Problem 4.1: For the damp pendulum equation

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\sin(\gamma\theta) = 0,\ \theta(0) = \theta_0,\ \dot{\theta}(0) = 1$$

find suitable rescaling for $\alpha, \gamma = \mathcal{O}(1)$ and $\beta >> 1$.

Solution:

Currently, the equation is in the form:

$$\frac{d^2\theta^*}{dt^{*^2}} + \alpha\frac{d\theta^*}{dt^*} + \beta\sin(\gamma\theta^*) = 0$$

Rescale t^*:

$$t^* = [t^*]\hat{t}$$

Use the scaled derivative:

$$\frac{d\theta^*}{dt^*} = \frac{1}{[t^*]}\frac{d\theta^*}{d\hat{t}}$$
$$\frac{d^2\theta^*}{dt^{*^2}} = \frac{1}{[t^{*^2}]}\frac{d^2\theta^*}{d\hat{t}^2}$$

Substitute into the original equation:

$$\frac{1}{[t^{*^2}]}\frac{d^2\theta^*}{d\hat{t}^2} + \alpha\frac{1}{[t^*]}\frac{d\theta^*}{d\hat{t}} + \beta\sin(\gamma\theta^*) = 0$$
$$\frac{d^2\theta^*}{d\hat{t}^2} + [t^*]\alpha\frac{d\theta^*}{d\hat{t}} + [t^{*^2}]\beta\sin(\gamma\theta^*) = 0$$

Define:

$$[t^*] = \frac{1}{\sqrt{\beta}}$$
$$[t^{*^2}] = \frac{1}{\beta}$$

Thus, the equation becomes:

$$\frac{d^2\theta^*}{d\hat{t}^2} + \frac{\alpha}{\sqrt{\beta}}\frac{d\theta^*}{d\hat{t}} + \sin(\gamma\theta^*) = 0$$