Problem 6: Matured insects lay eggs with per capita rate of r, which survive and hatch to immature population with survival rate $e^{-\psi x}$, where x is a number of eggs. The immature insects become matured with per capita maturation rate γ. Assume that δ and μ are per capita mortality rate of immature and mature insect populations, respectively.

1. Develop a patchy model with two patches, one representing immature insects and another representing mature insects.
2. Consider a control mechanism which results in the reduction of the egg laying rate, i.e., $r \to r(1-\theta)r$ with the control level θ. Perform bifurcation analysis of the model to identify the level of control mechanism for extinction and for persistence of the insect population.

Solution:

Let $M(t)$ represent the population of mature insects, and $I(t)$ represent the population of immature insects.

$$\frac{dI}{dt} = e^{-\psi r x} \cdot M - (\gamma + \delta)I$$
$$\frac{dM}{dt} = \gamma I - \mu M$$

The immature population goes to the mature population, and the mature population lays eggs that go to the immature population.

Let $r' = r(1-\theta)$ replace r in the model.

$$\frac{dI}{dt} = e^{-\psi r(1-\theta)x} M - (\gamma + \delta)I$$
$$\frac{dM}{dt} = \gamma I - \mu M$$

Find the equilibrium points by setting the DEs to zero:

$$e^{-\psi r(1-\theta)x} M - (\gamma + \delta)I = 0$$
$$\gamma I - \mu M = 0$$

The second equation gives:

$$I = \frac{\mu}{\gamma} M$$

Substitute this into the first equation:

$$e^{-\psi r(1-\theta)x} M - (\gamma + \delta)\left(\frac{\mu}{\gamma} M\right) = 0$$

Simplify:

$$e^{-\psi r(1-\theta)x}M - \frac{\mu(\gamma+\delta)}{\gamma}M = 0$$

Factor out M:

$$M\left(e^{-\psi r(1-\theta)x} - \frac{\mu(\gamma+\delta)}{\gamma}\right) = 0$$

This equation has two solutions:

1. $M = 0$ (trivial equilibrium) 2. $e^{-\psi r(1-\theta)x} = \frac{\mu(\gamma+\delta)}{\gamma}$

For the non-trivial equilibrium:

$$\begin{aligned}
-\psi r(1-\theta)x &= \ln\left(\frac{\mu(\gamma+\delta)}{\gamma}\right) \\
(1-\theta) &= -\frac{1}{\psi rx}\ln\left(\frac{\mu(\gamma+\delta)}{\gamma}\right) \\
-\theta &= -\frac{1}{\psi rx}\ln\left(\frac{\mu(\gamma+\delta)}{\gamma}\right) - 1 \\
\theta &= 1 + \frac{1}{\psi rx}\ln\left(\frac{\mu(\gamma+\delta)}{\gamma}\right)
\end{aligned}$$

This gives us the critical value of θ for which a non-trivial equilibrium exists.

For $\theta < 1 + \frac{1}{\psi rx}\ln\left(\frac{\mu(\gamma+\delta)}{\gamma}\right)$, the insect population will persist.

For $\theta > 1 + \frac{1}{\psi rx}\ln\left(\frac{\mu(\gamma+\delta)}{\gamma}\right)$, the insect population will go extinct.