Homework #1

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Problem 1:

Solve

$$\frac{dy}{dx} + 2xy = f(x), \quad y(0) = 2$$

where

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

Solution:

$$\frac{dy}{dx} + g(x)y = f(x), \quad y(0) = y_0$$

Define the integrating factor:

$$\begin{aligned} I(x,0) &= e^{\int_0^x g(u)du} \\ &= e^{\int_0^x 2udu} \\ &= e^{x^2 - 0^2} \\ &= e^{x^2} \end{aligned}$$

Evaluate:

$$\begin{aligned} \frac{dI}{dx} &= e^{\int_0^x g(u)du} \cdot \frac{d}{dx} \int_0^x g(u)du \\ &= e^{\int_0^x g(u)du} \cdot g(x) \\ &= g(x)I \end{aligned}$$

Multiply the differential equation by the integrating factor:

$$\begin{aligned} I\frac{dy}{dx} + Ig(x)y &= If(x) \\ I\frac{dy}{dx} + \frac{dI}{dt}\cdot y &= f(x)I \\ \frac{d}{dx}(I\cdot y) &= If(x) \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \int_0^x \frac{d}{du}(I(u,0)\cdot y)du &= \int_0^x I(u,0)f(u)du \\ I(u,0)y(u)\Big|_0^x &= \int_0^x I(u,0)f(u)du \\ I(x,0)y(x) - I(0,0)y(0) &= \int_0^x I(u,0)f(u)du \\ I(x,0)y(x) - 1\cdot y_0 &= \int_0^x I(u,0)f(u)du \\ I(x,0)y(x) - y_0 &= \int_0^x I(u,0)f(u)du \end{aligned}$$

Plug in values:

$$\begin{aligned} I(x,0)y(x) - y_0 &= \int_0^x I(u,0)f(u)du \\ e^{x^2}y(x) - 2 &= \int_0^x e^{u^2}f(u)du \end{aligned}$$

Consider $f(u)$:

$$f(u) = \begin{cases} u, & 0 \le u \le 1 \\ 0, & u > 1 \end{cases}$$

The integral is from 0 to x, and u is between these values:

$$\begin{aligned} e^{x^2}y(x) - 2 &= \int_0^x e^{u^2}f(u)du \\ &= \begin{cases} \int_0^x u\cdot e^{u^2}du, & 0 \le x \le 1 \\ \int_0^1 u\cdot e^{u^2}du, & x > 1 \end{cases} \end{aligned}$$

Evaluate the integrals:

$$\begin{aligned} \int_0^x u\cdot e^{u^2}du &= \frac{1}{2}e^{u^2}\Big|_0^x \\ &= \frac{1}{2}e^{x^2} - \frac{1}{2}e^{0^2} \\ &= \frac{1}{2}e^{x^2} - \frac{1}{2} \\ &= \frac{e^{x^2}-1}{2} \end{aligned}$$

and

$$\int_0^1 u \cdot e^{u^2} du = \frac{1}{2}e^{u^2}\Big|_0^1$$
$$= \frac{1}{2}e^{1^2} - \frac{1}{2}e^{0^2}$$
$$= \frac{1}{2}e - \frac{1}{2}$$
$$= \frac{e-1}{2}$$

Solve for $y(x)$ when $0 \leq x \leq 1$:

$$\begin{aligned} e^{x^2}y(x) - 2 &= \frac{e^{x^2}-1}{2} \\ e^{x^2}y(x) &= 2 + \frac{e^{x^2}-1}{2} \\ y(x) &= \frac{2}{e^{x^2}} + \frac{e^{x^2}-1}{2e^{x^2}} \\ &= \frac{4}{2e^{x^2}} + \frac{1}{2} - \frac{1}{2e^{x^2}} \\ &= \frac{3}{2e^{x^2}} + \frac{1}{2} \end{aligned}$$

Solve for $y(x)$ when $x > 1$:

$$\begin{aligned} e^{x^2}y(x) - 2 &= \frac{e-1}{2} \\ e^{x^2}y(x) &= 2 + \frac{e-1}{2} \\ y(x) &= \frac{2}{e^{x^2}} + \frac{e-1}{2e^{x^2}} \\ &= \frac{4}{2e^{x^2}} + \frac{e-1}{2e^{x^2}} \\ &= \frac{e+3}{2e^{x^2}} \end{aligned}$$

The final solution can be written as:

$$y(x) = \begin{cases} \frac{3}{2e^{x^2}} + \frac{1}{2}, & 0 \leq x \leq 1 \\ \frac{e+3}{2e^{x^2}}, & x > 1 \end{cases}$$

Problem 2: Write a first order, linear, inhomogenous differential equation whose solution $y(t)$ goes to 4 as $t \to \infty$.

Solution:

Consider a general first order linear inhomogenous differential equation:

$$\frac{dy}{dt} + g(t)y = f(t) \quad y(t_0) = y_0$$

This has a general solution of:

$$y(t) = e^{-\int_{t_0}^{t} g(s)ds} \cdot y_0 + \int_{t_0}^{t} f(u)e^{-\int_{u}^{t} g(s)ds} du$$

Define:

$$\begin{aligned} y_0 &= 4 \\ f(t) &= 4t \\ g(t) &= t \end{aligned}$$

Plug into the general solution:

$$\begin{aligned} y(t) &= e^{-\int_{t_0}^{t} sds} \cdot 4 + \int_{t_0}^{t} 4te^{-\int_{u}^{t} sds} du \\ &= 4 + \int_{t_0}^{t} 4te^{\frac{-(t^2-u^2)}{2}} du \\ &= 4 + (4 - 4e^{\frac{-(t^2-t_0^2)}{2}}) \end{aligned}$$

Take the limit as $t \to \infty$:

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} 4 + (4 - 4e^{\frac{-(t^2-t_0^2)}{2}}) \\ &= 4 + 4 - 4e^0 \\ &= 8 - 4 \\ &= 4 \end{aligned}$$

Therefore, the differential equation is:

$$\frac{dy}{dt} + ty = 4t \quad y(0) = 4$$

Problem 3: Find the general solution to

$$\dot{x} = x + \cos(t)$$

Show by choosing the initial condition appropriately that there is exactly one periodic solution to this problem. Remember, by periodic we mean that there is some T such that $x(t + T) = x(t)$.

Solution:

Assume the initial condition:

$$x_0 = 0$$

Rewrite the differential equation:

$$\dot{x} - x = \cos(t)$$

Response:

$$\begin{aligned}
x(t) &= \int_{t_0}^{t} \cos(s)e^{(t-s)}ds \\
&= e^t \int_{t_0}^{t} \cos(s)e^{-s}ds \\
&= e^t \left(-\cos(s)e^{-s}\Big|_{t_0}^{t} - \int_{t_0}^{t} \sin(s)e^{-s}ds \right) \\
&= e^t \left((-\cos(s)e^{-s} + \sin(s)e^{-s})\Big|_{t_0}^{t} - \int_{t_0}^{t} \cos(s)e^{-s}ds \right) \\
&= e^t \left(-\cos(s)e^{-s} + \sin(s)e^{-s} \right) \Big|_{t_0}^{t} - x(t) \\
2x(t) &= e^t \left(-\cos(s)e^{-s} + \sin(s)e^{-s} \right) \Big|_{t_0}^{t} \\
x(t) &= \frac{1}{2}e^t \left(-\cos(s)e^{-s} + \sin(s)e^{-s} \right) \Big|_{t_0}^{t} \\
&= \frac{1}{2}e^t \left(-e^{-t}\cos(t) + e^{-t}\sin(t) + e^{-t_0}\cos(t_0) - e^{-t_0}\sin(t_0) \right) \\
&= \frac{1}{2} \left(-\cos(t) + \sin(t) + e^{(t-t_0)}(\cos(t_0) - \sin(t_0)) \right) \\
&= \frac{1}{2} \left(-\cos(t) + \sin(t) \right) + \frac{1}{2}e^{(t-t_0)} \left(\cos(t_0) - \sin(t_0) \right) \\
&= \frac{-1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}\cos(t) - \frac{1}{\sqrt{2}}\sin(t) \right) + \frac{1}{2}e^{(t-t_0)} \left(\cos(t_0) - \sin(t_0) \right) \\
&= \frac{-1}{\sqrt{2}} \left(\cos(\frac{\pi}{4}) \cdot \cos(t) - \sin(\frac{\pi}{4}) \cdot \sin(t) \right) + \frac{1}{2}e^{(t-t_0)} \left(\cos(t_0) - \sin(t_0) \right) \\
&= \frac{-1}{\sqrt{2}} \left(\cos(t - \frac{\pi}{4}) \right) + \frac{1}{2}e^{(t-t_0)} \left(\cos(t_0) - \sin(t_0) \right)
\end{aligned}$$

With the initial condition $t_0 = \frac{\pi}{4}$, the solution becomes:

$$\begin{aligned}
x(t) &= \frac{-1}{\sqrt{2}} \left(\cos(t - \frac{\pi}{4}) \right) + \frac{1}{2}e^{(t-\frac{\pi}{4})} \left(\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4}) \right) \\
&= \frac{-1}{\sqrt{2}} \left(\cos(t - \frac{\pi}{4}) \right) + \frac{1}{2}e^{(t-\frac{\pi}{4})} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \\
&= \frac{-1}{\sqrt{2}} \left(\cos(t - \frac{\pi}{4}) \right)
\end{aligned}$$

This solution is periodic with period 2π.

Problem 4: Consider the equation

$$\dot{x} + p(t)x = 0$$

Suppose that $p(t)$ is continuous and periodic with period T, i.e. $p(t+T) = p(t)$. Show that the solution $x(t)$ for any initial condition is periodic if and only if

$$\int_0^T p(s)ds = 0$$

Said another way, you are showing that if $p(t)$ has zero average in time, then the solution will be periodic.

Solution:

The general solution to the equation $\dot{x} + p(t)x = 0$ is:

$$x(t) = x_0 e^{-\int_{t_0}^{t} p(s)ds}$$

Suppose the solution is periodic with period T for any initial condition:

$$x(t+T) = x(t) \quad \forall t, x_0$$

Use this equality to simplify the general solution:

$$\begin{aligned}
x(t+T) &= x(t) \\
x_0 e^{-\int_{t_0}^{t+T} p(s)ds} &= x_0 e^{-\int_{t_0}^{t} p(s)ds} \\
e^{-\int_{t_0}^{t+T} p(s)ds} &= e^{-\int_{t_0}^{t} p(s)ds} \\
-\int_{t_0}^{t+T} p(s)ds &= -\int_{t_0}^{t} p(s)ds \\
-\int_{t_0}^{t} p(s)ds - \int_{t}^{t+T} p(s)ds &= -\int_{t_0}^{t} p(s)ds \\
-\int_{t}^{t+T} p(s)ds &= 0 \\
\int_{t}^{t+T} p(s)ds &= 0
\end{aligned}$$

Use the fact that $p(t)$ is periodic with period T. The periodic function is being integrated over the length of the entire period, so the value of t is irrelevant:

$$\begin{aligned}
\frac{d}{dt}\int_{t}^{t+T} p(s)ds &= p(t+T) - p(t) = 0 \\
\implies \int_{t}^{t+T} p(s)ds &= c \\
\implies \int_{0}^{T} p(s)ds &= \int_{t}^{t+T} p(s)ds = 0
\end{aligned}$$

This proves $\implies$.

Now, assume:

$$\int_0^T p(s)ds = 0$$

We are given that $p(t)$ is periodic with period T. We can use this to show that the solution is periodic:

$$\begin{aligned}
x(t+T) &= x_0 e^{-\int_{t_0}^{t+T} p(s)ds} \\
&= x_0 e^{-\int_{t_0}^{t} p(s)ds - \int_t^{t+T} p(s)ds} \\
&= x_0 e^{-\int_{t_0}^{t} p(s)ds - \int_0^{T} p(s)ds} \quad \text{as shown above.} \\
&= x_0 e^{-\int_{t_0}^{t} p(s)ds - 0} \\
&= x_0 e^{-\int_{t_0}^{t} p(s)ds} \\
x(t+T) &= x(t)
\end{aligned}$$

This proves $\impliedby$.

Problem 5: For the system in the prior problem, show that if

$$\int_0^T p(s)ds = 0$$

then the solution is uniformly stable. Note, you'll need to use the fact that a continuous function, which is $x(t)$ in this case, is bounded, i.e. $\exists M > 0$ such that $|x(t)| \leq M$, over a finite interval.

Solution:

Assume:

$$\int_0^T p(s)ds = 0$$

From the previous problem, we know this implies the solution $x(t)$ is periodic with period T for any initial condition x_0. Since $x(t)$ is continuous on the closed, bounded interval $[0, T]$, $x(t)$ must have a max and min by the Extreme Value Theorem. Define:

$$M = \max |x(t)| \quad t \in [0, T]$$

$x(t)$ is periodic with period T, so:

$$|x(t)| \leq M \quad \forall t$$

Now, define $\gamma(t_0)$ such that

$$\gamma(t_0) = \frac{M}{|x_0|}$$
$$M = \gamma(t_0)|x_0|$$

Both $M > 0$ and $|y_0| > 0$, so $\gamma(t_0) > 0$.

We now have:

$$|x(t)| \leq M \quad \forall t \geq 0$$
$$|x(t)| \leq \gamma(t_0)|x_0| \quad \forall t \geq t_0$$

The system is uniformly stable:

$$\text{Remember } x(t) = x_0 e^{-\int_{t_0}^{t} p(s)ds}$$
$$\text{Let } p(t) \geq 0$$
$$\implies \int_{t_0}^{t} p(s)ds \geq 0$$
$$\implies e^{-\int_{t_0}^{t} p(s)ds} \leq 1$$
$$\implies |x(t)| \leq |x_0|$$

Therefore, $x(t)$ is uniformly stable.