Homework #1

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Problem 1: Use $A = UDV^T$ to recover the first column of A. Show detailed calculations of all the relevant matrices and vectors, and use space-time decomposition to describe your results. For extra credit: Describe the spatial and temporal modes, and their corresponding variances or energies.

Solution:

$$D = \begin{pmatrix} 1.414214 & 0 \\ 0 & 1.414214 \end{pmatrix} \quad \text{Diagonal Values}$$

$$U = \begin{pmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \end{pmatrix} \quad \text{Spatial Pattern Matrix}$$

$$V^T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Temporal Pattern Matrix}$$

Calculate UDV^T:

$$\begin{aligned}
UDV^T &= \begin{pmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \end{pmatrix} \begin{pmatrix} 1.414214 & 0 \\ 0 & 1.414214 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \end{pmatrix} \begin{pmatrix} 1.414214(-1) + 0(0) & 1.414214(0) + 0(-1) \\ 0(-1) + 1.414214(0) & 0(0) + 1.414214(-1) \end{pmatrix} \\
&= \begin{pmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \end{pmatrix} \begin{pmatrix} -1.414214 & 0 \\ 0 & -1.414214 \end{pmatrix} \\
&= \begin{pmatrix} -0.707107(-1.414214) + (-0.707107)(0) & -0.707107(0) + (-0.707107)(-1.414214) \\ -0.707107(-1.414214) + 0.707107(0) & -0.707107(0) + 0.707107(-1.414214) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\end{aligned}$$

So, the first column of A is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Problem 2: Use R and the updated Darwin and Tahiti standardized SLP data to reproduce the EOFs and PCs and to plot the EOF pattern maps and PC time series.

Solution:

R code:

```
setwd("C:/Users/sebas/OneDrive/Desktop/Homework/LinAlg")
Pda<-read.table("PSTANDdarwin.txt", header=F)
dim(Pda)
Pda
pdaDec<-Pda[,13] #Darwin Dec standardized SLP anomalies data
pdaDec

Pta<-read.table("PSTANDtahiti.txt", header=F)
ptaDec=Pta[,13] #Tahiti Dec standardized SLP anomalies
ptada1 = rbind(pdaDec, ptaDec) #space-time data matrix
ptada1
#Space-time data format
colnames(ptada1) <- 1951:2015

rownames(ptada1)<-c("Darwin", "Tahiti")
ptada1

dim(ptada1)
svdptd = svd(ptada1)

svdptd

U=round(svdptd$u, digits=2)
U

D = round(diag(svdptd$d), digits = 2)
D

V =round(svdptd$v, digits=2)
t(V)

eof1 = U[, 1]
eof2 = U[, 2]

PC1 = V[, 1]
PC2 = V[, 2]
```

```
x = c(eof1[1], eof2[1])
y = c(eof1[2], eof2[2])

# Plot EOFs with different colors
plot(x,y, col = c("blue", "red"), pch = 16,)

years = 1951:2015

# Plot PC over time
plot(years, PC1, type = 'l', col = 'blue', xlab = 'Year', ylab = 'Principal C
     main = 'Principal Components over Time', ylim = range(c(PC1, PC2)))
lines(years, PC2, type = 'l', col = 'red')
legend('topright', legend = c('PC1', 'PC2'), col = c('blue', 'red'), lty = 1)
```

Plots:

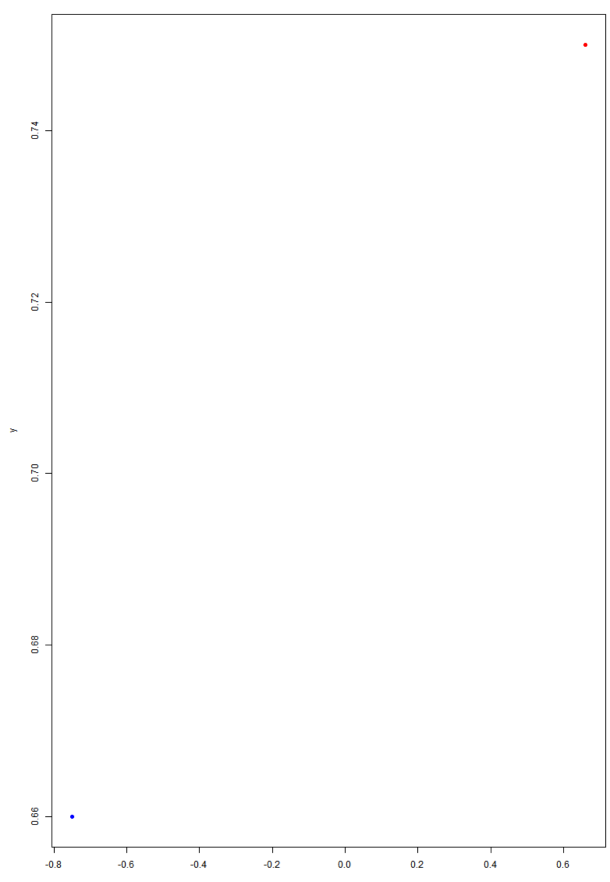

Figure 1: EOFs

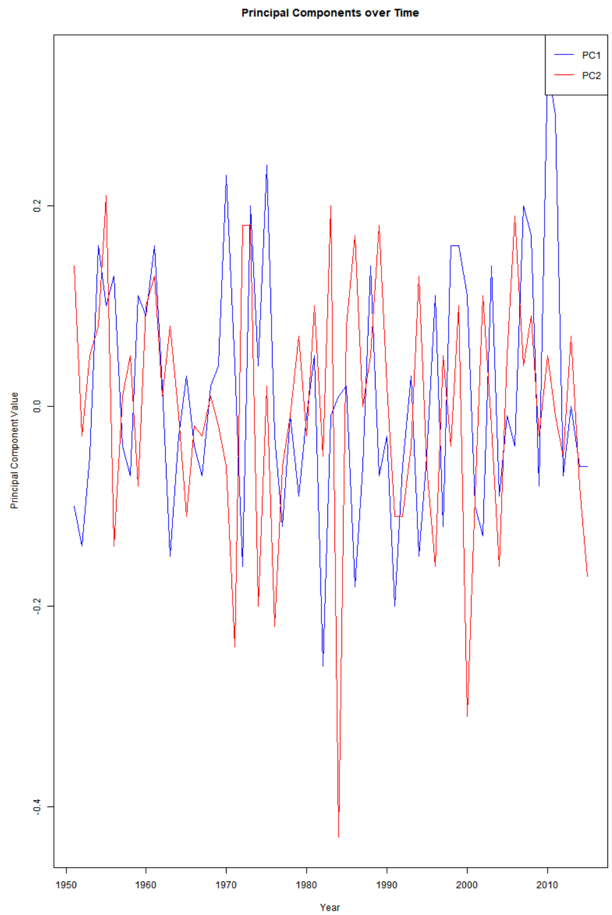

Figure 2: PCs

Problem 3:

(a) A covariance matrix C can be computed from a space-time observed anomaly data matrix X, which has N rows for spatial locations and Y columns for time in years:

$$C = \frac{X \cdot X^T}{Y}$$

This results in an $N \times N$ matrix. Select an X data matrix from the USHCN annual total precipitation data at three California stations, ordered from north to south: San Francisco, CA (040693); Santa Barbara, CA (047902); and San Diego, CA (042239). Use data from five years, spanning 2001 to 2005. Consider the anomaly data relative to the 2001-2005 mean and employ R to calculate the covariance matrix with $N = 3$ and $Y = 5$.

(b) Utilize R to determine the inverse matrix of the covariance matrix C.

(c) Use R to compute the eigenvalues and eigenvectors of C.

(d) Perform Singular Value Decomposition (SVD) on the data matrix X using R, such that $X = UDV^T$. Explicitly present the three resulting matrices U, D, and V.

(e) Explore the relationship between the eigenvalues of C and the matrix D using R.

(f) Compare the eigenvectors of C with the matrix U.

(g) Plot the Principal Component (PC) time series and provide a description of their behavior.

Solution:

(a) The covariance matrix C:

$$C = \begin{pmatrix} 0.8000000 & 0.3838680 & 0.7204883 \\ 0.3838680 & 0.8000000 & 0.2709303 \\ 0.7204883 & 0.2709303 & 0.8000000 \end{pmatrix}$$

R code:

```
SF = c(214.9, 107.7, 233.4, 337.5, 359.1)
SB = c(695.9, 389.7, 391.2, 446.8, 768.3)
SD = c(496.2, 475.1, 490.7, 685.7, 664.2)

mSF = mean(SF)
mSB = mean(SB)
mSD = mean(SD)

aSF = (SF - mSF)/sd(SF)
aSB = (SB - mSB)/sd(SB)
aSD = (SD - mSD)/sd(SD)

A = rbind(aSF, aSB, aSD)

C = A %*% t(A) / 5

C
```

(b) The inverse of the covariance matrix C^{-1}:

$$C^{-1} = \begin{pmatrix} 8.097171 & -1.5990410 & -6.7508602 \\ -1.599041 & 1.7277197 & 0.8549985 \\ -6.750860 & 0.8549985 & 7.0403385 \end{pmatrix}$$

R code:

```r
invC = solve(C)

invC
```

(c) Eigenvalues and eigenvectors of C:

- Eigenvalues: $\lambda_1 = 1.74763214$, $\lambda_2 = 0.58378101$, $\lambda_3 = 0.06858685$
- Eigenvectors:

$$\begin{pmatrix} 0.6495950 & 0.2108313 & 0.7304632 \\ 0.4403361 & -0.8875627 & -0.1354128 \\ 0.6197826 & 0.4096128 & -0.6693929 \end{pmatrix}$$

R code:

```r
eigC = eigen(C)

eigC
```

(d) SVD of C:

$$D = \begin{pmatrix} 2.9560380 & 0 & 0 \\ 0 & 1.7084803 & 0 \\ 0 & 0 & 0.5856059 \end{pmatrix}$$

$$U = \begin{pmatrix} -0.6495950 & 0.2108313 & -0.7304632 \\ -0.4403361 & -0.8875627 & 0.1354128 \\ -0.6197826 & 0.4096128 & 0.6693929 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.08105994 & -0.65102219 & -0.09272693 \\ 0.60884394 & 0.05280542 & 0.59658302 \\ 0.30415262 & 0.23730478 & -0.77171237 \\ -0.36257497 & 0.65581870 & 0.17907751 \\ -0.63148153 & -0.29490670 & 0.08877877 \end{pmatrix}$$

R code:

```r
X = svd(A)

X
```

(e) The relationship between the eigenvalues of C and the matrix D is $\lambda_i = d_i^2/5$. R code:

```r
2.9560380^2/5
1.7084803^2/5
0.5856059^2/5
```

(f) The eigenvectors of C are the same as the columns of U, but some of the signs are flipped. R code:

```r
eigC$vectors
X$u
```

(g) The PC time series R code:

```
pc1 = X$v[,1]
pc2 = X$v[,2]
pc3 = X$v[,3]

years = 2001:2005

plot(years, pc1, type='b', col='red', ylim=range(c(pc1,pc2,pc
xlab='Year', ylab='PC Value', main='Principal Component Time
lines(years, pc2, col='blue')
lines(years, pc3, col='green')
legend('topright', legend=c('PC1', 'PC2', 'PC3'),
col=c('red', 'blue', 'green'), lty=1, pch=1)
```

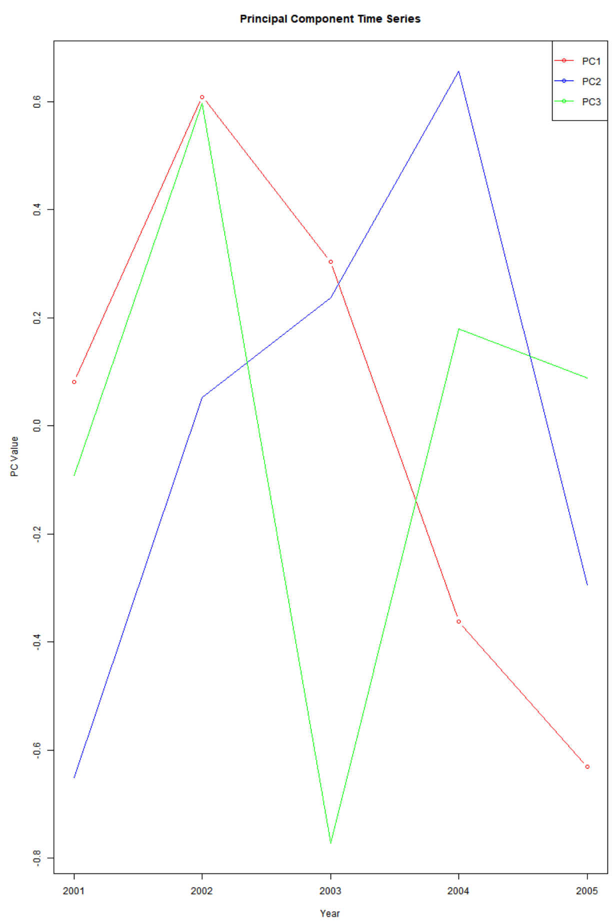

Figure 3: PC Time Series

PC1 is San Francisco, PC2 is Santa Barbara, PC3 is San Diego. They all vary in different ways. There does not seem to be any connection between the three.

Problem 4: The burning of propane C_3H_8 with oxygen O_2 produces water H_2O and carbon dioxide CO_2. Balance the chemical reaction equation.

Solution:

Write the equation:

$$C_3H_8 + O_2 \to CO_2 + H_2O$$

Write the equain terms of x, y, z:

$$C_3H_8 + xO_2 = yCO_2 + zH_2O$$

Balance the equation:

$$\begin{aligned} C &: 3 = y \\ H &: 8 = 2z \\ O &: 2x = 2y + z \end{aligned}$$

Solve the equations:

$$y = 3, \quad z = 4, \quad x = 5$$

Therefore, the balanced equation is:

$$C_3H_8 + 5O_2 = 3CO_2 + 4H_2O$$

Problem 5: Write a computer code to

(a) Read the NOAAGlobalTemp data file, and

(b) Generate a 4×8 space-time data matrix for the December mean surface air temperature anomaly data of four grid boxes and eight years.

Solution:

Rcode:

```
setwd("C:/Users/sebas/OneDrive/Desktop/Homework/LinAlg")

noaa_data <- read.csv("NOAAGlobalT.csv", header = TRUE)

noaa1 = noaa_data[1777,]
noaa2 = noaa_data[1778,]
noaa3 = noaa_data[1779,]
noaa4 = noaa_data[1780,]
```

```r
# Adjust DecDat to use data from 2001 to 2008
DecIndex = seq(1467, 1551, 12)
DecDat = noaa1[DecIndex]
DecDat2 = noaa2[DecIndex]
DecDat3 = noaa3[DecIndex]
DecDat4 = noaa4[DecIndex]

# Verify the data
DecDat

#rbind some stuff
Xm = rbind(DecDat, DecDat2, DecDat3, DecDat4)

rownames(Xm) = c('Lat-237.5/Lon-32.5', 'Lat-242.5/Lon-32.5', 'Lat-247.5/Lon-3
colnames(Xm) = 2001:2008
Xm
```

R code output:

```
                     2001    2002   2003    2004   2005    2006    2007
Lat 237.5/Lon 32.5 -0.4677  0.1699 0.1579 -0.0300 0.3653  0.3606 -1.0685
Lat 242.5/Lon 32.5 -0.8994 -0.2796 0.5777  0.2757 0.9472 -0.1739 -0.9703
Lat 247.5/Lon 32.5 -0.8168 -0.6909 0.4853 -0.1080 0.7542 -0.4942  0.0847
Lat 252.5/Lon 32.5  0.8871  0.1945 1.5203  0.2909 0.3835 -0.0018  0.7548
                     2008
Lat 237.5/Lon 32.5 -0.1625
Lat 242.5/Lon 32.5  0.5768
Lat 247.5/Lon 32.5  1.5718
Lat 252.5/Lon 32.5  0.5495
```

Problem 6: Write a computer code to find the inverse of the following matrix

$$\begin{pmatrix} 1.7 & -0.7 & 1.3 \\ -1.6 & -1.4 & 0.4 \\ -1.5 & -0.3 & 0.6 \end{pmatrix}$$

Solution:

Rcode:

```r
A1x = matrix(c(1.7,-0.7,1.3,-1.6,-1.4,0.4,-1.5,-0.3,0.6), nrow=3, byrow=T

solve(A1x)
```

Inverse of the matrix:

$$\begin{pmatrix} 0.2010050 & -0.008375209 & -0.4299274 \\ -0.1005025 & -0.829145729 & 0.7705193 \\ 0.4522613 & -0.435510888 & 0.9771078 \end{pmatrix}$$

Problem 7: Write a computer code to solve the following linear system of equations

$$Ax = b,$$

where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$$

$$x = \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}$$

$$b = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Solution:

Rcode:

```
A2x = matrix(c(1, 2, 3, 4, 5, 6, 7, 8, 0), nrow=3, byrow=TRUE)
b2 = c(-1, 0, 1)
solve(A2x, b2)
```

Problem 8: The following equation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has infinitely many solutions, and cannot be directly solved by a simple com- puter command, such as solve(A, b).

(a) Show that the three row vectors of the coefficient matrix are not linearly independent.

(b) Because of the dependence, the linear system has only two independent equations. Thus, reduce the linear system into two equations by treating x_3 as an arbitrary value while treating x_1 and x_2 as variables.

(c) Solve the two equations for x_1 and x_2 and express them in terms of x_3.

Solution:

$$x_1 + 2x_2 + 3x_3 = 0$$
$$4x_1 + 5x_2 + 6x_3 = 0$$
$$7x_1 + 8x_2 + 9x_3 = 0$$
$$2(4x_1 + 5x_2 + 6x_3) - (x_1 + 2x_2 + 3x_3) = 7x_1 + 8x_2 + 9x_3$$

Therefore, the row vectors are not linearly independent.

Now, let $x_3 = t$. Then, the system of equations becomes:

$$x_1 + 2x_2 + 3t = 0$$
$$4x_1 + 5x_2 + 6t = 0$$

Solve for x_1 and x_2 in terms of t:

$$x_1 = -2x_2 - 3t$$
$$x_2 = -4x_1 - 6t$$

Then

$$\begin{aligned} x_2 &= -4(-2x_2 - 3t) - 6t \\ &= 8x_2 + 12t - 6t \\ &= 8x_2 + 6t - 7x_2 \qquad = 6t \\ x_2 &= -\frac{6}{7}t \end{aligned}$$

So,

$$\begin{aligned} x_1 &= -2x_2 - 3t \\ &= -2(-\frac{6}{7}t) - 3t \\ &= \frac{12}{7}t - 3t \\ &= -\frac{9}{7}t \end{aligned}$$

Problem 9: Ethane is a gas similar to the greenhouse gas methane and can burn with oxygen to form carbon dioxide and water:

$$C_2H_6 + O_2 \to CO_2 + H_2O$$

Given two ethane molecules, how many molecules of oxygen, carbon dioxide and water should be in order for this chemical reaction equation to be balanced?

Solution:

Assign:

$$x : O_2 \quad y : CO_2, \quad z : H_2O$$

Write the equation:

$$2C_2H_6 + xO_2 = yCO_2 + zH_2O$$

Balance the equation:

$$\begin{aligned} C &: 2 \cdot 2 = y \\ H &: 2 \cdot 6 = 2z \\ O &: 2x = 2y + z \end{aligned}$$

Set up a matrix to find O:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$$

Put this into R and solve:

```
A3 = matrix(c(0, 1, 0, 0, 0, 1, 2, -2, -1), nrow=3, byrow=TRUE)
b3 = c(4, 6, 0)
solve(A3, b3)
```

The solution is:

$$\begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

So, the solution is:

$$2C_2H_6 + 7O_2 = 4CO_2 + 6H_2O$$

Problem 10:

(a) Use matrix multiplication to show that the vector

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is not an eigenvector of the following matrix

$$A = \begin{pmatrix} 0 & 4 \\ -2 & -7 \end{pmatrix}$$

(b) Find all the unit eigenvectors of matrix A in Part (a)

Solution:

(a) Multiply A by u:

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0 & 4\\-2 & -7\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\-9\end{pmatrix}$$

If you try scaling u by 4, you get:

$$4\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$$

This is not $\begin{pmatrix}4\\-9\end{pmatrix}$.

(b) Use R to find the unit eigenvectors of A:

```
A = matrix(c(0, 4, -2, -7), nrow=2, byrow=TRUE)
eigen_result = eigen(A)
eigenvectors = eigen_result$vectors
unit_eigenvectors = apply(eigenvectors, 2, function(v) v / sq
unit_eigenvectors
```

The unit eigenvectors are:

$$\begin{pmatrix}-0.5838895\\0.8118331\end{pmatrix}, \quad \begin{pmatrix}0.9410038\\-0.3383961\end{pmatrix}$$