Problem 4: The favorite food of the tiger shark is the sea turtle. A two-species prey-predator model is given by

$$\frac{dP}{dt} = P(a - bP - cS)$$
$$\frac{dS}{dt} = S(-k + \lambda P)$$

where P is the sea turtle, S is the shark, $a, b, c, k, \lambda > 0$.

1. Let $b = 0$ and the value of k is increased. Ecologically, what is the interpretation of increasing k and what is its effect on the non-zero equilibrium populations of sea turtles and sharks?
2. Obtain all the equilibrium solutions for $b = 0$ and $b \neq 0$.
3. Obtain the linearized system of differential equations about the equilibrium point $P^* = \frac{k}{\lambda}$ and $S^* = \frac{a}{c} - \frac{bk}{c\lambda} > 0$, which can be put in the form
$$\frac{dP_1}{dt} = \frac{k}{\lambda}(-bP_1 - cS_1)$$
$$\frac{dS_1}{dt} = \lambda P_1 \left(\frac{a}{c} - \frac{bk}{c\lambda}\right)$$
4. Obtain the conditions for which the linearized system is stable.
5. Draw the solution curves in the phase plane with $a = 0.5, b = 0.5, c = 0.01, k = 0.3, \lambda = 0.01$. What do you expect to happen to the dynamics of the model if $c = 0$?

Solution:

1. Ecological Interpretation and Effect of Increasing k

   When $b = 0$, the equation for $\frac{dP}{dt}$ is
$$\frac{dP}{dt} = P(a - cS)$$
   Increasing k in the equation $\frac{dS}{dt} = S(-k + \lambda P)$ can be interpreted as increasing the death rate of the shark population or decreasing the reproduction reproduction rate.

   Set the equations equal to zero to find the equilibrium points:
$$P(a - cS) = 0$$
$$S(-k + \lambda P) = 0$$
   Solving for non-zero equilibrium points gives
$$P^* = \frac{k}{\lambda}$$

$$S^* = \frac{a}{c}$$

As k increases, P^* increases, so the equilibrium population of sea turtles increases. S^* does not depend on k, so the equilibrium population of sharks does not change.

2. Equilibrium Solutions for $b = 0$ and $b \neq 0$

   For $b = 0$: The system becomes:

$$\frac{dP}{dt} = P(a - cS)$$
$$\frac{dS}{dt} = S(-k + \lambda P)$$

   The trivial equilibrium points are $P = 0$ and $S = 0$. The non-zero equilibrium points are found by solving:

$$a - cS = 0 \implies S = \frac{a}{c}$$
$$-k + \lambda P = 0 \implies P = \frac{k}{\lambda}$$

   The equilibrium points are:

$$(P_0^*, S_0^*) = (0, 0)$$
$$(P_1^*, S_1^*) = \left(\frac{k}{\lambda}, \frac{a}{c}\right)$$

   For $b \neq 0$: The equations are:

$$P(a - bP - cS) = 0$$
$$S(-k + \lambda P) = 0$$

   The trivial equilibrium points are $P = 0$ and $S = 0$. Solving for non-zero equilibria:

$$a - bP - cS = 0$$
$$-k + \lambda P = 0 \implies P = \frac{k}{\lambda}$$

   Substitute $P = \frac{k}{\lambda}$ into the first equation:

$$a - b\left(\frac{k}{\lambda}\right) - cS = 0 \implies S = \frac{a}{c} - \frac{bk}{c\lambda}$$

   Therefore, the non-zero equilibrium points are

$$(P_0^*, S_0^*) = (0, 0)$$
$$(P_1^*, S_1^*) = \left(\frac{k}{\lambda}, \frac{a}{c} - \frac{bk}{c\lambda}\right)$$

3. Linearization of the System Around the Equilibrium Point (P^*, S^*)

   Compute the Jacobian Matrix

$$J = \begin{pmatrix} \frac{\partial}{\partial P}\left(\frac{dP}{dt}\right) & \frac{\partial}{\partial S}\left(\frac{dP}{dt}\right) \\ \frac{\partial}{\partial P}\left(\frac{dS}{dt}\right) & \frac{\partial}{\partial S}\left(\frac{dS}{dt}\right) \end{pmatrix}$$
$$= \begin{pmatrix} a - 2bP - cS & -cP \\ \lambda S & -k + \lambda P \end{pmatrix}$$

   Evaluate the Jacobian at the equilibrium point

$$P^* = \frac{k}{\lambda} \quad S^* = \frac{a}{c} - \frac{bk}{c\lambda}$$

   Substitute P^* and S^* into the Jacobian matrix:

$$J(P^*, S^*) = \begin{pmatrix} a - 2b\left(\frac{k}{\lambda}\right) - c\left(\frac{a}{c} - \frac{bk}{c\lambda}\right) & -c\left(\frac{k}{\lambda}\right) \\ \lambda\left(\frac{a}{c} - \frac{bk}{c\lambda}\right) & -k + \lambda\left(\frac{k}{\lambda}\right) \end{pmatrix}.$$

   Simplify the matrix:

$$J(P^*, S^*) = \begin{pmatrix} a - \frac{2bk}{\lambda} - \left(a - \frac{bk}{\lambda}\right) & -\frac{ck}{\lambda} \\ \frac{\lambda a}{c} - \frac{bk}{c} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{bk}{\lambda} & -\frac{ck}{\lambda} \\ \frac{\lambda a}{c} - \frac{bk}{c} & 0 \end{pmatrix}$$

   Write out the matrix form:

$$\begin{pmatrix} \frac{dP_1}{dt} \\ \frac{dS_1}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{bk}{\lambda} & -\frac{ck}{\lambda} \\ \frac{\lambda a}{c} - \frac{bk}{c} & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ S_1 \end{pmatrix}$$

   Write out the system of equations:

$$\frac{dP_1}{dt} = -\frac{bk}{\lambda}P_1 - \frac{ck}{\lambda}S_1$$
$$\frac{dS_1}{dt} = \left(\frac{\lambda a}{c} - \frac{bk}{c}\right)P_1$$

   This can also be written as:

$$\frac{dP_1}{dt} = \frac{k}{\lambda}(-bP_1 - cS_1)$$
$$\frac{dS_1}{dt} = \lambda P_1\left(\frac{a}{c} - \frac{bk}{c\lambda}\right)$$

4. Conditions for Stability of the Linearized System

   Consider the Jacobian matrix evaluated at the equilibrium point (P^*, S^*):

$$J(P^*, S^*) = \begin{pmatrix} -\frac{bk}{\lambda} & -\frac{ck}{\lambda} \\ \frac{\lambda a}{c} - \frac{bk}{c} & 0 \end{pmatrix}$$

The system is stable when the trace of the Jacobian is negative and the determinant is positive.

$$\mathrm{Tr}(J) = -\frac{bk}{\lambda}$$
$$\det(J) = \left(0 - \left(-\frac{ck}{\lambda}\right)\left(\frac{\lambda a}{c} - \frac{bk}{c\lambda}\right)\right) = \frac{ck}{\lambda}\left(\frac{\lambda a}{c} - \frac{bk}{c\lambda}\right) = ak - \frac{bk^2}{c\lambda^2}$$

For stability, we need:

$$\mathrm{Tr}(J) < 0 \implies -\frac{bk}{\lambda} < 0$$
$$\det(J) > 0 \implies ak - \frac{bk^2}{c\lambda^2} > 0$$

5. Phase Plane

   Parameters:

   $$a = 0.5, \quad b = 0.5, \quad c = 0.01, \quad k = 0.3, \quad \lambda = 0.01$$

   Equilibrium Point:

   $$P^* = \frac{k}{\lambda} = \frac{0.3}{0.01} = 30$$
   $$S^* = \frac{a}{c} - \frac{bk}{c\lambda} = \frac{0.5}{0.01} - \frac{0.5 \cdot 0.3}{0.01 \cdot 0.01} = 50 - 1500 = -1450$$

   S^* cannot be negative because there can't be a negative number of sharks so this equilibrium point is not feasible. The only equilibrium point is the trivial equilibrium $(0, 0)$.

   Phase Plane Dynamics:

   The Jacobian at $(0, 0)$ is:

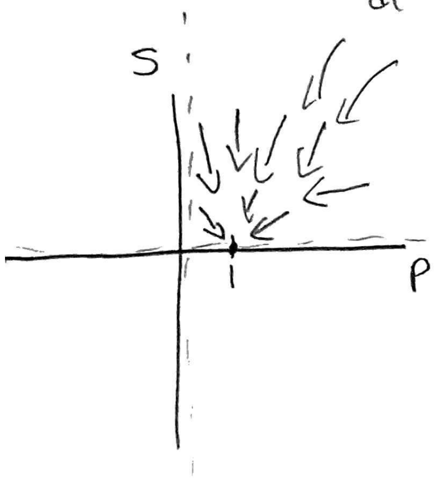
   $$J(0,0) = \begin{pmatrix} a & -c \cdot 0 \\ \lambda \cdot 0 & -k \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & -0.3 \end{pmatrix}$$

   The eigenvalues of this matrix are $\lambda_1 = 0.5$ and $\lambda_2 = -0.3$. Since one eigenvalue is positive and the other is negative, the origin is a saddle.

   The phase plan is on the next page.

   If $c = 0$, the turtle population P follows logistic growth unaffected by the sharks. The shark population still depends on the turtle population.

   Graphs:



$P > 0,\ S > 0$

$\frac{dP}{dt}\ P(0.5 - 0.5P - 0.01S)$

If $P > 1$: $\frac{dP}{dt} > 0 \ \rightarrow$

If $P < 1$: $\frac{dP}{dt} < 0 \ \leftarrow$

$\frac{dS}{dt} = S(-0.3 + 0.01S)$

$\frac{dS}{dt} < 0 \ \downarrow$