Midterm 1

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Problem 1: For all $t \geq 0$, solve the differential equation

$$\frac{dy}{dt} + y = f(t), \quad y(0) = y_0$$

where $f(t)$ is periodic with period T, i.e. $f(t+T) = f(t)$, and on $[0, T]$, $f(t)$ is defined to be

$$f(t) = \begin{cases} 1 & 0 \leq t < \frac{T}{2} \\ 0 & \frac{T}{2} \leq t < T \end{cases}$$

Solution:

Define the integrating factor,

$$\Phi(t, s) = e^{-\int_s^t 1dt} = e^{s-t}$$

Solution to the ODE:

$$\begin{aligned} y(t) &= \Phi(t, 0) \cdot y(0) + \int_0^t \Phi(t, s) \cdot f(s)ds \\ &= e^{-t} \cdot y_0 + \int_0^t e^{s-t} \cdot f(s)ds \end{aligned}$$

Evaluate the integral for each interval of $f(t)$:

- For $0 \leq t < \frac{T}{2}$:

$$\int_0^t e^{s-t} \cdot 1ds = \int_0^t e^{s-t}ds = e^{s-t}\Big|_0^t = e^{t-t} - e^{0-t} = 1 - e^{-t}$$

- For $\frac{T}{2} \leq t < T$:

$$\int_0^{\frac{T}{2}} e^{s-t} \cdot 1ds + \int_{\frac{T}{2}}^t e^{s-t} \cdot 0ds = \int_0^{\frac{T}{2}} e^{s-t}ds = e^{s-t}\Big|_0^{\frac{T}{2}} = e^{\frac{T}{2}-t} - e^{-t}$$

- For $T \leq t < \frac{3T}{2}$:

$$\begin{aligned}
\int_0^{\frac{T}{2}} e^{s-t} \cdot 1ds + \int_{\frac{T}{2}}^{T} e^{s-t} \cdot 0ds + \int_T^t e^{s-t} \cdot 1ds &= \int_0^{\frac{T}{2}} e^{s-t}ds + \int_T^t e^{s-t}ds \\
&= e^{\frac{T}{2}-t} - e^{-t} + e^{t-t} - e^{T-t} \\
&= e^{\frac{T}{2}-t} - e^{-t} + 1 - e^{T-t}
\end{aligned}$$

- For $\frac{3T}{2} \leq t < 2T$:

$$\begin{aligned}
\int_0^{\frac{T}{2}} e^{s-t} \cdot 1ds + 0 + \int_T^{\frac{3T}{2}} e^{s-t} \cdot 1ds + 0 &= \int_0^{\frac{T}{2}} e^{s-t}ds + \int_T^{\frac{3T}{2}} e^{s-t}ds \\
&= (e^{\frac{T}{2}-t} - e^{-t}) + (e^{\frac{3T}{2}-t} - e^{T-t})
\end{aligned}$$

Note:

$$\int_0^T = \int_0^{\frac{T}{2}}$$

There is a pattern:

$$\int_0^T e^{s-t}ds = e^{\frac{T}{2}-t} - e^{-t}$$

$$\int_T^{2T} e^{s-t}ds = e^{\frac{3T}{2}-t} - e^{T-t}$$

Generalizing,

$$\int_{kT}^{(k+1)T} e^{s-t}ds = e^{\frac{(2k+1)T}{2}-t} - e^{kT-t}$$

Therefore,

$$\int_0^t e^{s-t}ds = \sum_{j=0}^{n-1} \int_{jT}^{(j+1)T} e^{s-t}ds + \int_{nT}^t e^{s-t}ds$$

The solution is

$$\begin{aligned}
y(t) &= e^{-t}y_0 + \sum_{j=0}^{n-1} \int_{jT}^{(j+1)T} e^{s-t}ds + \int_{nT}^t e^{s-t}ds \\
&= e^{-t}y_0 + \sum_{j=0}^{n-1} (e^{\frac{(2j+1)T}{2}-t} - e^{jT-t}) + \int_{nT}^t e^{s-t}ds \\
&= e^{-t}y_0 + \sum_{j=0}^{n-1} (e^{jT} \cdot e^{\frac{T}{2}-t} - e^{jT} \cdot e^{-t}) + \int_{nT}^t e^{s-t}ds \\
&= e^{-t}y_0 + \sum_{j=0}^{n-1} e^{jT}(e^{\frac{T}{2}-t} - e^{-t}) + \int_{nT}^t e^{s-t}ds \\
&= e^{-t}y_0 + (e^{\frac{T}{2}-t} - e^{-t}) \sum_{j=0}^{n-1} e^{jT} + \int_{nT}^t e^{s-t}ds
\end{aligned}$$

Consider the identity

$$\sum_{j=0}^{n-1} e^{jT} = \frac{e^{nT}-1}{e^T-1}$$

Therefore,

$$\begin{aligned} y(t) &= e^{-t}y_0 + (e^{\frac{T}{2}-t} - e^{-t}) \cdot \frac{e^{nT}-1}{e^T-1} + \int_{nT}^{t} e^{s-t}ds \\ &= e^{-t}y_0 + (e^{\frac{T}{2}-t} - e^{-t}) \cdot \frac{e^{nT}-1}{e^T-1} + (1 - e^{nT-t}) \end{aligned}$$

where $n = \lfloor \frac{t}{T} \rfloor$.

Problem 2: Sketch the phase portraits corresponding to the matrices

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & -2 \\ 5 & -2 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Solution:

For,

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Find the eigenvalues,

$$\begin{aligned} \lambda &= \frac{1}{2}(\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}) \\ &= \frac{1}{2}(2 \pm \sqrt{4 - 4(-8)}) \\ &= \frac{1}{2}(2 \pm \sqrt{36}) \\ &= \frac{1}{2}(2 \pm 6) \\ &= 1 \pm 3 \\ &= -2, 4 \end{aligned}$$

This is a saddle. Find the eigenvectors for $\lambda = -2$,

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 + 3x_2 = 0$$

$$x_1 = -x_2$$

So the eigenvector for $\lambda = -2$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\lambda < 0$, so this goes towards the origin.

Find the eigenvectors for $\lambda = 4$,

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 3x_2 = 0$$

$$x_1 = x_2$$

So the eigenvector for $\lambda = 4$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\lambda > 0$, so this goes away from the origin.

For,

$$A = \begin{pmatrix} 3 & -2 \\ 5 & -2 \end{pmatrix}$$

Find the eigenvalues,

$$\begin{aligned}\lambda &= \frac{1}{2}(1 \pm \sqrt{1^2 - 4 \cdot 4}) \\ &= \frac{1}{2}(1 \pm \sqrt{-15}) \\ &= \frac{1}{2}(1 \pm i\sqrt{15})\end{aligned}$$

$\lambda \in \mathbb{C}$, so this is a spiral. $\mathrm{Tr}(A) > 0$ and $\det(A) > 0$, so this spirals outwards from the origin.

For,

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Find the eigenvalues,

$$\begin{aligned}\lambda &= \frac{1}{2}(-2 \pm \sqrt{(-2)^2 - 4 \cdot 0}) \\ &= \frac{1}{2}(-2 \pm 2) \\ &= -2, 0\end{aligned}$$

Find the eigenvectors for $\lambda = -2$,

$$\begin{aligned}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x_1 + x_2 &= 0 \\ x_1 &= -x_2\end{aligned}$$

So the eigenvector for $\lambda = -2$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\lambda < 0$, so this goes towards the origin.

Find the eigenvectors for $\lambda = 0$,

$$\begin{aligned}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ -x_1 + x_2 &= 0 \\ x_1 &= x_2\end{aligned}$$

So the eigenvector for $\lambda = 0$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\lambda = 0$, so this nothing moves on this vector.

Phase planes:

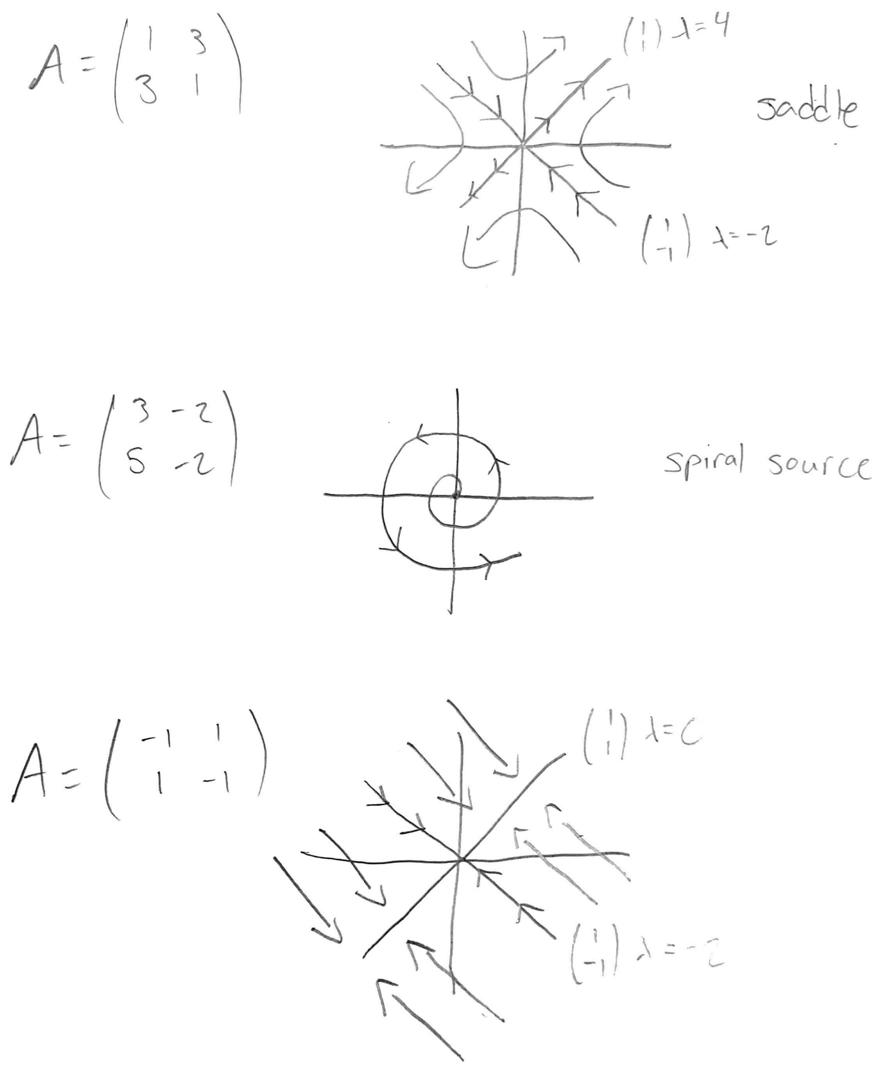

Figure 1: Phase planes

Problem 3: Solve the differential equation

$$\frac{d\vec{x}}{dt} = A\vec{x},\ x(0) = x_0$$

for,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

Solution:

For,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The matrix is diagonal, so the eigenvalues are

$$\lambda = 1, 2, 3$$

and the eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Both V and V^{-1} are the identity matrix, so the equation is

$$\frac{d\vec{x}}{dt} = V\Lambda V^{-1}\vec{x} = \Lambda\vec{x}$$

with general solution

$$\vec{x}(t) = V \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} V^{-1}\vec{x_0} = I \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} I\vec{x_0}$$

The solution is

$$\vec{x}(t) = \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \vec{x_0}$$

Now, for

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

One eigenvalue is $\lambda = 2$ with eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Now look at the 2×2 matrix,

$$\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$$

Find the eigenvalues,

$$\begin{aligned} \lambda &= \frac{1}{2}(8 \pm \sqrt{64-64}) \\ &= 4 \end{aligned}$$

This is a repeated eigenvalue. Find the eigenvectors for $\lambda = 4$,

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for the 2×2 matrix, and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for the 3×3 matrix.

Find $\vec{u}$ where $(A - 4I)\vec{u} = \vec{v}$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$x_2 - x_3 = 1$$

So $\vec{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The Jordan decomposition is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

The solution is

$$\begin{aligned} \vec{x}(t) &= V \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1} \vec{x_0} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{4t} & te^{4t} \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \vec{x_0} \end{aligned}$$

Problem 4: Assuming $||A|| < 1$ and using

$$(I - A)^{-1} = \sum_{j=0}^{\infty} A^j$$

prove the inequalities,

$$||(I - A)^{-1}|| \leq \frac{||A||}{1 - ||A||}$$

and

$$\left|\left|\sum_{j=m}^{\infty} \frac{1}{j!} A^j\right|\right| \leq \frac{||A||^m}{m!} \frac{1}{1 - ||A||}$$

Solution:

Proof.

$$\begin{aligned}
||(I - A)^{-1} - I|| &= \left|\left|\sum_{j=0}^{\infty} A^j - I\right|\right| \\
&= \left|\left|\sum_{j=1}^{\infty} A^j\right|\right| \\
&\leq \sum_{j=1}^{\infty} ||A^j|| \\
&\leq \sum_{j=1}^{\infty} ||A||^j \\
&= \frac{||A||}{1 - ||A||} \quad \text{because it's a geometric series and } ||A|| < 1
\end{aligned}$$

Therefore,

$$||(I - A)^{-1} - I|| \leq \frac{||A||}{1 - ||A||}$$

□

Now the other inequality where $j > m$,

Proof.
$$\begin{aligned}
\left|\left|\sum_{j=m}^{\infty}\frac{1}{j!}A^j\right|\right| &\leq \sum_{j=m}^{\infty}\frac{1}{j!}||A^j|| \\
&\leq \sum_{j=m}^{\infty}\frac{1}{j!}||A||^j \\
&= \frac{||A||^m}{m!} + \frac{||A||^{m+1}}{(m+1)!} + \frac{||A||^{m+2}}{(m+2)!} + \cdots \\
&= \frac{||A||^m}{m!}\left(1 + \frac{||A||}{m+1} + \frac{||A||^2}{(m+1)(m+2)} + \cdots\right) \\
&\leq \frac{||A||^m}{m!}(1 + ||A|| + ||A||^2 + \cdots) \\
&= \frac{||A||^m}{m!}\sum_{j=0}^{\infty}||A||^j \\
&= \frac{||A||^m}{m!}\frac{1}{1-||A||}
\end{aligned}$$

Therefore,
$$\left|\left|\sum_{j=m}^{\infty}\frac{1}{j!}A^j\right|\right| \leq \frac{||A||^m}{m!}\frac{1}{1-||A||}$$

□

Problem 5: For A a real 2×2 matrix and $\vec{x} \in \mathbb{R}^2$ and $\vec{f}(T) \in \mathbb{R}^2$, with each entry $f_j(t)$ assumed to be continuous

- Show that the solution to the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t), \ \vec{x}(0) = \vec{x}_0$$

  is given by

$$\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)ds$$

- Now let A be such that $A^T = -A$. Show that

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

  for some real value ω. Now suppose the forcing $\vec{f}(t)$ is such that

$$\vec{f}_j(t) = \cos(\lambda_j t), \ \lambda_j \in \mathbb{R}, \ j = 1, 2$$

  Show by direct computation that if either $\lambda_1 = \omega$ or $\lambda_2 = \omega$, then you get terms in your solution $\vec{x}(t)$ which grow linearly, and thus unboundedly, in t.

Solution:

Part 1

Define:

$$\Phi(t) = e^{-At}$$
$$\vec{z}(t) = \Phi(t)\vec{x}$$

Then,

$$\begin{aligned}
\frac{d}{dt}\vec{z}(t) &= \frac{d}{dt}\left(\Phi(t)\vec{x}\right) \\
&= \frac{d\Phi(t)}{dt}\vec{x} + \Phi(t)\frac{d\vec{x}}{dt} \\
&= -A\Phi(t)\vec{x} + \Phi(t)(A\vec{x} + \vec{f}(t)) \\
&= -A\vec{z}(t) + \Phi(t)A\vec{x} + \Phi(t)\vec{f}(t) \\
&= -A\vec{z}(t) + A\vec{z}(t) + \Phi(t)\vec{f}(t) \\
&= \Phi(t)\vec{f}(t)
\end{aligned}$$

Integrate both sides from 0 to t,

$$\begin{aligned}\vec{z}(t) - \vec{z}_0 &= \int_0^t \Phi(s)\vec{f}(s)\,ds \\ \vec{z}(t) &= \vec{z}_0 + \int_0^t \Phi(s)\vec{f}(s)\,ds \\ \Phi(t)\vec{x} &= \vec{z}_0 + \int_0^t \Phi(s)\vec{f}(s)\,ds\end{aligned}$$

The initial condition is

$$\vec{z}_0 = \Phi(0)\vec{x}_0 = I \cdot \vec{x}_0 = \vec{x}_0$$

So,

$$\begin{aligned}\Phi(t)\vec{x} &= \vec{x}_0 + \int_0^t \Phi(s)\vec{f}(s)\,ds \\ \vec{x}(t) &= \Phi(t)^{-1}\vec{x}_0 + \Phi(t)^{-1}\int_0^t \Phi(s)\vec{f}(s)\,ds\end{aligned}$$

Plug in $\Phi(t) = e^{-At}$,

$$\begin{aligned}\vec{x}(t) &= (e^{-At})^{-1}\vec{x}_0 + (e^{-At})^{-1}\int_0^t e^{-As}\vec{f}(s)ds \\ &= e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)ds\end{aligned}$$

Part 2a

Let $A^T = -A$.

If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then,

$$A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -A = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$$

So,

$$\begin{aligned}a_{11} &= -a_{11} \\ a_{12} &= -a_{21} \\ a_{21} &= -a_{12} \\ a_{22} &= -a_{22}\end{aligned}$$

First,

$$a_{11} = -a_{11} \text{ and } a_{22} = -a_{22} \implies a_{11} = a_{22} = 0$$

Let $a_{12} = \omega$. Then,

$$a_{21} = -a_{12} = -\omega$$

Therefore,

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

Part 2b

Now, let $\vec{f}(t) = \begin{pmatrix} \cos(\lambda_1 t) \\ \cos(\lambda_2 t) \end{pmatrix}$ and let $\lambda_1 = \omega$.

The solution to the initial value problem is given by,

$$\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)ds$$

Consider the power series form:

$$\begin{aligned}
\vec{x}(t) &= e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)ds \\
&= \left(\sum_{j=0}^{\infty} \frac{A^j t^j}{j!}\right)\vec{x}_0 + \left(\sum_{j=0}^{\infty} \frac{A^j t^j}{j!}\right)\int_0^t \left(\sum_{j=0}^{\infty} \frac{(-A)^j s^j}{j!}\right)\begin{pmatrix} \cos(\omega s) \\ \cos(\lambda_2 s) \end{pmatrix} ds
\end{aligned}$$

Compute powers of A:

$$\begin{aligned}
A^0 &= I \\
A^1 &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = A \\
A^2 &= \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = -\omega^2 I \\
A^3 &= \begin{pmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix} = -\omega^2 A \\
A^4 &= \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{pmatrix} = \omega^4 I \\
A^5 &= \begin{pmatrix} 0 & \omega^5 \\ -\omega^5 & 0 \end{pmatrix} = \omega^4 A \\
A^6 &= \begin{pmatrix} -\omega^6 & 0 \\ 0 & -\omega^6 \end{pmatrix} = -\omega^6 I \\
A^7 &= \begin{pmatrix} 0 & -\omega^7 \\ \omega^7 & 0 \end{pmatrix} = -\omega^6 A \\
A^8 &= \begin{pmatrix} \omega^8 & 0 \\ 0 & \omega^8 \end{pmatrix} = \omega^8 I
\end{aligned}$$

There is a pattern.

Write out the terms of e^{At}:

$$
\begin{aligned}
e^{At} &= \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} \\
&= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \frac{A^5t^5}{5!} + \frac{A^6t^6}{6!} + \cdots \\
&= I + At - \omega^2 I \frac{t^2}{2!} - \omega^2 A \frac{t^3}{3!} + \omega^4 I \frac{t^4}{4!} + \omega^4 A \frac{t^5}{5!} - \omega^6 I \frac{t^6}{6!} - \omega^6 A \frac{t^7}{7!} + \cdots \\
&= I \left(1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} - \frac{\omega^6 t^6}{6!} + \cdots \right) + A \left(t - \frac{\omega^2 t^3}{3!} + \frac{\omega^4 t^5}{5!} - \frac{\omega^6 t^7}{7!} + \cdots \right) \\
&= I \left(\sum_{j=0}^{\infty} \frac{(-1)^j \omega^{2j} t^{2j}}{(2j)!} \right) + A \left(\sum_{j=0}^{\infty} \frac{(-1)^j \omega^{2j} t^{2j+1}}{(2j+1)!} \right) \\
&= I\cos(\omega t) + \frac{A}{\omega}\sin(\omega t)
\end{aligned}
$$

Therefore,

$$
e^{At} = I\cos(\omega t) + \frac{A}{\omega}\sin(\omega t)
$$

Note:

$$
\frac{A}{\omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

Remember the solution formula:

$$
\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds
$$

Then,

$$
\vec{x}(t) = \left(I\cos(\omega t) + \frac{A}{\omega}\sin(\omega t) \right)\vec{x}_0 + \left(I\cos(\omega t) + \frac{A}{\omega}\sin(\omega t) \right)\int_0^t \left(I\cos(\omega s) - \frac{A}{\omega}\sin(\omega s) \right)\begin{pmatrix} \cos(\lambda_1 s) \\ \cos(\lambda_2 s) \end{pmatrix} ds
$$

Looking just at the integral term,

$$
\int_0^t \left(I\cos(\omega s) - \frac{A}{\omega}\sin(\omega s) \right)\begin{pmatrix} \cos(\omega s) \\ \cos(\lambda_2 s) \end{pmatrix} ds
$$

Compute the product inside the integral by breaking up A and I:

$$
\begin{aligned}
&\left(I\cos(\omega s) - \frac{A}{\omega}\sin(\omega s) \right)\begin{pmatrix} \cos(\omega s) \\ \cos(\lambda_2 s) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\omega s) & 0 \\ 0 & \cos(\omega s) \end{pmatrix}\begin{pmatrix} \cos(\omega s) \\ \cos(\lambda_2 s) \end{pmatrix} + \begin{pmatrix} 0 & -\sin(\omega s) \\ \sin(\omega s) & 0 \end{pmatrix}\begin{pmatrix} \cos(\omega s) \\ \cos(\lambda_2 s) \end{pmatrix} \\
&= \begin{pmatrix} \cos^2(\omega s) \\ \cos(\omega s)\cos(\lambda_2 s) \end{pmatrix} + \begin{pmatrix} -\sin(\omega s)\cos(\lambda_2 s) \\ \sin(\omega s)\cos(\omega s) \end{pmatrix} \\
&= \begin{pmatrix} \cos^2(\omega s) - \sin(\omega s)\cos(\lambda_2 s) \\ \cos(\omega s)\cos(\lambda_2 s) + \sin(\omega s)\cos(\omega s) \end{pmatrix}
\end{aligned}
$$

This is the integral term:

$$\int_0^t \begin{pmatrix} \cos^2(\omega s) - \sin(\omega s)\cos(\lambda_2 s) \\ \cos(\omega s)\cos(\lambda_2 s) + \sin(\omega s)\cos(\omega s) \end{pmatrix} ds$$

I set $\lambda_1 = \omega$, so there is a $\cos^2$ on the top, but you would get a $\cos^2$ on the bottom if you set $\lambda_2 = \omega$. It doesn't really change anything because you substitute $\cos^2 = 1 - \sin^2$ in either case.

Look just at the first term in the vector and take the integral:

$$\int_0^t \left(1 - \sin^2(\omega s) - \sin(\omega s)\cos(\omega s)\right) ds = \int_0^t 1\, ds - \int_0^t \sin^2(\omega s)\, ds - \int_0^t \sin(\omega s)\cos(\omega s)\, ds$$
$$= t - \frac{t}{2} + \dots$$

You get no other t terms in these integrals, so I just put dots because those other terms aren't important.

Therefore, you get at least one t term in the solution, so the solution is unbounded as $t \to \infty$.