Problem 3: A hypothetical reaction in the study of isothermal autocatalytic reactions was considered by Gray and Scott (1985), whose kinetics in dimensionless form are given as follows:

$$\frac{dx}{dt} = a(1-x) - xy^2$$
$$\frac{dy}{dt} = xy^2 - (a+k)y$$

where a and k are positive parameters. Show that the saddle node bifurcation occurs at $k = -a \pm \frac{\sqrt{a}}{a}$.

Solution:

Label the equations:

$$\frac{dx}{dt} = a(1-x) - xy^2 \quad (1)$$
$$\frac{dy}{dt} = xy^2 - (a+k)y \quad (2)$$

Set both derivatives to zero to find the equilibrium points.

From equation (2):

$$xy^2 - (a+k)y = 0$$
$$y(xy - (a+k)) = 0$$

This gives two cases:

1. $y = 0$
   Substitute $y = 0$ into equation (1):

   $$a(1-x) = 0 \Rightarrow x = 1$$

   So, one equilibrium point is $(1, 0)$.

2. $xy = a + k$
   Solve for x:

   $$x = \frac{a+k}{y}$$

   Substitute $x = \frac{a+k}{y}$ into equation (1):

   $$a\left(1 - \frac{a+k}{y}\right) - \left(\frac{a+k}{y}\right)y^2 = 0$$
   $$a - \frac{a(a+k)}{y} - (a+k)y = 0$$
   $$ay - a(a+k) - (a+k)y^2 = 0$$
   $$(a+k)y^2 - ay + a(a+k) = 0$$

Set up the quadratic equation for y:

$$y = \frac{a \pm \sqrt{a^2 - 4a(a+k)^2}}{2(a+k)}$$

A bifurcation occurs when the discriminant of the quadratic equation is zero.

Set $D = 0$:

$$\begin{aligned}
a^2 - 4a(a+k)^2 &= 0 \\
a - 4(a+k)^2 &= 0 \\
4(a+k)^2 &= a \\
(a+k)^2 &= \frac{a}{4} \\
a + k &= \pm\frac{\sqrt{a}}{2} \\
k &= -a \pm \frac{\sqrt{a}}{2}
\end{aligned}$$

The quadratic equation would change the sign of y, so this is a saddle node bifurcation, and it occurs at $k = -a \pm \frac{\sqrt{a}}{2}$.