Homework #9

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As in lecture, we look at the problem

$$\dot{x} = (A_0 + f(t)A_1)x, \quad x(t_0) = x_0 \tag{1}$$

where $f(t) > 0$ and we further suppose that $\exists\beta > 0$ such that

$$\forall(t, t_0) \quad t \geq t_0, \quad \int_{t_0}^{t} f(s)ds \leq \beta \tag{2}$$

Note, if we take $t \to \infty$, we see the requirement on $f(t)$ becomes

$$\int_{t_0}^{\infty} f(s)ds \leq \beta,$$

so we see that $f(t) \to 0$ as $t \to \infty$.

We now further suppose that A_0 is a stability matrix, meaning that for every eigenvalue λ of A, $Re\{\lambda\} < 0$. As we know, this leads to exponential stability by way of the fact that there exist positive constants $\tilde{c}, \tilde{\lambda}$ such that

$$\left\| e^{A_0(t-t_0)} \right\| \leq \tilde{c}e^{-\tilde{\lambda}(t-t_0)}$$

We now show that Equation (1) has a solution such that there exists $\tilde{t}_0 > 0$ and positive constants $\tilde{C}, \hat{\lambda}$ such that for $t \geq \tilde{t}_0$ we have

$$\|x(t)\| \leq \tilde{C}e^{-\hat{\lambda}(t-\tilde{t}_0)} \left\| x(\tilde{t}_0) \right\|$$

To do this, show that:

Problem 1: For any real symmetric matrix $A = O\Lambda O^T$ that

$$\langle Ax, x\rangle \leq \max_j \lambda_j \left\|x\right\|^2$$

where λ_j is an eigenvalue of A, or a diagonal entry of Λ.

Solution:

$$\begin{aligned}
\langle Ax, x\rangle &= \langle O\Lambda O^T x, x\rangle \\
&= \langle \Lambda O^T x, O^T x\rangle \\
&= \langle \Lambda y, y\rangle \\
&= \sum_{i=1}^{n} \lambda_i y_i^2 \\
&\leq \max_j \lambda_j \sum_{i=1}^{n} y_i^2 \\
&= \max_j \lambda_j \left\|x\right\|^2
\end{aligned}$$

Problem 2: There exists a positive definite matrix P such that

$$A_0^T P + P A_0 = -Q < 0,$$

and if we define $E(t) = \langle Px(t), x(t) \rangle$ then

$$\frac{dE}{dt} = -\langle Qx, x \rangle + f(t)\langle (A_1^T P + P A_1)x, x \rangle.$$

Solution:

A is positive definite, so by definition, there exists,

$$A_0^T P + P A_0 = -Q$$

Now,

$$\begin{aligned}
E(t) &= \langle Px, x \rangle \\
&= x^T P x \\
\frac{dE}{dt} &= \frac{d}{dt} x^T P x \\
&= \dot{x}^T P x + x^T P \dot{x} \\
&= x^T (A_0 + f(t)A_1)^T P x + x^T P (A_0 + f(t) A_1) x \\
&= x^T (A_0^T P + P A_0) x + f(t) x^T (A_1^T P + P A_1) x \\
&= -\langle Qx, x \rangle + f(t) \langle (A_1^T P + P A_1)x, x \rangle
\end{aligned}$$

Problem 3: There exists a constant γ (could have any sign or even be zero) such that

$$\langle (A_1^T P + P A_1)x, x \rangle \leq \gamma \|x\|^2$$

Solution:

$$\begin{aligned}
\langle (A_1^T P + P A_1)x, x \rangle &= \langle O\Lambda O^T x, x \rangle \\
&= \langle \Lambda O^T x, O^T x \rangle \\
&= \langle \Lambda y, y \rangle \\
&= \sum_{i=1}^{n} \lambda_i y_i^2 \leq \max_j \lambda_j \|y\|^2 \leq \max_j \lambda_j \|x\|^2
\end{aligned}$$

Here $\gamma = \max_j \lambda_j$ from the problem statement.

Problem 4: Given the inequalities

$$\begin{aligned} 0 &< \lambda_m^{(Q)} \\ \lambda_m^{(Q)} \|x\|^2 \leq \langle Qx, x\rangle &\leq \lambda_M^{(Q)} \|x\|^2 \\ 0 &< \lambda_m^{(P)} \\ \lambda_m^{(P)} \|x\|^2 \leq \langle Px, x\rangle &\leq \lambda_M^{(P)} \|x\|^2 \end{aligned}$$

show that there exists some time $\tilde{t}_0$ such that for $t \geq \tilde{t}_0$,

$$\frac{dE}{dt} \leq -\frac{1}{\lambda_M^{(P)}} \left(\lambda_m^{(Q)} - \gamma f(t)\right) E(t).$$

Remember, we know that $\lim_{t\to\infty} f(t) = 0$.

Solution:

Remember:

$$\frac{dE}{dt} = -\langle Qx, x\rangle + f(t)\langle (A_1^T P + PA_1)x, x\rangle$$

Now,

$$\begin{aligned} \frac{dE}{dt} &\leq -\langle Qx, x\rangle + \gamma f(t) \|x\|^2 \\ &\leq -\lambda_m^{(Q)} \|x\|^2 + \gamma f(t) \|x\|^2 \\ &\leq \left(-\lambda_m^{(Q)} + \gamma f(t)\right) \|x\|^2 \end{aligned}$$

$f(t)$ goes to 0, so we don't need to worry about the $\gamma f(t)$ term messing with signs. Also,

$$-\|x\|^2 \leq -\frac{\langle Px, x\rangle}{\lambda_M^{(P)}}$$

So,

$$\left(-\lambda_m^{(Q)} + \gamma f(t)\right) \|x\|^2 \leq -\frac{\langle Px, x\rangle}{\lambda_M^{(P)}} \left(-\lambda_m^{(Q)} + \gamma f(t)\right)$$

So,

$$\frac{dE}{dt} \leq -\frac{1}{\lambda_M^{(P)}} \left(\lambda_m^{(Q)} - \gamma f(t)\right) E(t).$$

Problem 5: The prior inequality becomes

$$\|x(t)\|^2 \leq \frac{\lambda_M^{(P)}}{\lambda_m^{(P)}} e^{-\frac{1}{\lambda_M^{(P)}}\left(\lambda_m^{(Q)}(t-\tilde{t}_0)-\gamma\int_{\tilde{t}_0}^{t} f(s)ds\right)} \left\|x(\tilde{t}_0)\right\|^2$$

Solution:

$$\begin{aligned}
E(t) &\leq \frac{1}{\lambda_m^{(P)}} e^{-\frac{1}{\lambda_M^{(P)}}\left(\lambda_m^{(Q)}(t-\tilde{t}_0)-\gamma\int_{\tilde{t}_0}^{t} f(s)ds\right)} E(\tilde{t}_0) \\
\lambda_m^{(P)} \|x(t)\|^2 &\leq \lambda_M^{(P)} e^{-\frac{1}{\lambda_M^{(P)}}\left(\lambda_m^{(Q)}(t-\tilde{t}_0)-\gamma\int_{\tilde{t}_0}^{t} f(s)ds\right)} \left\|x(\tilde{t}_0)\right\|^2 \\
\|x(t)\|^2 &\leq \frac{\lambda_M^{(P)}}{\lambda_m^{(P)}} e^{-\frac{1}{\lambda_M^{(P)}}\left(\lambda_m^{(Q)}(t-\tilde{t}_0)-\gamma\int_{\tilde{t}_0}^{t} f(s)ds\right)} \left\|x(\tilde{t}_0)\right\|^2
\end{aligned}$$

Problem 6: Using Equation (2) and by examining the cases $\gamma \leq 0$ and $\gamma > 0$, show that there exist positive constants $C, \hat{\lambda}$ such that

$$\|x(t)\| \leq Ce^{-\hat{\lambda}(t-\tilde{t}_0)} \left\|x(\tilde{t}_0)\right\|$$

Thus we have shown that the LTV system in Equation (1) inherits the exponential stability from A_0 since $f(t) \to 0$ as $t \to \infty$.

Solution:

Case 1: $\gamma \leq 0$

In this case, $-\gamma f(t) \geq 0$ for all t, so we can just ignore the $-\gamma f(t)$ term.

$$\|x(t)\|^2 \leq \frac{\lambda_M^{(P)}}{\lambda_m^{(P)}} e^{-\frac{\lambda_m^{(Q)}}{\lambda_M^{(P)}}(t-\tilde{t}_0)} \left\|x(\tilde{t}_0)\right\|^2$$

$$\|x(t)\| \leq \sqrt{\frac{\lambda_M^{(P)}}{\lambda_m^{(P)}} e^{-\frac{\lambda_m^{(Q)}}{2\lambda_M^{(P)}}(t-\tilde{t}_0)}} \left\|x(\tilde{t}_0)\right\|$$

Case 2: $\gamma > 0$

Since $f(t) \to 0$ as $t \to \infty$, there exists $T > 0$ such that $|f(t)| < \frac{\lambda_m^{(Q)}}{2\gamma}$ for all $t > T$.

Therefore, for $t > T$:

$$\begin{aligned}
\gamma \int_{\tilde{t}_0}^{t} f(s)ds &= \gamma \int_{\tilde{t}_0}^{T} f(s)ds + \gamma \int_{T}^{t} f(s)ds \\
&\leq \gamma M + \gamma \int_{T}^{t} \frac{\lambda_m^{(Q)}}{2\gamma} ds \\
&= \gamma M + \frac{\lambda_m^{(Q)}}{2}(t-T)
\end{aligned}$$

where $M = \int_{\tilde{t}_0}^{T} |f(s)|ds$.

Then,

$$\begin{aligned}
\|x(t)\|^2 &\leq \frac{\lambda_M^{(P)}}{\lambda_m^{(P)}} e^{-\frac{1}{\lambda_M^{(P)}}\left(\lambda_m^{(Q)}(t-\tilde{t}_0)-\gamma M-\frac{\lambda_m^{(Q)}}{2}(t-T)\right)} \left\|x(\tilde{t}_0)\right\|^2 \\
&= \frac{\lambda_M^{(P)}}{\lambda_m^{(P)}} e^{-\frac{\lambda_m^{(Q)}}{2\lambda_M^{(P)}}(t-\tilde{t}_0)} e^{\frac{\gamma M}{\lambda_M^{(P)}}} \left\|x(\tilde{t}_0)\right\|^2
\end{aligned}$$

$$\|x(t)\| \leq \sqrt{\frac{\lambda_M^{(P)}}{\lambda_m^{(P)}}} e^{\frac{\gamma M}{2\lambda_M^{(P)}}} e^{-\frac{\lambda_m^{(Q)}}{4\lambda_M^{(P)}}(t-\tilde{t}_0)} \left\|x(\tilde{t}_0)\right\|$$