Problem 4.2: For the damp pendulum equation

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\sin(\gamma\theta) = 0, \ \theta(0) = \theta_0, \ \dot{\theta}(0) = 1$$

find suitable rescaling for $\alpha, \beta = \mathcal{O}(1)$ and $\gamma \gg 1$.

Solution:

Currently, the equation is in the form:

$$\frac{d^2\theta^*}{dt^{*2}} + \alpha\frac{d\theta^*}{dt^*} + \beta\sin(\gamma\theta^*) = 0$$

Rescale θ^*:
$$\theta^* = [\theta^*]\hat{\theta}$$

Use the scaled derivatives:
$$\frac{d\theta^*}{dt^*} = [\theta^*]\frac{d\hat{\theta}}{dt^*}$$
$$\frac{d^2\theta^*}{dt^{*2}} = [\theta^*]\frac{d^2\hat{\theta}}{dt^{*2}}$$

Substitute into the original equation:

$$[\theta^*]\frac{d^2\hat{\theta}}{dt^{*2}} + \alpha[\theta^*]\frac{d\hat{\theta}}{dt^*} + \beta\sin(\gamma[\theta^*]\hat{\theta}) = 0$$

Define:
$$[\theta^*] = \frac{1}{\gamma}$$

Then the equation becomes:

$$\frac{1}{\gamma}\frac{d^2\hat{\theta}}{dt^{*2}} + \frac{\alpha}{\gamma}\frac{d\hat{\theta}}{dt^*} + \beta\sin(\hat{\theta}) = 0$$