Problem 1: The dynamics of the number of photons $n(t)$ in a laser field is given by

$$\frac{dn}{dt} = (GN_0 - k)n - \alpha G n^2$$

where G is the gain coefficient for a simulated emission, k is the decay rate due to photon loss by scattering, α is the rate at which atoms drop back to their ground states and in the absence of a laser field, the number of excited atoms is kept fixed at N_0.

1. Find the equilibrium points of the system and comment on their stability.
2. Show that the system undergoes a transcritical bifurcation at $N_0 = \frac{k}{G}$.

Solution:

The equilibrium points are found by setting $\frac{dn}{dt} = 0$:

$$(GN_0 - k)n - \alpha G n^2 = 0$$

Solving for n, we get:

$$n_1^* = 0$$
$$n_2^* = \frac{GN_0 - k}{\alpha G}$$

To analyze the stability of the equilibrium points, we examine the derivative of $\frac{dn}{dt}$ with respect to n:

$$\frac{d}{dn}\left((GN_0 - k)n - \alpha G n^2\right) = GN_0 - k - 2\alpha G n$$

Equilibrium Point $n_1^* = 0$:

Substitute $n = 0$:

$$\left.\frac{d}{dn}\left(\frac{dn}{dt}\right)\right|_{n=0} = GN_0 - k$$

- If $GN_0 - k < 0$, then $n_1^* = 0$ is stable.

- If $GN_0 - k > 0$, then $n_1^* = 0$ is unstable.

Equilibrium Point $n_2^* = \frac{GN_0 - k}{\alpha G}$:

Substitute $n = n_2^*$:

$$
\begin{aligned}
\frac{d}{dn}\left(\frac{dn}{dt}\right)\bigg|_{n=n_2^*} &= GN_0 - k - 2\alpha G\left(\frac{GN_0 - k}{\alpha G}\right) \\
&= (GN_0 - k) - 2(GN_0 - k) \\
&= -(GN_0 - k)
\end{aligned}
$$

- If $GN_0 - k > 0$, then n_2^* is stable.

- If $GN_0 - k < 0$, then n_2^* is unstable.

Both equalibria change stability at $N_0 = \frac{k}{G}$, so a transcritical bifurcation occurs at this point.