Homework #6

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October 25, 2024

Problem 1: Given a transfer function $\hat{G}(s)$, let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be a realization for its transpose $\tilde{G}(s) := \hat{G}(a)^T$. Show that (A, B, C, D) is a realization for $\hat{G}(s)$ with

$$A = \tilde{A}^T, \quad B = \tilde{C}^T, \quad C = \tilde{B}^T, \quad D = \tilde{D}^T$$

Solution:

$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a realization for $\tilde{G}(s) := \hat{G}(s)^T$. So

$$\begin{aligned}
\hat{G}(s)^T &= \tilde{C}(s - \tilde{A})^{-1}\tilde{B} + \tilde{D} \\
(\hat{G}(s)^T)^T &= (\tilde{C}(s - \tilde{A})^{-1}\tilde{B} + \tilde{D})^T \\
\hat{G}(s) &= (\tilde{C}(s - \tilde{A})^{-1}\tilde{B})^T + \tilde{D}^T \\
\hat{G}(s) &= \tilde{B}^T(s - \tilde{A}^T)^{-1}\tilde{C}^T + \tilde{D}^T, \quad sI \text{ is diagonal so the transpose does not change anything} \\
\hat{G}(s) &= C(s - A)^{-1}B + D
\end{aligned}$$

Problem 2:

(a) Compute the controllable canonical form realization for the transfer function

$$\hat{g}(s) = \frac{k}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n}$$

(b) For the realization in (a), compute the transfer function from the input u to the new output $y = x_i$, where x_i is the ith element of the state x.

(c) Compute the controllable canonical form realization for the transfer function:

$$\hat{g}(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n}$$

Solution:

(a) $\hat{g}(s)$ is 1×1, A has to be $n \times n$ because of the α's. So C has to be $1 \times n$ and B has to be $n \times 1$.

$$\hat{g}(s) = \frac{1}{d(s)}(k)$$

Using the known matrices:

$$A = \begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & k \end{pmatrix}$$

$$D = \begin{pmatrix} 0 \end{pmatrix}$$

(b) State-space system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

For the new output $y = x_i$, we need to modify the output matrix C to select the ith state. The new C matrix, denoted as C_i, will be:

$$C_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

where the 1 is in the ith position. The new transfer function from the input u to the output $y = x_i$ is given by:

$$\hat{G}_i(s) = C_i(s - A)^{-1}B + D$$

Substituting the known matrices A, B, C_i, and D, we get:

$$\hat{G}_i(s) = \begin{pmatrix} 0 & 0 & \cdots & k \end{pmatrix} \begin{pmatrix} s+\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ -1 & s & 0 & \cdots & 0 \\ 0 & -1 & s & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(c) Like before but with β's,

$$A = \begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$C = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{pmatrix}$$
$$D = \begin{pmatrix} 0 \end{pmatrix}$$

(d) Transpose and swap some stuff:

$$A = \begin{pmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ -\alpha_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$
$$D = \begin{pmatrix} 0 \end{pmatrix}$$

Problem 3: Consider the following two systems:

$$\dot{x} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} x$$
$$\dot{x} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} x$$

(a) Are these systems zero-state equivalent?

(b) Are they algebraically equivalent?

Solution:

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$$
$$D = \begin{pmatrix} 0 \end{pmatrix}$$

Compute:

$$\begin{aligned} \hat{g}_1(s) &= C(s-A)^{-1}B + D \\ &= \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{(s-2)^2} \end{aligned}$$

The other one:

$$\begin{aligned} \hat{g}_2(s) &= C(s-A)^{-1}B + D \\ &= \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} s-2 & -1 & -1 \\ 0 & s-2 & -1 \\ 0 & 0 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{(s-2)^2} \end{aligned}$$

They are zero-state equivalent. They are not algebraically equivalent because the eigenvalues are different.

Problem 4: Consider the following system:

$$\dot{x} = Ax + Bu, \quad y = Cx$$

with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

and the feedback state control

$$u = -\begin{pmatrix} f_1 & f_2 \end{pmatrix} x + v$$

where v denotes a disturbance input and f_1 and f_2 two scalar constants to be specified later.

(a) Compute the close-loop state-space model with input v and output y, leaving your answer as a function of the constants f_1 and f_2.

(b) Compute the closed loop transfer function from v to y, leaving your answer as a function of the constants f_1 and f_2.

(c) Determine values for f_1 and f_2 so as to obtain the following transfer function from v to y:

$$\frac{1}{s+1}$$

Solution:

(a)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= Ax + B(-Fx + v) \\ &= (A - BF)x + Bv \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} f_1 & f_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \\ &= \begin{pmatrix} 0 & 1 \\ -f_1 & -f_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \end{aligned}$$

The output equation remains:

$$y = Cx = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

Therefore, the closed-loop state-space model is:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -f_1 & -f_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{aligned}$$

(b) New transfer function:

$$G(s) = C(s - (A - BF))^{-1}B$$

Find $(s - (A - BF))$:

$$s - (A - BF) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -f_1 & -f_2 \end{pmatrix} = \begin{pmatrix} s & -1 \\ f_1 & s + f_2 \end{pmatrix}$$

New transfer function:

$$G(s) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} s & -1 \\ f_1 & s + f_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Simplify:

$$G(s) = \frac{s+1}{s^2 + f_2 s + f_1}$$

(c)

$$\frac{s+1}{s^2 + f_2 s + f_1} = \frac{1}{s+1}$$
$$s^2 + 2s + 1 = s^2 + f_2 s + f_1$$

So,

$$f_1 = 1, \quad f_2 = 2$$