Problem 4: The dynamics of two bacterial species, W and M can be modeled as the following system of differential equations.

$$\frac{dW}{dt} = r_W W(1 - W - \alpha M)$$
$$\frac{dM}{dt} = r_M M(1 - M - \beta W)$$

1. Obtain all the equilibrium solutions.
2. Identify conditions for the existence of non-zero equilibrium point (both species non-zero).
3. Pick a case with one of the conditions (any one) for the existence of non-zero equilibrium point (both species non-zero) that you identified above. Draw phase plane diagram and conclude the long-term dynamics of this system.

Solution:

Equilibrium Solutions

Set the derivatives to zero:

$$r_W W(1 - W - \alpha M) = 0$$
$$r_M M(1 - M - \beta W) = 0$$

From the first equation:

$$W = 0 \quad \text{or} \quad 1 - W - \alpha M = 0$$

From the second equation:

$$M = 0 \quad \text{or} \quad 1 - M - \beta W = 0$$

Thus, the equilibrium solutions are:

1. $W = 0, M = 0$
2. $W = 0, M = 1$
3. $W = \frac{1}{1+\beta}, M = 0$
4. Non-trivial equilibrium:

   $$1 - W - \alpha M = 0$$
   $$1 - M - \beta W = 0$$

   Solving the system:

   $$W^* = \frac{1 - \alpha}{1 - \alpha\beta}, \quad M^* = \frac{1 - \beta}{1 - \alpha\beta}$$

Equilibrium points conditioms:

Assuming $\alpha, \beta > 0$:

For the non-trivial equilibrium (W^*, M^*) to exist and be positive:

$$1 - \alpha\beta \neq 0 \quad \text{and} \quad 1 - \alpha > 0, \quad 1 - \beta > 0$$

Thus:

$$\alpha\beta \neq 1, \quad \alpha < 1, \quad \beta < 1$$

Phase Plane Diagram and Long-term Dynamics

Phase plane diagram:

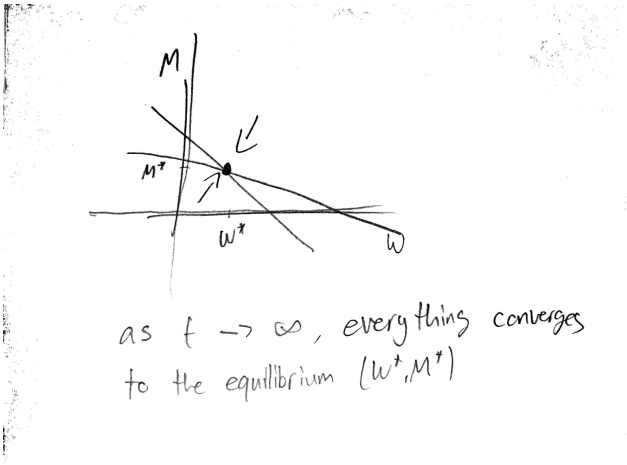

Figure 1: Phase Plane Diagram