Problem 5.2: Consider the following model equation

$$\begin{aligned} \frac{\partial y}{\partial t} &= D\frac{\partial^2 y}{\partial x^2} + \gamma y^3 \\ y(x,0) &= 0,\ 0 < x < \infty \\ y(0,t) &= y_0,\ t > 0,\ y(\infty,t) = 0,\ t > 0 \end{aligned}$$

Non-dimensionalize the equation.

Solution:

Rescale the variables:

$$\begin{aligned} x &= [x]x^* \\ t &= [t]t^* \\ y &= [y]y^* \end{aligned}$$

Scale the derivatives:

$$\begin{aligned} \frac{\partial y}{\partial t} &= \frac{[y]}{[t]}\frac{\partial y^*}{\partial t^*} \\ \frac{\partial^2 y}{\partial x^2} &= \frac{[y]}{[x^2]}\frac{\partial^2 y^*}{\partial x^{*2}} \end{aligned}$$

Substitute the rescaled variables and derivatives into the original equation:

$$\begin{aligned} \frac{[y]}{[t]}\frac{\partial y^*}{\partial t^*} &= D\frac{[y]}{[x^2]}\frac{\partial^2 y^*}{\partial x^{*2}} + \gamma[y^3]y^{*3} \\ \frac{\partial y^*}{\partial t^*} &= D\frac{[t]}{[x^2]}\frac{\partial^2 y^*}{\partial x^{*2}} + \gamma[t][y^2]y^{*3} \end{aligned}$$

Define:

$$\begin{aligned} [x] &= \sqrt{D} \\ [x^2] &= D \\ [y] &= \frac{1}{\sqrt{\gamma}} \\ [y^2] &= \frac{1}{\gamma} \\ [t] &= 1 \end{aligned}$$

Substitute these into the equation:

$$\frac{\partial y^*}{\partial t^*} = \frac{\partial^2 y^*}{\partial x^{*2}} + y^{*3}$$