vec{b}$.

Let $\vec{y} \in \ker(A^T)$.

Consider $\langle \vec{b}, \vec{y} \rangle$:

$$\begin{aligned} \langle \vec{b}, \vec{y} \rangle &= \langle A\vec{x}, \vec{y} \rangle \\ &= \langle \vec{x}, A^T\vec{y} \rangle \\ &= \langle \vec{x}, 0 \rangle \\ &= 0 \end{aligned}$$

Therefore, $\langle \vec{b}, \vec{y} \rangle = 0$ for all $\vec{y} \in \ker(A^T)$.

Part 2: Assume $\langle \vec{b}, \vec{y} \rangle = 0$ for all $\vec{y} \in \ker(A^T)$.

If $\ker(A^T)$ is trivial, then $\ker(A)$ is also trivial, so A spans all of $\mathbb{R}^2$, and thus $\vec{b}$ is in the range of A.

If $\ker(A^T)$ is not trivial, then there exists a nonzero $\vec{y}$ such that $A^T\vec{y} = 0$.

$\langle \vec{b}, \vec{y} \rangle = 0$ implies that $\vec{b} \perp \vec{y}$.

The Fundamental Theorem of Linear Algebra says $\mathbb{R}^2 = \operatorname{rng}(A) \oplus \ker(A^T)$.

Since $\vec{b} \perp \vec{y}$ and $\vec{y} \in \ker(A^T)$, then $\vec{b} \in \operatorname{rng}(A)$.

Problem 7: For real 2×2 matrix A suppose that $\text{Tr}(A) = 4\det(A)$, and $A \neq cI$, so that A has a repeated, degenerate, eigenvalue λ_0, i.e.

$$\det(A - \lambda I) = (\lambda - \lambda_0)^2 .$$

- Let $A\vec{v} = \lambda_0\vec{v}$ and suppose $\vec{w} \cdot \vec{v} = 0$ (which we can always find via Gram-Schmidt). Show that
$$A^T\vec{w} = \lambda_0\vec{w}$$
Hint: Using the results from the prior problems, if A only has one eigenvalue, then so does A^T. Likewise, we're in $\mathbb{R}^2$, so two non-trivial orthogonal vectors form a basis.

- From this and the previous problem, show that $\vec{v} \in \text{rng}(A - \lambda_0 I)$. Therefore we can find $\vec{z}$ such that
$$(A - \lambda_0 I)\vec{z} = \vec{v}$$

- Show finally that
$$A(\vec{v}\vec{z}) = (\vec{v}\vec{z})\begin{pmatrix}\lambda_0 & 1 \\ 0 & \lambda_0\end{pmatrix}$$

- What ensures that $(\vec{v}\vec{z})^{-1}$ exists?

Solution:

Part 1:

Let λ_0 be the only eigenvalue of A and $\vec{v}$ the only eigenvector of A. Let $\vec{w} \cdot \vec{v} = 0$.

$\vec{v} \in \ker(A - \lambda_0 I)$

$\vec{v} \cdot \vec{w} = 0 \implies \vec{v} \perp \vec{w}$.

FTLA says $\mathbb{R}^2 = \ker(A - \lambda_0 I) \oplus \text{rng}(A^T - \lambda_0 I)$.

$\vec{v} \in \ker(A^T - \lambda_0 I) \implies \vec{w} \in \text{rng}(A^T - \lambda_0 I)$.

$$\begin{aligned}(A - \lambda_0 I)\vec{v} &= 0 \\ \langle (A - \lambda_0 I)\vec{v}, \vec{w}\rangle &= 0 \\ \langle \vec{v}, (A^T - \lambda_0 I)\vec{w}\rangle &= 0 \quad (*)\end{aligned}$$

$\implies \vec{v} \perp (A^T - \lambda_0 I)\vec{w} \quad$ where $\vec{v} \perp \vec{w}$

$\implies (A^T - \lambda_0 I)\vec{w} = c\vec{w}$

$c = 0$ because of (*)

So, $(A^T - \lambda_0 I)\vec{w} = 0$

Then, $\vec{w} \in \ker(A^T - \lambda_0 I)$

Therefore, $A^T\vec{w} = \lambda_0\vec{w}$

Part 2:

$\vec{w} \in \ker(A^T - \lambda_0 I)$

FTLA says $\mathbb{R}^2 = \ker(A^T - \lambda_0 I) \oplus \operatorname{rng}(A - \lambda_0 I)$

$\vec{v} \in \operatorname{rng}(A - \lambda_0 I)$

Therefore, $\exists \vec{z}$ such that $(A - \lambda_0 I)\vec{z} = \vec{v}$

Part 3:

$(A - \lambda_0 I)\vec{z} = \vec{v}$

$A\vec{z} = \lambda_0\vec{z} + \vec{v}$

$$\begin{aligned} A\vec{z} &= \lambda_0\vec{z} + \vec{v} \\ A(\vec{v}\vec{z}) &= (\lambda_0\vec{v} \mid \lambda_0\vec{z} + \vec{v}) \\ &= (\vec{v} \mid \vec{z}) \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix} \end{aligned}$$

Part 4:

$(\vec{v}\vec{z})^{-1}$ exists because $\vec{v} \cdot \vec{z} = 0$. Two non-zero orthogonal vectors form a basis, so $(\vec{v}\vec{z})$ is invertible.