Problem 1: The population, $N(t)$, of a plant species grows with per capita rate μ per week and its growth is controlled by crowding effect of the form $-\beta N^\alpha, \alpha > 1$ and $\beta > 0$.

1. Develop a model for the population dynamics of this plant species.
2. Perform scaling to reduce equation into the form of $\frac{dy}{dt} = \mu y - y^\alpha$.
3. For $\alpha = 3$, perform bifurcation analysis and sketch a bifurcation diagram with μ as a bifurcation parameter.

Solution:

1. **Population Model**

   Assuming exponential growth, and including the crowding effect, the model is:

$$\frac{dN}{dt} = \mu N - \beta N^\alpha$$

2. **Scaling**

$$\frac{dN}{dt} = \mu N - \beta N^\alpha$$

   Define:

$$N = [N]N^*$$
$$t = [t]t^*$$

   Substituting into the model equation:

$$\begin{aligned}\frac{[N]}{[t]}\frac{dN^*}{dt^*} &= \mu[N]N^* - \beta([N]N^*)^\alpha \\ \frac{1}{[t]}\frac{dN^*}{dt^*} &= \mu N^* - \beta([N]^{\alpha-1})(N^{*\alpha}) \\ \frac{dN^*}{dt^*} &= \mu[t]N^* - \beta[t]([N]^{\alpha-1})(N^{*\alpha})\end{aligned}$$

   To achieve the desired form $\frac{dy}{dt} = \mu y - y^\alpha$, we need:

$$\begin{aligned}[t] &= 1 \\ \beta[t][N]^{\alpha-1} &= 1\end{aligned}$$

   Solving these equations:

$$[t] = 1$$

$$[N] = \left(\frac{1}{\beta}\right)^{\frac{1}{\alpha-1}}$$

Let $y = N^*$. Then the scaled equation becomes:

$$\frac{dy}{dt} = \mu y - y^\alpha$$

3. **Bifurcation Analysis**

For $\alpha = 3$:

$$\frac{dy}{dt} = \mu y - y^3$$

Equilibrium Points:

$$\mu y - y^3 = 0 \implies y(\mu - y^2) = 0$$

So:

$$y = 0, \quad y = \sqrt{\mu}, \quad y = -\sqrt{\mu}$$

Stability Analysis:

$$f(y) = \mu y - y^3, \quad f'(y) = \mu - 3y^2$$

- At $y = 0$:

$$f'(0) = \mu$$

If $\mu > 0$, $y = 0$ is unstable. If $\mu < 0$, $y = 0$ is stable.

- At $y = \sqrt{\mu}$:

$$f'\left(\sqrt{\mu}\right) = \mu - 3(\mu) = -2\mu$$

If $\mu > 0$, $y = \sqrt{\mu}$ is stable. If $\mu < 0$, $y = \sqrt{\mu}$ is unstable.

- At $y = -\sqrt{\mu}$:

$$f'\left(-\sqrt{\mu}\right) = \mu - 3(\mu) = -2\mu$$

If $\mu > 0$, $y = -\sqrt{\mu}$ is stable. If $\mu < 0$, $y = -\sqrt{\mu}$ is unstable.

Bifurcation Diagram:

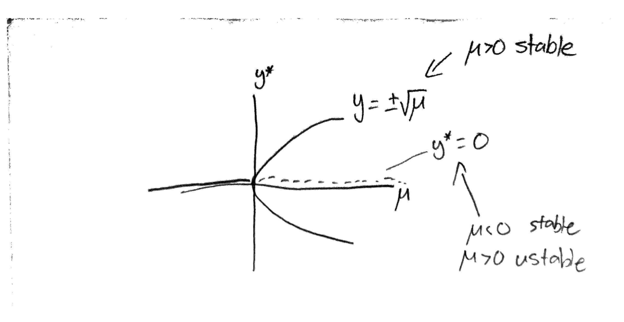

Figure 1: Bifurcation Diagram