Problem 2: The following system of differential equations describe the motions of a certain pendulum:

$$\frac{d\theta}{dt} = y$$
$$\frac{dy}{dt} = -5\sin(\theta) - \frac{9}{13}y$$

where θ is the angle between the rod and the downward vertical direction and $\frac{d\theta}{dt}$ is the speed at which the angle changes. Find all the steady state solutions for the system. Also, identify if the steady state solutions are stable.

Solution:

Given the system:

$$\frac{d\theta}{dt} = y$$
$$\frac{dy}{dt} = -5\sin(\theta) - \frac{9}{13}y$$

Set the DEs to zero for steady-state:

$$0 = y$$
$$0 = -5\sin(\theta) - \frac{9}{13}y$$

From the first equation, $y = 0$.

Substitute $y = 0$ into the second equation:

$$0 = -5\sin(\theta) \implies \sin(\theta) = 0$$

The solutions for θ where $\sin(\theta) = 0$ are:

$$\theta = n\pi \quad n \in \mathbb{Z}$$

Steady-State Solutions:

$$(\theta, y) = (n\pi, 0) \quad n \in \mathbb{Z}$$

Compute the Jacobian matrix of the system at each equilibrium point:

$$J = \begin{bmatrix} \frac{\partial}{\partial\theta}\left(\frac{d\theta}{dt}\right) & \frac{\partial}{\partial y}\left(\frac{d\theta}{dt}\right) \\ \frac{\partial}{\partial\theta}\left(\frac{dy}{dt}\right) & \frac{\partial}{\partial y}\left(\frac{dy}{dt}\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5\cos(\theta) & -\frac{9}{13} \end{bmatrix}$$

Plug in $(\theta, y) = (n\pi, 0)$:

$$\cos(n\pi) = (-1)^n$$

So, the Jacobian becomes:

$$J = \begin{bmatrix} 0 & 1 \\ -5(-1)^n & -\frac{9}{13} \end{bmatrix}$$

The trace of the Jacobian is always negative, so the equilibrium points are saddles. Saddles are not stable, so none of the equilibrium points are stable.