

Problem 4.3: For the damp pendulum equation

$$\ddot{\theta} + \alpha \dot{\theta} + \beta \sin(\gamma \theta) = 0, \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = 1$$

find suitable rescaling for $\alpha \sim \gamma \beta \sim \frac{1}{\gamma} \gg 1$.

Solution:

$$\frac{1}{\gamma} \gg 1 \implies \gamma \ll 1.$$

Currently, the equation is in the form:

$$\frac{d^2 \theta^*}{dt^{*2}} + \alpha \frac{d\theta^*}{dt^*} + \beta \sin(\gamma \theta^*) = 0$$

Rescale θ^* and t^* :

$$\begin{aligned} \theta^* &= [\theta^*] \hat{\theta} \\ t^* &= [t^*] t \end{aligned}$$

Use the scaled derivatives:

$$\begin{aligned} \frac{d\theta^*}{dt^*} &= \frac{[\theta^*]}{[t^*]} \frac{d\hat{\theta}}{dt^*} \\ \frac{d^2 \theta^*}{dt^{*2}} &= \frac{[\theta^*]}{[t^*]^2} \frac{d^2 \hat{\theta}}{dt^{*2}} \end{aligned}$$

Substitute into the original equation:

$$\begin{aligned} \frac{[\theta^*]}{[t^*]^2} \frac{d^2 \hat{\theta}}{dt^{*2}} + \alpha \frac{[\theta^*]}{[t^*]} \frac{d\hat{\theta}}{dt^*} + \beta \sin(\gamma [\theta^*] \hat{\theta}) &= 0 \\ [\theta^*] \frac{d^2 \hat{\theta}}{dt^{*2}} + \alpha [t^*] [\theta^*] \frac{d\hat{\theta}}{dt^*} + [t^{*2}] \beta \sin(\gamma [\theta^*] \hat{\theta}) &= 0 \end{aligned}$$

Define:

$$\begin{aligned} [t^*] &= \frac{1}{\sqrt{\beta}} \\ [t^{*2}] &= \frac{1}{\beta} \\ [\theta^*] &= \frac{1}{\alpha} \end{aligned}$$

The equation becomes:

$$\frac{1}{\alpha} \frac{d^2 \hat{\theta}}{dt^{*2}} + \frac{1}{\sqrt{\beta}} \frac{d\hat{\theta}}{dt^*} + \sin\left(\frac{\gamma}{\alpha} \hat{\theta}\right) = 0$$

$$\frac{dM_1(t)}{dt} = r_1 M_1(t) \left(1 - \frac{M_1(t)}{K_1} \right) - \sigma_1 M_1(t) + \gamma_2 M_2(t - \tau) - \gamma_1 M_1(t)$$

$$\frac{dM_2(t)}{dt} = -\sigma_2 M_2(t) + \gamma_1 M_1(t - \tau) - \gamma_2 M_2(t)$$