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Graphs and Combinatorics

On the Number of Plane Geometric Graphs

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Abstract. We investigate the number of plane geometric, i.e., straight-line, graphs, a set S of n points in the plane admits. We show that the number of plane geometric graphs and connected plane geometric graphs as well as the number of cycle-free plane geometric graphs is minimized when S is in convex position. Moreover, these results hold for all these graphs with an arbitrary but fixed number of edges. Consequently, we provide a unified proof that the cardinality of any family of acyclic graphs (for example spanning trees, forests, perfect matchings, spanning paths, and more) is minimized for point sets in convex position.

In addition we construct a new maximizing configuration, the so-called double zig-zag chain. Most noteworthy this example bears $\Theta^*\left(\sqrt{72}^{\,n}\right) = \Theta^*\left(8.4853^n\right)$ triangulations (omitting polynomial factors), improving the previously known best maximizing examples.

1. Introduction

Let us denote by $\mathcal{S}_n$ the set of sets of n points in the plane in general position, that is, no three points of a set $S \in \mathcal{S}_n$ lie on a common line. With $\Gamma_n \in \mathcal{S}_n$ we denote any set of n points in convex position. Throughout this paper we consider *plane geometric graphs* G on top of $S \in \mathcal{S}_n$. That means that the set of vertices of G is S, edges of G are straight-line segments spanned by points from S and two edges of G do not intersect in their interior but might have endpoints in common. From now on we use the term graph to denote plane geometric graphs, unless otherwise noted.

In other words, we consider the rectilinear drawing of the complete graph K_n with vertex set $S \in \mathcal{S}_n$ and study its crossing-free subgraphs. The problem of how large the number of such subgraphs may be has been attracting a lot of attention; many references can be found in the handbook [14] and in the lately published book [10]. It has also been proved recently that the set of crossing-free subgraphs can be realized as a polytope [16].

A fundamental contribution was given by Ajtai et al. [9]: the number of plane graphs on top of any $S \in \mathcal{S}_n$ is bounded from above by some fixed exponential

c^n; the bound $c \leq 10^{13}$ was given there and has been successively improved up to $c \leq 344$ [21]. It is worth mentioning that a main tool developed in [9] is the nowadays famous "Crossing Lemma": every plane drawing of a graph with n vertices and $m > 4n$ edges contains at least cm^3/n^2 crossings, for some constant c. This result, independently proved by Leighton [15], has later found many applications.

In fact, the motivation in [9] was to provide an upper bound *for the number of polygonizations* (crossing-free spanning cycles) on top of any $S \in \mathcal{S}_n$. Obviously the bound for generic plane graphs applies, yet better specific bounds have been obtained for polygonizations as well as for plane triangulations, perfect matchings, spanning trees and many other classes of plane graphs; precise references are given later in this paper.

To describe the asymptotic growth of the number of graphs we use the $\mathcal{O}^*()$-, $\Omega^*()$-, and $\Theta^*()$-notation. In these notations we neglect polynomial factors and just give the dominating exponential term. Moreover when the base of the exponent is explicitly given as a numerical value, this has to be seen as an approximation up to the given precision.

Maximal plane graphs, i.e., triangulations, are a case of special interest, because any plane graph can be completed to a triangulation. Hence any upper bound $\mathcal{O}^*(\alpha^n)$ on the number of triangulations implies a corresponding upper bound $\mathcal{O}^*(2^{3n}\alpha^n) = \mathcal{O}^*((8\alpha)^n)$ on the number of generic plane graphs, because every triangulation has at most $3n - 6$ edges and therefore contains at most 2^{3n} subgraphs, see also Table 2. The current best upper bound for the number of triangulations of 43^n was very recently obtained by Sharir and Welzl [21], improving the $\mathcal{O}^*(59^n)$ bound of Santos and Seidel [19]. The aforementioned bound of $\mathcal{O}^*(344^n)$ for plane graphs is derived from that[1].

On the opposite direction, it is also known that every $S \in \mathcal{S}_n$ admits at least $\Omega^*(2.33^n)$ triangulations, and it has been conjectured that the number of triangulations is minimized when S is the point set called *double circle*, that has $\Theta^*\left(\sqrt{12}^{\,n}\right)$ triangulations [5].

In this paper we obtain new lower and upper bounds for the maximum and minimum, respectively, number of plane geometric graphs of different types. All given bounds will be exponential bounds of the form α^n where the goal is to optimize the base α.

More precisely, in Sect. 2 we prove that the number of plane graphs of several classes is minimized by point sets in convex position, a fact that was known for perfect matchings, spanning trees and spanning paths [13, 20]. Here we provide a unified approach that encompasses those results and extends to many more classes.

In Sect. 3 and 4 we turn our attention to upper bounds and, in particular, we prove the existence of a certain point set that has $\Theta^*\left(\sqrt{72}^{\,n}\right) = \Theta^*\left(8.4853^n\right)$ triangulations. By this we improve the previously known best maximizing examples and disprove the common belief that the tight upper bound for the number of triangulations would be $\Theta^*\left(8^n\right)$.

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[1] In [21] the authors mention further possible improvements for the base constant for the number of triangulations to a value below 43. This will directly implicate further improvements for all related bounds.

We use the following notation for the indicated classes of graphs on top of $S \in \mathcal{S}_n$: $\mathsf{sc}(S)$: spanning cycles (Hamiltonian cycles, polygonizations); $\mathsf{pm}(S)$: perfect matchings; $\mathsf{sp}(S)$: spanning paths (Hamiltonian paths); $\mathsf{tr}(S)$: triangulations; $\mathsf{ppt}(S)$: pointed pseudo triangulations; $\mathsf{pt}(S)$: pseudo triangulations; $\mathsf{st}(S)$: spanning trees; $\mathsf{cf}(S)$: cycle-free graphs (forests); $\mathsf{cg}(S)$: connected graphs; $\mathsf{pg}(S)$: all plane graphs. For the corresponding maximal value we will use the notation $\mathsf{sc}(n)$ (similar for the other classes), i.e., $\mathsf{sc}(n) = \max_{|S|=n} \mathsf{sc}(S)$. The following partial hierarchy shows these classes of plane graphs.

$$\left.\begin{array}{l}
\left.\begin{array}{l}\left.\begin{array}{l}\text{triangulations } \mathsf{tr}(S)\\ \text{pointed pseudo triangulations } \mathsf{ppt}(S)\end{array}\right\}\text{pseudo triangulations } \mathsf{pt}(S)\\ \text{spanning cycles } \mathsf{sc}(S)\\ \left.\begin{array}{l}\left.\text{spanning paths } \mathsf{sp}(S)\right\}\text{ spanning trees } \mathsf{st}(S)\\ \text{perfect matchings } \mathsf{pm}(S)\end{array}\right\}\end{array}\right\}\begin{array}{l}\text{connected graphs } \mathsf{cg}(S)\\ \text{cycle-free graphs (forests) } \mathsf{cf}(S)\end{array}
\end{array}\right\}\text{all plane graphs } \mathsf{pg}(S)$$

With the exception of triangulations, the cardinality of all these classes is minimized for point sets in convex position. This result is obvious for spanning cycles and has been shown for (pointed) pseudo triangulations in [3]. For all remaining classes we will develop a unified framework to prove minimality for the convex case. The accurate statement for triangulations would be that the number of plane graphs with $k = 2n - 3$ edges is minimized for sets in convex position.

2. Convexity Minimizes

In the following subsections we provide injective mappings of all plane graphs of Γ_n to any set of $\mathcal{S}_n$ such that the number of edges is retained.

2.1. Injective Mappings

Consider the set of all plane graphs $\mathsf{pg}(\Gamma_n)$ on top of Γ_n, and an arbitrary set $S \in \mathcal{S}_n$ together with its set of plane graphs $\mathsf{pg}(S)$. We will show that we can map $G \in \mathsf{pg}(\Gamma_n)$ to a graph $G' \in \mathsf{pg}(S)$ in an injective way such that the number of edges of G and G' is the same. We will provide different mappings to utilize the special properties of connected or cycle-free graphs.

2.1.1. Mapping for Plane Geometric Graphs. Let G, Γ_n, and S be given as defined above. We first fix root vertices $r \in \Gamma_n$ and $r' \in S$. Each root vertex has to be chosen as an arbitrary, but unique extreme vertex[2]. We label the remaining vertices of Γ_n in clockwise order around r and the remaining vertices of S around r', respectively, see Fig. 1. Consider the (possibly empty) fan of all vertices of Γ_n connected to r in G and connect the vertices with corresponding labels in S to r' in G'. By extending the inserted edges to rays as indicated in the right part of Fig. 1, we are left with subsets of equal number of vertices for both, Γ_n and S. We proceed on these subsets

[2] For example, we can choose the vertex with the smallest y-coordinate, and in case there are ties, the one with the minimum x-coordinate among them. Note that we use this method in all given Figures.

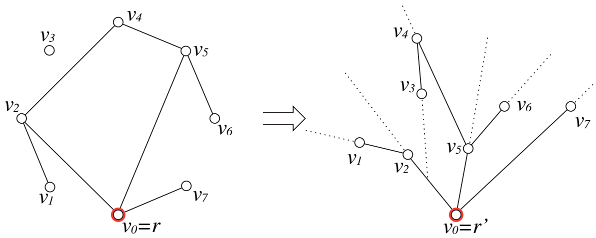

Fig. 1. Injective mapping of a plane geometric graph G of Γ_n

in a recursive manner, using the just connected vertices as new, 'local' root vertices. Note that the extension of edges to rays is limited to the interior of each subset. Each local root conquers the set of all vertices in the wedge to its left, including the left-neighbored local root if it exists. The last vertex added to r (in clockwise order) plays a double-role, as it is the local root for its left and right wedge, respectively.

Note that in each recursive step the vertices of a subset are locally relabeled in both sets according to their clockwise order around the local root vertex. As all vertices of a subset lie in a half-plane defined by a line through the local root vertex, there exists a canonical vertex of the subset to start the labeling from. For example, in a second recursive step in Fig. 1, v_4 is the local root of the set $\{v_2, v_3, v_4\}$. As the vertices are locally resorted around v_4 the edge v_2v_4 of G maps to the edge v_3v_4 of G'. It follows that our mapping is in general not isomorphic. The most simple example of a non isomorphic mapping is given in Fig. 3. In Sect. 2.1.3 we will develop isomorphic mappings for cycle-free plane graphs.

If a (local) root vertex is not connected to any interior vertex of the subset, in the next recursive step a new root vertex is chosen similar to the first step. For example, in Fig. 1, v_7 is neither connected to v_5 nor v_6, such that in the next step v_6 in Γ_n and v_5 in S are the corresponding local root vertices of the subset $\{v_5, v_6\}$ and the edge v_5v_6 is inserted in G'.

The recursion stops if the local root vertex is the only vertex of a subset. As each subset has a strictly smaller cardinality than the previous set (the previous root vertex can never be part of a subset), the process terminates.

It is instructive to see how the mapping works for example for all triangulations of Γ_5 to different point sets, see Fig. 2. The interested reader might want to check why some plane graphs with 7 edges on top of the non-convex sets are not generated.

Theorem 1. *For any fixed number k,* $0 \le k \le 2n-3$*, the number of plane geometric graphs with k edges on top of a set of n points is minimized for sets in convex position.*

Proof. To prove the theorem it is sufficient to show that the above mapping is injective. First observe that all recursive steps are independent in the sense that no edges constructed in G' cross the rays separating the subsets of S. Thus the image of G is in fact a plane graph G'. Intuitively the injectivity of our mapping follows then from this independency and the fact that each root vertex under consideration is connected to a uniquely determined subset of vertices.

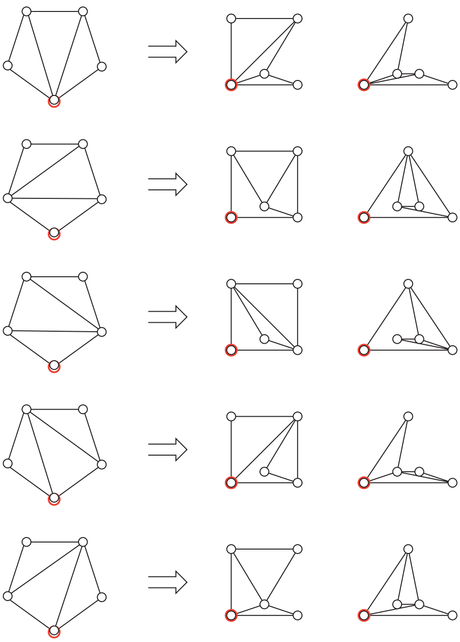

Fig. 2. Mapping of all triangulations of Γ_5 to plane graphs with 7 edges on non-convex sets with 5 points

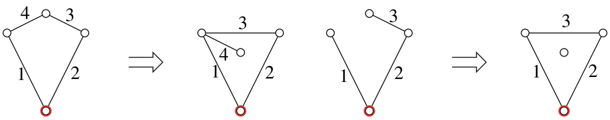

Fig. 3. Transformations resulting in non-isomorphic graphs.Corresponding edges are labeled in order of their consideration

More formal we prove the statement that there exists an injective mapping by induction over the number of points. The root vertex is chosen in a unique way and splits the problem into smaller subproblems. Note that this is still true if the chosen root of Γ_n is not connected to other vertices, as the cardinality of the remaining problem is reduced by at least one. Thus we can apply the induction hypothesis to get injective mappings for each subset. These sub-mappings are combined in a unique way via the root vertex, and the theorem follows. □

Corollary 1. *The number of plane geometric graphs on top of a set of n points is minimized for sets in convex position.*

The left side of Fig. 3 shows a connected graph with a cycle which is transformed into a non-isomorphic graph. Thus our transformation is not suited for degree preservation, bipartite graphs etc. Also connectivity is not preserved by our mapping as

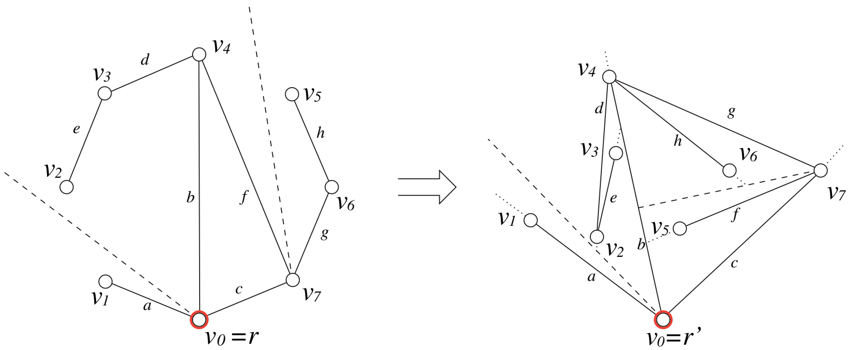

Fig. 4. Using linear separators (*dashed*) to preserve connectedness. Corresponding edges are labeled with the same letter

can be seen from the right part of Fig. 3. In the next section, we will give a variation of the mapping which preserves connectivity and in SubSect. 2.1.3 we will extend this to an isomorphic mapping for cycle-free graphs.

2.1.2. Mapping for Connected Plane Geometric Graphs. In this section, we consider $\mathsf{cg}(\Gamma_n)$, the set of connected graphs on top of Γ_n. Let $G \in \mathsf{cg}(\Gamma_n)$.

From the right part of Fig. 3 it can be seen that troubles with connectivity occur when, within the same fan, two neighbored local root vertices r_1 and r_2 of S get connected in G'. This is caused by the resorting of the subset V of vertices between r_1 and r_2 (including r_1) around r_2.

Observe that if there would have already been a connection between r_1 and r_2 in V (not necessarily a direct edge) then connectivity would not have changed. To solve this problem we allow an edge between r_1 and r_2 in G' only if they are connected in the subgraph induced by V. Note that this only preserves connectedness for connected graphs G, but does not guarantee isomorphism.

For a subset $\tilde{S}$ of Γ_n and a graph $\tilde{G}$ of $\tilde{S}$ we call a straight line through a vertex of $\tilde{S}$ which does not intersect any edge of $\tilde{G}$ a linear separator for $\tilde{S}$. For our new mapping we insert a linear separator into a wedge of G formed by two neighbored root vertices r_1 and r_2 of a fan whenever r_1 and r_2 are not connected within the wedge, see Fig. 4. In this case the subset of vertices is split into two independent parts and we have two separated recursive steps for the wedge, one with local root r_1, and one with local root r_2, respectively. Thus no edge inserted in G' crosses the linear separator.

Observe that in the example of Fig. 4, vertices v_3 and v_6 would be singletons in G' using the mapping of Sect. 2.1.1, that is, without using linear separators.

That our new mapping indeed respects connectedness can be seen from the fact that each (local) root vertex is properly connected to its subset, and not to a neighbored root vertex of the same fan. Thus the claim follows from the recursive approach by induction.

Note that connectivity plays a crucial role when using linear separators. We can guarantee injectivity of the mapping only if, in the case that there is no linear

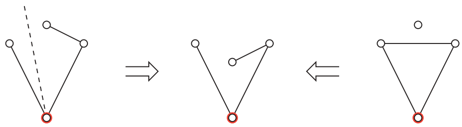

Fig. 5. Using linear separators in combination with non-connected graphs causes loss of injectivity

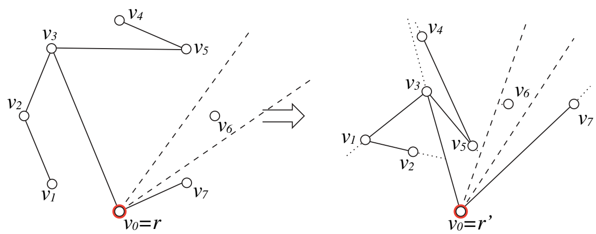

Fig. 6. Cycle-free graphs: using multiple linear separators guarantees isomorphic mapping

separator, all vertices of the subset V are connected within the subgraph induced by V. The right part of Fig. 5 shows a non-connected graph, where no linear separator exists. As the subset V does not induce a connected graph, the suggested approach does not work. Thus the example shows that we can use this method only for connected graphs.

Theorem 2. *For any fixed number k, $n-1 \leq k \leq 2n-3$, the number of connected plane geometric graphs with k edges on top of a set of n points is minimized for sets in convex position.*

Corollary 2. *The number of connected plane geometric graphs on top of a set of n points is minimized for sets in convex position.*

2.1.3. Isomorphic Mapping for Cycle-Free Plane Geometric Graphs. Recall that with $\mathsf{cf}(\Gamma_n)$ we denote the set of all cycle-free graphs on top of Γ_n. Let $G \in \mathsf{cf}(\Gamma_n)$ and let us point out that G is not necessarily connected.

From Fig. 5 it can be seen that the mapping, which we used in the last section, is in general non-isomorphic. This might be the case if a cycle in G gets 'opened' by mapping it to G', or the other way around, if a new cycle appears in G'. Therefore we now extend the mapping of the previous section. As G has been a connected graph in the last section, we had at most one linear separator for each wedge. Now, as G is cycle-free but possibly not connected, we are going to use multiple linear separators per wedge. We insert a linear separator between two vertices of the wedge whenever these two vertices are not connected within the wedge, see Fig. 6.

As G is cycle-free there is always at least one linear separator per inner wedge. In addition we get a linear separator for each connected component which lies within a wedge but is not connected to the root vertex. Thus, situations like in Fig. 5 cannot occur. Moreover, any recursive subset is independent from other subsets not only in the sense that no other edges cross, but also that there are no other incident edges from outside V except the ones from the (local) root. Thus, the mapping indeed generates isomorphic graphs.

The local labeling in recursive steps might still change. Although we get an isomorphic graph G', the final labeling of S needs not to be the one obtained by sorting the vertices around r'. See, for example, v_1 and v_2 in Fig. 6.

Theorem 3. *There exists an injective mapping of all cycle-free plane geometric graphs on top of a set of n points in convex position to isomorphic plane geometric graphs on top of any set of n points.*

Corollary 3. *The number of (plane geometric) spanning trees, spanning paths, perfect matchings, and cycle-free graphs (forests) is minimized for point sets in convex position.*

Note that for $k \geq 2$ the minima stated in Theorems 1 and 2 as well as in Corollaries 1, 2 and 3 are strict, except for perfect matchings. That is, for any set S of n points, $S \neq \Gamma_n$, there exist strictly more graphs of the respective class than on top of Γ_n. To see this, consider the following disjoint pair of edges. The first edge e_1 connects the global root of S to an inner vertex of S, and e_2 is the convex hull edge of S intersected by the ray extending e_1. Obviously the pair (e_1, e_2) can never be obtained by one of our mappings from Γ_n, as all mappings will split the sets according to the supporting line of e_1. Moreover, we can complete (e_1, e_2) to obtain a graph on top of S for any given class of plane graphs (except for perfect matchings). This graph is not covered by any mapping from Γ_n and the statement follows. For perfect matchings a regular 5-gon with an additional point in the center bears 5 perfect matchings, the same number as a convex 6-gon.

That the number of crossing-free perfect matchings and spanning trees is minimized for convex sets has been shown in [13]. A similar result has been obtained for the number of spanning paths [20].

To obtain the numbers in Table 1 we made use of the exhaustive data base of all point sets in the plane [1, 6, 7]. It is interesting to observe that there seems to be a close relation between maximizing the number of plane graphs and connected graphs on the one hand and minimizing the rectilinear crossing number (see, e.g. [2, 7]) on the other hand. For $n \leq 6$ there is always one unique example which minimizes the rectilinear crossing number. At the same time it maximizes the number of plane graphs and connected graphs. For $n = 7, \ldots, 11$ there are (2, 2, 1, 2, 3) sets which maximize the number of plane graphs and among them there is for each n one set which is the only one maximizing the number of connected graphs[3]. All these sets stem from the (3, 2, 10, 2, 374), $n = 7, \ldots, 11$, sets minimizing the rectilinear crossing number.

[3] As indicated in Table 1, for $n = 11$ the given maxima are only lower bounds, as we did check only sets minimizing the rectilinear crossing number. Anyway, we conjecture that these bounds are tight.

Table 1. Minimum and maximum numbers of plane graphs and connected graphs, respectively, for small sets. The last row gives the sequence identification codes in Sloans Integer Sequence encyclopedia [23]. See Table 3 for asymptotic bounds

n	All plane graphs		Connected graphs	
	Minimum	Maximum	Minimum	Maximum
3	8	8	4	4
4	48	64	23	38
5	352	768	156	494
6	2 880	13 824	1 162	9 482
7	25 216	270 336	9 192	187 318
8	231 168	6 443 008	75 819	4 478 792
9	2 190 848	164 429 824	644 908	113 290 236
10	21 292 032	4 612 423 680	5 616 182	3 145 329 136
11	211 044 352	$\geq$ 131 922 001 920	49 826 712	$\geq$ 88 754 921 232
	A054726		A007297	

The mentioned observations indicate that sets maximizing the number of plane graphs or the number of connected graphs, respectively, have to minimize the rectilinear crossing number. Nevertheless, there is no (inverse) monotonous relation between these two properties: there exist sets which, when compared to other sets, have a larger rectilinear crossing number and a higher number of plane graphs at the same time.

Moreover, there is no monotonous relation between the number of plane graphs and the number of connected graphs. For $n \geq 7$ there exist point sets S_1 and S_2, such that there exist more plane graphs on top of S_1 than on top of S_2, but for connected graphs the situation is converse.

The second column of Table 4 (Sect. 4) shows the asymptotic growth of several types of graphs for the convex set Γ_n. Except for triangulations the given bounds are lower bounds for these types of graphs.

3. On Upper Bounds

In order to show the bound of $\mathcal{O}^*(344^n)$ for the number of plane graphs, the upper bound on the number of triangulations has been used. As any triangulation of S has at most $3n-6$ edges, it contains at most 2^{3n-6} plane subgraphs. Therefore we get the bound $|\mathsf{pg}(n)| \leq 2^{3n-6}|\mathsf{tr}(n)| \leq 344^n$. The last inequality comes from the currently best upper bound for $|\mathsf{tr}(n)| \leq 43^n$ given in [21].

For the maximum number of cycle-free graphs in a triangulation we get an upper bound of $\mathcal{O}(\binom{3n-6}{n-1}) = \mathcal{O}^*(6.75^n)$. For the special case of spanning trees this has recently been improved to $\mathcal{O}^*(5.\dot{3}^n)$ by observing that the dual tree has maximum degree at most three [18].

Taking the average vertex degree in a triangulation into account, a bound of $\mathcal{O}^*(3^n)$ for the number of spanning paths in a given triangulation has been shown in [20]. Using their arguments one can also show a bound of $\mathcal{O}^*(\sqrt{5}^n)$ for the number

Table 2. Number of different types of graphs per triangulation on top of n points. For easier comparison all expressions are given by their numerical values, see the related references for exact formulas

Type	Per triangulation of n points	Type	Per triangulation of n points
sc	$\mathcal{O}^*(\sqrt{6}^n)$ [20]	pt	$\mathcal{O}^*(3^n)$ [17]
pm	$\mathcal{O}^*(\sqrt{5}^n)$ Sect. 3	st	$\mathcal{O}^*(5.\dot{3}^n)$ [18]
sp	$\mathcal{O}^*(3^n)$ [20]	cf	$\mathcal{O}^*(6.75^n)$ Sect. 3
tr	1	cg	$\mathcal{O}^*(8^n)$
ppt	$\mathcal{O}^*(3^n)$ [17]	pg	$\mathcal{O}^*(8^n)$ [13]

Table 3. Asymptotic bounds for various classes of graphs on top of a point set S with $|S| = n$. All types except triangulations are minimized for sets in convex position. References for upper bounds give either the presented bounds or their relation to the number of triangulations

Type	Minimal value	Number for Γ_{10}	Maximal value	
sc(S)	1	1	$\Omega^*(4.64^n)$ [13]	$\mathcal{O}^*(86.81^n)$ [20][4]
pm(S)	$\Theta^*(2^n)$ [12, 13]	42	$\Omega^*(3^n)$ [13]	$\mathcal{O}^*(10.05^n)$ [20][4]
sp(S)	$\Theta^*(2^n)$	1 280	$\Omega^*(4.64^n)$ [13]	$\mathcal{O}^*(100.88^n)$ [20][4]
tr(S)	$\Omega^*(2.33^n)$ $\mathcal{O}^*(3.47^n)$ [5]	250	$\Omega^*(8.48^n)$ Thm. 4	$\mathcal{O}^*(43^n)$ [21]
ppt(S)	$\Theta^*(4^n)$ [3]	1 430	$\Omega^*(12^n)$ [8]	$\mathcal{O}^*(129^n)$ [17]
pt(S)	$\Theta^*(4^n)$ [3]	1 430	$\Omega^*(20^n)$ [8]	$\mathcal{O}^*(129^n)$ [17]
st(S)	$\Theta^*(6.75^n)$ [12]	246 675	$\Omega^*(10.42^n)$ [11]	$\mathcal{O}^*(229.33^n)$ [21, 22]
cf(S)	$\Theta^*(8.22^n)$ [12]	2 117 283	$\Omega^*(11.62^n)$ Thm.7	$\mathcal{O}^*(290.25^n)$ Sect.3
cg(S)	$\Theta^*(10.39^n)$ [12]	5 616 182	$\Omega^*(35.49^n)$ Thm.6	$\mathcal{O}^*(344^n)$ [21]
pg(S)	$\Theta^*(11.65^n)$ [12]	21 292 032	$\Theta^*(39.80^n)$ [13], Thm.5	$\mathcal{O}^*(344^n)$ [21]

of crossing-free perfect matchings in a triangulation: Any perfect matching can be obtained by iteratively matching the leftmost unused vertex to one of its incident (right) neighbors. For each vertex the number of possible ways to find a matching is bounded by the number of incident edges going to the right. Following [20] we call this the effective degree of a vertex. The sum of the effective degrees of all vertices is the number of edges of the triangulation, bounded by $3n - 6$. Asymptotically the number of different perfect matchings can be bounded by a sum over products of the effective degrees. The dominating term in this expression is obtained by uniformly distributing the effective degrees over all considered vertices—see [20] for a proper argumentation of these facts. As in a triangulation every vertex (except the rightmost) has at least one incident edge going to the right, we can uniformly distribute the remaining (at most) $2n - 5$ edges to the $\frac{n}{2}$ left endpoints of the matchings. This gives an average effective degree of 5 for these vertices and the given bound $\mathcal{O}^*(\sqrt{5}^n)$ follows.

The bound on the number of spanning paths also implies an upper bound for the number of spanning cycles: For every spanning cycle $C \in \mathsf{sc}(S)$, $S \in \mathcal{S}_n$, we get n spanning paths by omitting one edge of C. As any two elements of $\mathsf{sc}(S)$ differ by at least two edges, this implies $|\mathsf{sp}(S)| \geq n \cdot |\mathsf{sc}(S)|$. An improved bound of $\mathcal{O}^*(\sqrt{6}^n)$ spanning cycles per triangulation was provided by Raimund Seidel and reported in [20].

Table 4. Special configurations of n points and their asymptotic number of graphs

Type	Convex set	Double circle	Double chain	Double zig-zag chain
sc	1	$\mathcal{O}^*(4.83^n)$ [4]	$\Omega^*(4.64^n)$ $\mathcal{O}^*(5.61^n)$ [13]	
pm	$\Theta^*(2^n)$ [13]	$\Theta^*(2.20^n)$ [4]	$\Theta^*(3^n)$ [13]	
sp	$\Theta^*(2^n)$	$\mathcal{O}^*(4.83^n)$ [4]	$\Omega^*(4.64^n)$ [13]	
tr	$\Theta^*(4^n)$	$\Theta^*(\sqrt{12}^n)$ [5, 8]	$\Theta^*(8^n)$ [13]	$\Theta^*(8.48^n)$ Thm. 4
ppt	$\Theta^*(4^n)$	$\Theta^*(\sqrt{28}^n)$ [8]	$\Theta^*(12^n)$ [8]	
pt	$\Theta^*(4^n)$	$\Theta^*(\sqrt{40}^n)$ [8]	$\Theta^*(20^n)$ [8]	
st	$\Theta^*(6.75^n)$ [12]	$\Omega^*(7.07^n)$ [4]	$\Omega^*(10.42^n)$ [11]	
cf	$\Theta^*(8.22^n)$ [12]	$\Omega^*(8.55^n)$ [4]	$\Omega^*(11.62^n)$ Thm. 7	
cg	$\Theta^*(10.39^n)$ [12]	$\Omega^*(11.83^n)$ [4]	$\Omega^*(35.49^n)$ Thm. 6	
pg	$\Theta^*(11.65^n)$ [12]		$\Theta^*(39.80^n)$ [13], Thm. 5	

Most of the upper bound constructions for various classes of graphs are based on the just mentioned relation to triangulations. For example, the bound on the number of cycle-free graphs in a triangulation implies the upper bound of $\mathcal{O}^*(290.25^n)$ for the number of cycle-free graphs given in Table 3. Only recently different approaches have been investigated, an outstanding result being the $\mathcal{O}^*(10.0438^n)$ bound for crossing-free perfect matchings [20]. Syntactically, their proof follows the proof of [19] bounding the number of triangulations, but uses novel ideas and refined observations.

Interestingly the result for crossing-free perfect matchings can be used to bound the number of spanning cycles and spanning paths [20]: Observe that, for even n, every spanning cycle is the union of two crossing-free perfect matchings. Thus $|\mathsf{sc}(n)| \leq |\mathsf{pm}(n)|^2$. Similarly every spanning path contains a crossing-free perfect matching on n points and a crossing free perfect matching on $n-2$ points (omit start and endpoint). We thus get an upper bound of $\mathcal{O}^*(100.88^n)$ for both structures. Recently this bound has been improved to $\mathcal{O}^*(86.81^n)$ for spanning cycles [20][4].

4. Special Configurations

Besides points in convex position, which other specific configurations of points may achieve the minimum or the maximum number of geometric non-crossing graphs in every class has also been the subject of intensive research. In the following subsections, we describe two configurations that have been previously studied, namely the double circle (DC) [5] and the double chain [13], as well as a new configuration, the so-called double zig-zag chain. We will show that the double zig-zag chain is an improved (and up to now best known) example for maximizing the asymptotic number of triangulations on top of a given point set.

[4] Further improvements on the constants have been made very recently, and will be reported in [22]. To our knowledge the currently best bounds are $|\mathsf{sc}(n)| = \mathcal{O}^*(74.60^n)$, $|\mathsf{pm}(n)| = \mathcal{O}^*(9.22^n)$, and $|\mathsf{sp}(n)| = \mathcal{O}^*(85.01^n)$.

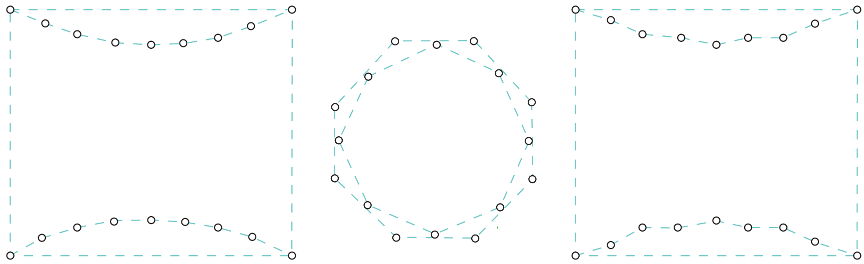

Fig. 7. Double chain (*left*), double circle (*center*) and double zig-zag chain (*right*)

4.1. Double Circle, Double Chain and Double Zig-Zag Chain

Let S_E be a set of $\frac{n}{2}$ points in convex position, where n is a given even integer, and let S_I be a set of $\frac{n}{2}$ points interior to $CH(S_E)$, each one infinitesimally close to a different midpoint of an edge of $CH(S_E)$; here E stands for *external* and I for *internal*. The configuration $S_E \cup S_I$ is called the *double circle* [5] and is denoted by DC_n (Fig. 7, center).

Somehow surprisingly, point sets in convex position fail to minimize the number of plane triangulations, as the vertices of a convex n-gon admit $\Theta^*\left(4^n\right)$ triangulations while the number of triangulations of DC_n is only $\Theta^*\left(\sqrt{12}^n\right)$; in fact, the double circle has been conjectured to minimize this number [5].

The configuration called the *double chain* is shown in Fig. 7, left; its convex hull is a convex quadrilateral. The point set consists of $n = 2m$ points; half of them form a concave chain (the *upper chain*) connecting the two upper extreme points, the other half of them form a concave chain (the *lower chain*) connecting the two lower extreme points. In addition, any line connecting two vertices of the upper chain leaves all points of the lower chain below, and reversely.

For most of the graph classes we consider, no point sets are known to have a larger number of graphs than the double chain has; in other words, the double chain seems to achieve maximal values. In particular, the double chain was up to now the configuration with the asymptotically highest number of triangulations, namely $\Theta^*\left(8^n\right)$ [13]. It was widely believed (including most of the authors) that this could be the true upper bound for the number of triangulations. Nevertheless in this section we combine the double circle and the double chain in order to obtain a new maximizing configuration, the so-called double zig-zag chain, that has more triangulations than the double chain.

To obtain the double zig-zag chain, take two distorted double circles with $\frac{n}{2}$ points each and combine them within a convex quadrilateral as indicated in Fig. 7, right. The example is similar to the double chain, but instead of two concave chains we now have two zig-zag chains of edges splitting the area of the quadrilateral into three parts. These edges are unavoidable in the sense that they are not crossed by any other edge spanned by the point set. For example, the edges of the zig-zag chains will have to be part of any triangulation. Note that the two zig-zag configurations are at sufficient distance from each other such that any vertex of a chain can 'see' all vertices of the opposite chain, that is, an edge connecting a vertex of the lower chain

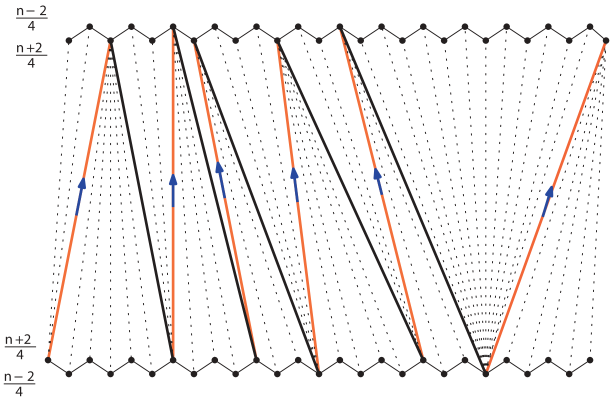

Fig. 8. Counting triangulations in the interior of the double zig-zag chain

to a vertex of the upper chain does not cross a zig-zag edge. Both, upper and lower part, are double circles with $\frac{n}{2}$ vertices and thus $\Theta^*\big(\sqrt{12}^{\frac{n}{2}}\big)$ triangulations each. To be precise, there is one interior vertex near the horizontal convex-hull edges missing in each subset, which does not influence the asymptotic counting arguments for the upper and lower parts. This can be seen by, on the one hand, mapping any triangulation of this part to a triangulation of the (full) double circle of size $\frac{n}{2}$, where the additional point has fixed degree three. On the other hand we map any triangulation of the (full) double circle of size $\frac{n}{2}-2$ to this set by 'opening' an edge incident to a fixed convex hull vertex, to a triangle incident to the horizontal edge.

We start by proving that the double zig-zag chain has more triangulations than the double chain.

Theorem 4. *The double zig-zag chain of n points contains $\Theta^*\big(8.48528^n\big)$ triangulations.*

Proof. The double zig-zag chain consists of three parts separated by the two zig-zag chains. The number of graphs can be counted independently for each part.

We first count the number of triangulations in the interior of the double zig-zag chain (the area between the two zig-zag chains). To this end we are going to choose m points, $0 \le m \le \frac{n}{2}-1$, from each zig-zag chain. The points of each zig-zag chain can be viewed as lying on a smaller and a larger circle. For the upper zig-zag chain we choose i_1 points from the larger and j_1 points from the smaller circle, $i_1 + j_1 = m$, where we can choose any points except the rightmost one. Similarly we choose i_2 and j_2 points from the lower zig-zag chain, except the leftmost point.

There are $\sum_{0\le m\le\frac{n}{2}-1}\sum_{i_1,i_2,i_1+j_1=i_2+j_2=m}\binom{\frac{n-2}{4}}{i_1}\binom{\frac{n-2}{4}}{j_1}\binom{\frac{n-2}{4}}{i_2}\binom{\frac{n-2}{4}}{j_2}$ ways to do so. Then we connect the chosen points by pairing a point from the upper zig-zag chain with a point from the lower zig-zag chain by scanning them from left to right. We say that these pairs are connected by black edges, shown as dark edges in Fig. 8. Next we draw red edges (marked by arrows in Fig. 8) connecting the black edges in a way that we connect the lower endpoint of the 'left' black edge to the upper endpoint of the 'right' black edge. The first red edge uses the leftmost point of the lower chain to start with, and the last red edge ends at the right most point of the upper chain.

We complete the drawing to a triangulation of the interior by connecting the two zig-zag chains in a fan like manner to the endpoints of the red and black edges. Let us color these edges gray (dotted edges in Fig. 8). Note that we can flip some of the gray edges (connected to the smaller circle of a zig-zag chain) to obtain different triangulations, still containing the red and black edges. Here a flip exchanges the two diagonals of the convex quadrilateral formed by the two adjacent triangles.

If we consider all the triangulations that contain the black segments we see that for a flippable gray edge we have two possibilities, while non-flippable gray edges and red edges do not provide any multiplicative factor. It is important to observe that every triangulation is uniquely assigned to its given set of black edges. In other words, for a given triangulation the black edges can be uniquely restored: we can always detect the red and black edges: By starting from the leftmost point of the lower zig-zag chain we take the edge to the rightmost visible point on the upper zig-zag chain. This gives us the first red edge and the starting point of the next black edge. The black edge is determined as the rightmost edge incident to this vertex and going back to the lower zig-zag chain. Continuing in the same manner this gives us the remaining red and black edges. Note that the above argumentation still goes through when flippable gray edges have been flipped arbitrarily. Therefore the number of triangulations is

$$\sum_{0\le m\le\frac{n}{2}-1}\ \sum_{i_1,i_2,i_1+j_1=i_2+j_2=m}\binom{\frac{n-2}{4}}{i_1}\binom{\frac{n-2}{4}}{j_1}\binom{\frac{n-2}{4}}{i_2}\binom{\frac{n-2}{4}}{j_2}2^{\frac{n}{2}-1-(j_1+j_2)}.$$

Neglecting polynomial factors, the asymptotic of this sum is determined by its largest element, since we have only a polynomial number of terms. As $i_1 + j_1$ is independent from $i_2 + j_2$, the maximum is obtained for $i_1 = i_2 = c\frac{n-2}{4}$ and $j_1 = j_2 = d\frac{n-2}{4}$, by symmetry, and we have to maximize the term $\binom{\frac{n-2}{4}}{c\frac{n-2}{4}}^2\binom{\frac{n-2}{4}}{d\frac{n-2}{4}}^2 2^{(1-d)\frac{n}{2}}$ for $0 \le c, d \le 1$.

Utilizing Stirling's formula $n! \approx \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$ one derives the well known estimate $\binom{n}{\alpha n} = \Theta(n^{-\frac{1}{2}}2^{H(\alpha)n})$ where $H(\alpha) = -(\alpha\log_2\alpha + (1-\alpha)\log_2(1-\alpha))$ denotes the binary entropy function.

Using this estimate we conclude that $c = \frac{1}{2}$ gives the maximum for the first factor, namely $\Theta^*\left(2^{\frac{n}{2}}\right)$. For d we have to maximize $2^{H(d)\frac{n}{2}+(1-d)\frac{n}{2}}$, which is equivalent to maximizing $H(d) - d$. It is well known that the maximum is obtained for $d = \frac{1}{3}$, resulting in $\Theta^*\left(2^{\frac{\log_2 3}{2}n}\right)$. Combining both factors gives $\Theta^*\left(2^{H(\frac{1}{3})\frac{n}{2}+\frac{5n}{6}}\right)$ triangulations, which can be simplified to $\Theta^*\left(\sqrt{6}^n\right) = \Theta^*\left(2.44949^n\right)$ triangulations for the interior part.

The two outer parts of the double zig-zag chain are double circles without their convex hull edges, and each of them has $\Theta^*\left(\sqrt{12}^{\frac{n}{2}}\right)$ triangulations. In total we thus get $\Theta^*\left(\sqrt{12}^{\frac{n}{2}}\right)^2 \cdot \Theta^*\left(\sqrt{6}^n\right) = \Theta^*\left((6\sqrt{2})^n\right) = \Theta^*\left(8.48528^n\right)$ triangulations for the double zig-zag chain. □

It is plausible that the double zig-zag chain has more graphs than the double chain for some other graph classes than just the case of triangulations, yet the countings are quite difficult and more substantial research effort may be required to prove or disprove that. In general, counting graphs on top of the double zig-zag chain also requires good estimates on the number of graphs on top of the double circle, as is the case in the preceding result. We do not elaborate more on the latter numbers to focus on the results in this paper; the interested reader may check reference [4], where more bounds for this configuration are obtained (see Table 4).

Even for the simpler case of the double chain not all exact numbers are known, that's why the original results in [13] have often been revisited. In the remainder of this section we present new bounds for the number of plane graphs, connected plane geometric graphs and cycle-free plane geometric graphs that the double chain has. The values we obtain are the best lower bounds to date for the maximum number of these graphs that any point set may have.

In [13] a lower bound of $\Omega^*(39.80^n)$ plane graphs of the double chain was shown. We first show that adapting the argumentation from the preceding theorem to this configuration, we can strengthen that result to $\Theta^*\left(39.80^n\right)$.

Theorem 5. *The double chain has $\Theta^*\left(39.80^n\right)$ plane geometric graphs.*

Proof. The basic difference from the preceding proof is that for a chain of the double chain we do not need to distinguish between points of the smaller and larger circles, but choose all vertices from the same chain. In addition, no edge in the interior part is flippable any more, but red and gray edges may be deleted and therefore provide now a factor of 2 when we consider all plane graphs that would have the same black edges. Thus if we choose $c \cdot \frac{n}{2}$ points of each chain for the black edges[5], the general term of the sum simplifies to $\binom{\frac{n}{2}}{c\frac{n}{2}}^2 \cdot 2^{n-c\frac{n}{2}}$. This term is maximized for $c = \sqrt{2} - 1$; this is combined with the fact that a convex $\frac{n}{2}$-gon has $\Theta^*\left(11.65^{\frac{n}{2}}\right)$ plane connected graphs [12], and leads to $\Theta^*\left(39.79898^n\right)$ plane geometric graphs for the double chain, the exact base being $20 + 14\sqrt{2}$. □

Theorem 6. *The double chain has $\Omega^*(35.49^n)$ connected plane geometric graphs.*

Proof. Consider all graphs of the double chain such that the subgraphs of both convex $\frac{n}{2}$-gons of the double chain are connected. Adding one vertical edge of the convex hull of the double chain gives a connected graph. From [13] to Theorem 5 we know that there are $\Theta^*\left(39.80^n\right)$ plane graphs in the double chain. Moreover, a convex n-gon has $\Theta^*\left(11.65^n\right)$ plane graphs and $\Theta^*\left(10.39^n\right)$ connected graphs [12].

[5] Note that for simplification of presentation in the sequel we ignore small constants in this terms, when they do not effect the asymptotics, that is, we us, e.g., $\frac{n}{4}$ instead of $\frac{n-2}{4}$ for all remaining argumentations of this kind.

We thus have to correct the number 39.80^n by the factor $(\frac{10.39}{11.65})^n$ and get a lower bound of $\Omega^*(35.49^n)$ for the number of connected graphs of the double chain. □

Theorem 7. *The double chain has* $\Omega^*(11.6268^n)$ *cycle-free plane geometric graphs.*

Proof. Let $F_{n,k}$ be the number of cycle-free graphs (forests) consisting of k components on a convex n-point set Γ_n. Flajolet and Noy [12] provided a formula for $F_{n,k}$.

$$F_{n,k} = \frac{1}{2n-k}\binom{n}{k-1}\binom{3n-2k-1}{2n-k-1}.$$

We count the number of cycle-free graphs of the double chain (there are $\frac{n}{2}$ points on each chain) by first counting the number of cycle-free graphs $F_{\frac{n}{2},k}$ on the two convex sets for a suitable value of k. Then we choose one point of each tree, and we count the number of plane graphs in the *interior part* (the area between the two concave chains) of the double chain, thereby only using chosen points. Proceeding along the lines of the proof of Theorem 4 we get $\Theta^*(3.41421356^{2k})$ plane graphs in the interior part of a double chain with k points on each chain. Observe that any triangulation of the (open) interior of the double chain forms a spanning tree and therefore we do not obtain cycles. Furthermore, the number of cycle-free graphs of Γ_n is at least $F_{n,k}$ for any fixed value of k. Hence, for asymptotic counting we can restrict our attention to $F_{n,\alpha n}$ (respectively, $F_{\frac{n}{2},\alpha\frac{n}{2}}$). Then the relevant terms of the equation above are $\binom{n}{\alpha n}\binom{n(3-2\alpha)}{n(2-\alpha)}$. Combining the three terms we get the lower bound on the number of cycle-free graphs $\left(\binom{\frac{n}{2}}{\alpha\frac{n}{2}}\binom{\frac{n}{2}(3-2\alpha)}{\frac{n}{2}(2-\alpha)}\right)^2 3.41421356^{\alpha n}$. Using the estimate for the binomial coefficients as in the preceding results we get $\left(2^{H(\alpha)\frac{n}{2}}\cdot 2^{(3-2\alpha)H(\frac{2-\alpha}{3-2\alpha})\frac{n}{2}}\right)^2 3.41421356^{\alpha n} = 2^{\left(H(\alpha)+(3-2\alpha)H(\frac{2-\alpha}{3-2\alpha})+(\log_2(3.41421356)\alpha)\right)n}$.

The expression is maximized for $\alpha = 0.40338$, which yields $\Omega^*(11.6268^n)$ cycle-free graphs for the double chain. □

Observe that the above method results in a lower bound of $\Omega^*(9.65^n)$ on the number of spanning trees on the double chain, which is worse than the bound $\Omega^*(10.42^n)$ of Dumitrescu [11].

5. Further Work and Open Problems

The most challenging question is of course to close the gap between maximizing examples and upper bounds. Here Sharir and Welzl [20] recently have made enormous progress on the upper bounds.

Obviously further work is required to improve (or even fill in) the entries in Tables 2,3,4. In Tables 2 and 3 the goal is to close or at least narrow the gaps between lower and upper bounds, while for Table 4 several entries, especially for the double zig-zag chain, are missing.

Concerning our lower bound construction an interesting question is the following: does there exist an example that shows that an isomorphic mapping from any plane graph on top of a convex point set to any other point set (of the same cardinality) is not possible? If such an example does not exist, can we find a unified isomorphic mapping that works for all graphs?

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