Problem 2: For a given model equation:

$$\frac{dN(t)}{dt} = r_B N(t) e^{-\Psi r_B (N(t))} - B\frac{N(t)}{A + N(t)}$$

perform dimensional analysis to reduce the equation to the form

$$\frac{du}{d\tau} = rue^{-qu} - \frac{u}{1+u}$$

Use graphical techniques to perform the stability analysis and the bifurcation analysis with the parameter q fixed and the parameter r as a bifurcation parameter. Also sketch the bifurcation diagram.

Solution:

Let $N = [N]N^*$ and $t = [t]t^*$. Then,

$$\begin{aligned}\frac{[N]}{[t]}\frac{dN^*}{dt^*} &= r_B[N]N^* e^{-\Psi r_B [N]N^*} - B\frac{[N]N^*}{A + [N]N^*}\\ \frac{dN^*}{dt^*} &= [t]r_B N^* e^{-\Psi r_B [N]N^*} - \frac{B[t]N^*}{A + [N]N^*}\end{aligned}$$

Set:

$$\begin{aligned}[N] &= A\\ [t] &= \frac{A}{B}\end{aligned}$$

Substituting these values:

$$\begin{aligned}\frac{dN^*}{dt^*} &= \frac{r_B A}{B}N^* e^{-\Psi r_B A N^*} - \frac{B\frac{A}{B}N^*}{A + AN^*}\\ &= \frac{r_B A}{B}N^* e^{-\Psi r_B A N^*} - \frac{N^*}{1 + N^*}\end{aligned}$$

Let $u = N^*$, $\tau = t^*$, $r = \frac{r_B A}{B}$, and $q = \Psi r_B A$. Then:

$$\frac{du}{d\tau} = rue^{-qu} - \frac{u}{1+u}$$

Stability and Bifurcation Analysis:

Set:

$$rue^{-qu} - \frac{u}{1+u} = 0$$

Trivial equilibrium is $u = 0$. For non-trivial equilibria:

$$re^{-qu} = \frac{1}{1+u}$$

Let:

$$f(u) = re^{-qu}$$
$$g(u) = \frac{1}{1+u}$$

The intersections of these curves give the equilibrium points.

1. For $r < 1$, there's only one intersection at $u = 0$.

2. For $r > 1$, there are two equilibria: 0 and $u^* > 0$.

Here is the picture. These are the graphs with u factored out from the equations. In it, I found that 0 is unstable and $u^* > 0$ is stable.

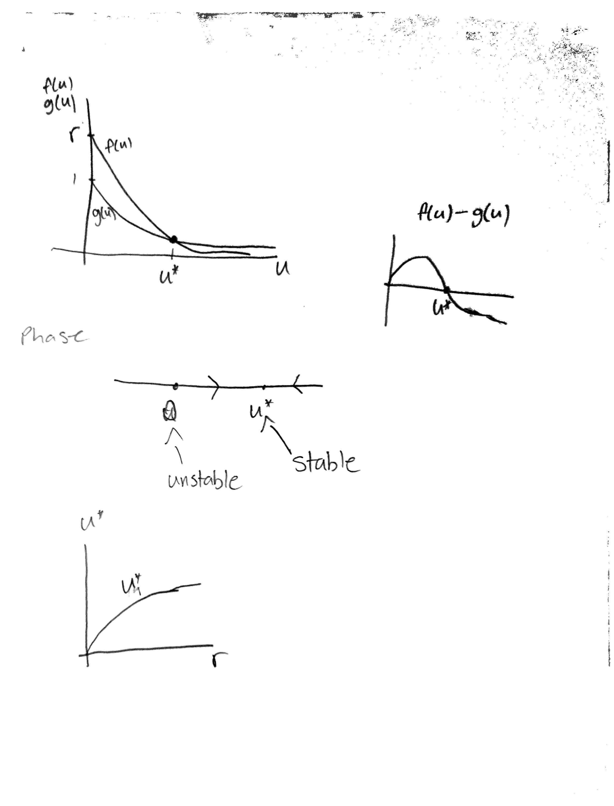

Figure 1: Bifurcation Diagram