

Model Predictive Control

Algorithm, Feasibility and Stability

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Infinite Time Constrained Optimal Control (what we would like to solve)

$$\begin{aligned}
 J_0^*(x(0)) &= \min \sum_{k=0}^{\infty} q(x_k, u_k) \\
 \text{s.t. } x_{k+1} &= Ax_k + Bu_k, k = 0, 1, 2, \dots \\
 x_k &\in \mathcal{X}, u_k \in \mathcal{U}, k = 0, 1, 2, \dots \\
 x_0 &= x(0)
 \end{aligned}$$

- **Stage cost** $q(x, u)$ describes “cost” of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties...
... but we can't compute it: there are an **infinite number of variables**

Receding Horizon Control

(what we can sometimes solve)

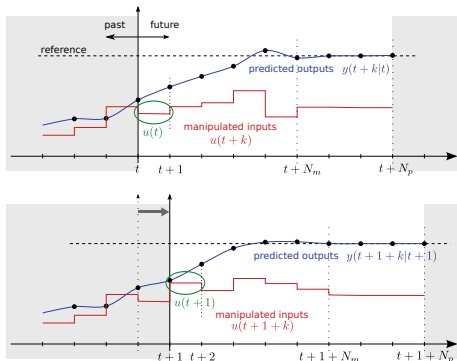
$$\begin{aligned}
 J_t^*(x(t)) = & \min_{U_t} \quad \textcolor{red}{p(x_{t+N})} + \sum_{k=0}^{\textcolor{red}{N-1}} q(x_{t+k}, u_{t+k}) \\
 \text{subj. to} \quad & x_{t+k+1} = Ax_{t+k} + Bu_{t+k}, \quad k = 0, \dots, N-1 \\
 & x_{t+k} \in \mathcal{X}, \quad u_{t+k} \in \mathcal{U}, \quad k = 0, \dots, N-1 \\
 & \textcolor{red}{x_{t+N} \in \mathcal{X}_f} \\
 & x_t = x(t)
 \end{aligned} \tag{1}$$

where $\mathcal{U}_t = \{u_t, \dots, u_{t+N-1}\}$.

Truncate after a finite horizon:

- $p(x_{t+N})$: Approximates the 'tail' of the cost
- \mathcal{X}_f : Approximates the 'tail' of the constraints

On-line Receding Horizon Control



- 1 At each sampling time, solve a **CFTOC**.
- 2 Apply the optimal input **only during** $[t, t+1]$
- 3 At $t+1$ solve a CFTOC over a **shifted horizon** based on new state measurements
- 4 The resultant controller is referred to as **Receding Horizon Controller (RHC)** or **Model Predictive Controller (MPC)**.

On-line Receding Horizon Control

- 1) MEASURE the state $x(t)$ at time instance t
- 2) OBTAIN $U_t^*(x(t))$ by solving the optimization problem in (1)
- 3) IF 'problem infeasible' THEN STOP
- 4) APPLY the first element u_t^* of U_t^* to the system
- 5) WAIT for the new sampling time $t + 1$, GOTO 1)

Note that, we need a constrained optimization solver for step 2).

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History of MPC

- **A. I. Propoi, 1963**, “Use of linear programming methods for synthesizing sampled-data automatic systems”, *Automation and Remote Control*.
- **J. Richalet et al., 1978** “Model predictive heuristic control- application to industrial processes”. *Automatica*, 14:413-428.
 - known as **IDCOM (Identification and Command)**
 - impulse response model for the plant, linear in inputs or internal variables (**only stable plants**)
 - quadratic performance objective over a finite prediction horizon
 - future plant output behavior specified by a reference trajectory
 - **ad hoc** input and output constraints
 - optimal inputs computed using a heuristic iterative algorithm, interpreted as the dual of identification
 - controller was not a transfer function, hence called **heuristic**

History of MPC

- 1970s: Cutler suggested MPC in his PhD proposal at the University of Houston in 1969 and introduced it later at Shell under the name Dynamic Matrix Control. **C. R. Cutler, B. L. Ramaker, 1979** “Dynamic matrix control – a computer control algorithm”. *AIChE National Meeting*, Houston, TX.
 - successful in the petro-chemical industry
 - linear step response model for the plant
 - quadratic performance objective over a finite prediction horizon
 - future plant output behavior specified by trying to follow the set-point as closely as possible
 - input and output constraints included in the formulation
 - optimal inputs computed as the solution to a least-squares problem
 - **ad hoc** input and output constraints. Additional equation added online to account for constraints. Hence a **dynamic matrix** in the least squares problem.
- **C. Cutler, A. Morshedi, J. Haydel, 1983**. “An industrial perspective on advanced control”. *AIChE Annual Meeting*, Washington, DC.
 - Standard QP problem formulated in order to systematically account for constraints.

History of MPC

- Mid 1990s: extensive theoretical effort devoted to provide conditions for guaranteeing feasibility and closed-loop stability
- 2000s: development of tractable robust MPC approaches; nonlinear and hybrid MPC; MPC for very fast systems
- 2010s: stochastic MPC; distributed large-scale MPC; economic MPC

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RHC Notation

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

$$x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}, \quad \forall t \geq 0$$

The CFTOC Problem

$$J_t^*(x(t)) = \min_{U_{t \rightarrow t+N|t}} p(x_{t+N|t}) + \sum_{k=0}^{N-1} q(x_{t+k|t}, u_{t+k|t})$$

$$\begin{aligned}\text{subj. to } \quad & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, \quad k = 0, \dots, N-1 \\ & x_{t+k|t} \in \mathcal{X}, \quad u_{t+k|t} \in \mathcal{U}, \quad k = 0, \dots, N-1 \\ & x_{t+N|t} \in \mathcal{X}_f \\ & x_{t|t} = x(t)\end{aligned}$$

with $U_{t \rightarrow t+N|t} = \{u_{t|t}, \dots, u_{t+N-1|t}\}$.

RHC Notation

- $x(t)$ is the state of the system at time t .
- $x_{t+k|t}$ is the state of the model at time $t+k$, predicted at time t obtained by starting from the current state $x_{t|t} = x(t)$ and applying to the system model

$$x_{t+1|t} = Ax_{t|t} + Bu_{t|t}$$

the input sequence $u_{t|t}, \dots, u_{t+k-1|t}$.

- For instance, $x_{3|1}$ represents the predicted state at time 3 when the prediction is done at time $t=1$ starting from the current state $x(1)$. It is different, in general, from $x_{3|2}$ which is the predicted state at time 3 when the prediction is done at time $t=2$ starting from the current state $x(2)$.
- Similarly $u_{t+k|t}$ is read as “the input u at time $t+k$ computed at time t ”.

RHC Notation

- Let $U_{t \rightarrow t+N|t}^* = \{u_{t|t}^*, \dots, u_{t+N-1|t}^*\}$ be the optimal solution. The first element of $U_{t \rightarrow t+N|t}^*$ is applied to system

$$u(t) = u_{t|t}^*(x(t)).$$

- The CFTOC problem is reformulated and solved at time $t + 1$, based on the new state $x_{t+1|t+1} = x(t + 1)$.

Receding horizon control law

$$f_t(x(t)) = u_{t|t}^*(x(t))$$

Closed loop system

$$x(t + 1) = Ax(t) + Bf_t(x(t)) \triangleq f_{cl}(x(t)), \quad t \geq 0$$

RHC Notation: Time-invariant Systems

As the system, the constraints and the cost function are time-invariant, the solution $f_t(x(t))$ becomes a time-invariant function of the initial state $x(t)$. Thus, we can simplify the notation as

$$\begin{aligned}
 J_0^*(x(t)) = \min_{U_0} \quad & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\
 \text{subj. to} \quad & x_{k+1} = Ax_k + Bu_k, \quad k = 0, \dots, N-1 \\
 & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k = 0, \dots, N-1 \\
 & x_N \in \mathcal{X}_f \\
 & x_0 = x(t)
 \end{aligned}$$

where $U_0 = \{u_0, \dots, u_{N-1}\}$.

The control law and closed loop system are **time-invariant** as well, and we write $f_0(x_0)$ for $f_t(x(t))$.

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MPC Features

Pros

- Any model
 - linear
 - nonlinear
 - single/multivariable
 - time delays
 - constraints
- Any objective:
 - sum of squared errors
 - sum of absolute errors (i.e., integral)
 - worst error over time
 - economic objective

Cons

- Computationally demanding in the general case
- May or may not be stable
- May or may not be feasible

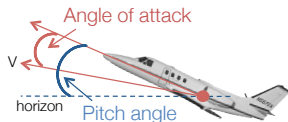
Example: Cessna Citation Aircraft

Linearized continuous-time model:

(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1 : angle of attack, x_2 : pitch angle, x_3 : pitch rate, x_4 : altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle $\pm 0.262\text{rad}$ ($\pm 15^\circ$), elevator rate $\pm 0.524\text{rad/s}$ ($\pm 60^\circ/\text{s}$), pitch angle ± 0.349 ($\pm 39^\circ$)

Open-loop response is unstable (open-loop poles: 0, 0, $-1.5594 \pm 2.29i$)

LQR and Linear MPC with Quadratic Cost

- Quadratic cost
- Linear system dynamics
- Linear constraints on inputs and states

LQR

$$J_{\infty}(x(t)) = \min \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k$$

s.t. $x_{k+1} = A x_k + B u_k$
 $x_0 = x(t)$

Assume: $Q = Q^T \succeq 0$, $R = R^T \succ 0$

MPC

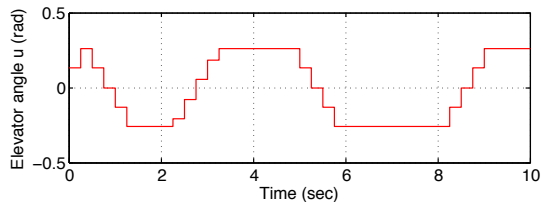
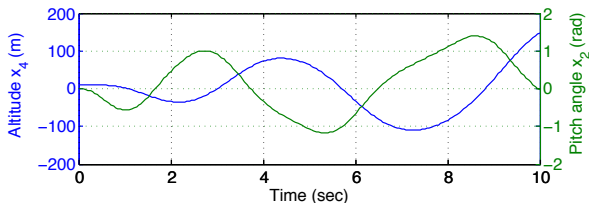
$$J_0^*(x(t)) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k$$

s.t. $x_{k+1} = A x_k + B u_k$
 $x_k \in \mathcal{X}$, $u_k \in \mathcal{U}$
 $x_0 = x(t)$

Example: LQR with saturation

Linear quadratic regulator with saturated inputs.

At time $t = 0$ the plane is flying with a deviation of 10m of the desired altitude, i.e. $x_0 = [0; 0; 0; 10]$



Problem parameters:

Sampling time 0.25sec,
 $Q = I$, $R = 10$

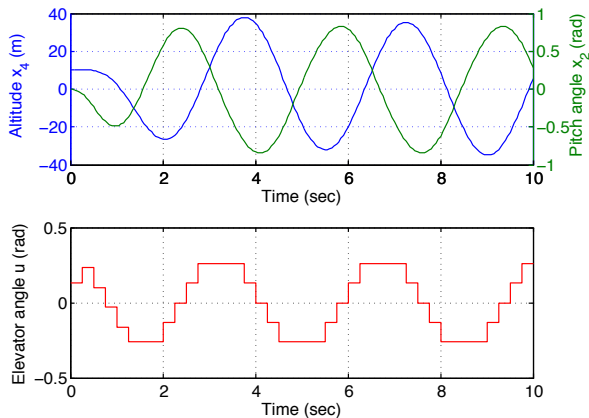
- Closed-loop system is unstable
- Applying LQR control and saturating the controller can lead to instability!

Example: MPC with Bound Constraints on Inputs

MPC controller with input constraints $|u_i| \leq 0.262$

Problem parameters:

Sampling time 0.25sec,
 $Q = I$, $R = 10$, $N = 10$



The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

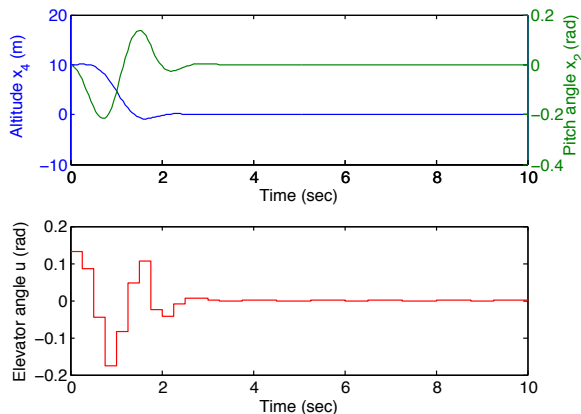
⇒ System does not converge to desired steady-state but to a limit cycle

Example: MPC with all Input Constraints

MPC controller with input constraints $|u_i| \leq 0.262$
 and rate constraints $|\dot{u}_i| \leq 0.349$
 approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

Problem parameters:

Sampling time 0.25sec,
 $Q = I$, $R = 10$, $N = 10$



The MPC controller considers all constraints on the actuator

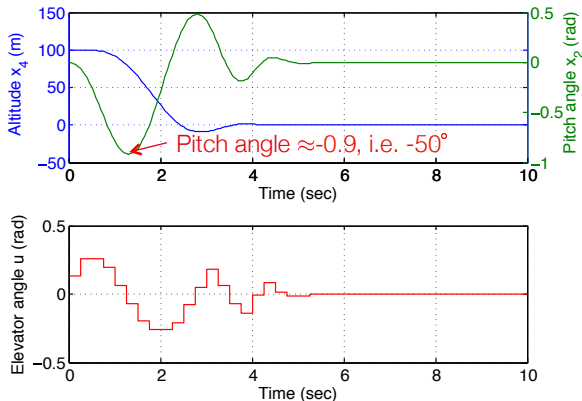
- Closed-loop system is stable
- Efficient use of the control authority

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$
 and rate constraints $|\dot{u}_i| \leq 0.349$
 approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

Problem parameters:

Sampling time 0.25sec,
 $Q = I$, $R = 10$, $N = 10$



Increase step:

At time $t = 0$ the plane is flying with a deviation of 100m of the desired altitude, i.e. $x_0 = [0; 0; 0; 100]$

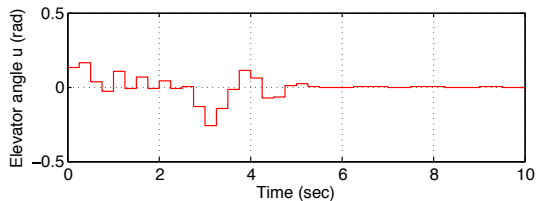
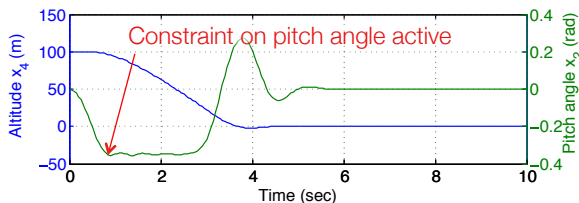
- Pitch angle too large during transient

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$
 and rate constraints $|\dot{u}_i| \leq 0.349$
 approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

Problem parameters:

Sampling time 0.25sec,
 $Q = I$, $R = 10$, $N = 10$



Add state constraints for passenger comfort:

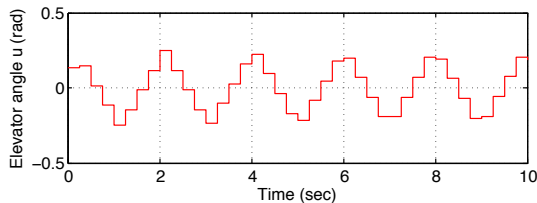
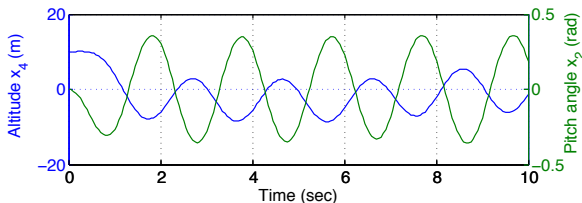
$$|x_2| \leq 0.349$$

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$
 and rate constraints $|\dot{u}_i| \leq 0.349$
 approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

Problem parameters:

Sampling time 0.25sec,
 $Q = I$, $R = 10$, $N = 4$



Decrease in the prediction horizon causes loss of the stability properties

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Loss of Feasibility and Stability

What can go wrong with “standard” MPC?

- No feasibility guarantee, i.e., the MPC problem may not have a solution
- No stability guarantee, i.e., trajectories may not converge to the origin

Example: Loss of feasibility - Double Integrator

Consider the double integrator

$$\begin{cases} x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to the input constraints

$$-0.5 \leq u(t) \leq 0.5$$

and the state constraints

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Compute a receding horizon controller with quadratic objective with

$$N = 3, \quad P = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 10.$$

Example: Loss of feasibility - Double Integrator

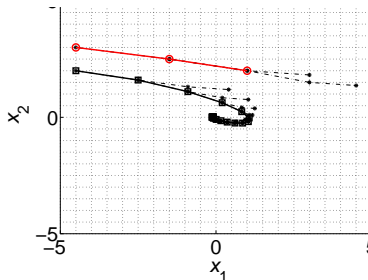
The QP problem associated with the RHC is

$$H = \begin{bmatrix} 13.50 & -10.00 & -0.50 \\ -10.00 & 22.00 & -10.00 \\ -0.50 & -10.00 & 31.50 \end{bmatrix}, \quad F = \begin{bmatrix} -10.50 & 10.00 & -0.50 \\ -20.50 & 10.00 & 9.50 \end{bmatrix}, \quad Y = \begin{bmatrix} 14.50 & 23.50 \\ 23.50 & 54.50 \end{bmatrix}$$

$$G_0 = \begin{bmatrix} 0.50 & -1.00 & 0.50 \\ -0.50 & 1.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ -0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & 0.50 \\ -1.00 & 0.00 & 0.00 \\ 0.00 & -1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 \\ -0.50 & 0.00 & 0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.50 & 0.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 0.50 & 0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ -0.50 & -0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 1.00 & 1.00 \\ -0.50 & -0.50 \\ -1.00 & -1.00 \\ 0.50 & 0.50 \\ -0.50 & -1.50 \\ 0.50 & 1.50 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.00 & -1.00 \end{bmatrix}, \quad w_0 = \begin{bmatrix} 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \end{bmatrix}$$

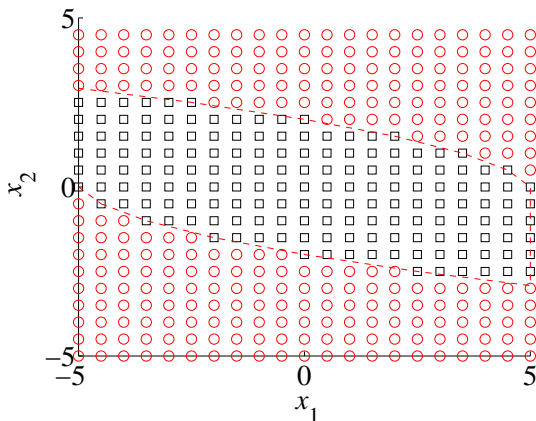
Example: Loss of feasibility - Double Integrator

- 1) MEASURE the state $x(t)$ at time instance t
- 2) OBTAIN $U_0^*(x(t))$ by solving the optimization problem in (1)
- 3) IF $U_0^*(x(t)) = \emptyset$ THEN 'problem infeasible' STOP
- 4) APPLY the first element u_0^* of U_0^* to the system
- 5) WAIT for the new sampling time $t + 1$, GOTO 1)



Depending on initial condition, closed loop trajectory may lead to states for which optimization problem is infeasible.

Example: Loss of feasibility - Double Integrator



Boxes (Circles) are initial points leading (not leading) to feasible closed-loop trajectories

Example: Feasibility and stability are function of tuning

Unstable system $x(t+1) = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

Input constraints $-1 \leq u(t) \leq 1$

Parameters: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

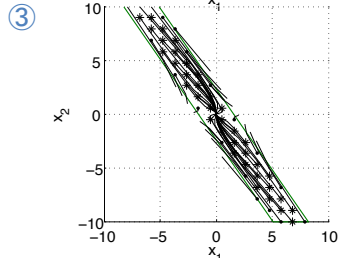
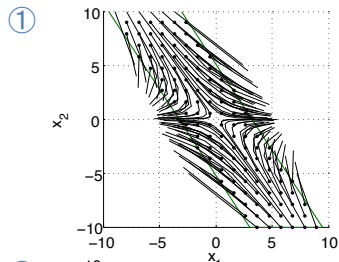
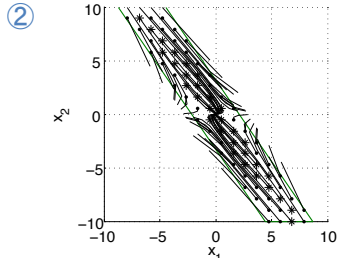
State constraints $\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

Investigate the stability properties for different horizons N and weights R by solving the finite-horizon MPC problem in a receding horizon fashion...

Example: Feasibility and stability are function of tuning

- 1 $R = 10, N = 2$: all trajectories unstable.
- 2 $R = 2, N = 3$: some trajectories stable.
- 3 $R = 1, N = 4$: more stable trajectories.

- * Initial points with convergent trajectories
- Initial points that diverge

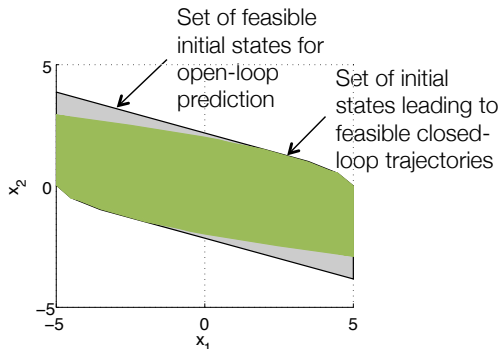
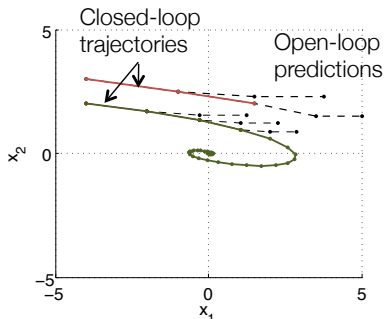


Green lines denote the set of all feasible initial points. They depend on the horizon N but not on the cost $R \implies$ Parameters have complex effect and trajectories.

Summary: Feasibility and Stability

Problems originate from the use of a 'short sighted' strategy

⇒ Finite horizon causes deviation between the open-loop prediction and the closed-loop system:



Ideally we would solve the MPC problem with an infinite horizon, but that is computationally intractable

⇒ Design finite horizon problem such that it approximates the infinite horizon

Summary: Feasibility and Stability

■ Infinite-Horizon

If we solve the RHC problem for $N = \infty$ (as done for LQR), then the open loop trajectories are the same as the closed loop trajectories. Hence

- If problem is feasible, the closed loop trajectories will be always feasible
- If the cost is finite, then states and inputs will converge asymptotically to the origin

■ Finite-Horizon

RHC is “short-sighted” strategy approximating infinite horizon controller. But

- **Feasibility.** After some steps the finite horizon optimal control problem may become infeasible. (Infeasibility occurs without disturbances and model mismatch!)
- **Stability.** The generated control inputs may not lead to trajectories that converge to the origin.

Feasibility and stability in MPC - Solution

Main idea: Introduce terminal cost and constraints to explicitly ensure feasibility and stability:

$$\begin{aligned}
 J_0^*(x_0) = & \min_{U_0} \quad \textcolor{red}{p(x_N)} + \sum_{k=0}^{\textcolor{red}{N-1}} q(x_k, u_k) && \textcolor{red}{\text{Terminal Cost}} \\
 & \text{subj. to} \\
 & x_{k+1} = Ax_k + Bu_k, \quad k = 0, \dots, N-1 \\
 & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k = 0, \dots, N-1 \\
 & \textcolor{red}{x_N \in \mathcal{X}_f} && \textcolor{red}{\text{Terminal Constraint}} \\
 & x_0 = x(t)
 \end{aligned}$$

$p(\cdot)$ and \mathcal{X}_f are chosen to **mimic an infinite horizon**.

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Feasibility and Stability of MPC: Proof

Main steps:

- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point
- Prove stability by showing that the optimal cost function is a Lyapunov function

Two cases:

- 1 Terminal constraint at zero: $x_N = 0$
- 2 Terminal constraint in some (convex) set: $x_N \in \mathcal{X}_f$

General notation:

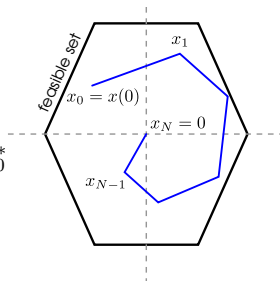
$$J_0^*(x_0) = \min_{U_0} \underbrace{p(x_N)}_{\text{terminal cost}} + \sum_{i=0}^{N-1} \underbrace{q(x_i, u_i)}_{\text{stage cost}}$$

Quadratic case: $q(x_i, u_i) = x_i^T Q x_i + u_i^T R u_i$, $p(x_N) = x_N^T P x_N$

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

- Assume feasibility of x_0 and let $\{u_0^*, u_1^*, \dots, u_{N-1}^*\}$ be the optimal control sequence computed at x_0 and $\{x(0), x_1, \dots, x_N\}$ be the corresponding state trajectory
- Apply u_0^* and let system evolve to $x(1) = Ax_0 + Bu_0^*$
- At $x(1)$ the control sequence $\{u_1^*, u_2^*, \dots, u_{N-1}^*, 0\}$ is feasible (apply 0 control input $\Rightarrow x_{N+1} = 0$)



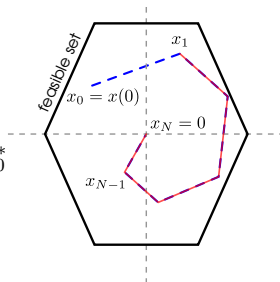
\Rightarrow *Recursive feasibility* ✓

$\Rightarrow J_0^*(x)$ is a Lyapunov function \rightarrow (Lyapunov) Stability ✓

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

- Assume feasibility of x_0 and let $\{u_0^*, u_1^*, \dots, u_{N-1}^*\}$ be the optimal control sequence computed at x_0 and $\{x(0), x_1, \dots, x_N\}$ be the corresponding state trajectory
- Apply u_0^* and let system evolve to $x(1) = Ax_0 + Bu_0^*$
- At $x(1)$ the control sequence $\{u_1^*, u_2^*, \dots, u_{N-1}^*, 0\}$ is feasible (apply 0 control input $\Rightarrow x_{N+1} = 0$)



\Rightarrow *Recursive feasibility* ✓

$\Rightarrow J_0^*(x)$ is a Lyapunov function \rightarrow (Lyapunov) Stability ✓

Stability of MPC - Zero terminal state constraint

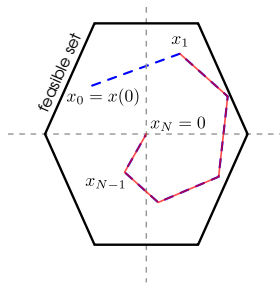
Terminal constraint: $x_N \in \mathcal{X}_f = 0$

Goal: Show $J_0^*(x_1) < J_0^*(x_0) \quad \forall x_0 \neq 0$

$$J_0^*(x_0) = \underbrace{p(x_N)}_{=0} + \sum_{i=0}^{N-1} q(x_i, u_i^*)$$

$$\begin{aligned} J_0^*(x_1) &\leq \tilde{J}_0(x_1) = \sum_{i=1}^N q(x_i, u_i^*) \\ &= \sum_{i=0}^{N-1} q(x_i, u_i^*) - q(x_0, u_0^*) + q(x_N, u_N) \\ &= J_0^*(x_0) - \underbrace{q(x_0, u_0^*)}_{\substack{\text{Subtract cost} \\ \text{at stage 0}}} + \underbrace{q(0, 0)}_{\substack{=0, \text{ Add cost} \\ \text{for staying at 0}}} \end{aligned}$$

$\Rightarrow J_0^*(x)$ is a Lyapunov function \rightarrow (Lyapunov) Stability ✓



Example: Impact of Horizon with Zero Terminal Constraint

System dynamics:

$$x_{k+1} = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

Constraints:

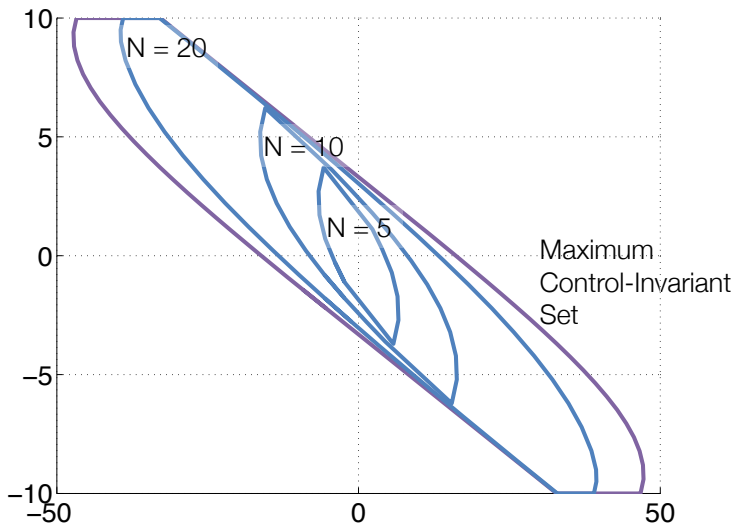
$$\mathcal{X} := \{x \mid -50 \leq x_1 \leq 50, -10 \leq x_2 \leq 10\} = \{x \mid A_x x \leq b_x\}$$

$$\mathcal{U} := \{u \mid \|u\|_\infty \leq 1\} = \{u \mid A_u u \leq b_u\}$$

Stage cost:

$$q(x, u) := x' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^T u$$

Example: Impact of Horizon with Zero Terminal Constraint



The horizon can have a strong impact on the region of attraction.

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6. Feasibility and Stability

6.1 Proof for $\mathcal{X}_f = 0$

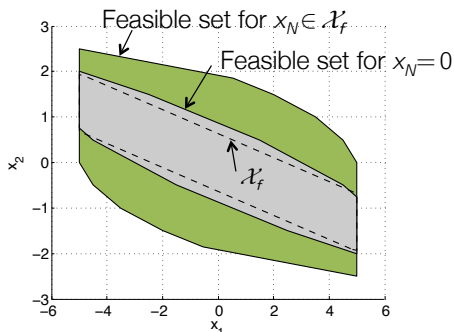
6.2 General Terminal Sets

6.3 Example

Extension to More General Terminal Sets

Problem: The terminal constraint $x_N = 0$ reduces the size of the feasible set

Goal: Use convex set \mathcal{X}_f to increase the region of attraction



Double integrator

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$-0.5 \leq u(t) \leq 0.5$$

$$N = 5, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 10$$

Goal: Generalize proof to the constraint $x_N \in \mathcal{X}_f$

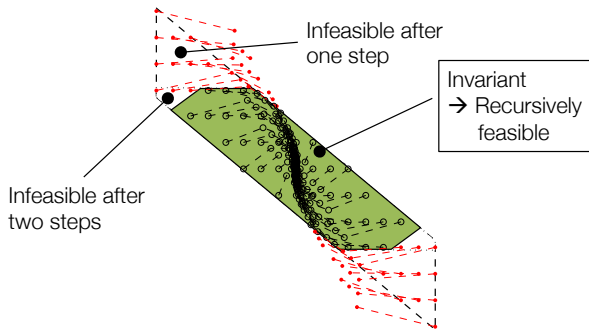
Invariant sets

Definition: Invariant set

A set \mathcal{O} is called *positively invariant* for system $x(t+1) = f_{cl}(x(t))$, if

$$x(0) \in \mathcal{O} \Rightarrow x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}_+$$

The positively invariant set that contains every closed positively invariant set is called the maximal positively invariant set \mathcal{O}_∞ .



Stability of MPC - Main Result

Assumptions

- 1 Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
- 2 Terminal set is **invariant** under the local control law $v(x_k)$:

$$x_{k+1} = Ax_k + Bv(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in \mathcal{X}_f :

$$\mathcal{X}_f \subseteq \mathcal{X}, \quad v(x_k) \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

- 3 Terminal cost is a continuous **Lyapunov function** in the terminal set \mathcal{X}_f and satisfies:

$$p(x_{k+1}) - p(x_k) \leq -q(x_k, v(x_k)), \quad \text{for all } x_k \in \mathcal{X}_f$$

Under those 3 assumptions:

Theorem

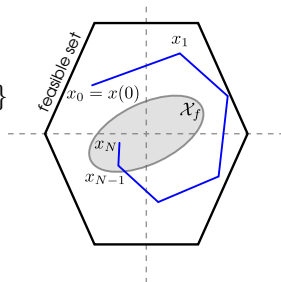
The closed-loop system under the MPC control law $u_0^*(x)$ is asymptotically stable and the set \mathcal{X}_f is positive invariant for the system $x(k+1) = Ax + Bu_0^*(x)$.

Stability of MPC - Outline of the Proof

- Assume feasibility of $x(0)$ and let $\{u_0^*, u_1^*, \dots, u_{N-1}^*\}$ be the optimal control sequence computed at $x(0)$ and $\{x(0), x_1, \dots, x_N\}$ the corresponding state trajectory

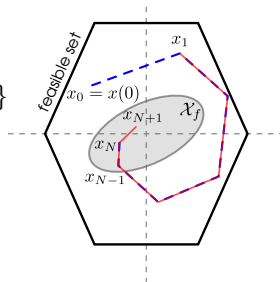
- At $x(1)$, $\{u_1^*, u_2^*, \dots, v(x_N)\}$ is feasible:
 x_N is in $\mathcal{X}_f \rightarrow v(x_N)$ is feasible
 and $x_{N+1} = Ax_N + Bv(x_N)$ in \mathcal{X}_f

\Rightarrow *Terminal constraint provides recursive feasibility*



Stability of MPC - Outline of the Proof

- Assume feasibility of $x(0)$ and let $\{u_0^*, u_1^*, \dots, u_{N-1}^*\}$ be the optimal control sequence computed at $x(0)$ and $\{x(0), x_1, \dots, x_N\}$ the corresponding state trajectory
- At $x(1)$, $\{u_1^*, u_2^*, \dots, v(x_N)\}$ is feasible:
 - x_N is in $\mathcal{X}_f \rightarrow v(x_N)$ is feasible
 - and $x_{N+1} = Ax_N + Bv(x_N)$ in \mathcal{X}_f



\Rightarrow *Terminal constraint provides recursive feasibility*

Asymptotic Stability of MPC - Outline of the Proof

$$J_0^*(x_0) = \sum_{i=0}^{N-1} q(x_i, u_i^*) + p(x_N)$$

Feasible, sub-optimal sequence for $x_1 : \{u_1^*, u_2^*, \dots, v(x_N)\}$

$$\begin{aligned} J_0^*(x_1) &\leq \sum_{i=1}^N q(x_i, u_i^*) + p(Ax_N + Bv(x_N)) \\ &= \sum_{i=0}^{N-1} q(x_i, u_i^*) + p(x_N) - q(x_0, u_0^*) + p(Ax_N + Bv(x_N)) \\ &\quad - p(x_N) + q(x_N, v(x_N)) \\ &= J_0^*(x_0) - q(x_0, u_0^*) + \underbrace{p(Ax_N + Bv(x_N)) - p(x_N) + q(x_N, v(x_N))}_{p(x) \leq 0} \\ &\implies J_0^*(x_1) - J_0^*(x_0) \leq -q(x_0, u_0^*), \quad q > 0 \end{aligned}$$

$J_0^(x)$ is a Lyapunov function decreasing along the closed loop trajectories*

\Rightarrow The closed-loop system under the MPC control law is asymptotically stable

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

$$\begin{aligned} J_0^*(x_0) = & \min_{U_0} \quad \textcolor{red}{x'_N P x_N} + \sum_{k=0}^{\textcolor{red}{N-1}} \textcolor{red}{x'_k Q x_k} + \textcolor{red}{u'_k R u_k} && \textcolor{red}{\text{Terminal Cost}} \\ & \text{subj. to} && \\ & x_{k+1} = A x_k + B u_k, \quad k = 0, \dots, N-1 && \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k = 0, \dots, N-1 && \\ & \textcolor{red}{x_N} \in \textcolor{red}{\mathcal{X}_f} && \textcolor{red}{\text{Terminal Constraint}} \\ & x_0 = x(t) && \end{aligned}$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

- Design unconstrained LQR control law

$$F_{\infty} = -(B'P_{\infty}B + R)^{-1}B'P_{\infty}$$

where P_{∞} is the solution to the discrete-time algebraic Riccati equation:

$$P_{\infty} = A'P_{\infty}A + Q - A'P_{\infty}B(B'P_{\infty}B + R)^{-1}B'P_{\infty}A$$

- Choose the terminal weight $P = P_{\infty}$
- Choose the terminal set \mathcal{X}_f to be the maximum invariant set for the closed-loop system $x_{k+1} = (A + BF_{\infty})x_k$:

$$x_{k+1} = Ax_k + BF_{\infty}(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in \mathcal{X}_f :

$$\mathcal{X}_f \subseteq \mathcal{X}, \quad F_{\infty}x_k \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

- 1 The stage cost is a positive definite function
- 2 By construction the terminal set is **invariant** under the local control law $v = F_{\infty}x$
- 3 Terminal cost is a continuous **Lyapunov function** in the terminal set \mathcal{X}_f and satisfies:

$$\begin{aligned}x'_{k+1}Px_{k+1} - x'_kPx_k &= x'_k(-P_{\infty} + A'P_{\infty}A - A'P_{\infty}B(B'P_{\infty}B + R)^{-1}B'P_{\infty}A)x_k \\ &= -x'_kQx_k\end{aligned}$$

All the Assumptions of the Feasibility and Stability Theorem are verified.

Example: Unstable Linear System

System dynamics:

$$x_{k+1} = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

Constraints:

$$\mathcal{X} := \{x \mid -50 \leq x_1 \leq 50, -10 \leq x_2 \leq 10\} = \{x \mid A_x x \leq b_x\}$$

$$\mathcal{U} := \{u \mid \|u\|_\infty \leq 1\} = \{u \mid A_u u \leq b_u\}$$

Stage cost:

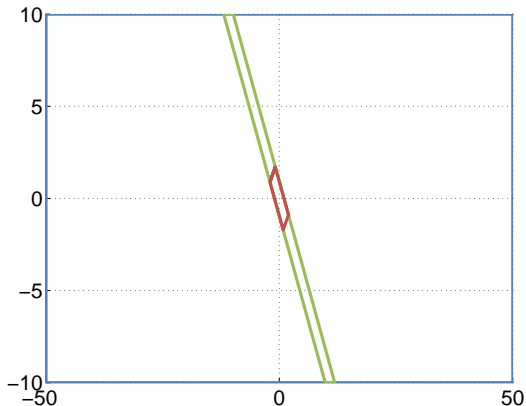
$$q(x, u) := x' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^T u$$

Horizon: $N = 10$

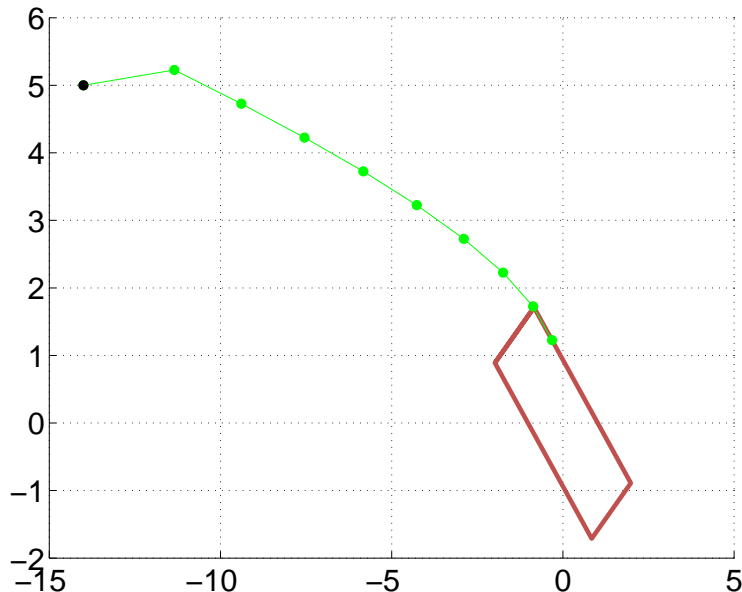
Example: Designing MPC Problem

- 1 Compute the optimal LQR controller and cost matrices: F_∞, P_∞
- 2 Compute the maximal invariant set \mathcal{X}_f for the closed-loop linear system $x_{k+1} = (A + BF_\infty)x_k$ subject to the constraints

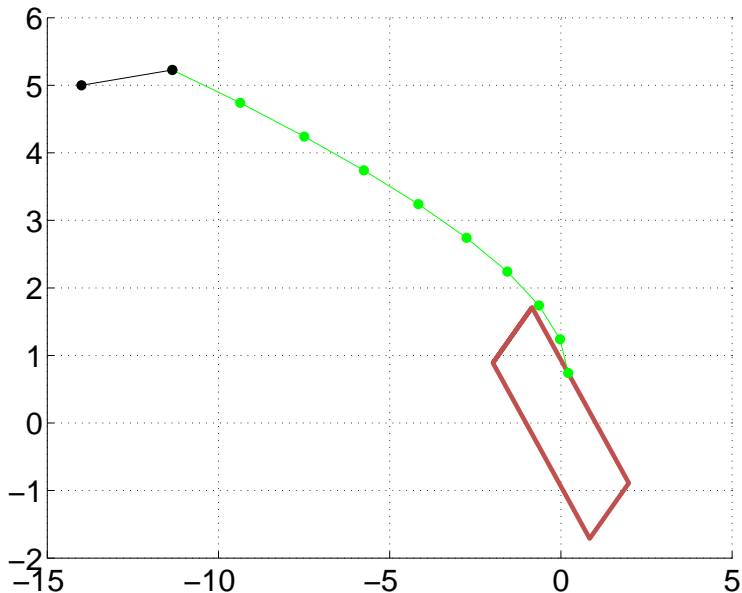
$$\mathcal{X}_{cl} := \left\{ x \mid \begin{bmatrix} A_x \\ A_u F_\infty \end{bmatrix} x \leq \begin{bmatrix} b_x \\ b_u \end{bmatrix} \right\}$$



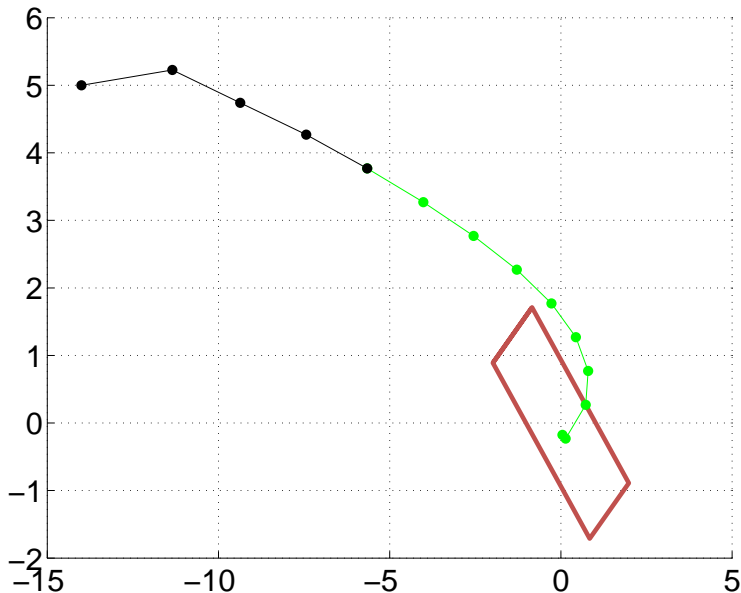
Example: Closed-loop behaviour



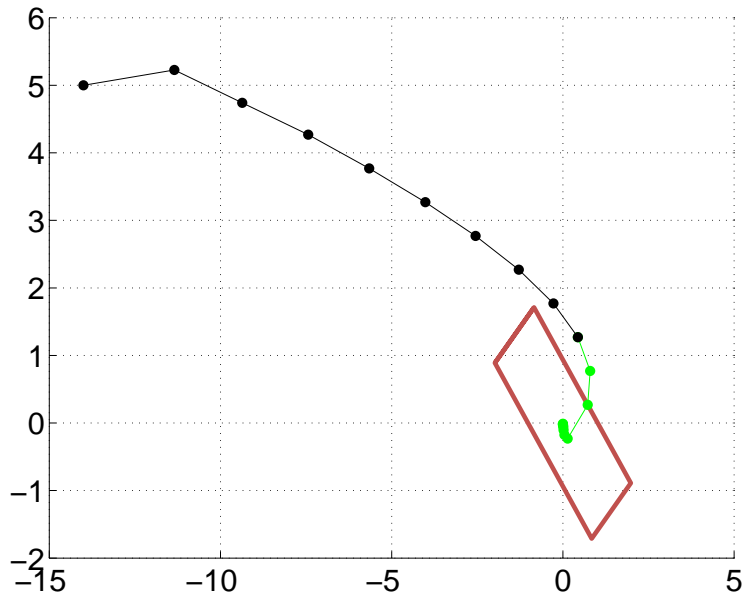
Example: Closed-loop behaviour



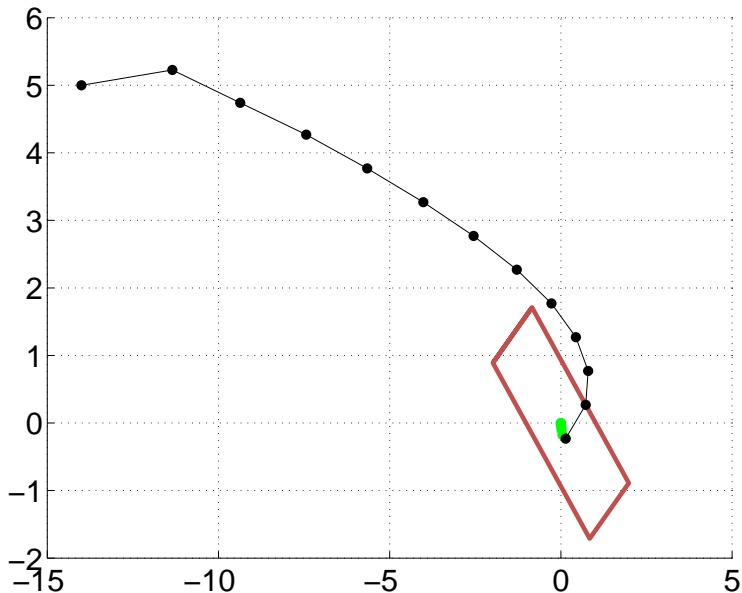
Example: Closed-loop behaviour



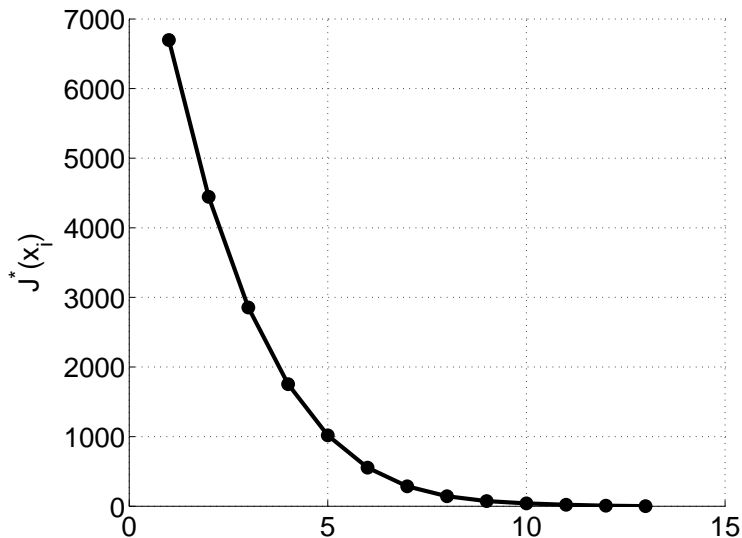
Example: Closed-loop behaviour



Example: Closed-loop behaviour



Example: Lyapunov Decrease of Optimal Cost



Stability of MPC - Remarks

- The terminal set \mathcal{X}_f and the terminal cost ensure recursive feasibility and stability of the closed-loop system.
But: the terminal constraint reduces the region of attraction.
(Can extend the horizon to a sufficiently large value to increase the region)

Are terminal sets used in practice?

- Generally not...
 - Not well understood by practitioners
 - Requires advanced tools to compute (polyhedral computation or LMI)
- Reduces region of attraction
 - A 'real' controller must provide *some* input in *every* circumstance
- Often unnecessary
 - Stable system, long horizon \rightarrow will be stable and feasible in a (large) neighbourhood of the origin

Choice of Terminal Set and Cost: Summary

- Terminal constraint provides a sufficient condition for stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attraction for small N
- Solution for linear systems with quadratic cost
- In practice: Enlarge horizon and check stability by sampling
- With larger horizon length N , region of attraction approaches maximum control invariant set

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6. Feasibility and Stability

6.1 Proof for $\mathcal{X}_f = 0$

6.2 General Terminal Sets

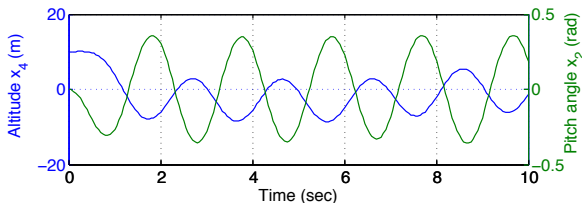
6.3 Example

Example: Short horizon

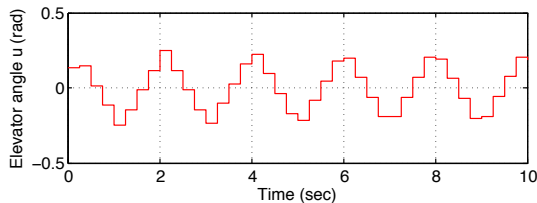
MPC controller with input constraints $|u_i| \leq 0.262$
 and rate constraints $|\dot{u}_i| \leq 0.349$
 approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

Problem parameters:

Sampling time 0.25sec,
 $Q = I$, $R = 10$, $N = 4$



Decrease in the prediction horizon causes loss of the stability properties

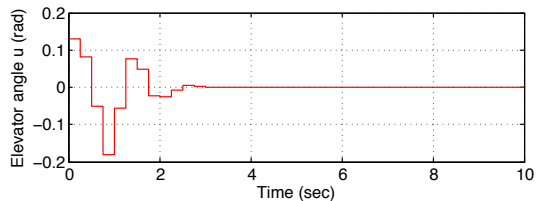
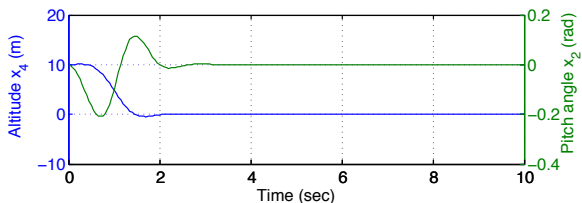


Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$
 and rate constraints $|\dot{u}_i| \leq 0.349$
 approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

Problem parameters:

Sampling time 0.25sec,
 $Q = I$, $R = 10$, $N = 4$



Inclusion of terminal cost and constraint provides stability

Summary

Finite-horizon MPC may not be stable!

Finite-horizon MPC may not satisfy constraints for all time!

- An infinite-horizon provides stability and invariance.
- We ‘fake’ infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- These ideas extend to non-linear systems, but the sets are difficult to compute.

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3. Receding Horizon Control Notation
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 - 6.2 General Terminal Sets
 - 6.3 Example
7. Extension to Nonlinear MPC

Extension to Nonlinear MPC

Consider the nonlinear system dynamics: $x(t+1) = g(x(t), u(t))$

$$\begin{aligned}
 J_0^*(x(t)) = \min_{U_0} \quad & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\
 \text{subj. to} \quad & x_{k+1} = g(x_k, u_k), \quad k = 0, \dots, N-1 \\
 & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k = 0, \dots, N-1 \\
 & x_N \in \mathcal{X}_f \\
 & x_0 = x(t)
 \end{aligned}$$

- Presented assumptions on the terminal set and cost did not rely on linearity
 - Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems
- Results can be directly extended to nonlinear systems.

However, computing the sets \mathcal{X}_f and function p can be very difficult!