Homework #2

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Problem 1: Solve

$$\frac{dy}{dx} + 2xy = f(x), \quad y(0) = 2$$

where

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2x, & 1 \leq x < 2 \\ 3x, & 2 \leq x < 3 \\ 0, & x \geq 3 \end{cases}$$

Solution:

$$\frac{dy}{dx} + 2xy = f(x), \quad y(0) = 2$$

Define the integrating factor:

$$\Phi(x,s) = e^{-\int_s^x 2t\,dt} = e^{-(x^2-s^2)}$$

Solution to the ODE:

$$\begin{aligned} y(x) &= \Phi(x,0)\cdot y(0) + \int_0^x \Phi(x,s)\cdot f(s)\,ds \\ &= 2e^{-x^2} + \int_0^x e^{-(x^2-s^2)}\cdot f(s)\,ds \end{aligned}$$

Evaluate the integral for each interval of $f(x)$:

- For $0 \leq x < 1$:

$$\begin{aligned}
\int_0^x e^{-(x^2-s^2)} \cdot s\,ds &= \int_0^x e^{-(x^2-s^2)} \cdot s\,ds \\
&= \int_0^x e^{s^2-x^2} \cdot s\,ds \\
&= \frac{1}{2}\int e^u\,du \quad u = s^2 - x^2 \\
&= \frac{e^u}{2} \\
&= \frac{e^{s^2-x^2}}{2}\bigg|_0^x \\
&= \frac{e^{x^2-x^2}}{2} - \frac{e^{0^2-x^2}}{2} \\
&= \frac{1-e^{-x^2}}{2} \\
&= \frac{-e^{-x^2}}{2} + \frac{1}{2}
\end{aligned}$$

- For $1 \leq x < 2$:

$$\begin{aligned}
\int_0^1 e^{-(x^2-s^2)} \cdot s\,ds + \int_1^x e^{-(x^2-s^2)} \cdot 2s\,ds &= \int_0^1 e^{-(x^2-s^2)} \cdot s\,ds + 2\int_1^x e^{-(x^2-s^2)} \cdot s\,ds \\
&= \frac{e^{s^2-x^2}}{2}\bigg|_0^1 + 2 \cdot \frac{e^{s^2-x^2}}{2}\bigg|_1^x \\
&= \frac{e^{1-x^2} - e^{-x^2}}{2} + 1 - e^{1-x^2} \\
&= \frac{-e^{-x^2}}{2} - \frac{e^{1-x^2}}{2} + 1
\end{aligned}$$

- For $2 \leq x < 3$:

$$\begin{aligned}
&\int_0^1 e^{-(x^2-s^2)} \cdot s\,ds + 2\int_1^2 e^{-(x^2-s^2)} \cdot s\,ds + 3\int_2^x e^{-(x^2-s^2)} \cdot s\,ds \\
&= \frac{e^{1-x^2} - e^{-x^2}}{2} + 2 \cdot \frac{e^{s^2-x^2}}{2}\bigg|_1^2 + 3 \cdot \frac{e^{s^2-x^2}}{2}\bigg|_2^x \\
&= \frac{e^{1-x^2} - e^{-x^2}}{2} + \frac{2e^{4-x^2} - 2e^{1-x^2}}{2} + \frac{3 - 3e^{4-x^2}}{2} \\
&= \frac{-e^{-x^2}}{2} - \frac{e^{1-x^2}}{2} - \frac{e^{4-x^2}}{2} + \frac{3}{2}
\end{aligned}$$

- For $x \geq 3$:

$$\begin{aligned}
&\int_0^1 e^{-(x^2-s^2)} \cdot s\,ds + \int_1^2 e^{-(x^2-s^2)} \cdot 2s\,ds + \int_2^3 e^{-(x^2-s^2)} \cdot 3s\,ds + \int_3^x e^{-(x^2-s^2)} \cdot 0\,ds \\
&= \int_0^1 e^{-(x^2-s^2)} \cdot s\,ds + \int_1^2 e^{-(x^2-s^2)} \cdot 2s\,ds + \int_2^3 e^{-(x^2-s^2)} \cdot 3s\,ds \\
&= \frac{e^{1-x^2} - e^{-x^2}}{2} + \frac{2e^{4-x^2} - 2e^{1-x^2}}{2} + 3 \cdot \frac{e^{s^2-x^2}}{2}\Big|_2^3 \\
&= \frac{e^{1-x^2} - e^{-x^2}}{2} + \frac{2e^{4-x^2} - 2e^{1-x^2}}{2} + \frac{3e^{9-x^2} - 3e^{4-x^2}}{2} \\
&= \frac{-e^{-x^2}}{2} - \frac{e^{1-x^2}}{2} - \frac{e^{4-x^2}}{2} + \frac{3e^{9-x^2}}{2}
\end{aligned}$$

Remember, the solution is

$$y(x) = 2e^{-x^2} + \int_0^x e^{-(x^2-s^2)} \cdot f(s)\,ds$$

The leading term of all the previous integrals is $\frac{-e^{-x^2}}{2}$, so adding $2e^{-x^2}$ to them gives the leading term of the solution, $\frac{3e^{-x^2}}{2}$. All the other terms stay the same. This gives a final solution of

$$y(x) = \begin{cases} \frac{3e^{-x^2}}{2} + \frac{1}{2}, & 0 \leq x < 1 \\ \frac{3e^{-x^2}}{2} - \frac{e^{1-x^2}}{2} + 1, & 1 \leq x < 2 \\ \frac{3e^{-x^2}}{2} - \frac{e^{1-x^2}}{2} - \frac{e^{4-x^2}}{2} + \frac{3}{2}, & 2 \leq x < 3 \\ \frac{3e^{-x^2}}{2} - \frac{e^{1-x^2}}{2} - \frac{e^{4-x^2}}{2} + \frac{3e^{9-x^2}}{2}, & x \geq 3 \end{cases}$$

Problem 2: Find the solution of the initial value problem $\frac{d\vec{x}}{dt} = A\vec{x},\ \vec{x}(0) = \vec{x_0}$ for A given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Solution:

Start with

$$\frac{d\vec{x}}{dt} = A\vec{x},\ \vec{x}(0) = \vec{x_0}$$

The solution is given by

$$\vec{x}(t) = V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1}\vec{x_0}$$

Find eigenvalues and eigenvectors of A:

$$\begin{aligned} p(\lambda) &= \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) \\ &= \lambda^2 - 3\lambda + 1 \end{aligned}$$

Set $p(\lambda) = 0$ and solve for λ:

$$\begin{aligned} \lambda^2 - 3\lambda + 1 &= 0 \\ \lambda &= \frac{1}{2}\left(3 \pm (9 - 4 \cdot 1)^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(3 \pm 5^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(3 \pm \sqrt{5}\right) \end{aligned}$$

So, the eigenvalues are $\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$ and $\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$.

Find the eigenvectors:

- For $\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$:

  $$A - \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right) I = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & -1 \\ -1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix}$$

  Set $(A - \lambda I)\vec{v} = 0$:

  $$\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & -1 \\ -1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

  This gives

  $$-v_1 + \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right) v_2 = 0 \implies v_1 = \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right) v_2$$

  Set $v_2 = 1$. Then

  $$v_1 = \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

  So, the eigenvector is

  $$\vec{v} = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

- For $\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$:

  $$A - \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right) I = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & -1 \\ -1 & -\frac{1}{2} + \frac{\sqrt{5}}{2} \end{pmatrix}$$

  Set $(A - \lambda I)\vec{v} = 0$:

  $$\begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & -1 \\ -1 & -\frac{1}{2} + \frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

This gives

$$-v_1 + \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right) v_2 = 0 \implies v_1 = \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right) v_2$$

Set $v_2 = 1$. Then

$$v_1 = \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$$

So, the eigenvector is

$$\vec{v} = \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

V is given by

$$V = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2} & -\frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$$

Find V^{-1}:

$$\begin{aligned} V^{-1} &= \frac{1}{\det(V)} \begin{pmatrix} 1 & \frac{1}{2} - \frac{\sqrt{5}}{2} \\ -1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} \\ &= \frac{-1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{1}{2} - \frac{\sqrt{5}}{2} \\ -1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} \end{aligned}$$

Final solution:

$$\begin{aligned} \vec{x}(t) &= V\Lambda V^{-1}\vec{x_0} \\ &= \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2} & -\frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)t} & 0 \\ 0 & e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t} \end{pmatrix} \frac{-1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{1}{2} - \frac{\sqrt{5}}{2} \\ -1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} \vec{x_0} \end{aligned}$$

Problem 3: For the real 2×2 matrix A, consider the initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0$$

Show that if $A = V\Lambda V^{-1}$ where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

then the solution is given by

$$x(t) = V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1} x_0$$

Solution:

Start with

$$\frac{d\vec{x}}{dt} = A\vec{x},\ \vec{x}(0) = \vec{x_0}$$

We want to find the eigenvalues and eigenvectors of A to decompose it into

$$A = V\Lambda V^{-1} = V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V^{-1}$$

Find eigenvalues and eigenvectors of A:

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)$$

Set $p(\lambda) = 0$ and solve for λ:

$$\begin{aligned} p(\lambda) &= \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \\ \lambda &= \frac{1}{2}\left(\mathrm{Tr}(A) \pm \left((\mathrm{Tr}(A))^2 - 4\det(A)\right)^{\frac{1}{2}}\right) \end{aligned}$$

Assume λ_1 and λ_2 are distinct. Decompose A into

$$\begin{aligned} A &= V\Lambda V^{-1} \\ &= V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V^{-1} \end{aligned}$$

Go back to the differential equation:

$$\begin{aligned} \frac{d\vec{x}}{dt} &= A\vec{x} \\ \frac{d\vec{x}}{dt} &= V\Lambda V^{-1}\vec{x} \\ V^{-1}\frac{d\vec{x}}{dt} &= \Lambda V^{-1}\vec{x} \\ \frac{d}{dt}(V^{-1}\vec{x}) &= \Lambda(V^{-1}\vec{x}) \end{aligned}$$

Let $\vec{y} = V^{-1}\vec{x}$. Then

$$\begin{aligned} \frac{d\vec{y}}{dt} &= \Lambda\vec{y} \\ \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{aligned}$$

This gives

$$\begin{aligned} \frac{dy_1}{dt} &= \lambda_1 y_1 \\ \frac{dy_2}{dt} &= \lambda_2 y_2 \end{aligned}$$

Solve each differential equation:

$$\begin{aligned} y_1(t) &= y_1(0) \cdot e^{\lambda_1 t} \\ y_2(t) &= y_2(0) \cdot e^{\lambda_2 t} \end{aligned}$$

Decompose this back into matrix form:

$$\vec{y}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \vec{y}(0)$$

Given $\vec{y}(t) = V^{-1}\vec{x}$, we have $\vec{x}(t) = V\vec{y}(t)$:

$$\begin{aligned} \vec{x}(t) &= V\vec{y}(t) \\ &= V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \vec{y}(0) \\ &= V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1}\vec{x_0} \end{aligned}$$

Problem 4: Show that if $A = VJV^{-1}$ where

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

then the solution is given by

$$x(t) = V \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} V^{-1}x_0$$

Solution:

Start with

$$\frac{d\vec{x}}{dt} = A\vec{x}, \ \vec{x}(0) = \vec{x_0}$$

From the previous problem, the eigenvalues are

$$\lambda = \frac{1}{2}\left(\text{Tr}(A) \pm \left((\text{Tr}(A))^2 - 4\det(A)\right)^{\frac{1}{2}}\right)$$

Assume the eigenvalues are equal. Then the matrix A can be decomposed as

$$A = VJV^{-1}$$

where

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Go back to the differential equation:

$$\begin{aligned} \frac{d\vec{x}}{dt} &= A\vec{x} \\ \frac{d\vec{x}}{dt} &= VJV^{-1}\vec{x} \\ V^{-1}\frac{d\vec{x}}{dt} &= JV^{-1}\vec{x} \\ \frac{d}{dt}(V^{-1}\vec{x}) &= J(V^{-1}\vec{x}) \end{aligned}$$

Let $\vec{y} = V^{-1}\vec{x}$. Then

$$\frac{d\vec{y}}{dt} = J\vec{y}$$
$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

This gives

$$\frac{dy_1}{dt} = \lambda y_1 + y_2$$
$$\frac{dy_2}{dt} = \lambda y_2$$

Solve the second equation:

$$y_2(t) = y_2(0) \cdot e^{\lambda t}$$

Substitute and solve the first equation:

$$\frac{dy_1}{dt} = \lambda y_1 + y_2(0) \cdot e^{\lambda t}$$
$$y_1(t) = y_1(0) \cdot e^{\lambda t} + y_2(0) \cdot te^{\lambda t}$$

We get

$$y_1(t) = y_1(0) \cdot e^{\lambda t} + y_2(0) \cdot te^{\lambda t}$$
$$y_2(t) = y_2(0) \cdot e^{\lambda t}$$

Decompose this back into matrix form:

$$\begin{aligned} \vec{y}(t) &= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}\vec{y}(0) \end{aligned}$$

Given $\vec{y}(t) = V^{-1}\vec{x}$, we have $\vec{x}(t) = V\vec{y}(t)$:

$$\begin{aligned} \vec{x}(t) &= V\vec{y}(t) \\ &= V\begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}\vec{y}(0) \\ &= V\begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}V^{-1}\vec{x}_0 \end{aligned}$$