Homework #6

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Problem 1: Consider the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & t \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x, \quad x \in \mathbb{R}^2, \quad u, y \in \mathbb{R}$$

a) Compute its state transition matrix.

b) Compute the system's output to the constant input $u(t) = 1, \forall t \geq 0$, for an arbitrary initial condition $x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$.

Solution:

a) Want to find $\Phi(t, s)$.

$$\vec{x}(t) = \Phi(t, t_0)\vec{x}(0) + \int_{t_0}^{t} \Phi(t, s)\vec{B}(s)u(s)ds$$

$$\dot{x_1}(t) = tx_2(t)$$
$$\dot{x_2}(t) = 2x_2(t)$$

So,

$$x_2(t) = x_{2,0}e^{2t-t_0}$$

Now,

$$\begin{aligned}
x_1(t) &= x_{1,0} + x_{2,0}\int_{t_0}^{t}(s-t_0+t_0)e^{2(s-t_0)}ds \\
&= x_{1,0} + x_{2,0}\int_{0}^{t-t_0}(u+t_0)e^{2u}du \\
&= x_{1,0} + x_{2,0}\left(\frac{u+t_0}{2}e^{2u}\right)_0^{t-t_0} - x_{2,0}\int_0^{t-t_0}\frac{1}{2}e^{2u}du \\
&= x_{1,0} + x_{2,0}\left(\frac{t-t_0+t_0}{2}e^{2(t-t_0)} - \frac{t_0}{2}\right) - x_{2,0}\left(\frac{1}{4}e^{2(t-t_0)} - \frac{1}{4}\right) \\
&= x_{1,0} + x_{2,0}\left(\frac{t}{2}e^{2(t-t_0)} - \frac{t_0}{2} - \frac{1}{4}e^{2(t-t_0)} + \frac{1}{4}\right)
\end{aligned}$$

So,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & \Phi(t,t_0) \\ 0 & e^{2(t-t_0)} \end{pmatrix}\begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$$

So,

$$\Phi(t,s) = \begin{pmatrix} 1 & \frac{t}{2}e^{2(t-s)} - \frac{s}{2} - \frac{1}{4}e^{2(t-s)} + \frac{1}{4} \\ 0 & e^{2(t-s)} \end{pmatrix}$$

b) part b

The soltion is

$$\vec{x}(t) = \Phi(t,t_0)\vec{x}(0) + \int_{t_0}^{t}\Phi(t,s)\vec{B}(s)u(s)ds$$

Here, $u(s) = 1$, so

$$\vec{x}(t) = \Phi(t,t_0)\vec{x}(0) + \int_{t_0}^{t}\Phi(t,s)\vec{B}(s)ds$$

It is given that $B(s) = \begin{pmatrix} 0 \\ s \end{pmatrix}$, so

$$\vec{x}(t) = \Phi(t,t_0)\vec{x}(0) + \int_{t_0}^{t}\Phi(t,s)\begin{pmatrix} 0 \\ s \end{pmatrix}ds$$

So,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & \frac{t}{2}e^{2(t-t_0)} - \frac{s}{2} - \frac{1}{4}e^{2(t-t_0)} + \frac{1}{4} \\ 0 & e^{2(t-t_0)} \end{pmatrix}\begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} + \int_{t_0}^{t}\begin{pmatrix} 1 & \frac{s}{2}e^{2(s-t_0)} - \frac{s}{2} - \frac{1}{4}e^{2(s-t_0)} + \frac{1}{4} \\ 0 & e^{2(s-t_0)} \end{pmatrix}\begin{pmatrix} 0 \\ s \end{pmatrix}ds$$

Wolfram Alpha said this was too many characters when I put it in. So, here is Chat-GPT's answer (probably wrong):

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_{1,0} + \left(\frac{t}{2} - \frac{1}{4}\right)e^{2(t-t_0)} - \frac{t_0}{2} + \frac{1}{4} \\ \left(x_{2,0} + \frac{t_0}{2} + \frac{1}{4}\right)e^{2(t-t_0)} - \frac{t}{2} - \frac{1}{4} \end{pmatrix}$$

Problem 2: Consider the homogeneous linear time-varying system

$$\dot{\vec{x}}(t) = A(t)\vec{x}(t), \quad \vec{x}(t_0) = \vec{x}_0$$

with state transition matrix $\Phi(t, \tau)$. Consider also the non-homogeneous system

$$\dot{\vec{z}}(t) = A(t)\vec{z}(t) + \vec{x}(t), \quad \vec{z}(t_0) = \vec{z}_0$$

a) Compute $\vec{x}(t)$ and $\vec{z}(t)$ as a function of $\vec{x}_0, \vec{z}_0$, and $\Phi(t, \tau)$. No integrals should appear in your answer.

b) For a given time $T > 0$, how should x_0 and z_0 be related to have $z(T) = 0$?

Solution:

a) Start with $\vec{x}(t)$

This has a straightforward solution:

$$\vec{x}(t) = \Phi(t, t_0)\vec{x}_0$$

Now for $\vec{z}(t)$:

$$\dot{\vec{z}}(t) = A(t)\vec{z}(t) + \vec{x}(t), \quad \vec{z}(t_0) = \vec{z}_0$$

Start with the solution with an integral and simplify:

$$\begin{aligned}
\vec{z}(t) &= \Phi(t, t_0)\vec{z}_0 + \int_{t_0}^{t} \Phi(t, s)\vec{x}(s)\, ds \\
&= \Phi(t, t_0)\vec{z}_0 + \int_{t_0}^{t} \Phi(t, s)\Phi(s, t_0)\vec{x}_0\, ds \\
&= \Phi(t, t_0)\vec{z}_0 + \int_{t_0}^{t} \Phi(t, t_0)\vec{x}_0\, ds \\
&= \Phi(t, t_0)\vec{z}_0 + \Phi(t, t_0)\vec{x}_0 \int_{t_0}^{t} ds \\
&= \Phi(t, t_0)\vec{z}_0 + \Phi(t, t_0)\vec{x}_0(t - t_0)
\end{aligned}$$

b) To achieve $\vec{z}(T) = 0$, set the solution at $t = T$ to zero:

$$\begin{aligned}
\vec{z}(T) &= \Phi(T, t_0)\vec{z}_0 + \Phi(T, t_0)\vec{x}_0(T - t_0) \\
&= \Phi(T, t_0)(\vec{z}_0 + \vec{x}_0(T - t_0)) = 0
\end{aligned}$$

Assuming $\Phi(T, t_0)$ is invertible (I think it has to be), we can solve for $\vec{z}_0$:

$$\vec{z}_0 + \vec{x}_0(T - t_0) = 0 \quad \Rightarrow \quad \vec{z}_0 = -\vec{x}_0(T - t_0)$$

Therefore, the initial conditions must satisfy

$$\vec{z}_0 = -(T - t_0)\vec{x}_0\, .$$

Problem 3: Compute A^n and e^{An} for the following matrices:

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Solution:

$$\begin{aligned} A_1 &= I + N \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$NI = IN$ because identity.

$$N^2 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 & 0 \\ 0 & (0)^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Binomial theorem:

$$\begin{aligned} (I+N)^n &= \sum_{k=0}^{n} \binom{n}{k} I^{n-k} N^k \\ &= \sum_{k=0}^{1} \binom{n}{k} I^{n-k} N^k \\ &= I + nN \\ &= \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Now A_2:

$$\begin{aligned} A_2 &= D + N \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

where

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = 0$$

$$D^n = D$$

$$DN + ND = N$$

Look at the pattern:

$$\begin{aligned}(D+N)^2 &= D^2 + 2DN + N^2 = D + N + N^2 \\ (D+N)^3 &= D + N + N^2 + DN^2 = D + N + 2N^2\end{aligned}$$

Induction:

$$(D+N)^1 = D + N$$

Assume:

$$(D+N)^n = D + N + (n-1)N^2$$

Then:

$$\begin{aligned}(D+N)^{n+1} &= (D+N)^n + (D+N)N^2 \\ &= D + N + (n-1)N^2 + DN^2 \\ &= D + N + nN^2\end{aligned}$$

Now A_3:

$$\begin{aligned}A_3^n &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}^n \\ &= \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}^n & 0 \\ 0 & \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}^n \end{pmatrix} \\ &= \begin{pmatrix} 2^n \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n & 0 \\ 0 & 3^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \end{pmatrix} \\ &= \begin{pmatrix} 2^n \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} & 0 \\ 0 & 3^n \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \end{pmatrix}\end{aligned}$$

Now $e^{A_1 t}$:

$$\begin{aligned}e^{A_1 t} &= \sum_{j=0}^{\infty} \frac{(tA_1)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{t^j (I+N)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \begin{pmatrix} 1 & j & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= e^t \sum_{j=0}^{\infty} \begin{pmatrix} 1 & j & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

For $e^{A_2 t}$:

$$\begin{aligned} e^{A_2 t} &= \sum_{n=0}^{\infty} \frac{(tA_2)^n}{n!} \\ &= \sum_{j=0}^{\infty} \frac{t^j (D+N)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{t^j (D+N+(j-1)N^2)}{j!} \end{aligned}$$

For $e^{A_3 t}$:

$$\begin{aligned} e^{A_3 t} &= \sum_{j=0}^{\infty} \frac{(tA_3)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \begin{pmatrix} 2^j & 0 & 0 & 0 \\ j2^j & 2^j & 0 & 0 \\ 0 & 0 & 3^j & j3^j \\ 0 & 0 & 0 & 3^j \end{pmatrix} \\ &= e^t \sum_{j=0}^{\infty} \begin{pmatrix} 2^j & 0 & 0 & 0 \\ j2^j & 2^j & 0 & 0 \\ 0 & 0 & 3^j & j3^j \\ 0 & 0 & 0 & 3^j \end{pmatrix} \end{aligned}$$

Problem 4: Consider an upper triangular matrix A.

a) Show that e^{At} is also upper triangular.

b) Relate the diagonal elements of A with those of e^{At}.

Solution:

a) Suppose $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$.

$$A = D + N$$

where D is diagonal and N is strictly upper triangular.

Put e^{At} in same form:

$$e^{At} = e^{Dt}e^{Nt} = \sum_{j=0}^{\infty} \frac{(Dt)^j}{j!} \left(\sum_{k=0}^{\infty} \frac{(Nt)^k}{k!} \right)$$

D remains diagonal always, and N stays upper triangular. So, e^{At} is upper triangular.

b) Relate the diagonal elements of A with those of e^{At}.

The diagonal elements of e^{At} are given by:

$$e^{Dt} = \sum_{j=0}^{\infty} \frac{(a_{ii}t)^j}{j!} = e^{a_{ii}t}$$

Each diagonal element of e^{At} is the exponential of the corresponding diagonal element of A multiplied by t.

$$A = \begin{pmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix},$$

$$e^{At} = \begin{pmatrix} e^{a_{11}t} & * & \cdots & * \\ 0 & e^{a_{22}t} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_{nn}t} \end{pmatrix}.$$

Problem 5: For $n \times n$ matrices A, B, show that if $AB = BA$, then $e^{A+B} = e^A e^B$.

Using this result, show then that $(e^A)^{-1} = e^{-A}$.

Solution:

$$\begin{aligned}
e^{A+B} &= \sum_{j=0}^{\infty} \frac{(A+B)^j}{j!} \\
&= \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \frac{1}{j!} \binom{j}{\ell} A^{\ell} B^{j-\ell} \\
&= \sum_{\ell=0}^{\infty} \sum_{j=\ell}^{\infty} \frac{1}{j!} \binom{j}{\ell} A^{\ell} B^{j-\ell} \\
&= \sum_{\ell=0}^{\infty} \sum_{j=\ell}^{\infty} \frac{j!}{j!\ell!(j-\ell!)} A^{\ell} B^{j-\ell} \\
&= \sum_{\ell=0}^{\infty} \sum_{j=\ell}^{\infty} \frac{1}{\ell!(j-\ell!)} A^{\ell} B^{j-\ell} \\
&= \sum_{\ell=0}^{\infty} \sum_{j-\ell=0}^{\infty} \frac{1}{\ell!(j-\ell!)} A^{\ell} B^{j-\ell} \\
&= \left(\sum_{\ell=0}^{\infty} \frac{A^{\ell}}{\ell!} \right) \left(\sum_{j-\ell=0}^{\infty} \frac{B^{j-\ell}}{(j-\ell)!} \right) \\
&= e^A e^B
\end{aligned}$$

Showing $(e^A)^{-1} = e^{-A}$:

$$e^A e^{-A} = e^{A+(-A)} = e^{A-A} = e^0 = I$$

Therefore, e^{-A} is the inverse of e^A:

$$e^{-A} = (e^A)^{-1}$$