# Model Predictive Control
# Reachability and Invariance

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# Outline

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# Definitions (Polyhedra and polytopes)

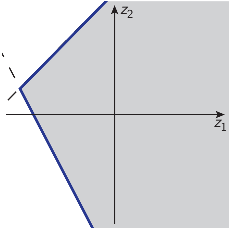
A polyhedron is the intersection of a *finite* number of closed halfspaces:

$$\begin{aligned} Z &= \{z \mid a_1^\top z \le b_1,\ a_2^\top z \le b_2, \dots, a_m^\top z \le b_m\} \\ &= \{z \mid Az \le b\} \end{aligned}$$

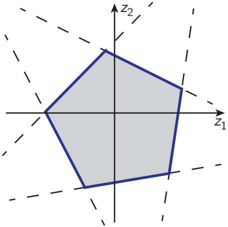
where $A := [a_1, a_2, \dots, a_m]^\top$ and $b := [b_1, b_2, \dots, b_m]^\top$.

A polytope is a *bounded* polyhedron.

Polyhedra and polytopes are always convex.




An (unbounded) polyhedron




A polytope

# General Set Definitions and Operations

- An $n$-dimensional ball $B(x_0, \rho)$ is the set $B(x_0, \rho) = \{x \in \mathbb{R}^n | \sqrt{\|x - x_0\|_2} \le \rho\}$. $x_0$ and $\rho$ are the center and the radius of the ball, respectively.
- The convex combination of $x_1, \dots, x_k$ is defined as the point $\lambda_1 x_1 + \dots + \lambda_k x_k$ where $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \ge 0$, $i = 1, \dots, k$.
- The convex hull of a set $K \subseteq \mathbb{R}^n$ is the set of all convex combinations of points in $K$ and it is denoted as $\mathrm{conv}(K)$:

$$\mathrm{conv}(K) \triangleq \{\lambda_1 x_1 + \dots + \lambda_k x_k \mid x_i \in K,\ \lambda_i \ge 0,\ i = 1, \dots, k,\ \sum_{i=1}^k \lambda_i = 1\}.$$

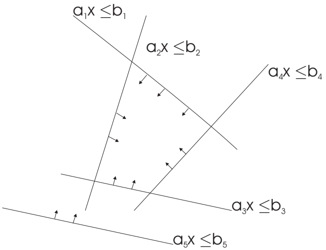
# Polyhedra Representations

An $\mathcal{H}$*-polyhedron* $\mathcal{P}$ in $\mathbb{R}^n$ denotes an intersection of a finite set of closed halfspaces in $\mathbb{R}^n$:

$$\mathcal{P} = \{x \in \mathbb{R}^n : \ Ax \leq b\}$$

In Matlab: P = Polytope(A,b) or P = Polyhedron(A,b)

A two-dimensional $\mathcal{H}$-polyhedron




Inequalities which can be removed without changing the polyhedron are called *redundant*. The representation of an $\mathcal{H}$-polyhedron is *minimal* if it does not contain redundant inequalities.

# Polyhedra Representations

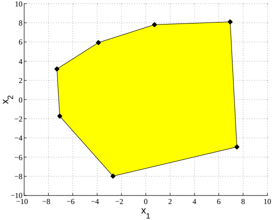
- A $\mathcal{V}$-polytope $\mathcal{P}$ in $\mathbb{R}^n$ is defined as

$$\mathcal{P} = \text{conv}(V)$$

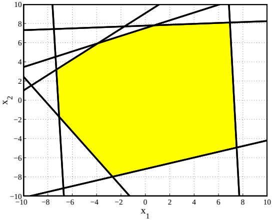
  for some $V = [V_1, \ldots, V_k] \in \mathbb{R}^{n \times k}$.
- Any $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope and viceversa.
- A polytope $\mathcal{P} \subset \mathbb{R}^n$, is full-dimensional if it is possible to fit a non-empty $n$-dimensional ball in $\mathcal{P}$
- If $\|A_i\|_2 = 1$, where $A_i$ denotes the $i$-th row of a matrix $A$, we say that the polytope $\mathcal{P}$ is *normalized*.

# Polyhedra Representations

- The faces of dimension 0 and 1 are called vertices and edges, respectively.




(a) $\mathcal{V}$-representation.




(b) $\mathcal{H}$-representation.

# Polytopal Complexes

A set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a P-collection (in $\mathbb{R}^n$) if it is a collection of a finite number of $n$-dimensional polytopes, i.e.

$$\mathcal{C} = \{\mathcal{C}_i\}_{i=1}^{N_C},$$

where $\mathcal{C}_i := \{x \in \mathbb{R}^n \ : \ C_i^x x \leq C_i^c\}$, $\dim(\mathcal{C}_i) = n$, $i = 1, \ldots, N_C$, with $N_C < \infty$.

In Matlab: Q = [P1, P2, P3], R = [P4, Q, [P5, P6], P7]

# Functions on Polytopal Complexes

## Definitions (PWA and PWQ)

- A function $h(\theta) : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise affine (PWA) if there exists a partition $R_1,\ldots,R_N$ of $\Theta$ and $h(\theta) = H^i\theta + k^i$, $\forall\theta \in R_i$.
- A function $h(\theta) : \Theta \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise quadratic (PWQ) if there exists a partition $R_1,\ldots,R_N$ of $\Theta$ and $h(\theta) = \theta' H^i\theta + k^i\theta + l^i$, $\forall\theta \in R_i$, $i = 1, \ldots, N$.

## Definitions (PPWA and PPWQ)

- A function $h(\theta) : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise affine on polyhedra (PPWA) if there exists a polyhedral partition $R_1,\ldots,R_N$ of $\Theta$ and $h(\theta) = H^i\theta + k^i$, $\forall\theta \in R_i$.
- A function $h(\theta) : \Theta \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise quadratic on polyhedra (PPWQ) if there exists a polyhedral partition $R_1,\ldots,R_N$ of $\Theta$ and $h(\theta) = \theta' H^i\theta + k^i\theta + l^i$, $\forall\theta \in R_i$.

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# Basic Operations on Polytopes

- Convex Hull of a set of points $V = \{V_i\}_{i=1}^{N_V}$, with $V_i \in \mathbb{R}^n$,

$$\mathrm{conv}(V) = \{x \in \mathbb{R}^n \ : x = \sum_{i=1}^{N_V} \alpha_i V_i,\ 0 \leq \alpha_i \leq 1,\ \sum_{i=1}^{N_V} \alpha_i = 1\}. \qquad (1)$$

  In Matlab: P=Polyhedron($V$), $V$ matrix containing vertices of the polytope P

- Vertex Enumeration of a polytope $\mathcal{P}$ given in $\mathcal{H}$-representation. (dual of the convex hull operation)
  In Matlab: V=P.V

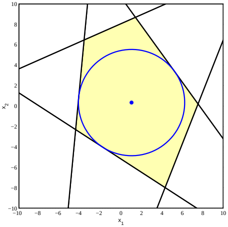
Used to switch from a $\mathcal{V}$-representation of a polytope to an $\mathcal{H}$-representation.

# Basic Operations on Polytopes

- Polytope reduction is the computation of the minimal representation of a polytope. A polytope $\mathcal{P} \subset \mathbb{R}^n$, $\mathcal{P} = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}$ is in a minimal representation if the removal of any row in $Ax \leq b$ would change it (i.e., if there are no redundant constraints).

  In Matlab: P = Polytope(A,b,normal,minrep), minrep=1
- The Chebychev Ball of a polytope $\mathcal{P}$ corresponds to the largest radius ball $\mathcal{B}(x_c, R)$ with center $x_c$, such that $\mathcal{B}(x_c, R) \subset \mathcal{P}$.

  In Matlab: P.chebyCenter.x, P.chebyCenter.r

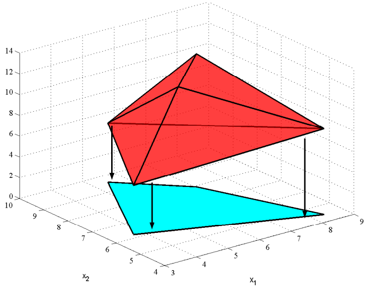


# Basic Operations on Polytopes

- Projection Given a polytope $\mathcal{P} = \{[x' y']' \in \mathbb{R}^{n+m} \ : \ A^x x + A^y y \leq b\} \subset \mathbb{R}^{n+m}$ the projection onto the $x$-space $\mathbb{R}^n$ is defined as

$$\mathrm{proj}_x(\mathcal{P}) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m : A^x x + A^y y \leq b\}.$$

In Matlab: Q = projection(P,dim)



# Affine Mappings and Polyhedra

- Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid Hx \leq k\}$, with $H \in \mathbb{R}^{n_P \times n}$ and an affine mapping $f(z)$

$$f: \; z \in \mathbb{R}^n \mapsto Az + b, \; A \in \mathbb{R}^{n \times n}, \; b \in \mathbb{R}^n$$

- Define the composition of $\mathcal{P}$ and $f$ as the following polyhedron

$$\mathcal{P} \circ f \triangleq \{z \in \mathbb{R}^n \mid Hf(z) \leq k\} = \{z \in \mathbb{R}^m \mid HAz \leq k - Hb\}$$

- Useful for backward-reachability

# Affine Mappings and Polyhedra

- Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid Hx \leq k\}$, with $H \in \mathbb{R}^{n_P \times n}$ and an affine mapping $f(z)$

$$f:\ z \in \mathbb{R}^n \mapsto Az + b,\ A \in \mathbb{R}^{n \times n},\ b \in \mathbb{R}^n$$

- Define the composition of $f$ and $\mathcal{P}$ as the following polyhedron

$$f \circ \mathcal{P} \triangleq \{y \in \mathbb{R}^n \mid y = Ax + b\ \forall x \in \mathbb{R}^n,\ Hx \leq k\}$$

- The polyhedron $f \circ \mathcal{P}$ in can be computed as follows. Write $\mathcal{P}$ in $\mathcal{V}$-representation $\mathcal{P} = \mathrm{conv}(V)$ and map the vertices $V = \{V_1, \ldots, V_k\}$ through the transformation $f$. Because the transformation is affine, the set $f \circ \mathcal{P}$ is the convex hull of the transformed vertices

$$f \circ \mathcal{P} = \mathrm{conv}(F),\ F = \{AV_1 + b, \ldots, AV_k + b\}.$$

- Useful for forward-reachability

# Outline

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# Set Definition

We consider the following two types of systems **autonomous systems:**

$$x(t+1) = f_a(x(t)), \tag{2}$$

and **systems subject to external inputs:**

$$x(t+1) = f(x(t), u(t)). \tag{3}$$

Both systems are subject to state and input constraints

$$x(t) \in \mathcal{X},\ \ u(t) \in \mathcal{U},\ \forall\ t \geq 0.$$

The sets $\mathcal{X}$ and $\mathcal{U}$ are polyhedra and contain the origin in their interior.

# Reach Set Definition

For the autonomous system (2) we denote the one-step reachable set as

$$\text{Reach}(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n : \ \exists\ x(0) \in \mathcal{S} \text{ s.t. } x = f_a(x(0))\}$$

For the system (3) with inputs we denote the one-step reachable set as

$$\text{Reach}(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n : \ \exists\ x(0) \in \mathcal{S},\ \exists\ u(0) \in \mathcal{U} \text{ s.t. } x = f(x(0), u(0))\}$$

# Pre Set Definition

"Pre" sets are the dual of one-step reachable sets. The set

$$\mathrm{Pre}(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n \ : \ f_a(x) \in \mathcal{S}\}$$

defines the set of states which evolve into the target set $\mathcal{S}$ in one time step for the system (2).

Similarly, for the system (3) the set of states which can be driven into the target set $\mathcal{S}$ in one time step is defined as

$$\mathrm{Pre}(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n \ : \ \exists u \in \mathcal{U} \text{ s.t. } f(x,u) \in \mathcal{S}\}$$

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# Pre Set Computation - Autonomous Systems

Assume the system is linear and autonomous

$$x(t+1) = Ax(t)$$

Let

$$\mathcal{S} = \{x \ : \ Hx \leq h\}, \tag{4}$$

Then the set Pre($\mathcal{S}$) is

$$\mathrm{Pre}(\mathcal{S}) = \{x \ : \ HAx \leq h\}$$

Note that by using polyhedral notation, the set $\mathrm{Pre}(\mathcal{S})$ is simply $\mathcal{S} \circ A$.

# Reach Set Computation - Autonomous Systems

The set $\mathrm{Reach}(\mathcal{S})$ is obtained by applying the map $A$ to the set $\mathcal{S}$.
Write $\mathcal{S}$ in $\mathcal{V}$-representation

$$\mathcal{S} = \mathrm{conv}(V) \tag{5}$$

and map the set of vertices $V$ through the transformation $A$.
Because the transformation is linear, the reach set is simply the convex hull of the transformed vertices

$$\mathrm{Reach}(\mathcal{S}) = A \circ \mathcal{S} = \mathrm{conv}(AV) \tag{6}$$

# Pre Set Computation - System with Inputs

Consider the system

$$x(t+1) = Ax(t) + Bu(t)$$

Let

$$\mathcal{S} = \{x \mid Hx \le h\}, \quad \mathcal{U} = \{u \mid H_u u \le h_u\}, \tag{7}$$

The Pre set is

$$\text{Pre}(\mathcal{S}) = \left\{ x \in \mathbb{R}^n \,|\, \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} h \\ h_u \end{bmatrix} \right\}$$

which is the projection onto the $x$-space (with dimension $\mathbb{R}^n$) of the polyhedron

$$\mathcal{T} := \{ \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} h \\ h_u \end{bmatrix} \}.$$

In Matlab: Q = projection($\mathcal{T}$,n)

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# Controllable Sets

## Definition ($N$-Step Controllable Set $\mathcal{K}_N(\mathcal{O})$)

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the $N$-step controllable set $\mathcal{K}_N(\mathcal{O})$ is defined as:

$$\mathcal{K}_N(\mathcal{O}) \triangleq \mathrm{Pre}(\mathcal{K}_{N-1}(\mathcal{O})) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{O}) = \mathcal{O}, \qquad N \in \mathbb{N}^+.$$

All states $x_0 \in \mathcal{K}_N(\mathcal{O})$ can be driven,through a time-varying control law, to the target set $\mathcal{O}$ in $N$ steps, while satisfying input and state constraints.

## Definition (Maximal Controllable Set $\mathcal{K}_\infty(\mathcal{O})$)

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the maximal controllable set $\mathcal{K}_\infty(\mathcal{O})$ for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints $x(t) \in \mathcal{X}, \; u(t) \in \mathcal{U}$ is the union of all $N$-step controllable sets contained in $\mathcal{X}$ ($N \in \mathbb{N}$).

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# $N$-Step Reachable Sets

**Definition ($N$-Step Reachable Set $\mathcal{R}_N(\mathcal{X}_0)$)**

For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the $N$-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ is

$$\mathcal{R}_{i+1}(\mathcal{X}_0) \triangleq \mathrm{Reach}(\mathcal{R}_i(\mathcal{X}_0)), \quad \mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0, \qquad i = 0, \ldots, N-1$$

All states $x_0 \in \mathcal{X}_0$ can will evolve to the $N$-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ in $N$ steps

Same definition of Maximal Reachable Set $\mathcal{R}_\infty(\mathcal{X}_0)$ can be introduced.

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# Invariant Sets

Invariant sets

- are computed for ***autonomous systems***
- for a *given* feedback controller $u = g(x)$, provide the set of initial states whose trajectory will never violate the system constraints.

## Definition (Positive Invariant Set)

A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a positive invariant set for the autonomous system $x(t+1) = f_a(x(t))$ subject to the constraints $x(t) \in \mathcal{X}$, if

$$x(0) \in \mathcal{O} \quad \Rightarrow \quad x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}^+$$

## Definition (Maximal Positive Invariant Set $\mathcal{O}_\infty$)

The set $\mathcal{O}_\infty$ is the maximal invariant set if $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all the invariant sets contained in $\mathcal{X}$.

# Invariant Sets

## Theorem (Geometric condition for invariance)

*A set $\mathcal{O}$ is a positive invariant set if and only if $\mathcal{O} \subseteq \mathrm{Pre}(\mathcal{O})$*

NOTE: $\mathcal{O} \subseteq \mathrm{Pre}(\mathcal{O}) \Leftrightarrow \mathrm{Pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$

## Algorithm

**Input:** $f_a$ , $\mathcal{X}$

**Output:** $\mathcal{O}_\infty$

*1* **let** $\Omega_0 = \mathcal{X}$,

*2* **let** $\Omega_{k+1} = \mathrm{Pre}(\Omega_k) \cap \Omega_k$

*3* **if** $\Omega_{k+1} = \Omega_k$ **then** $\mathcal{O}_\infty \leftarrow \Omega_{k+1}$

*4* **else go to** 2

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates when $\Omega_{k+1} = \Omega_k$ so that $\Omega_k$ is the maximal positive invariant set $\mathcal{O}_\infty$ for $x(t+1) = f_a(x(t))$.

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# Control Invariant Sets

***Control*** invariant sets

- are computed for systems ***subject to external inputs***
- provide the set of initial states for which *there exists* a controller such that the system constraints are never violated.

## Definition (Control Invariant Set)

A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set if

$$x(t) \in \mathcal{C} \quad \Rightarrow \quad \exists u(t) \in \mathcal{U} \text{ such that } f(x(t), u(t)) \in \mathcal{C}, \quad \forall t \in \mathbb{N}^+$$

## Definition (Maximal Control Invariant Set $\mathcal{C}_\infty$)

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints in $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$, if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$.

# Control Invariant Sets

Same geometric condition for control invariants holds: $\mathcal{C}$ is a control invariant set if and only if

$$\mathcal{C} \subseteq \mathrm{Pre}(\mathcal{C}) \tag{8}$$

## Algorithm

**Input:** $f$, $\mathcal{X}$ and $\mathcal{U}$

**Output:** $\mathcal{C}_\infty$

*1* **let** $\Omega_0 = \mathcal{X}$,

*2* **let** $\Omega_{k+1} = \mathrm{Pre}(\Omega_k) \cap \Omega_k$

*3* **if** $\Omega_{k+1} = \Omega_k$ **then** $\mathcal{C}_\infty \leftarrow \Omega_{k+1}$

*4* **else go to** 2

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates if $\Omega_{k+1} = \Omega_k$ so that $\Omega_k$ is the maximal control invariant set $\mathcal{C}_\infty$ for the constrained system.

# Invariant Sets and Control Invariant Sets

- The set $\mathcal{O}_\infty$ ($\mathcal{C}_\infty$) is ***finitely determined*** if and only if $\exists\ i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$.
- The smallest element $i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$ is called the ***determinedness index***.
- For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated.