Homework #4

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I put all my phase planes at the end of the document.

Problem 1: Find the general solution to

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \vec{x}$$

Plot the phase diagram for $a = \pm 1$. What happens when $a = 0$?

Solution:

Clearly, the eigenvalues of this matrix are $\lambda_1 = 1$ and $\lambda_2 = a$, and the eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, the general solution to this system is given by

$$\begin{aligned}
\vec{x}(t) &= V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1} \vec{x}(0) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{at} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x}(0) \\
&= \begin{pmatrix} e^{t} & 0 \\ 0 & e^{at} \end{pmatrix} \vec{x}(0) \\
&= x_1(0)e^{t} + x_2(0)e^{at}.
\end{aligned}$$

When $a = 1$, the solution is

$$\vec{x}(t) = x_1(0)e^{t} + x_2(0)e^{t} = (x_1(0) + x_2(0))e^{t}.$$

A source.

When $a = -1$, the solution is

$$\vec{x}(t) = x_1(0)e^{t} + x_2(0)e^{-t}.$$

A saddle.

When $a = 0$, the solution is

$$\begin{aligned}\vec{x}(t) &= x_1(0)e^t + x_2(0)e^{0t} \\ &= x_1(0)e^t + x_2(0).\end{aligned}$$

A source.

Phase planes at the end of the document.

Problem 2: Write
$$\ddot{x} + \dot{x} + 4x = 0$$
as a system, find its general solution, and sketch its phase diagram.

Solution:

Let $y = \frac{dx}{dt}$. Then, the equation becomes
$$\frac{dy}{dt} + y + 4x = 0$$
This gives the system
$$\begin{aligned}\frac{dy}{dt} &= -y - 4x \\ \frac{dx}{dt} &= y\end{aligned}$$
Write as a matrix equation
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
Eigenvalues:
$$\begin{aligned}\lambda &= \frac{1}{2}(\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}) \\ &= \frac{1}{2}(-1 \pm \sqrt{1-16}) \\ &= \frac{1}{2}(-1 \pm i\sqrt{15})\end{aligned}$$
Eigenvectors:
$$\begin{aligned}A - \lambda I &= \begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}(-1+i\sqrt{15}) & 0 \\ 0 & \frac{1}{2}(-1+i\sqrt{15}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} - \frac{i\sqrt{15}}{2} & 1 \\ -4 & \frac{-1}{2} - \frac{i\sqrt{15}}{2} \end{pmatrix}\end{aligned}$$
Solving for the eigenvector, we get
$$\begin{pmatrix} \frac{1}{2} - \frac{i\sqrt{15}}{2} & 1 \\ -4 & \frac{-1}{2} - \frac{i\sqrt{15}}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
Gives
$$-4v_1 + \left(\frac{-1}{2} - \frac{i\sqrt{15}}{2}\right)v_2 = 0$$
Let $v_2 = 1$. Then,
$$-4v_1 + \left(\frac{-1}{2} - \frac{i\sqrt{15}}{2}\right) = 0 \implies v_1 = \frac{-1}{8} - \frac{i\sqrt{15}}{8}$$

Thus, the eigenvector is

$$\vec{v}_1 = \begin{pmatrix} \frac{-1}{8} - \frac{i\sqrt{15}}{8} \\ 1 \end{pmatrix}$$

Second eigenvector:

$$\begin{pmatrix} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 1 \\ -4 & \frac{-1}{2} + \frac{i\sqrt{15}}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gives

$$-4v_1 + \left(\frac{-1}{2} + \frac{i\sqrt{15}}{2} \right) v_2 = 0$$

Let $v_2 = 1$. Then,

$$-4v_1 + \left(\frac{-1}{2} + \frac{i\sqrt{15}}{2} \right) = 0 \implies v_1 = \frac{-1}{8} + \frac{i\sqrt{15}}{8}$$

Thus, the eigenvector is

$$\vec{v}_2 = \begin{pmatrix} \frac{-1}{8} + \frac{i\sqrt{15}}{8} \\ 1 \end{pmatrix}$$

Define V and V^{-1} as follows:

$$V = \begin{pmatrix} \frac{-1}{8} - \frac{i\sqrt{15}}{8} & \frac{-1}{8} + \frac{i\sqrt{15}}{8} \\ 1 & 1 \end{pmatrix}$$

$$V^{-1} = \frac{1}{\frac{i\sqrt{15}}{4}} \begin{pmatrix} 1 & \frac{1}{8} - \frac{i\sqrt{15}}{8} \\ -1 & \frac{-1}{8} - \frac{i\sqrt{15}}{8} \end{pmatrix}$$

Then, the general solution is given by

$$\vec{x}(t) = \begin{pmatrix} \frac{-1}{8} - \frac{i\sqrt{15}}{8} & \frac{-1}{8} + \frac{i\sqrt{15}}{8} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\left(\frac{-1}{2} + \frac{i\sqrt{15}}{2}\right)t} & 0 \\ 0 & e^{\left(\frac{-1}{2} - \frac{i\sqrt{15}}{2}\right)t} \end{pmatrix} \frac{1}{\frac{i\sqrt{15}}{4}} \begin{pmatrix} 1 & \frac{1}{8} - \frac{i\sqrt{15}}{8} \\ -1 & \frac{-1}{8} - \frac{i\sqrt{15}}{8} \end{pmatrix} \vec{x}_0$$

Problem 3: Write

$$\ddot{x} + 4x = 0$$

as a system, find its general solution, and sketch its phase diagram.

Solution:

Let $y = \frac{dx}{dt}$. Then, the equation becomes

$$\frac{dy}{dt} + 4x = 0$$

This gives the system

$$\frac{dy}{dt} = -4x$$
$$\frac{dx}{dt} = y$$

Write as a matrix equation

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} \lambda &= \frac{1}{2}(\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}) \\ &= \frac{1}{2}(0 \pm \sqrt{0 - 16}) \\ &= \pm 2i \end{aligned}$$

Eigenvectors:

$$\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gives

$$-2iv_1 + v_2 = 0 \implies v_2 = 2iv_1$$

Let $v_1 = -i$. Then,

$$v_2 = 2i(-i) = 2$$

Thus, the eigenvector is

$$\vec{v}_1 = \begin{pmatrix} -i \\ 2 \end{pmatrix}$$

Second eigenvector:

$$\begin{pmatrix} 2i & 1 \\ -4 & 2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gives

$$2iv_1 + v_2 = 0 \implies v_2 = -2iv_1$$

Let $v_1 = i$. Then,

$$v_2 = -2i(i) = 2$$

Thus, the eigenvector is

$$\vec{v}_2 = \begin{pmatrix} i \\ 2 \end{pmatrix}$$

Define V and V^{-1} as follows:

$$V = \begin{pmatrix} -i & i \\ 2 & 2 \end{pmatrix}$$

$$V^{-1} = \frac{i}{4}\begin{pmatrix} 2 & -i \\ -2 & -i \end{pmatrix}$$

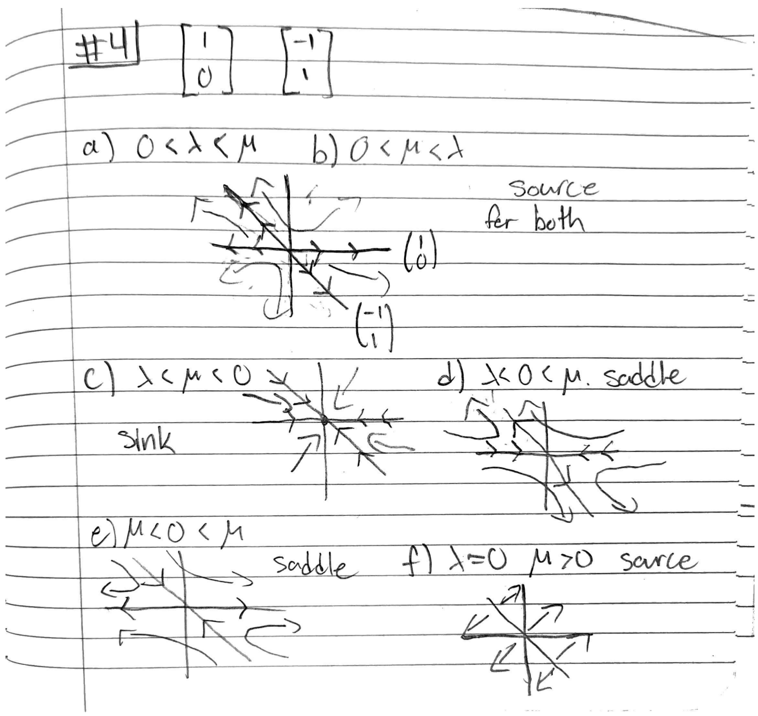
Then, the general solution is given by

$$\vec{x}(t) = \begin{pmatrix} -i & i \\ 2 & 2 \end{pmatrix}\begin{pmatrix} e^{(2i)t} & 0 \\ 0 & e^{(-2i)t} \end{pmatrix}\frac{i}{4}\begin{pmatrix} 2 & -i \\ -2 & -i \end{pmatrix}\vec{x}_0$$

Problem 4: Let the 2×2 matrix A have real, distinct eigenvalues λ and μ. Suppose that an eigenvector of λ is $(1,0)^T$ and an eigenvector of μ is $(-1,1)^T$. Sketch the phase portraits of $\dot{\vec{x}} = A\vec{x}$ for the following cases:

(a) $0 < \lambda < \mu$, (b) $0 < \mu < \lambda$, (c) $\lambda < \mu < 0$,
(d) $\lambda < 0 < \mu$, (e) $\mu < 0 < \lambda$, (f) $\lambda = 0$, $\mu > 0$.

Solution:



Problem 5: Find the general solution to

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \vec{x}$$

Plot the phase diagram.

Solution:

Eigenvalues:

$$\begin{aligned} \lambda &= \frac{1}{2}(\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}) \\ &= \frac{1}{2}(4 \pm \sqrt{16 - 20}) \\ &= 2 \pm i \end{aligned}$$

Eigenvectors:

$$\begin{pmatrix} 1-i & -2 \\ 1 & -1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gives

$$v_1 - (1+i)v_2 = 0 \implies v_1 = (1+i)v_2$$

Let $v_2 = 1$. Then,

$$v_1 = 1 + i$$

Thus, the eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

Second eigenvector:

$$\begin{pmatrix} 1+i & -2 \\ 1 & -1+i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gives

$$v_1 + (-1+i)v_2 = 0 \implies v_1 = (1-i)v_2$$

Let $v_2 = 1$. Then,

$$v_1 = 1 - i$$

Thus, the eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

Define V and V^{-1} as follows:

$$\begin{aligned} V &= \begin{pmatrix} 1+i & 1-i \\ 1 & 1 \end{pmatrix} \\ V^{-1} &= \frac{1}{2i} \begin{pmatrix} 1 & -1+i \\ -1 & 1+i \end{pmatrix} \end{aligned}$$

Then, the general solution is given by

$$\vec{x}(t) = \begin{pmatrix} 1+i & 1-i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(2+i)t} & 0 \\ 0 & e^{(2-i)t} \end{pmatrix} \frac{1}{2i} \begin{pmatrix} 1 & -1+i \\ -1 & 1+i \end{pmatrix} \vec{x}_0$$

Problem 6: Find the solution to

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} t \\ 1 \end{pmatrix} \ , \ \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution:

Find the eigenvalues:

$$\begin{aligned} \lambda &= \frac{1}{2}(\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}) \\ &= \frac{1}{2}(4 \pm \sqrt{4^2 - 4 \cdot 3}) \\ &= \frac{1}{2}(4 \pm \sqrt{4}) \\ &= 2 \pm 1 \end{aligned}$$

$$\lambda_1 = 3, \lambda_2 = 1$$

Eigenvectors:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gives

$$-v_1 + v_2 = 0 \implies v_1 = v_2$$

Let $v_1 = 1$. Then,

$$v_2 = 1$$

Thus, the eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Second eigenvector:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gives

$$v_1 + v_2 = 0 \implies v_1 = -v_2$$

Let $v_1 = 1$. Then,

$$v_2 = -1$$

Thus, the eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Define V and V^{-1} as follows:

$$V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$V^{-1} = \frac{-1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

Then, the general solution is given by

$$\vec{x}(t) = \Phi(t,0)\vec{x}_0 + \int_0^t \Phi(t,s)\vec{f}(s)ds$$

where

$$\Phi(t,s) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{3(t-s)} & 0 \\ 0 & e^{t-s} \end{pmatrix} \frac{-1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$\vec{f}(s) = \begin{pmatrix} s \\ 1 \end{pmatrix}$$

Simplify the integral

$$\begin{aligned}
\int_0^t \Phi(t,s)\vec{f}(s)ds &= \int_0^t \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{3(t-s)} & 0 \\ 0 & e^{t-s} \end{pmatrix} \frac{-1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} s \\ 1 \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} e^{3(t-s)} & e^{t-s} \\ e^{3(t-s)} & -e^{t-s} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} s \\ 1 \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} e^{3(t-s)} & e^{t-s} \\ e^{3(t-s)} & -e^{t-s} \end{pmatrix} \begin{pmatrix} \frac{s}{2} + \frac{1}{2} \\ \frac{s}{2} - \frac{1}{2} \end{pmatrix} ds \\
&= \frac{1}{2} \int_0^t \begin{pmatrix} e^{3(t-s)} & e^{t-s} \\ e^{3(t-s)} & -e^{t-s} \end{pmatrix} \begin{pmatrix} s+1 \\ s-1 \end{pmatrix} ds \\
&= \frac{1}{2} \int_0^t \begin{pmatrix} e^{3(t-s)}(s+1) + e^{t-s}(s-1) \\ e^{3(t-s)}(s+1) - e^{t-s}(s-1) \end{pmatrix} ds \\
&= \frac{1}{2} \begin{pmatrix} \int_0^t e^{3(t-s)}(s+1)ds + \int_0^t e^{t-s}(s-1)ds \\ \int_0^t e^{3(t-s)}(s+1)ds - \int_0^t e^{t-s}(s-1)ds \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \frac{1}{9}(-3t + 4te^{3t} - 4) + \frac{1}{9}(-3t - 2e^{3t} + 2) \\ \frac{1}{9}(-3t + 4te^{3t} - 4) - \frac{1}{9}(-3t - 2e^{3t} + 2) \end{pmatrix}
\end{aligned}$$

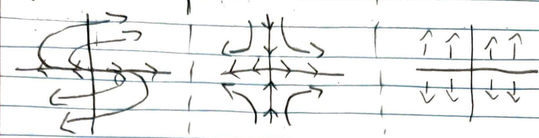
The solution is

$$\begin{aligned}
\vec{x}(t) &= \Phi(t,0)\vec{x}_0 + \int_0^t \Phi(t,s)\vec{f}(s)ds \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{1}{9}(-3t + 4te^{3t} - 4) + \frac{1}{9}(-3t - 2e^{3t} + 2) \\ \frac{1}{9}(-3t + 4te^{3t} - 4) - \frac{1}{9}(-3t - 2e^{3t} + 2) \end{pmatrix}
\end{aligned}$$

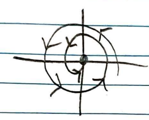
1 Phase Planes

#1

$a=1$ $\lambda=1>0$ degenerate source | $a=-1$ $\lambda_{1,2}=\pm 1$ saddle | $a=0$ $\lambda_1=1$ $\lambda_2=0$ source

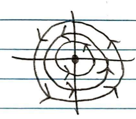


#2 $\lambda \in \mathbb{C}$, $Tr(A)<1$



Spiral sink

#3 $\lambda \in \mathbb{C}$, $Tr(A)=0$



#5 $\lambda \in \mathbb{C}$, $Tr(A)>0$ Spiral source