Homework #10

Sebastian Griego

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Problem 1: Consider the continuous-time LTI system

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-CLTI}$$

Prove the following two statements:

(a) The controllable subspace $\mathcal{C}$ of the system (AB-CLTI) is A-invariant.

(b) The controllable subspace $\mathcal{C}$ of the system (AB-CLTI) contains $Im(B)$.

Solution:

(a) Let $v \in \mathcal{C}$.

$$\mathcal{C} = Im(\tilde{C}) \implies \exists x \in \tilde{V} : v = \tilde{C}x.$$

Then

$$\begin{aligned} Av &= A\tilde{C}x \\ &= A\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix} x \\ &= \begin{pmatrix} AB & A^2B & \cdots & A^nB \end{pmatrix} x. \end{aligned}$$

By Cayley-Hamilton, $A^n = \sum_{i=0}^{n-1} a_i A^i$, so

$$Av = \begin{pmatrix} AB & A^2B & \cdots & A^{n-1}B & \sum_{i=0}^{n-1} a_i A^i B \end{pmatrix} x.$$

Therefore $Av \in \mathcal{C}$.

(b) Let $v \in Im(B)$. Then $\exists x \in \tilde{V} : v = Bx$.

Remember that $\mathcal{C} = Im(\tilde{C})$ and

$$\tilde{C} = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$$

Then $v = Bx = \tilde{C} \begin{pmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Therefore, $v \in Im(\tilde{C}) = \mathcal{C}$.

Therefore, $Im(B) \subseteq \mathcal{C}$.

Problem 2: Consider a system in controllable canonical form

$$A = \begin{bmatrix} -\alpha_1 I_{k\times k} & -\alpha_2 I_{k\times k} & \cdots & -\alpha_{n-1} I_{k\times k} & -\alpha_n I_{k\times k} \\ I_{k\times k} & 0_{k\times k} & \cdots & 0_{k\times k} & 0_{k\times k} \\ 0_{k\times k} & I_{k\times k} & \cdots & 0_{k\times k} & 0_{k\times k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{k\times k} & 0_{k\times k} & \cdots & I_{k\times k} & 0_{k\times k} \end{bmatrix}_{nk\times nk},$$

$$B = \begin{bmatrix} I_{k\times k} \\ 0_{k\times k} \\ \vdots \\ 0_{k\times k} \\ 0_{k\times k} \end{bmatrix}_{nk\times k},$$

$$C = \begin{bmatrix} N_1 & N_2 & \cdots & N_{n-1} & N_n \end{bmatrix}_{m\times nk}.$$

Show that such a system is always controllable.

Solution:

$$\begin{aligned} \tilde{C} &= \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix} \\ &= \begin{pmatrix} I_{k\times k} & -\alpha_1 I_{k\times k} & (\alpha_1^2 - \alpha_2) I_{k\times k} & \cdots & * \\ 0_{k\times k} & I_{k\times k} & -\alpha_1 I_{k\times k} & \cdots & * \\ 0_{k\times k} & 0_{k\times k} & I_{k\times k} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{k\times k} & 0_{k\times k} & 0_{k\times k} & \cdots & I_{k\times k} \end{pmatrix} \end{aligned}$$

This matrix has full rank since it is triangular with identity matrices on the diagonal. Therefore, the system is controllable.

Problem 3: Consider the SISO LTI system in controllable canonical form

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^1,$$

where

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{n\times n}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{n\times 1}.$$

(a) Compute the characteristic polynomial of the closed-loop system for

$$u = -Kx, \quad K := \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}.$$

Hint: Compute the determinant of $(sI - A + BK)$ by doing a Laplace expansion along the first line of this matrix.

(b) Suppose you are given n complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ as desired locations for the closed-loop eigenvalues. Which characteristic polynomial for the closed-loop system would lead to these eigenvalues?

(c) Based on the answers to parts (a) and (b), propose a procedure to select K that would result in the desired values for the closed-loop eigenvalues.

(d) Suppose that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

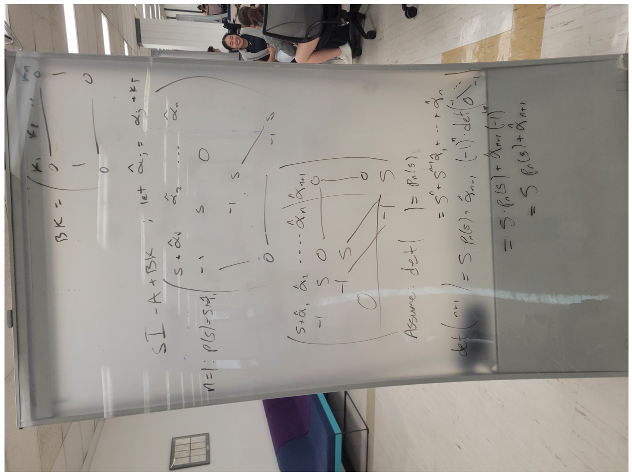
Find a matrix K for which the closed-loop eigenvalues are $\{-1, -1, -2\}$.

Solution:

(a) Find:

$$\begin{aligned} \det(sI - A + BK) &= \det \begin{pmatrix} s + \hat{\alpha}_1 & \hat{\alpha}_2 & \cdots & \hat{\alpha}_{n-1} & \hat{\alpha}_n \\ -1 & s & 0 & \cdots & 0 \\ 0 & -1 & s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & s \end{pmatrix} \\ &= s^n + \hat{\alpha}_1 s^{n-1} + \hat{\alpha}_2 s^{n-2} + \cdots + \hat{\alpha}_n \\ &= p(s) \end{aligned}$$

Matt and I proved this with induction. Here is a sideways picture:



(b) $f(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$

(c) Match coefficients of $f(s)$ and $p(s)$.

(d) Follow the process of the previous parts.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Choose K such that $\lambda = \{-1, -1, -2\}$.

Want $\det(sI - A + BK) = (s+1)^2(s+2) = s^3 + 4s^2 + 5s + 2$.

Compute $\det(sI - A + BK) = s^3 + (-1 + k_1)s^2 + (-2 + k_2)s + (-3 + k_3)$.

Set coefficients equal to get $k_1 = 5$, $k_2 = 7$, $k_3 = 5$.

Therefore,

$$K = \begin{bmatrix} 5 & 7 & 5 \end{bmatrix}.$$

Problem 4: The equations of motion of a satellite linearized around a steady-state solution, are given by

$$\dot{x} = Ax + Bu, \quad A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

where the state vector $x := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ includes the perturbation x_1 in the orbital radius, the perturbation x_2 in the radial velocity, the perturbation x_3 in the angle, and the perturbation x_4 in the angular velocity; and the input vector $u := \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ includes the radial thruster u_1 and a tangential thruster u_2.

(a) Show that the system is controllable from the input vector u.

(b) Can the system still be controlled if the radial thruster does not fire? What if it is the tangential thruster that fails?

Solution:

(a) Will show $\tilde{C}$ has full rank.

$$\begin{aligned} \tilde{C} &= \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 2\omega & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 1 & -2\omega & 1 & \dots \end{pmatrix} \end{aligned}$$

This already has full rank, so the system is controllable.

(b) Consider the eigenvectors of A^T and $\ker((Bu)^T)$.

$$Bu = \begin{pmatrix} 0 \\ u_1 \\ 0 \\ u_2 \end{pmatrix}$$

Excuse the notation.

$$\ker(Bu) = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{pmatrix}$$

One of the eigenvectors of A^T is $\begin{pmatrix} 2\omega \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

As is $\ker(Bu) \cap v = \{0\}$, so it is controllable.

If $u_1 = 0$, then $\ker(Bu) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix}$.

In this case, $\ker(Bu) \cap v = \{0\}$, so it is controllable.

If $u_2 = 0$, then $\ker(Bu) = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \end{pmatrix}$.

In this case, $\ker(Bu) \cap v \neq \{0\}$, so it is uncontrollable.

Problem 5: Consider an LTI system with realization

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Is this realization controllable? If not, perform a controllable decomposition to obtain a controllable realization of the same transfer function.

Solution:

$$\tilde{C} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\text{rank}(\tilde{C}) = 1$, so the system is uncontrollable.

$\tilde{V} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$\begin{aligned} \dot{\tilde{x}} &= T^{-1}ATx + T^{-1}Bu \\ y &= CT(T^{-1}x) + Du \end{aligned}$$

So,

$$\dot{\tilde{x}} = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} \tilde{x} + \begin{pmatrix} B_c \\ 0 \end{pmatrix} u$$

Find things:

$$\begin{aligned} A\tilde{V} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \tilde{V}A_c \implies A_c = -1 \\ B &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \tilde{V}B_c \implies B_c = 1 \\ CT &= T = \begin{pmatrix} C_c & C_u \end{pmatrix} \end{aligned}$$

Note,

$$T = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

So,

$$C_c = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Realiation:

$$\begin{aligned} \begin{pmatrix} C_c & C_u \end{pmatrix} (sI - \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix})^{-1} \begin{pmatrix} B_c \\ 0 \end{pmatrix} + D &= \begin{pmatrix} C_c & C_u \end{pmatrix} \begin{pmatrix} (sI - A_c)^{-1} & \tilde{A}_{21} \\ 0 & (sI - A_u)^{-1} \end{pmatrix} \begin{pmatrix} B_c \\ 0 \end{pmatrix} + D \\ &= \begin{pmatrix} C_c & C_u \end{pmatrix} \begin{pmatrix} (sI - A_c)^{-1}B_c \\ 0 \end{pmatrix} + D \\ &= C_c(s - A_c)^{-1}B_c + D \\ &= \frac{1}{s+1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \end{aligned}$$