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Tarea 4: Geometria computacional

Considere la superficie

$$X(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + u^2v,\ u^2 - v^2\right), \text{ con } u,v \in \mathbb{R}$$

Demuestre:

1. Los coeficientes de la Primera forma fundamental son:

$$E = G = (1+u^2+v^2)^2, \quad F = 0$$

Primero, encontramos X_u & X_v

$$X_u = (1 - u^2 + v^2,\ 2uv,\ 2u)$$

$$X_v = (2uv,\ 1 - v^2 + u^2,\ -2v)$$

Ahora sabiendo que:

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle$$

$$\begin{aligned}
\langle X_u, X_u \rangle &= (1-u^2+v^2)(1-u^2+v^2) + (2uv)(2uv) + (2u)(2u) \\
&= (1 - u^2 + v^2 - u^2 + u^4 - u^2v^2 + v^2 - u^2v^2 + v^4) + (4u^2v^2) + (4u^2) \\
&= (1 - 2u^2 + 2v^2 - 2u^2v^2 + u^4 + v^4) + (4u^2v^2) + (4u^2) \\
&= 1 + 2u^2 + u^4 + 2v^2 + 2u^2v^2 + v^4 \\
&= (1+u^2)^2 + 2(1+u^2)(v^2) + v^4 \\
&= (1+u^2+v^2)^2 \\
&= (1+u^2)^2 + 2(1+u^2)(v^2) + v^4 \\
&= 1 + 2u^2 + u^4 + 2v^2 + 2u^2v^2 + v^4 \\
&= 4u^2v^2 + (1 - 2v^2 + 2u^2 - 2v^2u^2 + v^4 + u^4) + 4v^2 \\
&= 4u^2v^2 + (1 - v^2 + u^2 - v^2 + v^4 - v^2u^2 + u^2 - v^2u^2 + u^4) + 4v^2 \\
\langle X_v, X_v \rangle &= (2uv)(2uv) + (1-v^2+u^2)(1-v^2+u^2) + (-2v)(-2v)
\end{aligned}$$

Por lo tanto $\langle X_u, X_u \rangle = \langle X_v, X_v \rangle$

$$E = G = (1+u^2+v^2)^2$$

$$\langle X_u, X_v\rangle = (1-u^2+v^2)(2uv) + (2uv)(1-v^2+u^2) + (2u)(-2v)$$

$$= 2uv - \cancel{2u^3v} + \cancel{2uv^3} + 2uv - \cancel{2uv^3} + \cancel{2u^3v} - 4uv$$

$$= 2uv + 2uv - 4uv$$

$$= 0$$

Por lo tanto $F=0$.

2. Los coeficientes de la segunda forma fundamental son:

$$e=2, \quad g=-2, \quad f=0$$

Encontramos entonces X_{uu} e X_{vv} & X_{uv}

$X_{uu} = (-2u, 2v, 2)$
$X_{vv} = (2u, -2v, -2)$
$X_{uv} = (2v, 2u, 0)$

y encontramos el vector normal

$$N = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}$$

$$X_u \wedge X_v = \begin{vmatrix} i & J & K \\ 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v \end{vmatrix} = \begin{array}{l} -4uv^2 - 2u - 2uv^2 + 2u^3 \, i \; - \\ -2v + 2vu^2 - 2v^3 - 2u^2v \, J + \\ 1 - 2u^2 + 2v^2 - 2u^2v^2 + u^4 + v^4 - 4u^2v^2 \, K \end{array}$$

$$= (-2u^3 - 2uv^2 - 2u)i + (2u^2v + 2v^3 + 2v)J + (-u^4 - 2u^2v^2 - v^4 + 1)K$$

$|X_u \wedge X_v|$: Sabemos que $|X_u \wedge X_v| = \sqrt{EG-F^2}$

$$= \sqrt{(1+u^2+v^2)} = (1+u^2+v^2)^2$$

$$N = \left(\frac{-2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{-(u^2+v^2-1)}{u^2+v^2+1} \right)$$

Ahora encontramos:

$$e = \langle N, X_{uu} \rangle$$
$$= \left(\frac{1}{u^2+v^2+1}\right)\left(4u^2+4v^2-2(u^2+v^2-1)\right)$$
$$= \frac{1}{u^2+v^2+1}\left(2u^2+2v^2+2\right) = 2$$

$$f = \langle N, X_{uv} \rangle$$
$$= \left(\frac{1}{u^2+v^2+1}\right)\left(-4uv+4uv+0\right)$$
$$= \left(\frac{1}{u^2+v^2+1}\right)(0)$$
$$= 0$$

$$g = \langle N, X_{vv} \rangle$$
$$= \left(\frac{1}{u^2+v^2+1}\right)\left(-4u^2-4v^2+2(u^2+v^2-1)\right)$$
$$= \left(\frac{1}{u^2+v^2+1}\right)\left(-2v^2-2u^2-2\right) = -2$$

3. Las curvaturas principales son:

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}$$

deduzca la curvatura gaussiana K y la curvatura promedio H. Diga si es una superficie mínima o no.

$$K = k_1 k_2 = \left(\frac{2}{(1+u^2+v^2)^2}\right)\left(\frac{-2}{(1+u^2+v^2)^2}\right) = \frac{-4}{(1+u^2+v^2)^2}$$

$$H = \frac{k_1+k_2}{2} = \frac{0}{(1+u^2+v^2)^2} = 0$$

Como $H=0$, es una superficie mínima.

4. Que las lineas de curvatura son las curvas coordenadas

Para esto Probamos que:

$$(fE-eF)(u')^2+(gE-eG)u'v'+(gF-fG)(v')^2=0$$

$$(0-0)(u')^2+(gE-eG)u'v'+(0-0)(v')^2=0$$

$$(gE-eG)u'v'=0$$

$$((1+u^2+v^2)^2(-2)-(1+u^2+v^2)^2(2))\,u'v'=0$$

$$(-4(1+u^2+v^2)^2)(0)=0 \quad \text{Recordemos que } \langle X_u,X_v\rangle=0$$

5. - Que las curvas asimptoticas son $u+v=$ const y $u-v=$ const

$u+v=$ const , $u-v=$ const

$$e(u')^2+2fu'v'+g(v')^2=0 \qquad t\in I$$

$$=(2)(u')^2-(2)(v')^2=0$$

$$=2(u'^2-v'^2)=0$$

$$=2(u'+v')(u'-v')=0$$

dos casos: C_1 $u'+v'=0$

C_2 $u'-v'=0$

y si integramos teniendo en cuenta que la derivada es 0, se sigue que la antiderivada es constante.

Por lo tanto, $u+v=$ const

$u-v=$ const