What The Funding?

A Theoretical Study of Stakeholder Impacts From Funding Costs and Funding Value Adjustments

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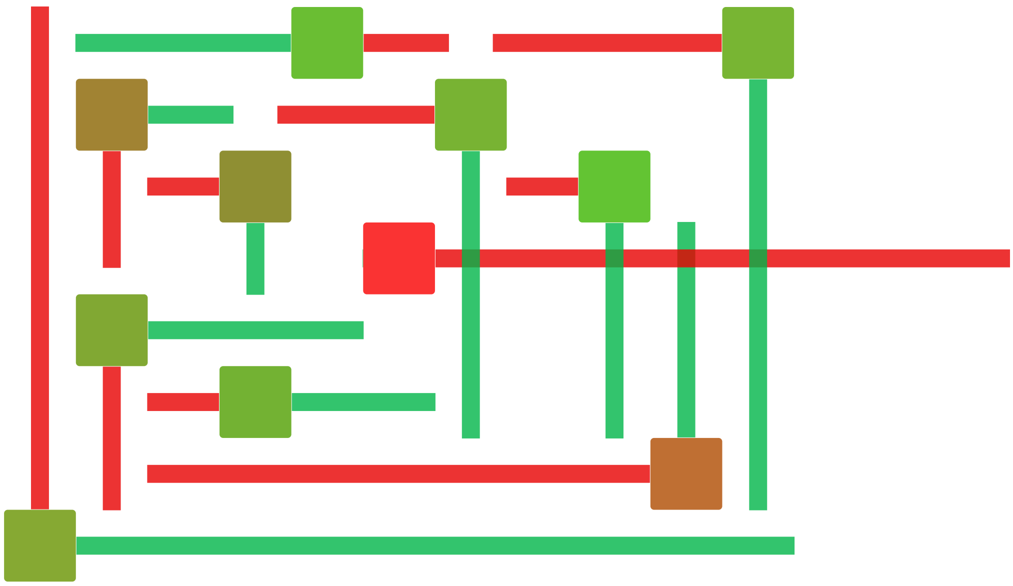

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Resumé

Efter finansielle institutioners vanskeligheder under finanskrisen er opfattelsen af deres fallitrisiko steget, og således er deres låneudgifter øget kraftigt. Som følge af dette har finansielle aktiver med finansieringsbehov oplevet et øget fokus, da udgående pengestrømme til deres vedligeholdelse potentielt kan koste dyrt i låneudgifter. For at tage højde for dette har finansielle institutioner taget *Funding Value Adjustments*, FVA, i anvendelse.

Sådanne justeringer har medført meget debat. På den ene side har teoretikere kaldt dem brud på traditionel prisfastsættelsesteori. Modsat har udøverne af justeringerne set sig nødsaget til at foretage dem, da de nødvendigvis må betale låneudgifterne og derfor tage hensyn til dem, når de bestemmer værdien på et aktiv.

Formålet med denne afhandling er at opsummere debatten samt at forstå, hvordan behovet for FVA opstår, og hvilke konsekvenser deres anvendelse har for en finansiel institution.

Afhandlingen konkluderer, at behovet for FVA opstår, når et finansielt aktiv har udgående pengestrømme, som kræver finansiering fra institutionen. Typisk vil aktivers markedsrisiko være afdækket, hvilket i sig selv kan reducere finansieringsbehovet, grundet de modsatrettede pengestrømme. Dog kan ufuldstændig risikoafdækning eller forskelle i kravene til sikkerhedsstillelse skabe uregelmæssigheder i disse pengestrømme, hvorfor institutionen må finansiere forskellen.

Gennem matematiske udledninger vises finansieringsudgifterne til et finansielt aktiv at blive betalt af institutionens aktionærer, der, grundet dette, samlet set ender med tab. FVA medregner aktionærernes tab i værdifastsættelsen af aktivet og er dermed en måde, hvorpå institutionen, der tager investeringsbeslutningen, kan strømline deres incitamenter med sine aktionærer. Resultatet for institutioner med sine aktionærers interesser forrest er investeringsbeslutninger, som bedre varetager deres mål og intentioner.

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1 Introduction

1.1 Background

Prior to the financial crisis, banks could obtain funding from the interbank market at very low lending rates, as they were considered almost free of credit risk. With the default of large financial institutions, it became clear that counterparties in the interbank market were not as safe as previously thought. Interbank lending would have to account for a higher default risk of the counterparty, which entailed an increase in the borrowing rates and therefore also an increase in the funding costs of banks.
As borrowing rates were low before the financial crisis, the funding costs were of negligible size and they were largely ignored in the derivatives pricing process. Soon after the crisis, it became apparent that funding costs had to be considered, much like institutions would consider their counterparty's default risk by credit value adjustments.

Valuation adjustments that accounts for the funding implications have been aptly named *Funding Value Adjustments*, FVA. These adjustments have been a source of great controversy in the financial field, as theoreticians strongly oppose them while practitioners deem them crucial for their operations. The most intense part of debate was sparked by Hull and White (2012) when they answered the question ”Is FVA a cost for derivatives desks?” with an unmistakable ”no”. They argue that FVAs go against traditional derivatives pricing frameworks and will create arbitrage opportunities if applied. The difference between a ”yes” and a ”no” could translate to a difference in the proximity of hundreds of millions of dollars for the largest financial institutions.
As practitioners turned out to very much disagree with Hull & White, the rejection of FVA as a valid adjustment lured out a vast number of counterarguments. In essence, practitioners argue that their derivatives valuation has to account for the actual costs of managing the derivative. They believe that funding is an important cost and therefore that FVAs are essential.

Regardless of how the debate about FVA ended, banks started universally accepting the inclusion of FVA in their derivatives pricing. To get a glimpse of the magnitude; when JP Morgan Chase first implemented an FVA framework for its over-the-counter derivatives it recorded a whopping 1.5 billion dollar loss for a one-time adjustment to its portfolio, according to JPMorgan Chase & Co. (2014). Since then, the debate has been less focused on the appropriateness of FVA and more focused on the way it should be accounted for. Still, there is really no universally accepted definition of FVA, and market practice differs significantly between dealers.

1.2 Research Questions

Evidently, FVA is a disputed topic but one that has been cemented as an important valuation adjustment by practitioners. However, as the debate about its validity has never been settled, it is a difficult topic to study with many sources pointing in different directions. Even if one decides to accept FVA, it is a complex matter, whose exact calculation, and perhaps even definition, is still under discussion.

This paper will attempt to collect the pieces of the FVA topic, and present them in a way that can properly explain what FVA is and what it is accounting for. This necessarily involves describing the debate about the relevance of the adjustment, by presenting arguments from both sides of the controversy and understanding the standpoint of each side.
To draw conclusions on the debate, it will also be essential to properly define the concept. However, to really grasp the topic, the paper will seek to define FVA in such simple terms that the essence is not lost in practicalities and implementation details.

These ideas, and the research they involve, can be expressed by the following research questions:

1. Why do funding costs appear in financial derivatives?
2. What has been the dispute concerning Funding Value Adjustments?
3. What are the implications of using- and not using Funding Value Adjustments in a simple structural model?

1.3 Delimitation

To answer the research questions, the paper will generally aim for simple models, that give acquaintance with- and insight into funding costs and FVAs. This familiarity with the topic could then be brought into more complex modelling in further research.

The first research question will be answered in a general, conceptual, and non-technical way. There are potentially many mechanisms that lead to funding costs, but it is not the intention to provide a comprehensive list of all possible collateralisation schemes, regulations, etc.

The answer to the second research question will merely be a description of what is considered the primary arguments of the debate about FVAs. The primary arguments will mainly be from Hull and White (2012) and Castagna (2012); some others will be included and cited when used.
The paper will not attempt to settle the debate or provide a definite solution, but the arguments will be challenged when presented.

For the third research question, the paper will use a corporate finance approach by considering a debt and equity financed institution making investment decisions. The concern will primarily be the impact on the institution's shareholders and creditors, from the financing and investment decisions it makes.

Prices and payoffs will be modelled using a discrete time approach with discrete random variables. An approach that would be more in line with reality would be to use stochastic processes; however, in favour of simplicity, the discrete framework is chosen since it conveys the basic ideas of FVAs better.

With the purpose of quantifying the funding implications from financial projects, some numerical computation will be presented. The values of the institutions's assets, and the payoff of the investments it obtains, do not aim to reflect a fully realistic image of an institutions's capital structure in the real world. Rather, they are chosen in a way that ease the interpretation of the funding implications.

With the above choice of procedure, it must be acknowledged that the paper will not be able to present a full-fledged valuation framework for use in financial institutions. The models do not capture the complexity of the real world, but their straightforwardness is an advantage in understanding this convoluted topic.

1.4 Outline

Part I, Credit Risk Theory, will introduce the necessary theory for the subsequent studies. In order to answer the first two research questions, Part II, The FVA Debate, will cover FVA without mathematical details, but by verbal discussion of the topics. The section will draw on a great amount of examples to be able to understand the matter, at least in a conceptual way. This includes describing how funding costs in derivatives come about and what some possible sources are. Following this, the debate on FVA will be summarised by presenting arguments from opposing sides.

Having an understanding of FVA on a conceptual level, the real world examples will be simplified with mathematical frameworks. These frameworks will provide a mean through which the workings of FVA can be studied. Part III presents a single-period model, which is extended by Part IV to a multi-period model. Examples and analyses using these mathematical frameworks will answer the third and last research question.

Part I

Credit Risk Theory

An integral component of the topics discussed in this paper is the tendency of firms to default. In any transaction that leaves a firm exposed to the counterparty, there is a risk that the counterparty does not honour its obligations to repay its debt. This risk, that the counterparty fails to make payments and defaults, is referred to as the counterparty credit risk or simply the credit risk. Many of the issues with which this paper is concerned exists because of credit risk and this part will therefore introduce its implications as well as possible mitigations.

The first subject will be concerned with credit risk mitigations, which intend to reduce the counterparty credit risk exposure.

2 Credit Risk Mitigation

Credit risk mitigation is actions taken by dealers with the purpose of reducing their exposure to counterparty credit risk. Reducing counterparty credit risk has important implications. From a risk management point of view, the amount of counterparty exposure a dealer is willing, or allowed, to take on is determined by the default risk of the counterparty as well as the risk appetite of the dealer's institution. When this limit is set, the dealer will be able to obtain some amount of trades, before the risk limit is met. However, if the dealer cleverly applies credit risk mitigations, she might be able to obtain an even larger amount of trades, increasing her capacity to do business. Even if the dealer does not increase her holdings after applying credit risk mitigations, she will have reduced her counterparty credit risk, which could lead to her capital requirements being reduced. Having motivated the use of credit risk mitigations, this section will discuss three methods of mitigating credit risk; namely, netting sets, collateralisation, and clearing through a central counterparty.

2.1 Netting

A netting set is a group of trades whose value can be *netted off*, when a counterparty defaults. When netting off a collection of trades, the trades' values are aggregated, offsetting payables with receivables, such that, only one single amount is owed if a counterparty defaults. Netting sets are important to credit exposure, as counterparty risk is settled at each set; therefore, netting is central to mitigate credit risk. To see this consider the following example.

> A derivatives dealer has two different netting sets, referred to as netting set X and netting set Y. The dealer sells a derivative with netting set X and, not wanting the

> exposure, buys the same derivative with netting set Y. Therefore, the market risk is eliminated, so the dealer is neutral from this point of view.
>
> However, the credit risk exposure metric for the dealer's entire portfolio will include the credit risk metric for both netting set X and Y considered separately. If instead the derivative is bought with netting set X, both the market risk and the credit risk will be offset.

Hence, the standalone credit risk of a trade might not give the full picture, as the trade will be part of a netting set with offsetting cash flows. Instead, the dealer can evaluate how much the credit risk measure would change if the trade was obtained; this is the incremental credit risk measure. By evaluating the incremental credit risk measures from the trade, potential netting benefits can be accounted for.

By strategically considering netting benefits the dealer can reduce her credit risk. Additionally, the dealer can further reduce her exposure by, in advance, taking custody of some of her receivables from the counterparty. Then, the receivables are protected from other creditors if the counterparty defaults.

2.2 Collateralisation and Securing Positions

Credit risk can be reduced significantly, and even eliminated completely, if the counterparty provides some form of security, which the dealer has priority to. Should the counterparty default, the dealer can use the security to cover any outstanding receivables from the counterparty. Likewise, the dealer can provide security to a counterparty and therefore receive better prices. The type of security used, and the circumstances under which they are available to the other entity, can vary significantly. This section will describe two common ways of providing security, namely *repo transactions* and *CSA agreements*.

2.2.1 Repurchase Agreements

A repo, short for repurchase agreement, is a trade in which a dealer sells bonds to a counterparty, with the promise of buying them back at a fixed price on a fixed date. The counterparty will be referred to as the lender. The dealer receives cash from the lender and the lender receives the bonds, which it uses as security for the money lent to the dealer. If the dealer defaults and fails to honour her obligation to buy back the bond, the lender can keep it and possibly liquidate it to recover the cash lent. Repos can be used as a form of short-term secured borrowing for the seller and an opportunity for the lender to invest funds for a customised period of time. Traders can use repos to apply leverage by buying long positions in bonds which they then post as the collateral in the repo.

Purchasing bonds through repos is beneficial to the dealer, as she can fund her purchase at the secured borrowing rate, which is going to be lower than the unsecured borrowing rate.

2.2.2 CSA Agreements

Derivatives transactions can be secured in a similar way if the counterparties agree upon and adhere to a Credit Support Annex, CSA, agreement. A CSA defines terms and conditions for collateral postings between counterparties, such that transactions between them are secured. Most CSA agreements require both entities in the transaction to post collateral if they have negative exposure to their counterparty. This is important for derivatives transactions that can have both positive and negative mark-to-market values, such that an entity can be both positively and negatively exposed to the counterparty during the lifetime of the derivative. When collateral is posted through a CSA, it provides a couple of benefits for the receiver. Most obvious, the receiver can use the collateral to cover losses incurred due to the default of the other party. However, another important feature of CSA agreements is the ability of dealers to *rehypothecate* collateral. To rehypothecate collateral refers to passing it on to other parties that require collateral from the dealer. By doing this, the dealer can reduce, or even eliminate, the need for providing her own funds for supporting other CSA agreements. This mechanism will prove to be very important in the context of Funding Value Adjustments.

Whether collateral should be posted, and how much should be posted, depends on a number of parameters defined in the CSA agreement as well as external values due to the general market conditions. The parameters deemed most important to this paper will be described in the following.

Threshold: the level of positive or negative exposure above which collateral will be posted. Allowing a higher threshold will also yield a higher exposure; however, only to the extent of the threshold size. Hence, the threshold is the amount of exposure the dealer can accept being uncollateralised. Typically, CSA agreements between banks will have zero thresholds, such that derivatives transactions are fully collateralised.

Minimum transfer amount: how much positive or negative exposure the counterparties can have before collateral is posted; hence, it is the smallest amount of collateral that can be exchanged between counterparties. It is often used to avoid the operational overhead from posting collateral for negligible amounts, but, like the exposure threshold, it comes at the cost of increased exposure.

This parameter is also a threshold and the difference between this and the actual threshold-parameter might seem subtle. The threshold refers to the exposure directly due to the mark-to-market value, however the minimum transfer amount considers the exposure after taking into account the mark-to-market, the current collateral posted, and the threshold.

Independent amount: an amount of collateral posted regardless of the derivative's mark-to-market value. Its purpose is to account for potential unexpected market moves that could lead to under-collateralisation. A party posting an independent amount decreases the exposure for the other party since it works as an initial buffer for credit risk mitigation. This quantity can also be referred to as *the initial margin*, however, strictly speaking, that term tends to be used when clearing through a central counterparty, which will be described in the following section. Conceptually, the two quantities are the same, and throughout this paper the two terms will be used interchangeably as the difference is subtle and irrelevant for the purposes of the analysis.

Collateral call frequency: how often a party can submit calls for collateral. Typically, CSA agreements between banks will have daily call frequency; other organizations will have a lower frequency simply due to the operational burden of posting collateral as often as daily. A lower call frequency means larger exposure to the counterparty, as the mark-to-market will have longer to diverge from the collateral posted at the last valuation date.

Besides the parameters mentioned, CSA agreements will have specifications about the type of collateral allowed, e.g. which currencies can be posted or if government bonds can be used as collateral, as well as a procedure to follow if one of the parties is downgraded to a specified credit rating. It should be emphasised that the parameters might not be identical for each party if one party has a different credit quality than the other. If the credit qualities differ a lot, the CSA agreement might even be one-directional, such that only one party posts collateral.

For the purpose of benchmarking collateralisation schemes, it will be helpful to define the theoretical concept of a *perfect* CSA agreement. A perfect CSA will have zero threshold and zero minimum transfer amount. In addition, it will have a, so to speak, infinitely high collateral call frequency, meaning that collateral calls are made continuously and instantly whenever there is the slightest change in the mark-to-market value. If the counterparty instantly meets collateral calls and have no cure period, a perfect CSA agreement would eliminate the exposure to the counterparty and therefore the credit risk. Of course, this is merely a theoretical construct, as all sorts of frictions introduce delays, which would break the assumptions. Nonetheless, the idea of a perfect CSA will be helpful to have as a reference, when studying Funding Value Adjustments.

As a final remark on this topic, it is important to note that this paper will have a rather idealised perception of collateralisation. It is assumed that each collateral payment perfectly offsets the mark-to-market value, which is not necessarily the case in the real world, where there might be disputes over the value of different instruments. In addition, it is assumed that calls for collateral will instantly be answered by either the required collateral or the counterparty's default. In reality, collateral will not arrive instantly, and a period of time will pass between the point where the counterparty does not post collateral to the point where it actually defaults. All of these frictions of the real world increase the credit risk to the counterparty and must be considered in practical applications. In the context of Funding Value Adjustments, they would simply be part of a large pool of contributors to funding frictions, and there is no value lost in assuming them non-existing; however, a great amount of simplicity is gained.

2.3 Clearing Through a Central Counterparty

A *Central Counterparty*, CCP, is an institution providing clearing for standardised OTC derivative contracts and therefore works as the intermediary in a transaction. A CCP is a counterparty to both the buyer and the seller in a transaction and guarantees the terms of the trade by providing compensation to one party when the other party defaults. It does so by collecting collateral from the buyer and seller to cover potential losses. From a collateralisation perspective, central clearing is very similar to trading through CSA agreements since collateral must also be posted to the CCP in order to support movements in the mark-to-market value. In the context of central clearing, this collateral is known as variation margin. Central clearing can also support the posting of an independent amount of collateral known as initial margin. Unlike the independent amount of a CSA, initial margin is possibly dynamic as it can depend on the riskiness of the parties, such that increases in risk measures might trigger calls for additional initial margin.
When using a CCP, rehypothecation of collateral is not possible, since the assets are in the possession of the clearing house.

For the purposes of this paper it is more meaningful to use CSA agreements where rehypothecation is allowed. Even though clearing through a CCP and trading under a CSA is very similar from a theoretical collateralisation perspective, the lack of rehypothecation with a CCP is very restrictive. Hence, throughout the paper, CSA agreements will be mentioned the most, but the implication of using central clearing instead should also be clear.

Many other forms of credit risk mitigation, than the ones mentioned here, exists. Some derivatives contain break clauses that allow a party to prematurely terminate the contract with a cash settlement corresponding to the derivative's value at that time. It is also possible for dealers to buy insurance against their counterparty's default, such that a notional amount is paid to the dealer if the counterparty defaults. The credit mitigations covered here are the ones deemed most relevant in studying funding costs.

The following section briefly introduces some theory on interest rates, which will later be necessary to understand how Funding Value Adjustments came about in the first place.

3 Overnight Index Swap Rates

An index swap is a contract that exchanges a fixed cash flow, determined at inception, for a floating cash flow tied to some price index. As implied by the name, an Overnight Index Swap, OIS, is a special case of an index swap that uses an overnight rate index to calculate the floating leg. The floating payments of an OIS are the daily compounded overnight rate over the floating coupon period; the fixed rate is a rate referred to as the OIS rate. The price index that specifies the reference rate is typically the overnight lending rate between banks published by the central bank, e.g. the Federal Funds Rate.

At maturity of the OIS, the parties exchange the difference between the interest accrued at the fixed rate and the interest accrued at the compounded floating index rate. There is no exchange of principal; therefore, an OIS generally carries very little credit risk.

The OIS rate can be combined with the LIBOR to form an indicative measure known as the LIBOR-OIS spread. The measure in itself is not too relevant for most of this paper, especially given the scandals that have led to the LIBOR rate being phased out, see Forbes (2021). However, its historical values will be very useful for explaining the origin of Funding Value Adjustments; and, of course, a prerequisite for this is understanding its composition.

3.1 The LIBOR-OIS Spread

The London Interbank Offered Rate, LIBOR, represents the average rate at which major London based banks charge each other for short-term unsecured borrowing. Multiple other reference rates for the interbank market exists, such as Euribor for Eurozone banks and TIBOR for Tokyo based banks. Some results in the following are not particularly dependent on the specific rate. When that is the case, the generic term xIBOR will be used to refer to these interest rate benchmarks.

A historically common measure of the banking system's health is the difference between the OIS rate and the LIBOR, known as the LIBOR-OIS spread. This spread can be used as an indication of how banks perceive the creditworthiness of other financial institutions. The LIBOR-OIS spread is a better indicator of credit risk in the interbank lending market than the LIBOR itself. The LIBOR is influenced both by the rates set by central banks and the general credit risk in the interbank lending market, while the OIS rate is only based on the rates set by central banks. Therefore, subtracting the OIS rate from LIBOR isolates the credit premium. A higher spread can be interpreted as a lower willingness to lend by major banks and therefore lower liquidity in the money market.

3.2 OIS as the Risk-Free Rate

Before the financial crisis in 2008, banks had typically been discounting derivatives using interest rate curves based on the various interbank borrowing rates. Typically, the interest rate curves were based on 3-Month xIBOR. Green (2015, Section 8.6) mentions two possible reasons for banks using these indices. First, the xIBOR discount curves were considered, and used, as proxies for the risk-free rate as, pre-crisis, the underlying banks in xIBOR generally had high credit ratings and were regarded as very safe counterparties. xIBOR rates were in fact very close to the yields of bonds issued by highly rated governments, and the xIBOR rates could be considered as being very close to risk-free while being associated with a highly liquid market. Second, the xIBOR curves represented the banks' own funding rate for derivatives, since xIBOR referenced unsecured borrowing rates.

In the beginning of the financial crisis, the xIBORs rose significantly which widened

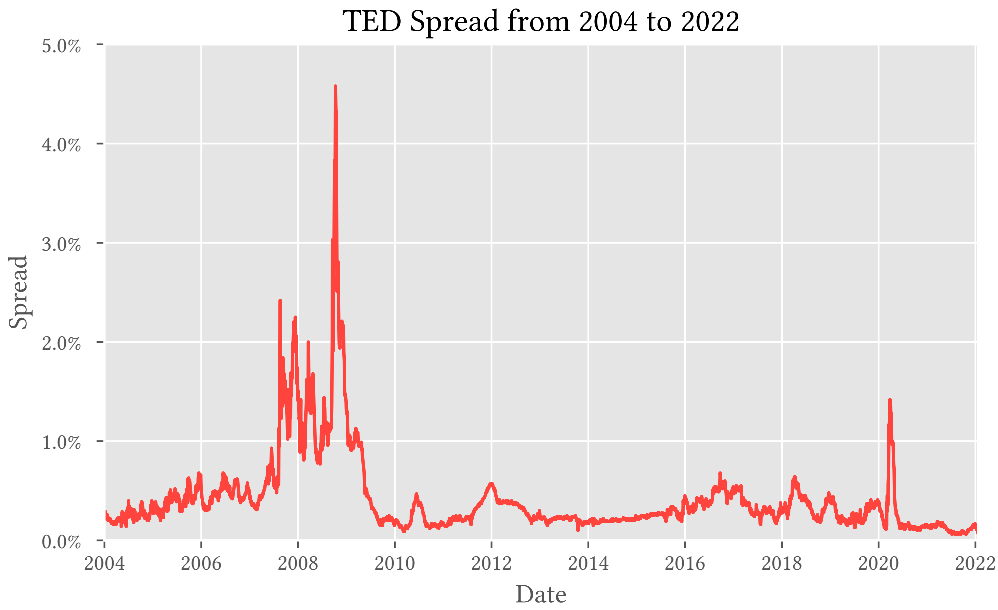

Figure 1: The spread between 3-Month LIBOR in USD and 3-Month T-Bill. The spread is usually relatively low, but blows up during recessions, such as the financial crisis of 2008 or the Covid-19 crisis in March 2020. (Source: Federal Reserve Bank of St. Louis)

spreads such as LIBOR-OIS or LIBOR against the Treasury bill, T-Bill. The latter spread is shown in figure 1. This clearly displays how the LIBOR against a very safe government bond blows up during recessions in the economy. While not directly comparing LIBOR and OIS, the TED spread conveys the conclusion very well nonetheless; the xIBOR contains other premiums than simply the risk-free rate. Figure 1 shows the TED spread instead of the LIBOR-OIS, simply because the data is readily available and since it reveals the same message. As the LIBOR's most commonly used tenor was 3-Month, actually calculating the LIBOR-OIS would require compounding the overnight rates to get the 3-Months OIS. These calculations have no benefit for this paper, hence the TED spread is used instead.

Figure 1 supports the conclusion from the previous section that spreads, such as LIBOR-OIS, isolate the credit premium. Therefore, their widening implies that the perceived credit risk of the banks in LIBOR increased.

The ballooning of these spreads had significant influence on the banks' pricing and valuation of derivatives as the previously mentioned arguments for xIBOR-discounting were disproven. The market was to find another curve to use for the risk-free rate, and, according to Green (2015), the choice have generally been the OIS rate.

In this paper the OIS rate will be used as the risk-free rate. As the precise values of the interest rates are not too important for this paper, neither is the choice of risk-free rate. Still, it will be easier conceptually to have decided on one, so the following paragraph presents arguments for using the OIS rate.

There are two sources of credit risk in an OIS. The first is the credit risk in the overnight borrowing of federal funds, which is very small. The second is the risk of one of the swap counterparties defaulting. Hull and White (2013) argue that this second risk is negligible in collateralised transactions, which standard swaps like OIS are due to legislation. Since credit risk is of inconsiderable size, it can be concluded that the OIS rate is a good proxy for the risk-free rate. Per these arguments, this paper will refer to the OIS rate as the risk-free rate and vice versa, depending on the context. As a consequence of this, the OIS rate will also be the rate earned on collateral postings.

When the unsecured borrowing rates, such as LIBOR, exploded during the financial crisis, so did the cost of financing. Suddenly, the unsecured funding costs went from negligible to significant, which prompted financial institutions to account for them, with the so called Funding Value Adjustments. This type of valuation adjustment, among others, is the topic of the following section.

4 X Value Adjustments

Following the 2008 financial crisis, financial institutions have increased their focus on adjustments to the risk-neutral price of derivatives; adjustments collectively known as X Value Adjustments, XVA.
This section will introduce a few of these, namely Credit-, Debit-, and Funding Value Adjustments. The former two will be of necessity to the exploration of the latter.

Deliberately free of mathematical details this section will be of introductory nature and lay ground for more in-depth discussions, when a framework allowing technical developments has been established.

4.1 Credit- and Debit Value Adjustments

A Credit Value Adjustment, CVA, is the price that a dealer is willing to pay to hedge the counterparty credit risk of a financial instrument. This concept is best introduced by an example.

> Suppose a credit risk-free dealer is exposed to some market risk which she wants to hedge by buying an OTC derivative. The derivative is offered by two different banks and the dealer can buy from either. One bank has a high credit rating, say AAA, and the other has a slightly lower credit rating, say BBB. The two banks offer the same derivative, except from the fact that their credit ratings differ. Thus, in one case the dealer's counterparty is rated AAA and in the other case the counterparty is rated BBB; so, if the banks offer the product at the same price, the dealer is clearly going to prefer doing business with the bank rated AAA.
>
> Since the BBB rated bank wants to be competitive and win the deal, it is going to submit a lower ask on the derivative than its competitor. If the BBB rated bank continues to lower the price, at some point the dealer will be indifferent between buying from

> the AAA rated bank and the BBB rated bank. At this price, the BBB rated bank has sufficiently compensated the dealer for taking on the additional credit risk; the discount offered by the BBB rated bank, is exactly the value that the dealer attributes to the additional credit risk. If the AAA rated bank is considered credit risk-free, this discount will correspond to the CVA.
>
> This motivates the definition of CVA as the difference between the price of the instrument including counterparty credit risk and the price of the instrument excluding counterparty credit risk.

From this example, it is clear that the credit risk of the BBB rated bank contributes to the CVA when the dealer calculates it. This principle can be extended further. In a transaction with two counterparties there might also be two sources of credit risk; so, suppose now that the dealer's institution is not credit risk-free but has a relatively low credit rating. Also, refer to the BBB rated bank as the counterparty of the dealer. The side from which CVA is considered is not unimportant; if the counterparty calculated the trade's CVA, it would account for the credit risk of the dealer. To not get lost in the terms, say that the dealer calculates CVA_D and that the counterparty calculates CVA_C.

While CVA_D is a negative contributor to the derivative's value perceived by the dealer, as it turns out, CVA_C is a benefit. If the dealer has outstanding liabilities with her counterparty and she defaults, she will not be able to fully pay her obligations. Surely, this possibility of receiving a lower payment is what the counterparty accounts for by calculating and applying CVA_C. However, from the dealer's point of view, this is a possibility of having to pay a lower amount, which is a benefit to her that should increase her valuation. This increase in the perceived value of the derivative, stemming from the dealer's own possibility of defaulting, is referred to as the Debit Value Adjustment, DVA.

Say that the dealer calculates DVA_D and the counterparty calculates DVA_C. Since both DVA_D and CVA_C are concerned with the default risk of the dealer, and vice versa for DVA_C and CVA_D, in theory, the following symmetric relations hold:

$$\begin{aligned}\text{CVA}_D &= \text{DVA}_C\\ \text{DVA}_D &= \text{CVA}_C\end{aligned}$$

These relations simply state the fact that the price of the counterparty's credit risk experienced by the dealer, CVA_D, is equal to the price of the counterparty's credit risk experienced by the counterparty itself, DVA_C, and vice versa. Including both the CVA and the DVA in a valuation is referred to as a bilateral adjustment, as opposed to a unilateral, where only the CVA is included. According to the symmetry above, both parties can agree on the necessary valuation adjustments in a bilateral setting, since each party is also calculating the counterparty's CVA and DVA. To this extent, the parties will reach symmetric valuations of the financial instrument. This result is however merely theoretical; in practice each institution will presumably apply different CVA models and be operating under different accounting rules, which will inevitably lead to the parties reaching different adjustments.

Eliminating CVA and DVA is possible by hedging the instrument's credit exposure, e.g. with a collateralisation scheme. Perfect collateralisation would ensure no credit exposure, however even strong CSA agreements will have collateral calls happening less frequently than what would be necessary to obtain perfect collateralisation. Still CSA agreements will reduce the credit exposure and hence CVA and DVA.

A dealer adjusting for her own credit risk with DVAs will surely not be offered risk-free rates when borrowing unsecured; rather, she will be paying a credit spread in excess of the risk-free rate. Hence, she will face costs when borrowing for financing, which motivates yet another valuation adjustment.

4.2 Funding Value Adjustments

A Funding Value Adjustment, FVA, is a quantity meant for accounting for the funding costs experienced by a dealer due to her acquisition of a financial instrument. Funding risks and costs arise wherever there is a cash flow in the instrument. Outgoing cash flows will need to be financed while incoming cash flows can be used to retire debt elsewhere in the dealer's institution, thus reducing the need for funding.

Again, funding costs, and the valuation adjustment that accounts for them, are best introduced by an example.

> Assume a dealer wants to purchase an asset with low risk and high liquidity, e.g. to keep the security for the purpose of always having access to liquid assets. Assume also, for simplicity, that the risk-free rate, OIS, is zero. Suppose the dealer purchases \$100 worth of face value in 1-Year T-Bills, at an upfront price of \$100. The upfront price is funded by the dealer issuing unsecured debt, for which she pays an unsecured credit spread of, say, 50 bp. One year later, the T-Bills mature and pay the face value of \$100. The dealer's debt also matures and the dealer pays $\$100 \cdot (1 + 50\,\text{bp}) = \$100.50$. Through this operation, the dealer has costed her shareholders a one year loss of \$0.50. At a glance, it seems like a rather poor decision for any institution to do this trade, but again it should be emphasised, that the dealer might have regulatory reasons to do so.
>
> Since the dealer does not want her shareholders to lose value on their claim, she will reduce her valuation of the T-Bills to account for the value lost. However, the price of the T-Bills is set by the market at \$100; if the dealer had no regulatory incentives, she would not purchase the T-Bills at the market price. By considering her financing costs when valuing the T-Bills, the dealer has made an FVA. By adjusting her valuation of the asset, the dealer is trying to align her shareholders' interests with her market operations.
>
> This valuation adjustment can also work in the opposite direction, for example if the dealer was to sell an instrument instead. The sale proceeds could be used by the dealer to lower the financing needs elsewhere in her organization, and therefore she would

value the instrument higher.

From the example, it is clear that FVA conceptually can be thought of as consisting of two elements, according to the following decomposition:

$$\text{FVA} = \text{FVA}_{\text{cost}} + \text{FVA}_{\text{benefit}}$$

FVA_{cost}, also known as Funding Cost Adjustment, FCA, is a negative contribution of funding costs to the value of the financial instrument. It captures the effect of outgoing cash flows, which will require funding and generate costs. On the other hand, $\text{FVA}_{\text{benefit}}$, also known as Funding Benefit Adjustment, FBA, captures the positive impact on the instrument value, from the incoming cash flows.

FVA could be decomposed in multiple other ways, than shown here. To mention one, KPMG (2013) suggests introducing another term in the decomposition, namely $\text{FVA}_{\text{buffer}}$. This term refers to an adjustment for the funding costs due to the maintenance of liquidity buffers. These buffers are in place for dealers to have access to funding, in the event that funding markets fail to function when the dealer faces unexpected funding requirements.

This also shows that there might be many sources of funding costs, attributed to different mechanisms. The simple decomposition of FVA into a positive and a negative contribution proves useful for this paper, and further investigation of FVAs will be focused only on the two components, FVA_{cost} and $\text{FVA}_{\text{benefit}}$. The exact sources of funding costs that this paper will be focused on will be introduced later.

Margin Requirements

Suppose an OTC derivatives transaction between a dealer and a counterparty requires the dealer to post collateral. The dealer may necessarily be obliged to fund the margin postings on top of the price of the derivative. To understand this setup, consider a single interest rate swap between a dealer and a counterparty that have in place a CSA agreement. If the value of the dealers's leg is lower than the counterparty's, she is required to post collateral, which she must necessarily fund by some method of financing. This type of funding is a category within the FVA framework, hence the use of it should be treated as a funding cost to the dealer. The funding costs occur due to the posting of margin as a consequence of the swap contract, and not directly from the price of an OTC derivatives transaction, as described in the earlier example. Later in this paper the issue is discussed further, where in section 10.5 a numerical example is analysed.

Assume that the dealer chooses to fund the collateral by obtaining debt that is borrowed at her unsecured borrowing rate. When posting the funds as collateral, the dealer earns the OIS rate. The asymmetry between the cost of borrowing and the rate earned on collateral postings adds additional costs to the swap transaction; these are funding costs and an FVA is a price alteration made to account for these.

On the contrary, if the dealer must receive collateral that can be rehypothecated, she pays the OIS rate and can receive a higher rate by reducing funding elsewhere in her organization. This asymmetry can be referred to as funding benefit rather than a cost.

Part II

The FVA Debate

Having established the relevant financial concepts, the paper can venture further into the topic of FVAs. This part will describe why these adjustments came about in the first place as well as covering some sources of funding costs; the quantities that an FVA tries to account for. Finally, with an even better understanding of FVAs, the controversy that has been surrounding them is summarised and analysed.

5 The Origin of FVAs

Before the financial crisis of 2008, the differences between yields on highly secure government bonds, such as T-Bills, and the interbank lending rates, e.g. LIBOR, were very low. As banks were considered almost risk-free, they could naturally fund themselves at the rates that were considered risk-free. Banks were seemingly similar to each other in this respect, and the notion of a financial product's fair value was very much driving the pricing of derivatives issued by banks, since mispricing would lead to arbitrage opportunities. If one bank offered a financial contract at a lower price than a financial contract with identical terms offered by another bank, arbitrageurs would exploit this apparent mispricing. The fact that banks contained more risk than perceived became clear in 2008 with the failure of some large banks, especially the bankruptcy of the investment bank Lehmann Brothers. These defaults made it clear that banks were in fact not as safe as believed; firms financing banks needed to do so only when receiving an even larger credit spread, i.e. a lending rate exceeding the interest rate that was considered risk-free.

As a consequence, banks are now forced to pay additional interest on their funding, which depends on their perceived default intensity. Due to the varying investment- and risk profiles of banks, the perceived default intensity differs between them, and each bank might face completely different funding schemes and costs than its competitors. Differing funding costs have massively challenged the concept of a derivative's fair value, since the price that different banks are willing to bid or offer might vary significantly. This has led many institutions to systematically apply FVAs to their prices in order to accommodate their funding costs. This practice is condemned by some theoreticians, which has ensued heavy debate.

In the following section the circumstances under which funding costs arise will be explored. This precedes diving into the implications of differing funding costs, or the lack thereof, and the discussion of whether applying FVAs is even appropriate.

6 Sources of Funding Costs

In the context of an OTC derivatives dealer, Ruiz (2013) mentions some of the most likely sources of funding costs as either asymmetry in collateral agreements or the payment liabilities due to the contract itself. Both of these sources will be discussed in the following sections. As will be concluded, funding costs contribute to the cost of trading under imperfect CSA agreements and trading derivatives whose market risk cannot be perfectly replicated.

6.1 Funding Costs From Asymmetrical Collateral Agreements

Imbalance between the collateralisation schemes of an OTC derivative and its market risk hedge is an origin of funding costs and the concern of this section. The conclusion will be that if a perfect market risk hedge exists, funding costs can be viewed as a component of the cost of trading derivatives that are not subject to a perfect CSA agreement.

After entering financial contracts with counterparties, a derivatives dealer will often enter into the opposite trades with the sole purpose of hedging market risk. Hedges will often be obtained on an exchange requiring full collateralisation, since the dealer will be trading with other financial institutions. Therefore, when the collateralisation scheme with the counterparty is imperfect, the collateral cash flows to and from the counterparty will likely not match the collateral cash flows from the hedge. This source of funding costs is best introduced in its most extreme case, where the dealer trades unsecured derivatives.

> Consider the situation depicted in figure 2. The derivatives dealer is selling an unsecured OTC derivative to a counterparty, and the counterparty pays the upfront price of an amount denoted by $$$. The particular type of derivative is not too important, but picture a derivative that can leave the dealer both negatively and positively exposed to the counterparty, e.g. an interest rate swap.
> In order to avoid the market risk of the investment, the dealer hedges the trade by performing the opposite, perhaps synthetic, trade on an exchange, In this example, it is assumed that the upfront price of the derivative matches exactly the upfront price of the market risk hedge.
> The exchange requires full collateralisation; when the dealer has positive exposure to the counterparty, she has negative exposure to the exchange and must post collateral. When she has positive exposure to the counterparty she receives collateral. The collateral posted at the exchange is secured by the actual investment demanding the collateral; therefore, any collateral posted earns the OIS rate at the receiver. The exchange could possibly charge a spread and pay less than the OIS rate, but for simplicity assume that this is not the case.
> Assume also that the dealer's institution has no excess cash on its balance sheet, such that any collateral that needs to be posted to the exchange must be borrowed from the dealer's funding institution, in this case her funding desk. This funding is unsecured

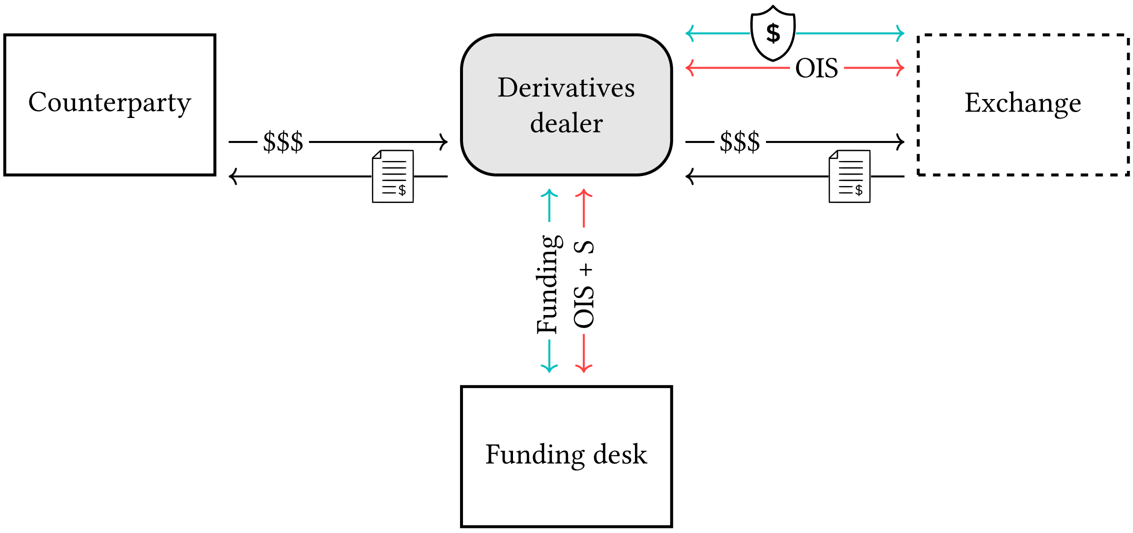

Figure 2: Illustration of funding costs from an unsecured OTC derivative.

and the funding desk charges a spread, here denoted by S, more specifically the dealer pays the rate OIS + S.

The excess interest rate charged by the funding desk is exactly the source of the funding cost. If the dealer has positive exposure to the counterparty, collateral must be posted to the exchange. The dealer then earns OIS from the posted collateral but pays OIS + S for funding; the difference, S, drives the funding cost. Alternatively, the exchange posts collateral to the dealer, for which she pays OIS. If rehypothecation is allowed, the dealer can retire debt and save the funding spread. In this case, S drives the funding benefit.

As mentioned, trading unsecured derivatives is the most extreme case, since any collateral call will lead to a funding surplus or deficit and therefore, respectively, a funding benefit or cost. The funding cost can be reduced if the dealer has in place a CSA agreement allowing rehypothecation with the counterparty. This would allow collateral to flow from the exchange to the counterparty, or the opposite direction; therefore, collateral postings from one party could, partly or fully, cover collateral needs from the other party.

Consider the example illustrated in figure 3. Again, the dealer sells a derivative to her counterparty; however, in this case, the two parties are trading under a CSA agreement, such that the trade is secured by collateralisation. The posting of collateral by the counterparty reduces the credit risk of the dealer, but she still faces the market risk from the derivative; hence, she creates a market risk hedge at the exchange.

Assume a perfect hedge, such that any market risk exposure to the counterparty is exactly offset by the opposite exposure to the exchange. Whenever the dealer calls for collateral from the counterparty, she can expect a call for collateral from the exchange. Since rehypothecation is allowed, the collateral posted by the counterparty

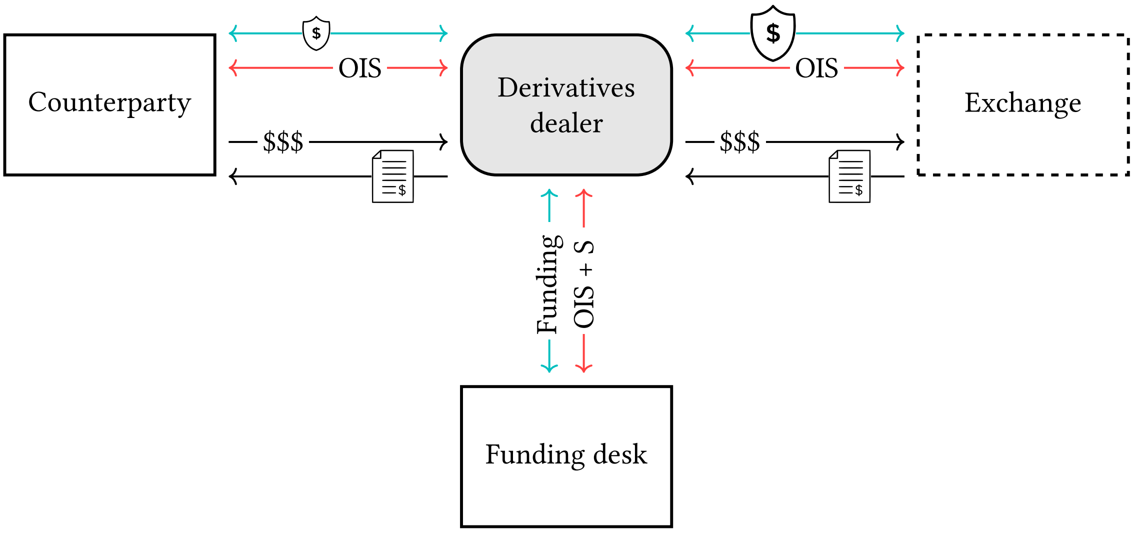

Figure 3: Illustration of funding costs from a secured OTC derivative.

can be passed on to the exchange. To a certain extent, this reduces the dealer's need for borrowing unsecured funds from the funding desk and ultimately reduces her funding costs.
However, funding costs are possibly still present as the collateral agreement with the counterparty is likely weaker than the one with the exchange. This is depicted in figure 3 by the shield on the counterparty side being smaller than the shield on the exchange side. The asymmetry has the implications that collateral posted by the counterparty might only partly cover collateral calls from the exchange and collateral posted by the exchange will more than cover collateral calls from the counterparty.

The former requires the dealer to finance the difference and generates a funding cost. The latter provides a funding benefit.

The differences between the CSA agreements, leading to a situation as described above, could vary. To mention a few, the CSA agreement with the counterparty might have higher exposure thresholds, higher minimum transfer amounts, or lower frequency of margin calls. All of these differences lead the collateral cash flows to and from the exchange to be at least as high as the cash flows to and from the counterparty. In case the two collateral agreements are identical, the funding costs will be eliminated, since the collateral received from the counterparty will completely suffice as collateral posted to the exchange.

To further grasp how unaligned collateralisation schemes can lead to cash shortfalls or surpluses, consider the example drawn by figure 4. This example displays the possible implications of hedging a secured derivative with a trade that calls for collateral more frequently.

The same dealer as previously trades an interest rate swap with a counterparty with whom a CSA is agreed upon. The value of the dealer's part of the contract is drawn as a function of time by the blue line with dots. To hedge the market risk, the dealer enters into the opposite trade on an exchange. The market risk is perfectly hedged, and the value of the two contracts are therefore identical but with the sign flipped. The value of the dealer's part of the hedge is indicated by the orange line. For simplicity, both swaps can take only one of five values; +\$, +\$\$, 0, −\$, and −\$\$.

Both swaps are secured by CSA agreements allowing rehypothecation and with no exposure thresholds or minimum transfer amounts. For the trade with the exchange, collateral calls can be submitted every time period; however, for the trade with the counterparty, it is assumed that collateral calls can only be submitted every second time period. These agreements lead to the collateral postings reported in the second row of the figure. Negative amounts are posted by the dealer, while positive amounts are posted to the dealer from the corresponding entity. In time periods marked with ✕ there can be no collateral calls between the counterparty and the dealer. Each period, the exchange considers the current mark-to-market and its collateral balance with the dealer. Subsequently, it submits a collateral call, posts collateral, or does nothing if the mark-to-market is unchanged since the last collateral posting. The same goes for the counterparty, except, only every second period.

In periods where collateral can be posted for both trades, the collateral posted by one trade will completely cover the collateral call from the other; hence, the dealer will neither have an excess nor a surplus of funds. However, in period 1, when the exchange calls for collateral, the counterparty is not obliged to post collateral to the dealer. If the dealer does not have excess cash, she will have to obtain the financing from her funding desk and therefore pay the unsecured borrowing rate of OIS + S. This debt can only be repaid in period 2 when the counterparty posts collateral. The asymmetry of the collateral cash flows in period 1 is thus a contributor to the funding costs of the derivative.

In period 5 the situation is the opposite. The exchange posts collateral to the dealer, but, even though the counterparty's trade is marked-to-market lower than in period 4, the counterparty cannot submit a collateral call. The dealer is therefore in excess of funding, which she can pass on to her funding desk. The funding desk can rehypothecate the collateral by passing it onto traders with a shortfall of collateral. The collateral shortfalls would otherwise have had to be covered by borrowing funds from the funding institution and the dealer is thereby saving the unsecured borrowing rate of OIS + S. The asymmetry in period 5 is therefore a funding benefit to the dealer since the excess collateral has reduced her institution's financing needs elsewhere. While this example is concerned with difference in frequency of collateral calls, examples with difference in exposure thresholds or higher minimum transfer amounts would convey the same message. When there is asymmetry in the collateral cash flows the dealer must provide or receive funding, which generates funding costs or benefits.

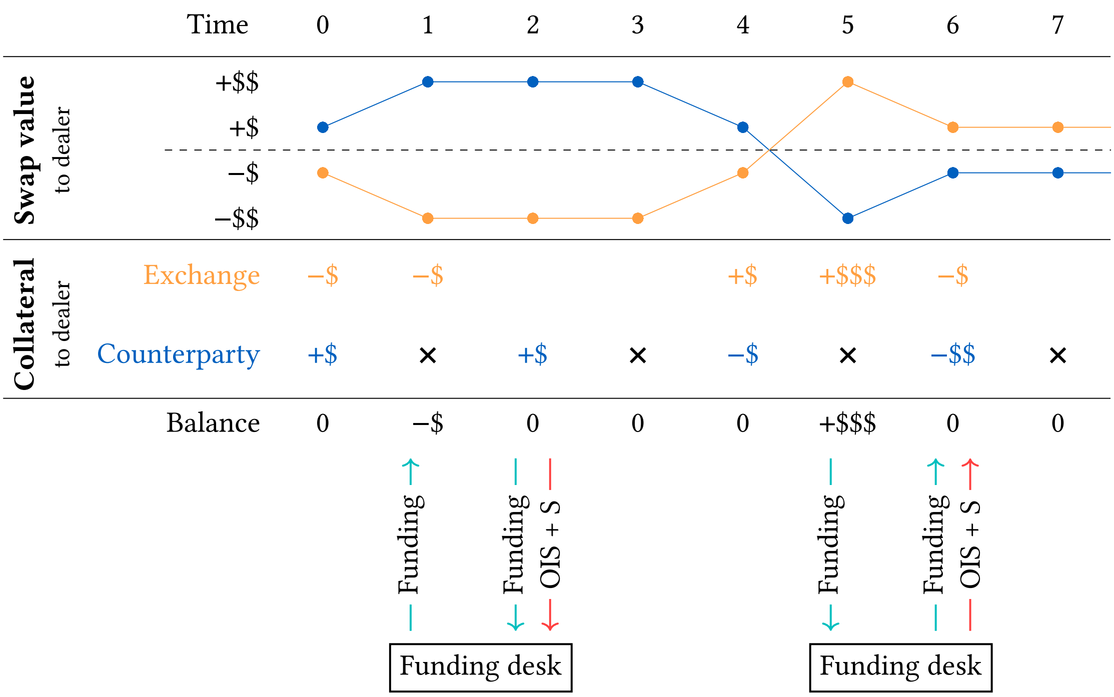

Figure 4: Illustration of funding costs from an interest rate swap with asymmetrical collateral agreements. The exchange is obliged to submit collateral calls or post collateral every period, while the counterparty must only do so every second period.

These examples should display how a derivatives dealer might face funding costs, when the lack of perfect symmetry in collateral needs calls for unsecured funding.

These type of funding costs are very likely to be incurred by dealers that hedge trades with corporate counterparties by trading with other financial institutions. Suppose a dealer has traded with a corporate client under a CSA agreement without an independent amount. If the dealer trades hedges with another financial institution and clears through a CCP, she will likely be required to post initial margin. If the variation margin from the CSA agreement perfectly offsets the variation margin from the CCP, there will be no offsetting cash flow for the initial margin, and the dealer will incur funding costs from obtaining financing.

Large financial institutions will be even more likely to face calls for initial margin when financial regulators put in measures to limit risk taking. This is, for example, the result of the reform BCBS and IOSCO (2020) that imposes initial margin requirements on non-centrally cleared derivatives, for institutions that trade a large amount of uncleared products.

It might be argued that the funding costs could be eliminated by trading derivatives, whose replication can be operated by repos. Seemingly, buying the hedging strategy at repo should eliminate the need for unsecured financing and thus the funding costs. However, even when repo market exists, hedging might require unsecured funding, which can be shown with an example by Castagna (2012).

> Consider a dealer buying a European call option, with the intention to hedge the market risk. To create the hedging strategy, the dealer must short an amount of the underlying asset corresponding to the option Delta of the call option. If the underlying asset can be sold at repo the dealer will receive the repo rate. However, to pay the premium for the call option the dealer must borrow unsecured funds from her funding institution, paying again the funding spread.

This section has shown how asymmetrical collateralisation schemes can lead to funding costs. When trades and their hedges have identical CSA agreements, the collateral cash flows can offset each other and there is no need for funding collateral. However, when the hedge has stronger collateralisation than the trade itself, the cash flows will not always align, which creates cash surpluses and deficits; and therefore funding costs or benefits.

6.2 Funding Costs From Derivative Cash Flows

Collateral postings is just one possible outgoing cash flow that a dealer might need to fund. Another is the contractually agreed on cash flows in a derivative, which must also be funded at the dealer's borrowing rate. However, as was the case with collateralisation, derivative cash flows can be used for offsetting other cash flows in the dealer's portfolio. This implies that with the existence of a perfect replication, the cash flows from a derivative can be completely offset by the hedging strategy, such that no additional funding is required and no funding costs are generated.

A perfect replication is a useful benchmark; however, in reality, replications of OTC derivatives are most often imperfect. Therefore, there is likely not a one-to-one correspondence in cash flows between the derivative and the hedge. Of course, the dealer might not even try to hedge a derivative directly, but rather let a netting set of cash flows offset each other. The principle is the same whether the derivative cash flows are offset by an imperfect hedge or another different derivative; both are again sources of funding cost, which can be illustrated by the following examples.

> Consider a dealer buying a simple option, without the intention to hedge its market risk. Buying the option requires paying an upfront price to the counterparty, which would seemingly require financing; however, this is only a source of funding costs if the dealer and the counterparty do not have a CSA agreement in place. If both the dealer and the counterparty are financial institutions, they will most likely maintain a CSA agreement such that any upfront price paid will immediately be met by a collateral posting in the opposite direction. Hence, the funding need for the upfront price will be satisfied and the funding costs eliminated. If no CSA exists and no collateral is posted, the upfront will still create funding costs, while selling the option instead will decrease the funding needs and create a funding benefit. This is not to say that a CSA agreement will eliminate funding costs completely. Hedging cash flows might also generate funding needs or surpluses, e.g. if a dealer must apply leverage to properly

> hedge a position and subsequently pays the higher rates of leverage due to financing spreads.

Another source of funding costs materialises in the next example when the dealer applies an imperfect hedge.

> The dealer has entered into a 10 year OTC swap with semi-annual payments, but for the replication the dealer is only able to obtain a 10 year swap with quarterly payments. As a result, the dealer will have a cash flow from the hedge every three months that does not correspond to the cash flow from the OTC derivative.

Both of these scenarios are examples of funding costs essentially stemming from the lack of perfect hedging. In conclusion, funding costs also occur when derivatives are not perfectly replicated since the outgoing cash flows will have to be financed.

6.3 Funding Costs From Other Sources

It is worth mentioning that there are multiple other mechanisms that can lead to funding costs. One, that has already been mentioned, is the maintenance of liquidity buffers, which requires capital to be locked up in liquid assets to protect against liquidity drying out. Most others fall into the category of over-collateralisation, which refers to providing collateral worth more than enough to cover losses in case of default. For example, CCPs typically require more types of margin than mentioned here, all with the purpose of protecting against the default of their clearing members. CSA agreements might also contain provisions for volatility buffers that work as over-collateralisation for the event that counterparties experience a drop in credit ratings. All of these are treated in detail by Green (2015, Chapter 10).

It is beyond the scope of this paper to cover all possible frictions that might provide funding costs, as that is not necessary to establish why derivatives in general have funding costs. The two previous sections have treated, in very broad terms, the two primary sources of funding costs, and other mechanisms can more or less fit into these.

Having described the general frictions in derivatives trading that leads to funding costs, the discussion can move on to the debate about whether accounting for them is appropriate for derivatives pricing.

7 Summary of the FVA Debate

Since the financial crisis, when banks started applying them, theoreticians and practitioners have been debating the use of FVAs when determining the prices of financial assets. The focal point of the discussion is specifically whether it is appropriate to apply FVAs in the valuation of financial assets. This section will provide a summary of the debate, by presenting arguments from what could be considered the three main parties;

namely dealers, accountants, and theoreticians.

7.1 FVA According to Dealers

The dealers in the derivatives desk selling or buying derivatives will incur the average funding rate of the bank, as charged by their funding desk. For derivatives that require financing the funding desk will charge the dealers. If the dealers do not take into account the funding cost when pricing the derivatives, a loss will be shown for trades that require funding. To see this, consider the following example.

> Assume a dealer buys a simple option at some upfront price. The option has a theoretical price or value that might have been calculated by assuming that financing could be obtained at risk-free rates. Should the dealer buy the option at this theoretical price, she will be making a negative net present value investment, since the trade will generate funding costs. If she is trading outside of a CSA agreement, there will be no collateral posted to offset the upfront cost and the payment will be subject to the funding rate, which is *above* the risk-free rate.
>
> Alternatively, she will be trading under a CSA agreement, and the collateral will offset the upfront cost. Still, she might want to hedge the position and, as the option and the hedge will not constitute a self-financing portfolio, unlike the assumptions in traditional pricing frameworks, the hedge might require leverage, which will cost the funding rate.

In conclusion, the funding desk will charge the actual funding rate; not the risk-free rate. Rather than accepting the theoretical price, the dealer should assess the funding cost of the trade and buy the option at the theoretical price adjusted by the funding cost, in order to break even on the value of the trade. In other words, the dealer should make an FVA, since the funding rate is a very real cost, which the dealer will be subject to.

”The dealers” in this argument might as well be other stakeholders that make decisions about new investments. In a less complex setup ”the funding desk” could simply be the creditors and ”the dealers” the company itself. The company might be making decisions in a way that optimises shareholder value, and therefore aligns its interests with them. Shareholders experience the same issue as dealers of being charged funding costs on investments and will therefore be just as eager to evaluate investments in light of their funding costs. This simple setup of creditors versus shareholders, will later provide a medium through which the impact of funding costs can be explored.

7.2 FVA According to Accountants

As financial statement auditors, accountants are concerned with providing objective valuations of derivatives, such that the valuation of the firm itself is fair and accurate. Therefore, accountants seek to value derivatives at their exit price, i.e. the price that clears the market, which is dependent on how other market participants price the transaction. To determine the value of derivatives, accountants use the notion of *fair value*. IFRS 13 (2013) defines the fair value of an asset as: *"the price that would be received from the sale of an asset or paid to transfer a liability in an orderly transaction between market participants at the measurement date"*. In addition, the fair value is clearly described as being market based and not entity based.

This description of fair value is used by Hull and White (2014) as an argument to rule out using the funding costs when evaluating prices. They claim that, since the funding rate, and therefore the funding costs, is entity specific, the fair value should be independent of these. While this might be true in isolation, according to KPMG (2013), the current market practice is for dealers, especially OTC derivatives dealers, to include FVA in their valuations. When evaluating the fair value, the dealers should consider the pricing practices that would be used by market participants if the derivative was being sold; if the market practice is to apply adjustments for funding costs, FVAs may very well be included in the fair value. The dealer should however consider the funding costs that market participants would consider, which is not necessarily equal to the dealer's own funding costs.

The original argument of Hull and White (2014) was that banks using their own funding costs to evaluate fair value, would lead to different banks pricing the same derivative differently, since the funding costs were entity specific. This is true, however, as mentioned in the previous paragraph, to evaluate fair value the bank should not be using its own funding costs, but rather the funding costs of another hypothetical market participant, to which the bank can sell its derivative. With that said, it is generally very difficult to determine the generic level of bank funding spread representing the funding rate for the hypothetical counterparty in a hypothetical transaction. KPMG (2013, Proposition 4) states that, for this reason, some banks have in fact been applying their own funding spreads in order to determine FVA under IFRS. Banks do this not for the sake of using their own funding spreads, but for it to act as a proxy for the market funding cost. If the bank can be assumed to have comparable funding costs to other market participants, KPMG believes that this practice is supportable.

In conclusion, accountants should accept adjusting fair value for funding costs, as long as it is market practice to make these adjustment, and as long as banks use the funding costs of other market participants, or proxies of those, as reference.

7.3 FVA According to Theoreticians

Again in the context of derivatives valuations, the arguments of theoreticians, like Hull and White themselves, are rooted in the assumptions of the valuation frameworks commonly used in derivatives pricing. The price of a derivative can be obtained by replicating the cash flows of the derivative with a self-financing trading strategy. In the absence of arbitrage, the value of the replicating portfolio and the derivative in question must be equal. The risk-neutral valuation principles require discounting at the risk-free rate, which is why theoreticians oppose using a different discount rate, claiming that there is no theoretical basis for making an FVA. The assumptions in this theoretical setup is however that the derivative exists in a very specific economic environment. In this economy, the replication of the derivative can be funded by borrowing and lending endlessly at the risk-free rate. As described in previous sections, this seems at odds with how the actual real markets behave and how much financing costs. Different entities will obtain funding at different costs above the risk-free rate due to size, default risk and multiple other market frictions, and interest rate spreads between collateralised and unsecured funding is significant.

Extending their arguments to other assets than derivatives, theoreticians argue that finance theory requires the discount rate for a project to be determined by the risk of the project. Using an FVA in derivatives pricing corresponds to replacing the risk-free rate by the higher funding cost when discounting cash flows. Hence, the value of the project will seemingly depend on the riskiness of the firm that undertakes the project. Hull and White (2012) use the following example to shed some light on this argument.

> Consider a dealer with an average excess funding rate of 200 bp. The dealer assesses the opportunity to buy a nearly risk-free bond with a promised excess yield of 30 bp, such that the theoretical price of the bond is the expected cash flow discounted by the risk-free rate plus 30 bp. However, for unspecified reasons, the bond is trading at a discount compared to the theoretical price. At the price offered it promises an excess yield of 80 bp.
>
> To put some numbers to this example, consider a dealer in a firm, financed by equity and debt, with a riskiness such that the firm's weighted average cost of capital is 200 bp. Assume a risk-free rate of 0 for simplicity. The dealer considers a new project with a present value of 100, and a discount rate of 30 bp. However, the project is actually trading at a price of 99.50, and therefore has an excess yield of approximately 80 bp, since $100 \cdot (1 + 30\,\text{bp})/99.50 - 1 \approx 80\,\text{bp}$. This situation is depicted in figure 5.
>
> Based on the information provided, Hull and White (2012) argue that the bond should be acquired, since it trades at a discount and therefore has a positive net present value. The dealer only needs to get financing either from equity- or debt issuance, and the provider of capital can receive a share of the net present value of the project. Since the riskiness of the project only corresponds to a yield of 30 bp, the dealer should be able to obtain financing at a lower rate than the actual return of 80 bp. Furthermore, as the dealer's average excess funding rate is higher than the excess yield of the bond, the average riskiness of the dealer's existing projects is higher than the riskiness of the new bond. Acquiring the bond would therefore reduce the dealer's riskiness, which should reduce the credit spread on new debt and therefore also reduce her subsequent funding rate.

From a theoretical perspective the conclusion of Hull and White (2012) does seem plausible. Obtaining the bond is a positive net present value investment, so surely the dealer should be able to find some distribution of the value to the relevant stakeholders,

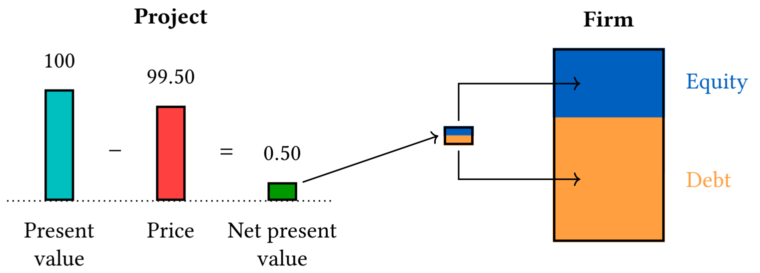

Figure 5: Depiction of Hull & White's example where a dealer in a debt and equity financed firm considers investing in a project with a positive net present value.

such that obtaining the bond is a positive net present value investment for all entities. The bond counterparty is essentially providing a donation, and some providers of funding should be willing to finance the bond for a share of the donation. However, the problem with this conclusion becomes apparent, when considering how the example would play out in reality. The bond might very well have such a low risk that an excess yield of 30 bp is justified; but, according to Castagna (2012), the financing cost of a firm will usually only gradually update to reflect change in riskiness. In reality the dealer will also be quite restrained in the way it can distribute the value obtained from the bond. The dealer can experience a free rider problem, where the profits from the bond might not be earned by the ones financing it, since the bond's payoffs will be a part of the asset pool of the firm, which different stakeholders have different claims on.

Further investigation into the mechanisms of this apparent firm friction, will be postponed to later sections where a framework has been developed, and a more concrete discussion can be ensured. For the time being, the argument will be ended with the result that, while the theoretical conclusion is simply that a project trading at a discount should be obtained, the practical conclusion is less clear-cut. Even when the project counterparty is making a donation through the discount, the firm might not be able to properly distribute the donation to entities, in a way that makes the project attractive.

Hull and White (2012) extend the previous argument by referencing an apparent inconsistency in financial institutions' application of FVAs. It is common for dealers to invest in government bonds, such as T-Bills, and generally other low-yielding assets. However, credit risky institutions, such as derivatives dealers, are clearly going to have a higher average funding cost than the yield of a T-Bill backed by the U.S. Government. Therefore, it would seem that the dealers are not applying FVAs to these assets.
The implication of the argument is that there is an inconsistency in dealer practice, as the low-yielding instruments would be unprofitable if an FVA was made. Hence, dealers must be recognising that it is the risk of the project that decides if a trade should be made; not the dealers' funding rate.

It is true that a simple comparison of a dealer's *average* funding rate to the yield of a T-Bill would result in the bond being deemed a losing trade. The problem with this argument is that dealers are not going to finance T-Bills using unsecured borrowing and paying their unsecured borrowing rate. For Treasury instruments and the like, very developed repo markets exist such that the purchase of these instruments can be financed with secured funding. Consequently, the funding rate of buying these instruments is much lower than the dealers' average funding rate, which explains why dealers willingly buy them. In previous sections, it was even explained that the lack of a repo market for derivatives is exactly the reason why derivatives require unsecured funding in the first place. On the contrary, the existence of a repo market for bonds is exactly the reason why dealers can buy bonds without suffering a loss.

In fact, Green (2015) uses the Semi-Replication model by Burgard and Kjaer (2013), to show that the existence of a derivatives repo market would eliminate the need for FVAs on derivatives. However, it is unlikely that a repo market for derivatives will be developed. Bonds are standardised and have high liquidity which make them easy to repo, but that is not the case for derivatives.

In conclusion, the summary of the debate can be distilled into two parts. Theoreticians are against FVAs, with the argument that they break the assumptions of traditional valuation frameworks. Finance theory states that the risk of a project should determine the discount rate for the cash flows, and that the funding rates are irrelevant. Dealers, however, are subject to the funding rates of their funding desk; if they do not account for funding costs, they will lose on trades that requires financing.

8 The Difference Between Price and Value

It would seem that if theoreticians are correct in their assessment of FVAs, the paper could be concluded at this point. However, the final judgement on this debate has yet to be reached, and, even at that point, the conclusion will likely be nuanced. In fact, it will soon be apparent that the two sides might be debating about applying FVAs to multiple different concepts. This section will come to a conclusion about whether FVAs should be used on, respectively, a derivative's market price and a derivative's value. These two quantities will be defined shortly. Still, this will, of course, not settle the entire debate, as there are many other aspects in which FVA's use are being discussed. This paper will not try to reach a conclusion on all of these other aspects. The following arguments are inspired by Ruiz (2015), and start off by defining the market price and value of a derivative.

When a dealer sells a derivative, she is going to receive some price at the inception date of the derivative. This price is the market price. Since she is selling at that price, the dealer will want it to be as high as possible.
"Creating" or "manufacturing" the derivative has an associated cost to the dealer, which is constituted by multiple different terms. Assume, after selling the derivative, the dealer is going to hedge the market risk by trading derivatives on an exchange. This market is highly collateralised and has a minimum of credit risk; for simplicity, assume that the dealer is able to hedge the market risk by paying the price of the derivative without credit risk. This price will be referred to as the risk-free price.

The dealer is also going to hedge the counterparty's default risk which costs CVA. Additionally, the dealer will be paying or receiving her funding rate on unsecured borrowing for the derivative cash flows, which costs FVA. There might be additional costs, but, for the purposes of this argument, only the ones mentioned will be considered.
In total, the cost of creating the derivative is:

$$\text{Cost} = \text{Risk Free Price} + \text{CVA} + \text{FVA}$$

The trade's value to the dealer is given by the difference in the trade's price and the cost to the dealer of manufacturing it:

$$\begin{aligned}\text{Value to Dealer} &= \text{Market Price} - \text{Cost} \\ &= \text{Market Price} - \text{Risk Free Price} - \text{CVA} - \text{FVA}\end{aligned}$$

The value of the trade is going to decide whether it will be economical for the dealer to enter the trade. If the value is negative, the dealer will lose money on the trade; if the value is positive, the dealer will make money. Setting Value to Dealer equal to zero in the above equation and solving for the market price, would yield the breakeven price:

$$\text{Breakeven Price} = \text{Risk Free Price} + \text{CVA} + \text{FVA}$$

The question is now how the market price and the FVA of the firm are related. Remember that the market price of a trade is set by supply and demand forces. It will therefore not necessarily adhere to some rigorous pricing scheme, but rather be influenced by many different factors in the market. One of these factors is the cost of funding the trade. If a trade requires a lot of funding, which will generate a lot of funding costs, the price of the trade will generally be relatively high. Since the trade would be expensive to operate, dealers will generally need to charge a higher price, in order for the value to stay above a level which they can accept. But, the price of the trade does not have to be high just because the cost of operating it is high. The price is set by the market by the invisible hand of supply and demand, and will therefore be driven by many other factors than the operating cost of the trade.

Therefore, there is no strict and rigorous FVA in the market price that always ensures the same price, but there is certainly an FVA in the value to the dealer.

A confusing case occurs if the dealer's institution has a very dominant position in the market. In that case, the dealer is able to influence the market price. Of course, she is going to increase the price to a level above her breakeven price for as long as there is a counterparty willing to buy the derivative at that price. Since the breakeven price is a lower bound for the price that the dealer is happy with, it might be said that the FVA is setting the price. However, that is inaccurate. If the dealer charges a price $\hat{P}$ that ensures the derivative's value is positive, she is only able to do so because the demand side accepts the price.
If, for some reason, the dealer's FVA drops to zero, all else being equal, she is clearly not

going to change its price from $\hat{P}$, simply because the demand side accepts the current price and the dealer will always charge as high a price as possible.

That FVA should not be included in the market price is part of the conclusion that Hull and White (2012) argue for: *"FVA should not be considered when determining the value of the derivatives portfolio, and it should not be considered when determining the prices the dealer should charge when buying or selling derivatives."* One reasoning from practitioners for using FVAs is that funding costs are very real costs to a dealer's operation. This is an argument for applying FVAs to the value of the trade, which is a very sensible thing to do. An FVA is necessary to arrive at the dealer's breakeven price; therefore, the dealer will not be able to make sound trading decisions without an FVA.

As such, theoreticians and practitioners are not disagreeing entirely on the subject. However, the debate is not limited to the difference between price and value that was accounted for here. There exists yet another price to consider, which is what the quote above refers to as *the value of the derivatives portfolio.*

Assume that the dealer is instead the buyer of a derivative. The dealer has calculated the value of the trade and decides to buy it. After obtaining the trade, at the price decided by the market, another price forms. The dealer is now interested in the price she would receive if she sold the trade again to a third party; namely, the exit price of the trade. This is the price that is marked-to-market for balance sheet accounting. The exit price is problematic because it is not observable in the market; it can only be estimated. If the dealer could, she would ask all potential buyers to reveal the price at which they would buy the trade, and use that information to calculate the actual exit price. That is not possible, hence, the dealer must instead try to estimate the price while conforming to the accounting rules. Accounting rules say that the dealer should account for her own credit risk when marking-to-market, which could prompt the dealer to making a DVA. However, any buyer of the trade will calculate the value of the trade based on their own funding rate to determine the price at which the trade is beneficial. The dealer is therefore left to choose between making adjustments to the exit price using her own funding rate or someone else's funding rate.
If the dealer decides on a funding rate, the problem expands further because each institution will have different funding rates, but no one will reveal their own. It is therefore impossible to really estimate the correct adjustment to make, which is why theoreticians like Hull and White (2012) oppose FVAs to exit prices.
Coming to a conclusion about whether to use FVAs when marking-to-market a trade is an interesting but vast topic. Asserting this could include research on the current market practice and the opinion of accounting institutions. However, this topic is too broad to fit within the scope of this paper, and it will therefore be left as a possible extension.

Still, with this discussion, a partial conclusion on the appropriateness of FVA can be reached. By the previous definitions, an FVA should be used to calculate the value of a trade. If the dealer does not make an FVA, she will not be accounting for the costs of the trade, and she will therefore not be able to make decisions about obtaining the project based on the full information. This conclusion will be the primer for the discussions following in the rest of the paper.

In the following sections the discussion will move beyond the validity of FVA, into the topic of how it should be defined and applied.

Part III

FVA in the Single-Period Model

In order to develop the necessary understanding of funding costs and their implications to a financial institution's stakeholders, this part will explore them using the single-period model. The model is a drastic simplification of reality but still it can provide useful results concerning FVAs. The first section will derive the relevant theory, which is later applied in practical examples.

Before diving into the mathematical details, consider the following example describing the problem that this part wants to address.

> A financial institution financed by debt and equity operates in a single-period binomial model. In the next time period, two possible states can have realised; an up-state where the value of the assets are high, or a down-state where the value of the assets are low. In the up-state the firm pays its debt to the creditors, and the remaining assets are liquidated and paid to the shareholders. In the down-state, the assets are not sufficient to pay the firm's creditors, and the firm defaults, leaving the creditors with a loss.
>
> This situation is depicted in figure 6 by the leftmost pair of blocks labelled "Pre-project". The red block shows the value of the assets in each state. The blue block shows the share of assets claimed by creditors, and the orange block the share claimed by shareholders.
>
> The firm considers investing in a new project, with a high payoff in the up-state, and a lower payoff in the down-state. The payoff of this project is shown by the striped red blocks.
>
> Consider the middle pair of blocks. Here, the firm finances the project by issuing debt to new creditors. The new creditors are promised a face value corresponding to the striped blue block in the up-state. Since the firm might default, the new creditors are offered a credit spread; therefore, in the up-state, the new creditors claims more than the project pays off. Hence, the shareholders' payoff is lower in the up-state. In the down-state, the legacy creditors gets a share of the projects payoff.
>
> In the rightmost pair of blocks, the firm finances the project by issuing equity to new shareholders. Since the new shareholders are not compensated in the down-state, they require an even higher return in the up-state than the new creditors did. The legacy shareholders' payoff is therefore even lower in the up-state. In the down-state the legacy creditors receive the entire payoff of the project.

Deliberately free of too many details, this example should show the overall workings of the setup, which the following sections will use. A firm considers a project, and its choice of financing will affect the payoff of shareholders and creditors. The payoff of

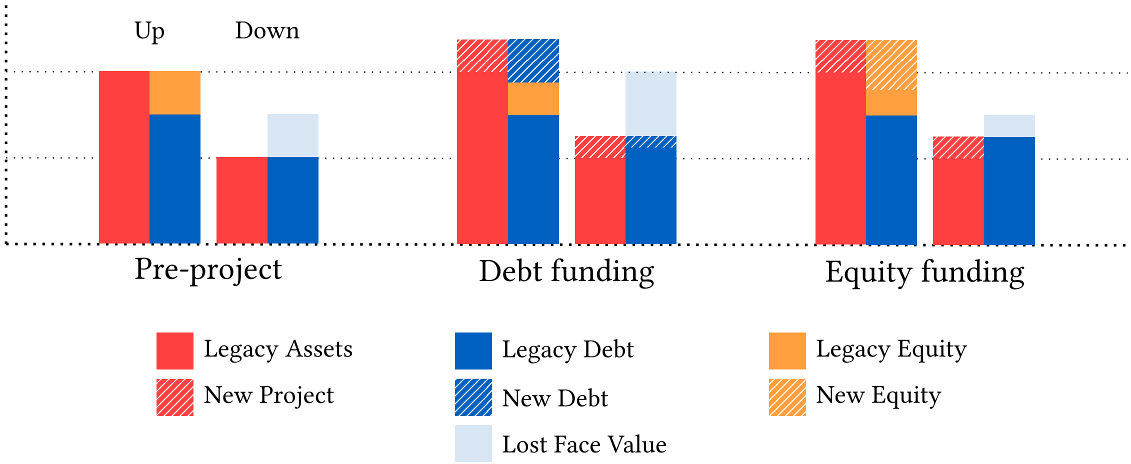

Figure 6: Distribution of assets to creditors and shareholders before entering a project, after entering a project financed by debt, and after entering a project financed by equity.

the project under consideration has a big influence on the funding implications and the project can have other important features such as collateralisation requirements.

9 Theorizing Funding Costs

This section will start off by defining the single-period model, followed by introducing a structural model of a firm that operates in such an economy. The firm will then be faced with a new potential project. The section will conclude with an mathematical assessment of how obtaining the new project affects the wealth of the firm's shareholders.

9.1 The Single-Period Model

The single-period framework is modelling what will happen in an economy as it transitions from the present, time 0, to one time period ahead, time 1. At time 1 the economy can materialise in one of N possible states, $\omega_1, \ldots, \omega_N$, and each state occur with a strictly positive probability. A state is defined by the price of an Arrow-Debreu security paying one unit of numeraire if the state is achieved and zero otherwise. The price of this security is referred to as the state price and denoted ψ_i. The vector of state prices, $\psi \equiv (\psi_1, \ldots, \psi_N)'$, can be used to price claims in the economy, since a state price defines the value of receiving a cash flow in a particular state. For simplicity, all investors are assumed to be in possession of the same market information.

To characterise the market valuation of financial derivatives represented on a firm's balance sheet, the finite set, $\mathcal{L}$, of payoffs at time 1 is assigned a fair value at time 0 by some "fair market value"-function $V : \mathcal{L} \to \mathbb{R}$. Two required assumptions are imposed on the market valuation assignment: (i) $V(\cdot)$ is linear, such that the value of a portfolio of cash flows is the sum of the values of the elements of the portfolio, and (ii) $V(\cdot)$ is increasing in payoffs, such that if payoff X is greater than or equal to payoff Y in all states, and if $X > Y$ in some states, then $V(X) > V(Y)$. These assumptions imply that V

is strictly positive, such that any $X > 0$ yields $V(X) > 0$.

The price of a payoff $d = (d_1, \ldots, d_N)'$ is given as:

$$\pi(d) = d'\psi = \sum_{i=1}^{N} d_i \psi_i$$

Especially, a zero coupon bond paying one unit of numeraire in every state has the price:

$$d_0 = \sum_{i=1}^{N} \psi_i$$

from which the risk-free rate can be defined as:

$$r_f = \frac{1}{d_0} - 1$$

The gross risk-free rate will be denoted by $R \equiv 1+r_f$. The product between the gross risk-free rate and the state price vector produces the risk-neutral probability distribution, $\mathbb{Q}$, i.e. $q_i = R\psi_i$.

In an OTC market, the market valuations of a derivative need not coincide with the price at which dealers are trading this derivative. At almost the same point in time, the same asset can be traded at several different prices reflecting distinct bids and asks of different dealers. The associated deviation in prices is partly explained by search costs, differences in dealer-client relationships, and differences in the dealers' capital structure.

Having defined how valuation is achieved in the single-period model, the following section will introduce a firm operating in such an economy.

9.2 Firm Capital Structure

This section will describe the mathematical structure of a firm in a single-period model. This precedes describing how the firm's structure changes after it has obtained a new investment,

Consider a firm operating in a single-period economy. The firm can invest in assets with risk but needs financing for supporting its investments. If the firm has no cash, financing can only be obtained at two different sources; namely, equity funding through distribution of the firm's own stocks or issuance of debt. The value of the firm's assets at time 1 is given by the random variable A. At time 1 the firm's creditors are promised to receive some cash flow denoted by L. This cash flow might be fixed, i.e. a face value, or random, i.e. depending on the realised state. In the event that the value of the firm, A, is not sufficient to pay the face value in its entirety, the firm defaults, and the creditors receive the remaining estate.
The firm's default event will be denoted by $\mathcal{D}$ and defined as $\{A < L\}$. The complement to the firm's default, its survival event, will be denoted by $\mathcal{D}^c$. In a default event, the

firm can suffer distress costs, e.g. due to liquidation of its assets. The remaining estate after a default depends on the recovery parameter $\kappa \in (0;1]$, such that the remaining estate is κA.

Given the possibility of a default, the creditors' payoff is random; it is equal to L in states where the firm does not default and equal to the remaining assets, κA in states where the firm does default.
The time 1 payoff to the creditors will be denoted by D, short for debt, and is mathematically given by:

$$D = \mathbb{1}_{\mathcal{D}} \kappa A + \mathbb{1}_{\mathcal{D}^c} L$$

The firm is owned by its shareholders who have a limited liability claim on the firm. Shareholders receive the remainder of the firm's value after the debt has been paid; in particular, they receive nothing when the firm defaults. Hence, the shareholders' payoff is also random; equal to $A - L$ in no-default states and equal to zero in default states.
The time 1 payoff to the shareholders will be denoted by E, short for equity, and is given by:

$$E = \mathbb{1}_{\mathcal{D}^c}(A - L)$$

The next section will describe this same firm structure, but after the firm has obtained some new investment.

9.3 Obtaining a New Financial Project

When a firm enters a new project, the new payoffs and other cash flows will alter the structure of the firm and change how much value shareholders and creditors assign to their own claim. This section will derive the mathematical details of the project, and the following sections will dive into the implications of obtaining the project.

The firm has an opportunity to engage in a new project, e.g. buying or selling a security, with a, possibly random, payoff denoted by Y. The firm considers obtaining an amount q of the project; therefore, the potential cash flow to the firm at time 1 will be qY. By entering into the project, the firm will have altered its default event, since assets and liabilities at time 1 have changed. The default event for the firm after it has obtained a quantity q of the project is denoted by $\mathcal{D}(q)$. The precise definition of the default event varies, depending on how the firm chooses to finance the project. The pre-project default event is essentially equal to $\mathcal{D}(0)$, but, to keep the notation simple, the argument is dropped leaving just $\mathcal{D} \equiv \mathcal{D}(0)$.

The purpose of the following derivations is to describe the cash flows to the firm of obtaining the project. For this purpose, assume that qY is the only default claim besides L. In that case, the definition of the firm's post-project default event is given by $\mathcal{D}(q) = \{A + qY < L\}$. Similarly, $\mathcal{D}^c(q) = \{A + qY \geq L\}$ defines the event of no default.

The payoff of the project, Y, can take on both positive and negative values depending on the state realised a time 1, but of course also depends on the specific type of project. A swap contract would be an example of a project where the payoff can take both signs, while the payoff of a bought call option can only take non-negative values. The treatment of the payoff in default events will depend on the sign of the payoff, and therefore it will be useful to be able to separate the two cases; positive payoff where the firm is owed cash, and negative payoff where the firm owes cash.

The positive part of the payoff, $Y^+ = \max(Y, 0)$, is perceived as an asset of the firm, and is measured net of any potential losses due to firm's counterparty credit risk. In addition, the negative part of the payoff, $Y^- = \max(-Y, 0)$, is a contingent liability, and is also interpreted as the contractual amount that the firm is obliged to pay at time 1, before taking the firm's risk of default into consideration.

If the contingent liability, Y^-, is not fully secured, a specification of how the associated counterparty recovers, in case of the firm defaulting, is needed. According to Andersen, Duffie, and Song (2019), in practice, a firm's swap-contingent liabilities are normally pari passu with its unsecured debt claims. When liabilities rank pari passu, they experience the same loss rate in a default. In other words, they receive a share of the remaining assets corresponding to their share of the total liabilities.
Since it is common in practice, throughout this paper it is assumed that the firm's unsecured liabilities rank pari passu with each other; therefore, the counterparty to the new project ranks pari passu with the creditors.

If the firm defaults, the share of the bankruptcy estate paid to the counterparty with the contingent liability Y^- is:

$$\rho = \frac{qY^-}{L + qY^-}$$

Then, at time 1, the random cash flow to the firm is given by:

$$C = \mathbb{1}_{\mathcal{D}^c(q)} qY + \mathbb{1}_{\mathcal{D}(q)} qY^+ - \mathbb{1}_{\mathcal{D}(q)} \rho \kappa A$$

The first term expresses the firm's payoff from the project in case it does not default. The positive payoff to the firm, Y^+, is paid in full if the firm defaults, which is represented by the second term. If the firm defaults, it pays a share, ρ, of the remaining assets, κA, to the counterparty of the project. This is represented by the third term. For clarification, note that $Y > 0$ implies $\rho = 0$, such that the firm does not pay anything if it does not owe anything.

Practically, it is possible for the contingent liability of the position to have both a secured as well as an unsecured component. For this motive, the payoff is divided in two, such that $Y = Y_1 + Y_2$, where Y_1 is the secured component and Y_2 is the unsecured component. Then, Y_1^- reflects the secured contingent liability, and Y_2^- reflects the unsecured contingent liability. The financial position is still assumed to pay a payoff, Y, at time 1 which is unrelated to the credit risk of the firm, and the unsecured contingent liability is still assumed to rank pari passu with all other unsecured creditor claims.
For the contingent liability, qY_1^-, to be secured, $A + qY_1 > 0$ is a necessary condition, and therefore assumed.

Then, the cash flow of the firm where the position can have both secured and unsecured parts is given by:

$$C = qY_1 + \mathbb{1}_{\mathcal{D}^c(q)} qY_2 + \mathbb{1}_{\mathcal{D}(q)} qY_2^+ - \mathbb{1}_{\mathcal{D}(q)} \rho\kappa(A + qY_1)$$

where $\rho = qY_2^- / (L + qY_2^-)$ is the pro rata share of the contingent liability. Notice that the secured position is now paid at time 1 no matter if the firm defaults or not. Only the unsecured position depends on the outcome of the total asset value.

This section has defined the firm's default event when considering the possibility of obtaining a quantity q of a project with payoff Y. The structure of the payoff has been elaborated on, and it has been assumed that the payoff can be both a liability and an asset to the firm as well as having both secured and unsecured parts. With these definitions in order, the analysis can now move on to consider the funding implications of the firm entering into the project. Specifically, the next section considers the impact on shareholders' valuation of obtaining the new project.

9.4 Shareholders' Financing Costs

When the firm invests in the new project it has implications for its stakeholders, specifically its shareholders and creditors. The upfront price requires some type of financing, which the firm will have to decide on. Obtaining this financing will have an influence on the value of the shareholders' claims; this section will derive the theoretical equations defining the impact of financing costs on the firm's shareholders. The theoretical results derived are mainly from Andersen, Duffie, and Song (2019) and will be followed by practical examples where they are applied.

For generality, the firm considers investing in q units of the project; the payoff is therefore qY while the upfront cost is a function of q, denoted by $U(q)$. The marginal investment cost per unit invested is given as $u = \lim_{q \to 0} U(q)/q$. Throughout, it is assumed that the asset value, A, equals the face value of debt, L, with probability zero, i.e. $\mathbb{P}(A = L) = 0$. This assumption is necessary to avoid singularities when calculating derivatives, and it will be used in a proof later. In addition, it is assumed that the random variables A, L, and Y, have finite expectations.

The following subsections will derive the marginal shareholder value from entering into the project under three different funding assumptions; specifically, debt-, equity-, and cash funding.

9.4.1 Funding by Debt Issuance

The firm can fund the upfront cost of the project by issuing new debt. This debt will be subject to an interest rate higher than the risk-free rate, however, in order to limit the complexity of the model, it is assumed that a yield spread on debt issuance can only occur due to credit risk. This is ensured by assuming that the market where the firm obtains its debt is fully efficient, such that creditors break even by offering the market value of the firms' debt claims. The debt obtained to finance the upfront cost, $U(q)$, is assumed to rank pari passu with existing debt.

The new creditors receive a credit spread denoted by $s(q)$; therefore, the face value of the newly issued debt is $U(q)(R+s(q))$. The firm defaults if it is not able to pay both the legacy liabilities and the new debt, therefore the default event is:

$$\mathcal{D}(q) = \{A + qY < L + U(q)(R + s(q))\}$$

Since creditors, by assumption, break even on the new debt issued to finance the cost $U(q)$, the credit spread must ensure that the present value of the debt claim equals the upfront cost. In other words, the credit spread, $s(q)$, must solve the following:

$$\begin{aligned} U(q) = &\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c(q)}U(q)(R+s(q))\right] \\ &+ \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}(q)}\frac{\kappa\left(A+qY_1+qY_2^+\right)}{L+U(q)(R+s(q))+qY_2^-}U(q)(R+s(q))\right] \end{aligned} \tag{9.1}$$

Hence, the present value of the expected payoff to the new creditors must equal the amount needed to fund the new project, $U(q)$. The expected payoff can be thought of as consisting of two legs; one leg that governs the payoff when the firm does not default, and another that governs the payoff when the firm does default.
The first term in this equation is the contribution to the expected value from the promised payoff, the face value, which is paid when the firm does not default. The face value equals the borrowed amount, $U(q)$, appreciated by the gross risk-free rate and the credit spread.

The second term is the contribution from the payoff when the firm defaults. The asset base consists of the legacy assets, A, the secured payoff, qY_1, and the unsecured payoff if it is in the firm's favour, qY_2^+. The secured payoff might be in the firm's favour, in which case the counterparty must pay it. Alternatively, it is a liability in which case it is posted as collateral before the default, and therefore not part of the estate and not available for the creditors. If the unsecured payoff is a receivable for the firm, the counterparty should pay it, like it should pay the secured payoff. If it is a liability, the counterparty ranks pari passu with the other claimants, and therefore experiences the loss rate.
The asset base is reduced by the distress costs, $1-\kappa$ so the estate available for claimants is a share, κ, of the assets. This is the quantity found in the numerator of the fraction.

The denominator represents the total amount owed at time 1, which is the sum of the pre-project liabilities, L, the face value of the new debt, and the liabilities due to the unsecured payoff of the project. The secured liabilities, if any, are already posted as collateral. The amount available for paying claimants divided by the amount owed to claimants, is the loss rate.

Since the credit spread depends on the decomposition of the payoff into a secured and unsecured part, it is easier to consider the limiting spread, $\lim_{q\to 0} s(q)$. This quantity is derived in the following paragraphs. Dividing first both sides of equation 9.1 by the face value $U(q)(R+s(q))$ and using the linearity of the expectation operator:

$$\frac{1}{R+s(q)} = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c(q)} + \mathbb{1}_{\mathcal{D}(q)}\frac{\kappa\left(A+qY_1+qY_2^+\right)}{L+U(q)(R+s(q))+qY_2^-}\right]$$

Since the events $\mathcal{D}^c(q)$ and $\mathcal{D}(q)$ are complements, the indicator of the no-default event, $\mathbb{1}_{\mathcal{D}^c(q)}$, can be rewritten as $1-\mathbb{1}_{\mathcal{D}(q)}$:

$$\frac{1}{R+s(q)} = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[1 + \mathbb{1}_{\mathcal{D}(q)}\left(\frac{\kappa\left(A+qY_1+qY_2^+\right)}{L+U(q)(R+s(q))+qY_2^-} - 1\right)\right]$$

If the firm defaults, the loss rate is sure to be greater than zero, since these two conditions are equivalent. Hence, the right hand side is strictly positive, as well as non-zero, and the multiplicative inverse transformation can be applied to both sides:

$$s(q) = R\left(\mathbb{E}^{\mathbb{Q}}\left[1 + \mathbb{1}_{\mathcal{D}(q)}\left(\frac{\kappa\left(A+qY_1+qY_2^+\right)}{L+U(q)(R+s(q))+qY_2^-} - 1\right)\right]^{-1} - 1\right)$$

Let S denote the limiting spread which is defined as the credit spread on debt charged for obtaining an infinitesimal quantity of the project, i.e. $S \equiv \lim_{q\to 0} s(q)$.

If the limits of two addends in a sum are finite, the algebraic limit theorem ensures that the limit of the summation operation can be expressed as the summation of the addends' limits. This also applies to two factors in a product. Since A, L, and Y have finite expectations, limits and integrals, e.g. the expectation in the above equation, can be interchanged.

Applying the limit of q approaching 0:

$$S = R\left(\mathbb{E}^{\mathbb{Q}}\left[1 + \lim_{q\to 0}\mathbb{1}_{\mathcal{D}(q)}\left(\frac{\kappa\left(A+qY_1+qY_2^+\right)}{L+U(q)(R+s(q))+qY_2^-} - 1\right)\right]^{-1} - 1\right)$$

The limit of $\mathcal{D}(q)$ is $\mathcal{D}$, since the post-project default event with an infinitesimal investment in the project corresponds to the pre-project default event. In addition, the upfront price of the project, $U(q)$, approaches zero as the quantity of the project approaches zero. Applying these results yields:

$$\begin{aligned} S &= R\left(\mathbb{E}^{\mathbb{Q}}\left[1 + \mathbb{1}_{\mathcal{D}}\left(\frac{\kappa A}{L} - 1\right)\right]^{-1} - 1\right) \\ &= R\left(\mathbb{E}^{\mathbb{Q}}\left[1 - \mathbb{1}_{\mathcal{D}}\frac{L-\kappa A}{L}\right]^{-1} - 1\right) \end{aligned} \tag{9.2}$$

For ease of notation and interpretation the loss rate can be defined as:

$$\phi = \frac{L-\kappa A}{L}\mathbb{1}_{\mathcal{D}} \tag{9.3}$$

Equation 9.2 can then be reformulated and finally result in the limiting spread:

$$\begin{aligned} S &= R\left(\frac{1}{1-\mathbb{E}^{\mathbb{Q}}\left[\phi\right]}-1\right) \\ &= \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi\right]R}{1-\mathbb{E}^{\mathbb{Q}}\left[\phi\right]} \end{aligned} \tag{9.4}$$

As anticipated, the limiting spread is invariant to the decomposition of the payoff into Y_1 and Y_2, as well as being independent of the quantity q. These properties will make the limiting spread very useful in the following derivation of the marginal shareholder valuations.

Having derived the spread on new debt for financing an infinitesimal project, the shareholders' valuation of obtaining this new project can be derived. This derivation starts of by defining the payoff to the shareholders of obtaining a project of size q, and then taking the derivative of that with respect to q.

Again, the face value of the new debt is the upfront including the interest rates paid; specifically, $U(q)(R+s(q))$. The shareholders receive the residual of the assets after debt claims have been paid unless the firm defaults, in which case the shareholders receive nothing. The marginal increase in the value of the firms equity per unit investment is therefore:

$$G_{\text{debt}} = \frac{\partial}{\partial q}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{R}\left(A+qY-L-U(q)(R+s(q))\right)^{+}\right]\Bigg|_{q=0}$$

Expressing this derivative as a difference quotient, using $U(0)=0$:

$$G_{\text{debt}} = \lim_{q\to 0}\frac{1}{R}\frac{\mathbb{E}^{\mathbb{Q}}\left[\left(A+qY-L-U(q)(R+s(q))\right)^{+}\right]-\mathbb{E}^{\mathbb{Q}}\left[(A-L)^{+}\right]}{q}$$

The argument in the first max-function can be recognised as the function governing the firm's default event after entering the project. Therefore it is positive exactly when the firm does not default and negative otherwise. Applying the max-function, yields zero when the firm defaults and the shareholders' payoff when the firm does not default. The same idea applies for the argument in the second max-function, but this argument instead governs the default of the firm *before* entering the project. Both max-functions can therefore be replaced by using indicators for the default events:

$$G_{\text{debt}} = \lim_{q\to 0}\frac{1}{R}\frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c(q)}\left(A+qY-L-U(q)(R+s(q))\right)\right]-\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\left(A-L\right)\right]}{q}$$

Using the linearity of the expectation operator to collect terms involving $A-L$:

$$= \lim_{q\to 0}\frac{1}{R}\frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c(q)}\left(qY-U(q)(R+s(q))\right)\right]+\mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{\mathcal{D}^c(q)}-\mathbb{1}_{\mathcal{D}^c}\right)(A-L)\right]}{q} \tag{9.5}$$

Using the algebraic limit theorem and deriving the limit of the first addend in the numerator:

$$\lim_{q \to 0} \frac{1}{R} \frac{\mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c(q)} \left(qY - U(q)(R + s(q)) \right) \right]}{q}$$

Using again the linearity of the expectation operator:

$$= \lim_{q \to 0} \frac{1}{R} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c(q)} \left(Y - \frac{U(q)}{q} (R + s(q)) \right) \right]$$

The assumption of finite expectations of A, L, and Y allows, again, for interchanging the limit and the expectation. The limit of $U(q)/q$ can be recognised as the marginal investment cost, u, while the limit of $s(q)$ is the limiting spread, S. The limit of the post-project no-default event, $\mathcal{D}^c(q)$, is the pre-project no-default event $\mathcal{D}^c$. The reformulation again uses the algebraic limit theorem, which applies since the limits of the factors exist:

$$= \frac{1}{R} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c} \left(Y - u(R + S) \right) \right] \tag{9.6}$$

Turning to the limit of the second term in the numerator of equation 9.5. Using again the assumption that A, L, and Y have finite expectations, the following expression is obtained:

$$\frac{1}{R} \mathbb{E}^{\mathbb{Q}} \left[\lim_{q \to 0} \frac{\mathbb{1}_{\mathcal{D}^c(q)} - \mathbb{1}_{\mathcal{D}^c}}{q} (A - L) \right]$$

Using the algebraic limit theorem on $\lim_{q \to 0} \left((\mathbb{1}_{\mathcal{D}^c(q)} - \mathbb{1}_{\mathcal{D}^c})/q \right)$ yields an indeterminate form, "0/0", since both the numerator and the denominator approach zero as q approaches zero. This limit can therefore be evaluated by applying L'Hôpital's rule. The derivative of the denominator, q, with respect to q, is 1, which is different from 0 as required.

Turning to the numerator, $\mathbb{1}_{\mathcal{D}^c(q)} - \mathbb{1}_{\mathcal{D}^c}$. The pre-project no-default event, $\mathbb{1}_{\mathcal{D}^c}$ is constant in q, and its derivative is zero. For an infinitesimal investment, the post-project no-default indicator, $\mathbb{1}_{\mathcal{D}^c(q)}$, can only change its value at the point $A = L$. However, the assumption $\mathbb{P}(A = L) = 0$ ensures that this does not occur; the derivative of the post-project no-default event is therefore surely zero in an interval around $q = 0$.

Hence, it can be concluded that:

$$\lim_{q \to 0} \frac{1}{R} \frac{\mathbb{E}^{\mathbb{Q}} \left[\left(\mathbb{1}_{\mathcal{D}^c(q)} - \mathbb{1}_{\mathcal{D}^c} \right) (A - L) \right]}{q} = 0 \tag{9.7}$$

Applying the results of equation 9.6 and equation 9.7 to equation 9.5 yields:

$$G_{\text{debt}} = \frac{1}{R} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c} \left(Y - u(R + S) \right) \right]$$

Using linearity of expectations and rearranging terms:

$$\begin{aligned} &= \frac{1}{R}\left(\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}Y\right] - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}u(R+S)\right]\right) \\ &= \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}Y\right] - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]u - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\frac{1}{R}uS \end{aligned} \tag{9.8}$$

The expected value of two dependent terms in the first term, $\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}Y\right]$, can be reformulated by applying the definition of the covariance operator. This reformulation will later make interpretation easier. Using the linearity of the expectation operator, the definition of covariance between two random variables W and Z can be rewritten:

$$\begin{aligned} \mathrm{Cov}\left(W, Z\right) &= \mathbb{E}\left[(W - \mathbb{E}\left[W\right])(Z - \mathbb{E}\left[Z\right])\right] \\ &= \mathbb{E}\left[WZ\right] - \mathbb{E}\left[W\right]\mathbb{E}\left[Z\right] \end{aligned}$$

According to this result, the aforementioned expectation can be rewritten as follows:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}Y\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\mathbb{E}^{\mathbb{Q}}\left[Y\right] + \mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}^c}, Y\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\mathbb{E}^{\mathbb{Q}}\left[Y\right] - \mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) \end{aligned}$$

Where the last equality uses $\mathbb{1}_{\mathcal{D}^c} = 1 - \mathbb{1}_{\mathcal{D}}$ combined with the standard properties of the covariance operator.
Substituting into equation 9.8 finally yields the marginal value to shareholders of debt financing:

$$\begin{aligned} G_{\text{debt}} &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\left(\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y\right] - u\right) - \frac{1}{R}\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\frac{1}{R}uS \\ &= p^{\mathbb{Q}}\mu - \frac{1}{R}\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) - \Phi \end{aligned} \tag{9.9}$$

Where $p^{\mathbb{Q}} = 1 - \mathbb{P}^{\mathbb{Q}}\left(\mathcal{D}\right)$ is the risk-neutral probability of the firm not defaulting. $\mu = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y\right] - u$ is the difference between the present value of the expected payoff and the upfront price, i.e. μ is the promised marginal profit on the new project. $\Phi = p^{\mathbb{Q}}\frac{1}{R}uS$ is the present value of the marginal excess return on the upfront price, discounted by the probability of the firm not defaulting. Φ can be interpreted as the present value to the shareholders of their share of the financing costs, uS, which they pay iff the firm does not default. If the firm's default event is positively correlated with the payoff of the project, a greater part of the projects price will be due to payoffs that are realised when the firm defaults. The opposite is the case when the default event is negatively correlated with the payoff. Since the shareholders get nothing in a default, the value of their claim decreases even more, when they are financing payoffs that they will never receive. This effect is captured by the term $\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)$.

9.4.2 Funding by Equity Issuance

In addition to obtaining debt, new projects can be financed by issuing additional shares that will provide a source of equity capital. Share issuances dilute the firm's stocks, which can reduce the value of the existing shareholders' stocks. This dilution will be the cause of the conflict of interest with this type of funding. It is assumed that the market for newly issued equity is sufficiently competitive, such that new shareholders break even when purchasing the newly issued shares; in other words, buying the firm's newly issued stock is a zero net present value investment.

With equity financing, the legacy shareholders still receive the difference between assets and debt claims, however they now have to split the bounty with the new shareholders, whom, at time 0, take an amount of shares worth $U(q)$. The new shareholders do not influence whether the firm defaults, so the default event is:

$$\mathcal{D}(q) = \{A + qY < L\}$$

Considering the newly issued shares of value $U(q)$, the marginal increase in the value of equity owned by legacy shareholders per unit investment is:

$$G_{\text{equity}} = \frac{\partial}{\partial q} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{R} (A + qY - L)^{+} \right] - U(q) \Bigg|_{q=0}$$

This derivative is very much like the derivative encountered under debt issuance. Therefore, the reformulation in this section will be less rigorous, as most details have already been covered in the previous section. Expressing first the derivative as a difference quotient:

$$G_{\text{equity}} = \lim_{q \to 0} \frac{1}{R} \frac{\mathbb{E}^{\mathbb{Q}} \left[(A + qY - L)^{+} \right] - U(q)R - \mathbb{E}^{\mathbb{Q}} \left[(A - L)^{+} \right]}{q}$$

Replacing max-functions by default indicators:

$$\begin{aligned}
&= \lim_{q \to 0} \frac{1}{R} \frac{\mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c(q)} (A + qY - L) \right] - U(q)R - \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c} (A - L) \right]}{q} \\
&= \lim_{q \to 0} \frac{1}{R} \frac{\mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c(q)} qY \right] - U(q)R + \mathbb{E}^{\mathbb{Q}} \left[\left(\mathbb{1}_{\mathcal{D}^c(q)} - \mathbb{1}_{\mathcal{D}^c} \right) (A - L) \right]}{q}
\end{aligned}$$

The limit of each term exists, and they are all known from and accounted for in the previous derivation. Applying the limit yields:

$$= \frac{1}{R} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c} Y \right] - u$$

The reformulation of $\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c} Y\right]$ is substituted, and the final result is therefore:

$$\begin{aligned} G_{\text{equity}} &= \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\mathbb{E}^{\mathbb{Q}}\left[Y\right] - \frac{1}{R}\text{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) - u \\ &= p^{\mathbb{Q}}\left(\mu + u\right) - \frac{1}{R}\text{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) - u \qquad (9.10) \\ &= p^{\mathbb{Q}}\mu - \frac{1}{R}\text{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) - \left(1 - p^{\mathbb{Q}}\right)u \qquad (9.11) \end{aligned}$$

The first two terms are recognised from the previous section in equation 9.9 and their interpretation are unchanged.

While equation 9.11 is clearly the easiest to compare to equation 9.9, equation 9.10 is easier to interpret, so that will be the focus. The first term represents the added value to the legacy shareholders of the firm's asset value increasing with the profit, μ, and the cash injection from the new shareholders amounting to u. These two cash flows are only valuable to the shareholders if the firm can actually maintain solvency; hence, they are discounted by the survival probability, $p^{\mathbb{Q}}$. In addition, the legacy shareholders do not actually receive the payment from the new shareholders, due to the stock dilution. This is represented by the third term $-u$. The new shareholders receive a share of equity worth u, and this is clearly a loss to the legacy shareholders. The interpretation of the second term has been covered previously.

9.4.3 Funding by Existing Cash

Instead of relying on external resources to fund the project, the firm might be in an excess of cash that can be used for financing. If the firm finances the new project with cash from its balance sheet, the equity increases by the present value of the expected payoff, offset by the upfront cost, and the cost of tying up capital in the project. The latter is an opportunity cost, since it locks up capital that could otherwise have been invested in bonds earning the risk-free rate. The firm therefore has to pay its liabilities with an asset base that is reduced by the upfront price appreciated by the risk-free rate. Hence, the default event is:

$$\mathcal{D}(q) = \{A - U(q)R + qY < L\}$$

and the marginal increase in the equity valuation per unit investment is:

$$G_{\text{cash}} = \left.\frac{\partial}{\partial q}\frac{1}{R}\frac{\mathbb{E}^{\mathbb{Q}}\left[(A - U(q)R + qY - L)^{+}\right]}{q}\right|_{q=0}$$

This derivative is, again, very similar to the derivatives already examined, so, for brevity, the calculations in this section will be omitted. The marginal shareholder valuation of funding a project with existing cash can be reformulated as the following:

$$
\begin{aligned}
G_{\text{cash}} &= \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c} Y\right] - u\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^c}\right] \\
&= \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\mathbb{E}^{\mathbb{Q}}\left[Y\right] + \frac{1}{R}\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}^c}, Y\right) - u\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^c}\right] \\
&= p^{\mathbb{Q}}\left(\mu + u\right) + \frac{1}{R}\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}^c}, Y\right) - p^{\mathbb{Q}}u \\
&= p^{\mathbb{Q}}\mu - \frac{1}{R}\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)
\end{aligned} \tag{9.12}
$$

The derived equation only contains terms involving expectations of future cash flows, namely the payoff from the project. This reflects the fact that at time 0 there is no immediate value lost or gained for the shareholders, since, as the project is fairly priced, there is no difference in owning cash worth u or a project worth u.

Having derived the impact on equity valuation for these different funding types, a natural question to ask is which type of funding is preferred from the shareholders perspective. This result can be obtained by comparing the three derived equations, which will be the objective of the next section.

9.4.4 The Pecking Order of Funding Preferences

The result of this section will be the inequality relations between the marginal shareholder valuations derived in the previous sections; namely, G_{debt}, G_{equity}, and G_{cash}. This will imply a ranking of the funding methods according to their utility to the shareholders. The direction of the inequalities will be determined by the sign of the marginal investment cost, u, i.e. whether the project initially provides or claims capital. Initially, the project is assumed to claim capital, such that u is strictly positive; relations between marginal shareholder valuations for other signs of u will follow directly after. Comparing first the marginal shareholder valuations between debt and equity funding:

$$
\begin{aligned}
G_{\text{debt}} - G_{\text{equity}} &= -\Phi - \left(-(1 - p^{\mathbb{Q}})u\right) \\
&= -p^{\mathbb{Q}}\frac{1}{R}uS + (1 - p^{\mathbb{Q}})u \\
&= u\left(1 - p^{\mathbb{Q}}\frac{1}{R}\left(R + S\right)\right)
\end{aligned} \tag{9.13}
$$

Recall the expression for the loss rate, ϕ, and the limiting spread, S:

$$
\phi = \frac{L - \kappa A}{L}\mathbb{1}_{\mathcal{D}} \qquad S = \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi\right] R}{1 - \mathbb{E}^{\mathbb{Q}}\left[\phi\right]}
$$

Since $(L - \kappa A)/L \leq 1$ the following holds:

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}}[\phi] \leq \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\mathcal{D}}] \\ \Leftrightarrow \quad & \mathbb{E}^{\mathbb{Q}}[\phi] \leq 1 - p^{\mathbb{Q}} \qquad (9.14) \\ \Leftrightarrow \quad & p^{\mathbb{Q}} \leq 1 - \mathbb{E}^{\mathbb{Q}}[\phi] \qquad (9.15) \end{aligned}$$

Returning to equation 9.13 and substituting the expression for S:

$$G_{\text{debt}} - G_{\text{equity}} = u\left(1 - p^{\mathbb{Q}} - \frac{p^{\mathbb{Q}}}{1 - \mathbb{E}^{\mathbb{Q}}[\phi]}\mathbb{E}^{\mathbb{Q}}[\phi]\right)$$

Due to equation 9.15 it holds that $\frac{p^{\mathbb{Q}}}{1-\mathbb{E}^{\mathbb{Q}}[\phi]} \leq 1$, which, combined with equation 9.14, ensures that:

$$\Rightarrow \qquad G_{\text{debt}} \geq G_{\text{equity}} \qquad (9.16)$$

Hence, shareholders weakly prefer debt funding over equity funding in terms of their perceived value lost. While shareholders will bear the cost of interest payments on loans obtained under debt funding, the dilution from share issuance is evidently less beneficial to them. If the probability of default is positive, all inequalities are strict, implying strong preference of debt funding over equity funding, $G_{\text{debt}} > G_{\text{equity}}$.

Turning now to the relation between the marginal shareholder valuations of cash and debt funding:

$$G_{\text{debt}} - G_{\text{cash}} = -p^{\mathbb{Q}}\frac{1}{R}uS$$

All factors in the product on the right hand side are non-negative, implying the relation:

$$\Rightarrow \qquad G_{\text{debt}} \leq G_{\text{cash}} \qquad (9.17)$$

Again, if the probability of default is non-zero, the limiting spread is positive and the preference is strict, $G_{\text{debt}} < G_{\text{cash}}$. As shown, shareholders prefer funding with cash over funding with debt. While thoroughly derived here, it does not come as much of a surprise that this relation holds.
The mechanism of using cash from the balance sheet is essentially identical to using cash obtained from debt, except for the fact that the funding cost of debt is higher than that of balance sheet cash. Clearly, shareholders will rather want to suffer the opportunity cost of the risk-free interest rate than both the risk-free interest rate *and* an additional spread due to the firms credit risk.

Equation 9.16 combined with equation 9.17, the assumption that the marginal investment cost is strictly positive, and the assumption that the firm's probability of default is non-zero yields the following relation:

$$G_{\text{equity}} < G_{\text{debt}} < G_{\text{cash}} \qquad (9.18)$$

If the investment cost, u, is negative and the project provides capital, the inequalities are reversed. Then, equity buyback is preferred over debt retiring which is preferred over buying bonds earning the risk-free rate. If there is no upfront payment to finance, $u = 0$, or if the default probability is zero, the marginal shareholder valuations are zero, regardless of funding method, such that $G_{\text{equity}} = G_{\text{debt}} = G_{\text{cash}} = 0$.

The equations describing the marginal valuations of projects to shareholders will be essential to the definition of FVA in the next section. In addition, they will be helpful for understanding the impact of projects to shareholders through examples in subsequent sections.

9.5 Defining FVA

Still remaining at this point is to properly define FVA, at least in a technical manner. According to Andersen, Duffie, and Song (2019) there are multiple ways of calculating FVA used in practice and theory, and three viable definitions will be explained in this section. The third, and last, definition will be the definition that is used throughout the remainder of this paper.

FVA as the Promised Excess Funding Cost

Since the firm obtains funds for the project at a spread in excess of the risk-free rate, the shareholders pay an additional rate when the firm does not default. This is a form of funding cost and argues in favour of defining the FVA as the present value of the costs paid in excess of the risk-free rate. This is the definition used by Hull and White (2014) who define FVA as *"the present value of the extra return required by lenders to compensate them for costs associated with possible defaults by the dealer on the funding."* Using the notation introduced in this paper, the adjustment will then be defined as:

$$\begin{aligned} \text{FVA} &= \frac{1}{R}\left(u(R+S) - uR\right) \\ &= \frac{1}{R}uS \end{aligned} \tag{9.19}$$

If the CDS-bond spread is zero, this definition of FVA is equal to the DVA. This fact is used by Hull and White (2014) as an argument against FVAs, since the DVA should already be accounted for and therefore including the adjustment for funding costs will be double counting. However, the shareholders do not pay the credit spread when the firm defaults, which this definition does not take into account. It will be more meaningful to account for the default risk, which is done by the next possible definition of FVA.

FVA as the Expected Excess Funding Cost

The FVA can also be defined by the quantity Φ in equation 9.9:

$$\text{FVA} = \Phi = p^{\mathbb{Q}}\frac{1}{R}uS \tag{9.20}$$

This value is equal to the wealth transfer from the shareholders to the legacy creditors of obtaining the new project. Compared to the previous definition of FVA, Φ captures the expected funding cost as opposed to the excess funding cost. Thus, the two quantities differ by a factor corresponding to the no default probability.

This approach seems more sensible than the previous, since it focuses more on the shareholders' actual costs of obtaining the project. Still, this adjustment might not be sufficient for the firm to be able to enter the project while preserving the shareholders' wealth. As value adjustments are generally made as compensation for some quantity, in such a way that the product being adjusted ends up as a zero net present value investment, it would be coherent to also define FVA as such.

FVA as the Adjustment for Shareholders' Breakeven

This suggests defining FVA as the difference between the project's price, u, and the price that makes the shareholders indifferent to engaging in the project. Phrased differently, the FVA is the donation needed from the project counterparty in order to preserve the shareholders' wealth.

Deriving the shareholders' breakeven price is a matter of setting the marginal value of entering the project equal to zero and solving for the marginal investment cost. With three different funding types considered, this will result in three different breakeven prices, as well as three different definitions of FVA. These will be derived in the following paragraphs.

Debt financing:

Under debt financing, the breakeven price is determined by setting equation 9.9 equal to zero and solving for the marginal investment cost:

$$\begin{aligned}
0 &= G_{\text{debt}} \\
&= p^{\mathbb{Q}}\left(\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y\right] - u\right) - \frac{1}{R}\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) - p^{\mathbb{Q}}\frac{1}{R}uS \\
&= \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y\right] - u - \frac{1}{R}\frac{\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)}{p^{\mathbb{Q}}} - \frac{1}{R}uS \\
\Leftrightarrow \qquad u^{*}_{\text{debt}} &\equiv \frac{1}{R+S}\left(\mathbb{E}^{\mathbb{Q}}\left[Y\right] - \frac{\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)}{p^{\mathbb{Q}}}\right) \qquad (9.21)
\end{aligned}$$

The expected value of the cash flow, $\mathbb{E}^{\mathbb{Q}}\left[Y\right]$, includes the counterparty credit risk, since that is inherited in the cash flow, but excludes the firm's own credit risk. In other words, $\mathbb{E}^{\mathbb{Q}}\left[Y\right]$ is adjusted for CVA but not DVA. The second term is the covariance between the default event and the cash flow, inflated by the probability of not defaulting. The shareholders' breakeven value can then be seen as the discounted expected value of the cash flow adjusted for the inflated covariance. The discounting rate used is however not the risk-free rate, but rather the funding rate, $R+S$, to account for the funding costs of the new debt. Having derived the shareholders' breakeven value, FVA under debt financing can be defined as the adjustment to the actual price needed to arrive at breakeven:

$$\mathrm{FVA}_{\mathrm{debt}} = u^*_{\mathrm{debt}} - u \tag{9.22}$$

Equity financing:

Likewise, the breakeven price under equity financing can be found by solving for u when equation 9.11 equals zero:

$$\begin{aligned}
& & 0 &= G_{\mathrm{equity}} \\
& & &= p^{\mathbb{Q}}\left(\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y\right] - u\right) - \frac{1}{R}\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) - \left(1 - p^{\mathbb{Q}}\right)u \\
\Leftrightarrow & & u^*_{\mathrm{equity}} &\equiv \frac{1}{R}\left(p^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}\left[Y\right] - \mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)\right) \\
& & &= \frac{1}{R}p^{\mathbb{Q}}\left(\mathbb{E}^{\mathbb{Q}}\left[Y\right] - \frac{\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)}{p^{\mathbb{Q}}}\right)
\end{aligned} \tag{9.23}$$

Again, the shareholders' breakeven value can be seen as the discounted expected value of the cash flow adjusted for the inflated covariance. However, the discount factor applied is different from the risk-free rate plus the credit spread as seen before. The discount factor in this case $\frac{1}{R}p^{\mathbb{Q}}$ corresponds to the price of an Arrow-Debreu security paying one unit of numeraire if the firm defaults and zero if not. The FVA under equity funding is then:

$$\mathrm{FVA}_{\mathrm{equity}} = u^*_{\mathrm{equity}} - u \tag{9.24}$$

Cash funding:

The breakeven price under cash funding is derived using equation 9.12:

$$\begin{aligned}
& & 0 &= G_{\mathrm{cash}} \\
& & &= p^{\mathbb{Q}}\left(\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y\right] - u\right) - \frac{1}{R}\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) \\
\Leftrightarrow & & u^*_{\mathrm{cash}} &\equiv \frac{1}{R}\left(\mathbb{E}^{\mathbb{Q}}\left[Y\right] - \frac{\mathrm{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)}{p^{\mathbb{Q}}}\right)
\end{aligned} \tag{9.25}$$

Once more, the breakeven value is the expected value of the cash flow adjusted for the inflated covariance, both discounted. Under cash funding the discount rate is the funding cost of using excess cash, i.e. the opportunity cost of tying up capital, which equals the risk-free rate. The FVA under cash funding is:

$$\mathrm{FVA}_{\mathrm{cash}} = u^*_{\mathrm{cash}} - u \tag{9.26}$$

These three adjustments to the breakeven price will constitute the definition of FVA used throughout this paper. If the firm has as objective to maximise shareholder value, it will not trade unless the upfront payment to the counterparty is at most the breakeven value, or, conversely, unless the upfront payment to the firm from the counterparty is at least the breakeven value.

Receiving a Donation From the Counterparty

Referring to the FVA as *the donation* needed from the counterparty to the dealer might be off-putting at first; derivatives dealers, and financial institutions in general, are seldom perceived as being charitable organizations. *Donation* is used rather figuratively in this context; it is meant to describe the counterparty agreeing to sell (buy) at a lower (higher) price than the theoretical, but not for any specific reason.
Though these reasons are not crucial for this paper, it will be helpful to have an idea about the situations that can motivate organizations to make a donation by price adjustments. To do so, it is worthwhile to recall section 8, where the difference between the derivative price and the derivative value was discussed. Any derivative has a theoretical price at which it would trade under ideal market conditions, but in reality, derivatives trade at a price set by an imperfect market.
In this market, dealers have different incentives and frictions that might lead to prices deviating from the theoretical price. These incentives are plentiful; they could, for instance, be regulatory, like stocking up liquidity buffers with high quality assets, as was mentioned in the example when introducing FVA. Another possible motivation that is worth mentioning, considering the topics of this paper, arises when the counterparty itself has a relatively high funding cost.

> Suppose that the counterparty is selling an unsecured derivative to the dealer, for some upfront price. For simplicity, the only cash flow considered is the upfront price. The dealer will have to fund the upfront price, for which she pays her funding rate, and she is going to take this funding cost into consideration when valuing the derivative. The counterparty is also going to apply an FVA, however, as it receives the upfront price, the adjustment will be due to the funding benefits. If the counterparty's funding rate is higher than the dealer's funding rate, all else being equal, it is going to adjust its own valuation upwards more than the dealer adjusts her valuation downwards. If the two funding rates are sufficiently different, the counterparty will be able to offer a price to the dealer that is low enough to make the dealer's shareholders indifferent to buying the derivative.
>
> This price reduction is *the donation*. It is simply driven by the fact that different organizations have different valuations or incentives such that they can agree on transactions even though the prices deviate from the theoretical.

As mentioned, there are many of such incentives, that could motivate organizations to making donations by reducing prices. An organization's particular motive for raising or reducing prices is not of interest in this paper and will be left unspecified. The cases mentioned above are merely a justification for using the term *donation*, in the definition of FVA.

This section has derived the marginal shareholder valuations of obtaining a new project with different types of funding. As was concluded, if an upfront price has to be financed, the shareholders prefer cash funding over debt funding which they prefer over equity funding. The definition of FVA to be utilised in this paper ended up as being the price adjustment needed for the shareholders' payoff to break even on obtaining the project.

10 Quantifying Funding Costs

Having derived a framework in which funding costs are defined, the paper can move on to applying the results in a practical application. The current section will consider the implications of funding costs and FVAs for a firm operating in a single-period economy. As evident from the previous theoretical derivations, obtaining projects might influence the value of the shareholders' and the creditors' claims.

Before considering any projects, the following section defines the capital structure of the firm in question.

10.1 A Firm in a Single-Period Economy

A firm is operating in a single-period economy defined by $N = 5$ states and the associated Arrow-Debreu prices given in table 1.

i	1	2	3	4	5
ψ_i	0.060	0.240	0.290	0.280	0.120

Table 1

This implies a discount factor of $\frac{1}{R} = 0.990$ and a risk-free interest rate of $r_f = 1.010\,\%$. The firm invests in risky assets which are funded by equity and debt deposits and return payoffs specified shortly. In the event of default, the firm faces no distress cost, i.e. $\kappa = 1$, such that the bankruptcy estate is distributed entirely to the creditors. The firm has been funded by debt such that the face value of debt is its total liabilities $L = 80.000$, which gives rise to the payoff structure defined in table 2.

i	1	2	3	4	5	Present value
$A(\omega_i)$	120.000	110.000	100.000	95.000	60.000	96.400
$D(\omega_i)$	80.000	80.000	80.000	80.000	60.000	76.800
$E(\omega_i)$	40.000	30.000	20.000	15.000	0.000	19.600

Table 2

The associated present values of the payoffs, A, D, and E, are the discounted expected values with respect to the risk-neutral probability measure:

$$\pi(A) = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[A] = 96.400$$

$$\pi(D) = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[D] = 76.800 \qquad \pi(E) = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[E] = 19.600$$

As seen in table 2, the creditors do not receive their entire face value in the default event, ω_5. The relative difference between the discounted payoff and the promised face value governs the credit spread, which compensates the creditors for this risk of default.

In addition to calculating the shareholders' marginal valuation of a new project, it is interesting to also evaluate the impact on the creditors' valuation of their claim. Therefore, it is useful to calculate the credit spread and loss rate of the firm's debt, since that can be used for comparison later.

The credit spread before entering into any projects is $80.000/76.800 - R = 3.157\,\%$. Using equation 9.3, the loss rate of the creditors in the default event is:

$$\phi(\omega_5) = \frac{80.000 - 60.000}{80.000} = 25.000\,\%$$

In all other states the loss rate is zero, as the firm does not default. Recall that the limiting spread is the spread on the firm's debt after obtaining an infinitesimal amount of a project. The loss rate can be substituted into equation 9.4 and the limiting spread can be calculated:

$$S = \frac{\mathbb{E}^{\mathbb{Q}}[\phi]\,R}{1 - \mathbb{E}^{\mathbb{Q}}[\phi]} = 3.157\,\%$$

The derivations in previous sections, which described limiting spreads and marginal shareholder valuations, were rooted in the assumption of the firm obtaining an infinitesimal project. This section will use the results in a practical application, where a larger than infinitesimal project will be obtained by the firm. Therefore, the results obtained from the equations will, to some extent, be inaccurate. This is solely due to the limiting spread; it is an estimate of the post-project credit spread, and therefore might be inaccurate to a degree that depends on the project. The magnitude of the inaccuracy will depend on how much the new project alters the creditors' loss rate, since that controls the credit spread. The scale of the impact on the creditors' loss rate depends on the actual size of the project compared to value of the firm's assets. The direction is determined by the payoff structure of the project compared to the structure of the firm's assets.

Fortunately, in the discrete framework used here, the actual marginal shareholder valuations can be obtained with numerical methods, such that the results of the equations can be verified and compared to the correct results.

Digression on The Limiting Spread

As was calculated above, in this example, the limiting spread is equal to the credit spread of the existing debt. This is no coincidence, and this digression will explain why it is so, and what the implications are. As it turns out, the framework presented here provides a helpful interpretation of the limiting spread.
Recall the definition of the limiting spread:

$$S = \frac{\mathbb{E}^{\mathbb{Q}}[\phi] R}{1 - \mathbb{E}^{\mathbb{Q}}[\phi]}$$

For a constant face value, the actual credit spread can be calculated by dividing the face value of the debt with the present value of the debt claim and subtracting the gross risk-free rate. Denote the face value by F. The present value of the debt claim is the discounted face value, $\frac{1}{R}F$, adjusted for the credit risk. As usual, the CVA corresponds to the discounted expected positive exposure multiplied by the loss rate given default, i.e. $\text{CVA} = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[F\phi]$. Hence the credit spread of existing debt is given by:

$$\frac{F}{\frac{1}{R}F - \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[F\phi]} - R = R\left(\frac{1}{1 - \mathbb{E}^{\mathbb{Q}}[\phi]} - 1\right) = \frac{\mathbb{E}^{\mathbb{Q}}[\phi] R}{1 - \mathbb{E}^{\mathbb{Q}}[\phi]}$$

Which can be recognised as the limiting spread. Hence, in the current setup, the definitions of the marginal shareholder valuations are based on the assumption that the post-project credit spread will equal the pre-project credit spread. However, these definitions are derived using infinitesimal projects.
In practical applications, it would be more appropriate to refer to the limiting spread as an approximation of the credit spread after obtaining a larger-than-infinitesimal project.

The mathematics have been accounted for, but the approximation does seem to be financial legitimate as well. First, the post-project credit spread is based on the risk of the entire firm, which likely encompasses many projects. Each new project is therefore going to be of relatively small size compared to the entire firm, and it will only have a small effect on the overall riskiness. Second, this effect is enhanced if the firm mostly invests in projects that match its overall riskiness in the first place. These two points work, since the credit spread does not change if the riskiness does not change.
Third and last, as was mentioned earlier, according to Castagna (2012), the cost of capital will only gradually update, such that obtaining a new project only have a very marginal effect on the credit spread.

In conclusion, the limiting spread is not merely a mathematical result but also a financially sound approximation of the incremental cost of obtaining new projects. With this interpretation in mind, the example can continue.

10.2 Obtaining a Risk-Free Project

Assume that the firm faces a new risk-free project in which it can invest, and that the project has a promised return of 1.000 at time 1 summarised in table 3. The project is risk-free in the sense that there is no market risk, and the payoff is known with certainty at time 0. Additionally, the counterparty of the project is assumed to be credit risk-free.

i	1	2	3	4	5	Present value
$Y(\omega_i)$	1.000	1.000	1.000	1.000	1.000	0.990

Table 3

Obtaining the payoff requires an upfront cost of the project. With a slightly redefined notation, since the project is not infinitesimal, the marginal investment cost of the project will be denoted by u. Assume that the counterparty offering the project makes a zero net present value investment, such that the price of the project equals the expected discounted value, i.e. $u = \pi(Y)$. When obtaining financing, the total asset value of the firm increases correspondingly. The time 1 value of the assets after obtaining the project will be denoted by $\tilde{A}$.
As discussed in previous sections, the firm has different means of obtaining funding for paying the investment cost of the project. Financing through debt issuance is examined in the following section and financing by equity issuance in the section after that.

10.2.1 Free Riders From Firm Frictions

In this section, the firm finances the new project by issuing new debt. In order to fund the upfront of the project, u, the firm must issue debt to new creditors such that the price of the debt, equals the price of the project.
As assumed previously, the new debt ranks pari passu with the legacy debt, such that all creditors experience the same loss rate in states where the firm defaults. The setup is illustrated in figure 7.
Denote the face value of the new debt claim by $\tilde{D}_1$, i.e. the value of the debt at time 1 if the firm does not default. A claim with face value $\tilde{D}_1$ which ranks pari passu to another claim with face value L has the random loss rate given by equation 9.3:

$$\begin{aligned}\phi &= \frac{\tilde{D}_1 + L - (A+Y)}{\tilde{D}_1 + L}\mathbb{1}_{\mathcal{D}} \\ &= \frac{\tilde{D}_1 + 80.000 - (A+Y)}{\tilde{D}_1 + 80.000}\mathbb{1}_{\{A+Y<\tilde{D}_1+80.000\}}\end{aligned}$$

The random payoff of the new pari passu debt is denoted by $\tilde{D}$ and given as:

$$\tilde{D} = (1-\phi)\tilde{D}_1$$

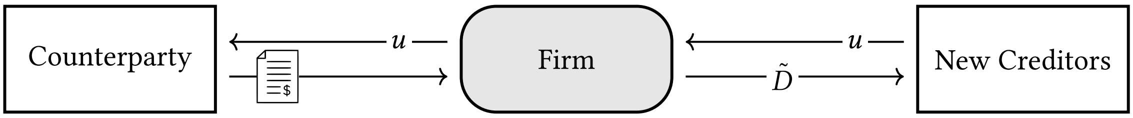

Figure 7: Illustration of funding a bond by issuing new debt.

In order for the firm to attract new creditors, they must offer a large enough face value on the debt, such that buying the debt is a zero net present value investment. The new debt should be able to cover the investment cost of the new project; hence, the present value of the new debt must equal u. Therefore, the face value must be chosen to solve the following equation:

$$\begin{aligned} & u = \pi(\tilde{D}) \\ \Leftrightarrow \qquad & 0.990 = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[1-\phi\right]\tilde{D}_1 \end{aligned}$$

While seemingly simple, this equation proves rather difficult to solve analytically for $\tilde{D}_1$. By substituting the loss rate, it is clear that the face value is quite entangled in the equation; both in the default event and in the fraction:

$$0.990 = \mathbb{E}^{\mathbb{Q}}\left[1 - \frac{80.000 + \tilde{D}_1 - A + Y}{80.000 + \tilde{D}_1}\mathbb{1}_{\{A+Y<80.000+\tilde{D}_1\}}\right]\tilde{D}_1$$

However, by studying this expanded equation, it is clear that the right hand side is increasing in $\tilde{D}_1$, as that holds true for all the terms that constitute it. Hence, the equation $u = \pi(\tilde{D})$ is very much suited for any simple numerical optimisation procedure for root finding, as long as the evaluation of the function itself is not too computationally heavy. In a single-period framework, especially one with as few states as the current, the evaluation of the expectation is by no means problematic. Since it is effective and simple, the equation will be solved by numerical procedures, which yields the following solution:

$$\Leftrightarrow \qquad \tilde{D}_1 = 1.031$$

Entering into the project and issuing new debt with face value 1.031 alters the payoffs associated with the firm described in table 4.

i	1	2	3	4	5	Present value
$\tilde{A}(\omega_i)$	121.000	111.000	101.000	96.000	61.000	97.390
$D(\omega_i)$	80.000	80.000	80.000	80.000	60.224	76.827
$E(\omega_i)$	39.969	29.969	19.969	14.969	0.000	19.573
$\tilde{D}(\omega_i)$	1.031	1.031	1.031	1.031	0.776	0.990

Table 4

The new project has therefore resulted in an expected default loss to creditors of

$\mathbb{E}^{\mathbb{Q}}[\phi] = 2.996\,\%$, and a new credit spread on debt of:

$$\frac{1.030889}{0.99} - R = \frac{76.826874}{80} - R = 3.120\,\%$$

The example will continue with this credit spread, instead of the limiting spread, to calculate the correct funding costs.
Recall that the limiting spread, i.e. the estimation of the post-project credit spread, was calculated as $S = 3.157\,\%$. Since the limiting spread is an overestimation of the actual credit spread, the funding costs will be overestimated as well. The overestimation is due to the project's risk compared to the firm's assets; since the project is risk-free, it reduces the overall riskiness of the firm as well as the loss rate and the credit spread. Thus, the examples use the actual credit spread instead of the estimation, such that the focus can remain on the funding implications without the uncertainty of estimations.

Investing in the project increases the present value of the firm's assets by $97.390 - 96.400 = 0.990$, which, not surprisingly, is the value of the project. More interesting is the impact on the shareholders' claim that decreases by an amount $19.573 - 19.600 = -0.027$; therefore, investing in the project is of negative value to the shareholders. If the firm defaults, the shareholders still receive nothing, but in all other states where the firm does not default, the shareholders pay the interest owed to the new creditors due to the credit risk in the firm. The legacy creditors observe their debt claim with face value 80.000 increase by an amount $76.827 - 76.800 = 0.027$.

Evidently, while the project is a negative net present value investment for the shareholders, the investment is a positive net present value investment for the legacy creditors. When the firm does not default, the creditors still receive their promised payoff corresponding to the face value of their debt. However, the new project has increased the asset base of the firm; if the firm defaults, the legacy creditors will share part of the payoff from the project with the new creditors. Hence, the legacy creditors are receiving a larger amount at the default compared to the pre-project amount, and their claim increases in value. Their increasing wealth also suggests that the conditions have improved in terms of their exposure to the firm's credit risk. This is also justified, as the loss rate given default has decreased from the pre-project value of 25.000 %. Correspondingly, the new credit spread is lower than the pre-project credit spread.

The Modigliani-Miller invariance proposition assures that making a zero net present value investment does not increase the value of the firm. That is to say that obtaining the new project should neither create nor destroy value in aggregate terms, which explains why the amount lost by shareholders is the amount gained by legacy creditors. The asset base also increases by the value of the project, but decreases by the value of the debt claim used to finance it; both are priced the same and therefore offset each other.
In this example, the wealth lost by the shareholders is entirely transferred to the legacy creditors, since the project is a zero net present value both to the new creditors and to the counterparty offering the project. Had the project been a negative net present value investment for the counterparty, or perhaps the new creditors, the shareholders would

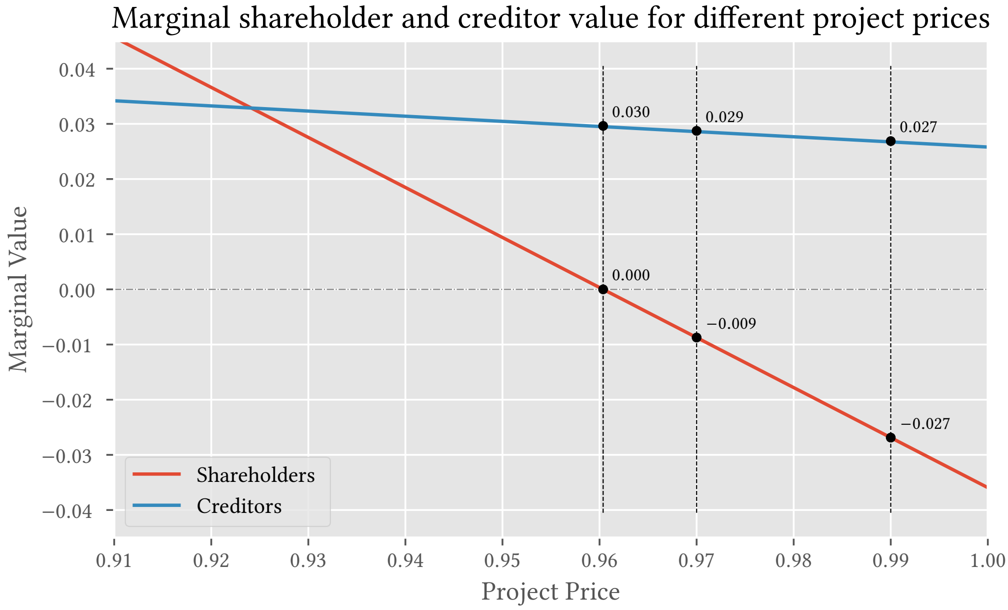

Figure 8: Marginal valuation of debt financing assuming that new creditors break even.

benefit from a share of this discount, and the value gained by legacy creditors would not be entirely from the shareholders.

The wealth transfer from shareholders to legacy creditors, due to the new project, suggests quantifying the FVA of the project. Using equation 9.21, the price needed for breakeven is:

$$\begin{aligned} u^*_{\text{debt}} &= \frac{1.000}{1.000 + 1.010\,\% + 3.120\,\%}(1.000 - 0.000) \\ &= 0.960 \end{aligned}$$

And, according to equation 9.22, the FVA is therefore:

$$\text{FVA} = 0.990 - 0.960 = 0.030$$

Any prices above the breakeven price, u^*_{debt}, will make the project a negative net present value investment for the shareholders, while any prices below will make it positive.

Consider figure 8 showing the marginal value of debt financing for creditors and shareholders for different values of the upfront price. The first price to consider is the price that was analysed so far, 0.990. At this price the entire gain for creditors can be attributed the loss for shareholders.

Suppose that the counterparty has some additional incentive to sell the derivative, such that it quotes a lower price than the theoretical, say 0.970. Selling the derivative is now a negative net present value investment for the counterparty; in other words, the counterparty has made a donation to the firm. The marginal valuation of the project to the shareholders is then −0.009, and to the creditors 0.029. The donation from the

counterparty has made an increase in the shareholders' *and* to the creditors' marginal valuation relative to the price of 0.990. Still, for a firm maximising shareholder value, the issue is the same as before; the project is a negative net present value investment for the shareholders, and therefore it will not be accepted.
Clearly, the donation from the counterparty can be distributed between the stakeholders in a non-wealth destructive way. For example, if the legacy creditors were willing to lower the face value of their claim to 79.980, that would leave a wealth gain for the creditors and shareholders of 0.011 and 0.009 respectively. This leaves all stakeholders economically happy, except of course from the counterparty who has other incentives.

Recall figure 5 describing the argument of Hull and White (2012). By their conviction, if a bond with a discount rate of risk-free rate plus 30 bp traded at an excess yield of 80 bp, it should be traded. This example shows exactly why that does not work in reality. At $u = 0.970$, the risk-free project in this example trades at an excess yield of approximately 100 bp. Even then, the firm will not be able to get its shareholders on board.

From the perspective of a social planner this problem seems to be a no-brainer. However, the operation relies solely on the willingness of creditors to reduce their face value, lower their seniority, or otherwise worsen the terms of their claim. In reality, it is very unlikely for the firm to be able to renegotiate the terms of their debt issuance, with multiple creditors all with different incentives.
The issue here is a free rider problem. The shareholders pay the funding costs of the new debt in the no-default states, but they do not reap the benefits. The creditors, on the other hand, pay nothing but still see the value of their claim increase.

Another observation from the graph in figure 8 is that the donation from the counterparty is distributed between creditors and shareholders. As evident from the slope, the shareholders receive a larger share than the creditors.
The creditors benefit from the donation since a lower price will decrease the face value of the new creditors, and therefore decrease the loss rate of the legacy creditors in the default state. The decrease of the new creditors' face value reduces the funding costs, which benefits the shareholders in the no-default state and increases their wealth.

This example has shown an apparent friction in the way firms obtain projects. Even when the project is heavily discounted the firm is unable to obtain it and still preserve its shareholders' wealth. It has also shown the problem with the argument of Hull and White (2012), since the firm is not able to distribute the positive net present value to the ones financing it.

10.2.2 Increasing Funding Costs With Equity Issuance

This section will consider how funding with equity issuance compares to results of the previous section, where the project was financed with debt issuance. Funding by share issuance will create a dilution, and the value of the already existing shares, owned by the legacy shareholders, will be reduced. To compare debt and equity funding, the project under consideration will be the same as in the previous section.

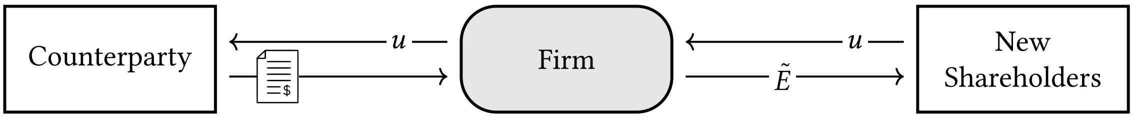

Figure 9: Illustration of funding a bond by issuing new equity.

A simple illustration of the new setup is depicted in figure 9.

The random payoff of the new shareholders will be denoted by $\tilde{E}$. The new shareholders will subscribe to the newly issued shares only to the extent that the investment has a net present value of zero. To be able to finance the project, the value of the new equity should therefore equal the upfront price of the project:

$$u = \pi(\tilde{E})$$

The new equity will lead to an increase in the total asset value of the firm, reducing the loss rate of the creditors in the default state. Therefore, the creditors should see an increase in the value of their claim.
The increase in wealth for the creditors is, however, at the expense of the legacy shareholders, since, like under debt financing, the shareholders bear the funding costs. As shareholders have an even higher required rate of return compared to creditors, the wealth transfer from the shareholders in this section should also be higher.

The payoff of the legacy shareholders is computed slightly differently than in the case of debt issuance. It is now essential to know the share of the equity the new shareholders are entitled to. This share will be denoted by α. The new shareholders' share equals the size of their investment, relative to the present value of total equity after the capital inflow:

$$\alpha = \frac{u}{\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[(A + Y - L)^{+}\right]}$$

The expected value of the equity is $\mathbb{E}^{\mathbb{Q}}\left[(A + Y - L)^{+}\right] = 20.677$, and with a project price of $u = 0.99$, the new shareholders are entitled to a share of $\alpha = 4.836\,\%$ of the total equity. The new shareholders then receive their share of the equity in each state, and the payoff to the legacy shareholders is the residual share of the total equity. The random variables denoting the shareholders' payoffs are therefore given by:

$$E = (1 - \alpha)(A + Y - L)^{+}$$
$$\tilde{E} = \alpha(A + Y - L)^{+}$$

After the transaction, the firm's balance sheet has increased with an amount equal to the upfront price. The firm has a new asset on the asset side, the project receivable worth 0.990, which is funded by 0.990 worth of new equity. The resulting payoffs of the example are shown in table 5, where the realised asset values include the expected payoff of the project.

i	1	2	3	4	5	Present value
$\tilde{A}(\omega_i)$	121.000	111.000	101.000	96.000	61.000	97.390
$D(\omega_i)$	80.000	80.000	80.000	80.000	61.000	76.920
$E(\omega_i)$	39.017	29.501	19.984	15.226	0.000	19.480
$\tilde{E}(\omega_i)$	1.983	1.499	1.016	0.774	0.000	0.990

Table 5

As anticipated, the project is a negative net present value investment for the legacy shareholders. The decrease in the value of their claim is calculated using equation 9.11:

$$G_{\text{equity}} = -(1.000 - 87.879\,\%) \cdot 0.990 = -0.120$$

This amount corresponds to the transfer of wealth from the legacy shareholders to the creditors, since the project is a zero net present value investment for the counterparty. Evidently, in accordance with the pecking order theory in equation 9.18, the value lost by the legacy shareholders is even higher for equity issuance than for debt issuance.

With the wealth transfer from the shareholders, the project is a positive net present value investment for the creditors; their wealth increases by:

$$-G_{\text{equity}} = 76.920 - 76.800 = 0.120$$

The value of the debt increases, since the creditors get the entire project receivable in the default state. Following this, the loss rate as well as the credit spread decreases significantly compared to the pre-project metrics.
The new loss rate in the default state is $\phi(\omega_5) = (80.000 - 61.000)/80.000 = 23.750\,\%$, and the new credit spread is $80.000/76.920 - R = 2.994\,\%$.

When trading decisions are made, the firm's preferences are assumed to be determined by the shareholders. It is clear that the legacy shareholders' position deteriorates, as they bear the funding costs from financing the project. So, to get the shareholders on board would require a donation by the counterparty, via a reduction in the upfront price of the project.

The breakeven price necessary to account for funding costs under equity funding is calculated using equation 9.23:

$$u^*_{\text{equity}} = \frac{1}{1 + 1.010\,\%} \cdot 87.879\,\% \cdot (1.000 - 0.000) = 0.870$$

and the adjustment needed is then:

$$\text{FVA} = u^*_{\text{equity}} - u = 0.990 - 0.870 = 0.120$$

Notice that this amount is the same as the wealth transfer. This happens because the donation from the counterparty does not affect the total asset value of the firm. Hence,

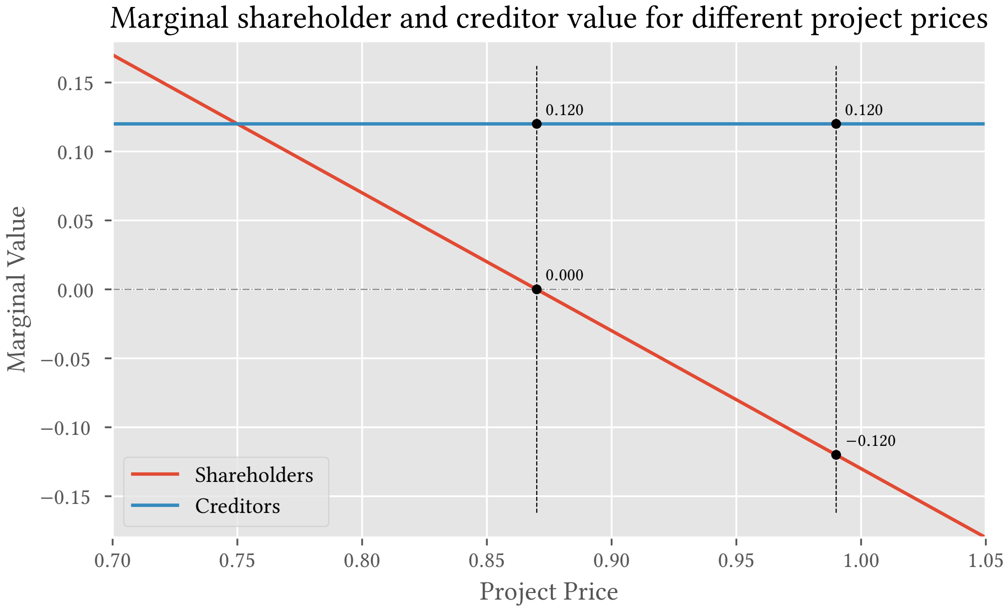

Figure 10: Marginal valuation of equity financing assuming that new shareholders break even.

the loss rate of the creditors remains unchanged from before the donation, and therefore so does the creditors' present value.

This is confirmed by figure 10 showing the marginal value for creditors and shareholders for different upfront prices. The creditors' marginal value is constant in the donation from the counterparty, but they benefit from a fixed wealth increase since the project increases the asset value. Therefore, under equity financing, the legacy shareholders receive the entire counterparty donation. As the donation from the counterparty increases, the new shareholders will finance a still cheaper project, and, since it is still assumed to be a zero net present value investment for them, their share of the firm's remaining capital is reduced. On the contrary, the legacy shareholders' share increases, and so does the value of their claim.

Compared to debt funding, issuing new equity is significantly worse for the legacy shareholders. The new shareholders have a much higher required rate of return than the new creditors, and as the legacy shareholders bear the funding cost, it is more expensive for them. A firm maximising its shareholders' wealth should therefore rather fund with debt issuance.

However, donations from the counterparty are more effective when the project is funded by equity issuance. When funded by debt, a share of the donation from the counterparty goes to the creditors; therefore, that share is not earned by the shareholders. As already covered, this is evident from figure 8 by the slope on the line showing the creditors' marginal value. Under equity financing the entire donation goes to the legacy shareholders; hence, the steepness of the line showing the shareholders' valuation in figure 10 is greater than the corresponding line in figure 8.

These two examples have shown how funding costs have influenced the values of shareholders' and creditors' claims. They clearly displayed the free rider problem that occurs since creditors reap the benefits of the project financed by the shareholders. This friction restricts which projects can be obtained by a firm; even positive net present value investments might deteriorate the shareholders' position, and will therefore not be obtained by the firm. The problem is even greater when financing with equity, since the new shareholders' required rate of return is higher than new creditors'.

The firm can use the breakeven price as a benchmark for projects. If the price of the project is higher than the breakeven price, the shareholders will lose wealth from obtaining it. Therefore, the breakeven price and an FVA provide a mechanism for the firm to align its interest with its shareholders'. If the firm's dealers apply FVAs in their valuation, they will only obtain projects that, at a minimum, preserve the shareholders' wealth.

The section has not covered the case of financing with existing cash since the main results of doing that can be summarised briefly here. If the project in question is a zero net present value investment, there is no wealth transfer under cash funding, and all stakeholders' wealth are preserved. When receiving a donation from the counterparty, some of the donation is obtained by the creditors, similar to the mechanism under debt funding. The same graphic for cash funding, as drawn for debt- and equity funding, can be found in Appendix A.

10.3 Selling Corporate Bonds With Funding Benefits

In the previous section, the outgoing cash flows, due to the upfront cost of the project, demanded funding from the firm, which led to funding costs. The firm would ultimately have to do an FVA, to account for the funding expenses that obtaining the project would lead to. On the contrary, when an instrument has cash flows going into the firm, it might have funding implications that are beneficial to the firm. Recall, from the introduction of FVA, the decomposition of FVA into two elements:

$$\mathrm{FVA} = \mathrm{FVA}_{\mathrm{cost}} + \mathrm{FVA}_{\mathrm{benefit}}$$

In this section, the firm is assumed to sell a single-period corporate bond with a face value of 1.000. The bond is unsecured, and will now be treated as a new liability to the firm, which ranks pari passu with the already existing debt.

Since the counterparty buys the bond from the credit-risky firm, the counterparty must account for this exposure. If the firm defaults it will not be able to meet its obligations, and the face value will not be fully paid out.
For the counterparty's incentives to be re-established, the price of the bond needs a valuation adjustment, i.e. CVA_C. By symmetry, $\mathrm{CVA}_C = \mathrm{DVA}_F$, where DVA_F is the debit value adjustment made by the firm in order to account for its own risk of default. Since the firm can only have positive exposure, CVA_F equals 0, and the theoretical market

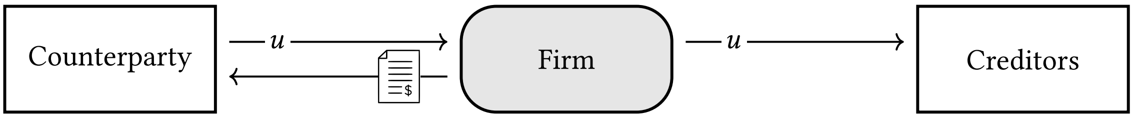

Figure 11: Illustration of the bond premium being used to retire existing debt.

price of the bond should therefore be given by:

$$P = \frac{1}{R} \cdot 1 - \mathrm{CVA}_C = \frac{1}{R} \cdot 1 - \mathrm{DVA}_F \tag{10.1}$$

Now consider the funding issue. On the transaction date of the bond, the firm receives a premium from the counterparty that can be invested providing a funding benefit. This section will study two possible investments. One where the upfront price is used for retiring existing debt that trades at a credit spread over the risk-free rate. Another where the upfront price is used for buying back equity.

10.3.1 Retiring Legacy Debt

In the following example, the firm uses the upfront price, for retiring existing debt from its legacy creditors. As it turns out, this will lead to the DVA accounting for all funding benefits, thus eliminating the need for an FVA. Before seeing this from calculations, it can be argued conceptually and the reasons for it can be highlighted.
By the assumptions in these examples, the only source of the firm's credit spread is the firm's own credit risk, which implies that there is no difference between the CDS spread referencing the firm, and the rate that the firm pays for funding. The CDS spread determines the cost to the counterparty of hedging its credit exposure to the firm. The funding rate determines the benefits to the firm of cash surpluses. Therefore there is a strong relation between the two adjustments.
Having hinted the possible concerns in the current setup, the example can continue and the issues can be quantified.

Notice that when the upfront cost is used to purchase debt at a fair market value, a new liability, the bond, is replacing a part of the existing liability, the legacy debt. Remembering the pari passu assumption, this debt-for-debt substitution leaves the total liability unchanged.
An outline of the setup is shown in figure 11.

The implications of debt buyback are the following. First, the net present value for the creditors who tender is zero. Hence, the loss rate as well as the credit spread has to be left unchanged after partly retiring the debt, otherwise the creditors would not tender. Second, the counterparty charges for the firm's credit risk to the extent that it is a zero net present value for the counterparty. The cash flow following the bond can be viewed as newly issued debt with a fair credit spread offered to the creditors. Third, the tendering creditors stay unaffected of the transaction, as the loss rate and credit spread is left intact.

Since the bond is a zero net present value for the firm, the counterparty, and the creditors, there is no wealth to transfer anywhere, and the contract must also have zero net present value to the shareholders. Hence, the bond fair value must be given by equation 10.1 as any other value would influence the wealth of the parties involved.

Assume now that the firm sells the bond to a counterparty for an upfront price of 0.960, which is spent on retiring debt from the firm's creditors. The bond payable is now a liability to the firm, ranking pari passu with the existing debt. Recall that the pre-project face value of the debt is 80.000.
The payoffs to the firm's stakeholders after selling the bond are shown in table 6, where the payoff to the counterparty is denoted by Y_C. The loss rate in the default state remains unchanged and is identical for the creditors and the counterparty $\phi(\omega_5) = 1 - 60.000/80.000 = 25.000\,\%$. Similarly, the credit spread remains the same as the pre-project value $79.000/75.840 - R = 3.157\,\%$.

i	1	2	3	4	5	Present value
$\tilde{A}(\omega_i)$	120.000	110.000	100.000	95.000	60.000	96.400
$D(\omega_i)$	79.000	79.000	79.000	79.000	59.250	75.840
$E(\omega_i)$	40.000	30.000	20.000	15.000	0.000	19.600
$Y_C(\omega_i)$	1.000	1.000	1.000	1.000	0.750	0.960

Table 6

The liability side of the firm's balance sheet will then look like:

	Book Value	**DVA**	**Market Value**
Equity	19.600	0.000	19.600
Debt	79.200	3.360	75.840
Derivative Payable	0.990	0.030	0.960
Total	99.790	3.390	96.400

The transaction is a zero net present value investment for the shareholders. As can be seen in table 6, the shareholders receive a payoff at time 1 equal to the pre-project value. The counterparty's payoff coming from the bond, Y_C, has a market value of:

$$\pi(Y_C) = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y_C\right] = 0.960$$

This concludes that the bond premium that the firm receives at time 0 is equal to 0.960, which is then the amount used for retiring debt. On top of this, the amount can be recognised as the price of the riskless bond adjusted for the firm's credit risk:

$$V = \frac{1}{R} \cdot 1 - \mathrm{DVA}_F = 0.990 - 0.030 = 0.960$$

which validates equation 10.1.

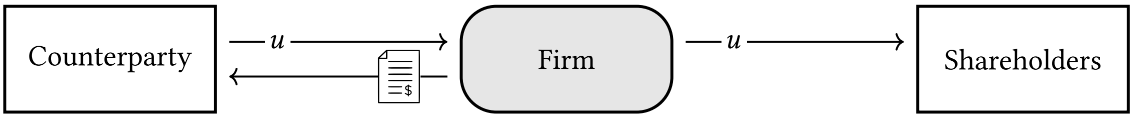

Figure 12: Illustration of the bond premium being used to buy back equity from shareholders.

All funding benefits have already been accounted for by the DVA; therefore, there is no FVA to make.
Take the first possible definition of FVA, where it was defined as the promised excess funding cost. Calculating this quantity yields $\frac{1}{R}uS = 0.990 \cdot -0.960 \cdot 3.157\,\% = -0.030$, which is exactly equal to the DVA with the sign flipped. Recall that Hull and White (2012) use this exact argument to argue against FVAs. As hinted in the beginning, this is not a coincidence as the CDS spread determining DVA is identical to the funding rate determining FVA. Adjusting the DVA-adjusted price with the first definition of FVA would be double counting, and would lead to the firm undervaluing the bond.

Consider also the last, and chosen, definition of FVA, where it was defined as the adjustment needed to the shareholders' breakeven. As already mentioned, the marginal shareholder value of the bond is zero, meaning that the price is already adjusted to the shareholders' breakeven. The fact is that the bond in this example needs no FVA, as the DVA accounts for all funding benefits.
Differences between DVA and an adjustment for funding benefits can only occur either because of a mismatch between the CDS spread and the funding spread or because of a difference in the portfolio on which the two quantities are calculated. The former has been assumed non-existing. The latter could occur if the firm had offsetting trades in its portfolio, since the FVA takes into account all sorts of netting benefits. In this example there is no offsetting trades, which is why there is no need for an FVA, as the DVA have already accounted for everything.

10.3.2 Buying Back Equity

In this section the firm still sells its corporate bond with a face value of 1.000. Now, the upfront price is used for buying back equity from the shareholders. This setup is illustrated in figure 12.

In the previous section, selling the bond and retiring debt with the upfront, simply corresponded to exchanging some existing debt for some other debt. When buying back equity, the bond value adds to the aggregate liabilities of the firm, and ranks pari passu with the existing creditors. This section studies the beneficial impacts on the firm's shareholders of doing this transaction. At time 0, the firm allocates some cash to the shareholders; therefore, if the firm defaults, the shareholders have already received some payoff. This leaves a smaller amount to the creditors compared to their pre-project payoff.

The firm's balance sheet is left unchanged, meaning that the sum of the market values of debt, equity, and bond payable is the same after the transaction. The values of the different parties are shown in table 7. The loss rate increases from the pre-project value and is computed as: $\phi = 1 - 60.000/81.000 = 25.926\,\%$. Similarly, the credit spread increases to $80.000/76.711 - R = 3.277\,\%$.

i	1	2	3	4	5	Present value
$\tilde{A}(\omega_i)$	120.000	110.000	100.000	95.000	60.000	96.400
$D(\omega_i)$	80.000	80.000	80.000	80.000	59.259	76.711
$E(\omega_i)$	39.000	29.000	19.000	14.000	0.000	18.730
$Y_C(\omega_i)$	1.000	1.000	1.000	1.000	0.741	0.959

Table 7

The project is again assumed to be a zero net present value investment for the counterparty. In other words, the firm receives an upfront of the credit risk adjusted market value, i.e. $u = -0.959$, as the bond premium at time 0, which is immediately spend on equity buyback.

Buying back equity leaves two payoffs for the shareholders: (i) the immediate payoff corresponding to the bond premium, and (ii) the present value of the expected payoff at time 1.

To establish whether entering into this project is attractive for the shareholders, the expression for the marginal valuation of equity funding, equation 9.11, is used. The payoff for the firm is here the negative bond payoff, i.e. Y. The survival probability is $p^{\mathbb{Q}} = 87.879\,\%$, and the covariance between the indicator of default states and Y is 0 as the promised payoff is known with certainty. The change of wealth for the shareholders is then calculated as:

$$G_{\text{equity}} = 0.089$$

Again, as the price of the bond is assumed to be the credit risk adjusted market value, the bond is a zero net present value for the counterparty. The positive marginal valuation for the shareholders then implies a wealth transfer from the creditors to the shareholders. Hence, the bond contract is a positive net present value investment for the shareholders, where the change of wealth is verified by the change in the present value of the expected payoff:

$$(18.730 + 0.959) - 19.600 = 0.089$$

On the other hand, the transaction is a negative net present value investment for the creditors:

$$76.711 - 76.800 = -0.089$$

Notice that, even though the price received on the bond is lower than in the previous section, the shareholders see an increase in their wealth. In the last section, the price was 0.960 and the shareholders broke even. In this example, the upfront payment is lower, but the shareholders' marginal valuation is positive. This confirms the pecking order theory for $u < 0$; shareholders prefer equity buybacks over debt retiring.

The positive marginal valuation of the shareholders does suggest, that they are willing to accept an even lower price than 0.959. The lowest price the shareholders will accept, is the breakeven price, which, under equity financing, can be computed from equation 9.23:

$$u^*_{\text{equity}} = -\frac{1}{R}87.879\,\% = -0.870$$

which suggest an FVA given by:

$$\text{FVA} = u^*_{\text{equity}} - u = -0.870 - (-0.959) = 0.089$$

The positive valuation adjustment suggests an overall funding benefit.

Similar to the case of buying the bond by issuing new shares analysed in section 10.2.2, the FVA amount is equal to the wealth transfer. Evidently, the price cut from the firm is translated directly to a wealth loss for the shareholders while the creditors wealth is maintained. The donation from the firm to the counterparty does not affect the total asset value; therefore, the loss rate on the debt and the bond payable remains unchanged, and so does the present value of the creditors' and counterparty's payoffs.
The creditors still lose wealth worth 0.089 from the firm obtaining the project. The shareholders are however willing to receive a lower price of the project than 0.959. For all prices down to 0.870, the shareholders will still consider the transaction a non-negative net present value investment.

In conclusion, when financing with equity issuance or buybacks, the price of the project is irrelevant to the creditors, as price adjustments are transferred directly to shareholders. The wealth of the creditors is solely dependent on the payoff structure of the project; their wealth increases when the asset value increases and vice versa.

10.4 No Funding Buyback

Now suppose, instead of buying back debt or equity, the firm spends the cash proceeds from the upfront price in riskless assets. If the derivative traded with the counterparty has a known payoff at time 1, this procedure is virtually the same as if the firm borrows money to buy the risk-free asset as described in section 10.2.1. To deviate from the exact same conclusions, the derivative is now assumed to have a payoff structure at time 1 as described in table 8. The derivative will still be fairly priced, i.e. the price is the discounted expected payoff for the counterparty, such that the credit risk of the firm is accounted for in the price. At time 1 the firm will pay the counterparty accordingly, hence the derivative payable, is a contingent liability to the firm and ranks pari passu with the already existing debt.
This section analyses whether this capital structure setup causes reasons for funding benefits. Intuitively, this could seem appropriate as the firm receives cash proceeds from the upfront price that can be invested beneficially. The setup is depicted in figure 13.

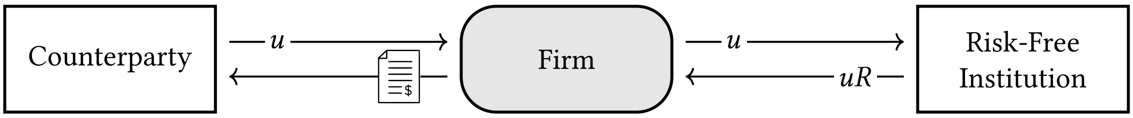

Figure 13: Illustration of derivative premium being used to invest in riskless assets.

i	1	2	3	4	5
$Y(\omega_i)$	−4.000	−3.500	−2.500	−1.500	−1.000

Table 8

After receiving the upfront price in the form of cash, the firm immediately invests in the risk-free asset. This implies that the firm's total asset value increases with an amount equal to the forward discounted upfront price. The capital structure, after this transaction, is described in table 9. The derivative's credit risk adjusted price and loss rate given default are found by solving the following system of equations for $\pi(Y_C)$ and $\phi(\omega_5)$:

$$\begin{cases} \pi(Y_C) = \dfrac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y_C\right] \\ A(\omega_5) + uR = D(\omega_5) + Y_C(\omega_5) \end{cases} \tag{10.2}$$

with the second equation ensuring that the sum of the payoffs to the creditors and the counterparty equals the asset value in case of default, i.e. state ω_5. The realised asset value at maturity here includes the value of the derivative adding to the total asset base.

The loss rate given default obtained from solving the above problem is $\phi(\omega_5) = 23.036\,\%$, and the new credit spread is then $80.000/76.989 - R = 2.901\,\%$. Compare the new credit spread with the pre-project credit spread of 3.157 %; by obtaining the new project, the credit spread has decreased quite significantly. The reduction can be explained by the project's payoff structure, as the states where the firm's asset value is the highest are also the states where the project's payoff is the highest and vice versa. Therefore, by selling the project, the firm has reduced its riskiness as well as the loss rate and credit spread of the creditors.

i	1	2	3	4	5	Present value
$\tilde{A}(\omega_i)$	122.341	112.341	102.341	97.341	62.341	98.717
$D(\omega_i)$	80.000	80.000	80.000	80.000	61.571	76.989
$E(\omega_i)$	38.3408	28.841	19.841	15.841	0.000	19.411
$Y_C(\omega_i)$	4.000	3.500	2.500	1.500	0.770	2.317

Table 9

The market value of the derivative including adjustments for credit risk would deem a fair price of the project, as it would be a zero net present value investment for the counterparty. With such pricing, however, a transfer of wealth will occur between the shareholders and the creditors.

Selling a derivative and using the sale proceeds for buying the risk-free asset, is equivalent, but with opposite signs, to buying a derivative and funding the upfront price with existing cash. In the latter case, the existing cash could have earned the risk-free rate and it has an opportunity cost; in the former case, the sale proceeds actually earn the risk-free rate. This is taken care of by equation 9.12, which defines the marginal change in the shareholders' wealth using existing cash financing.
The marginal change in the equity valuation with $u = -2.317$ is therefore:

$$G_{\text{cash}} = p^{\mathbb{Q}}\pi - \frac{1}{R}\text{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) = -0.189$$

The change in wealth for the shareholders can be verified by the difference between the new present value of the equity and the old present value of the equity:

$$19.411 - 19.600 = -0.189$$

The change in wealth for the creditors is similarly verified by:

$$76.989 - 76.800 = 0.189$$

The increment happens as the risk-free asset, which the firm invested in with the upfront price, makes the asset value increase, and a larger total amount is recovered in case of default. If the firm does not default, the creditors are not affected by the new project. Hence, this is a negative net present value investment for the shareholders and a positive net present value investment for the creditors.
Due to the risk in the derivative, the shareholders do have a chance to gain from the project, where in state ω_4, the derivative payable is less than the return on the risk-free asset.

On the basis of the wealth transfer there seems to be some funding implications. The FVA is, as always, the adjustment to the project's apparent value to the firm that makes the shareholders indifferent to taking on the project. As the firm is funding the risk-free project with the premium proceeds, no funding benefits actually occur, and the FVA is constituted entirely by the funding costs from paying the derivative payable.

Using equation 9.25, which describes the shareholders' price of breakeven with cash financing:

$$u^*_{\text{cash}} = \frac{1}{R}\left(\mathbb{E}^{\mathbb{Q}}\left[Y\right] - \frac{\text{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)}{p^{\mathbb{Q}}}\right) = 2.532$$

yielding a valuation adjustment of:

$$\text{FVA} = u^*_{\text{cash}} - u = 2.532 - 2.317 = 0.215$$

Alternatively, this value can be computed by solving the related equation system for FVA_{cost} as described above, where the value to the firm is the funding cost adjusted market value:

$$\begin{cases} V = \pi(Y_C) + \text{FVA} \\ A(\omega_5) + uR = D(\omega_5) + Y_C(\omega_5) \\ G_{\text{cash}} = 0 \end{cases}$$

The first equation is the adjusted derivative value to the firm. This will replaces the fair market value pricing assumption, the second equation in equation 10.2, in the new setup. Furthermore, the third equation ensures that there is no wealth transfer from the shareholders.
The new problem solved yields a loss rate of $\phi = 22.769\,\%$, which is slightly lower, and the donation needed from the counterparty to make the project of zero net present value to the shareholders is FVA = 0.215.

As this example have shown, selling a derivative is not a sure-fire way to gain funding benefits. The investment that the firm makes with the upfront price has a huge impact on the shareholders' valuation, in accordance with the pecking order theory in equation 9.18. Investing in the risk-free asset is the least favourable investment the firm can make. The payoff structure of this specific derivative is clearly also a great driver, in the direction of the shareholders' welfare loss.
Using a structural model analogy; since the shareholders' claim can be modelled as a European call option, they would like for the riskiness of the underlying to increase due to the positiveness of the option Vega. Since the derivative in question reduces the riskiness of the firm's total assets, so does the value of the shareholders' claim.

10.5 Funding Secured Derivatives

Consider again the risk-free contract that had a promised payoff of 1.000 in every state. In this section, the firm sells the contract to a counterparty. Assume now that the firm and the counterparty have in place a CSA agreement that requires collateralisation with no threshold or minimum transfer amount. Therefore, at time 0, when the counterparty pays the upfront price, the firm posts the same amount as collateral.

In addition to the variation margin, the CSA agreement requires an independent amount to be posted, corresponding to 50 % of the purchase price at time 0.
This setup is illustrated in figure 14.

The independent amount will be a source of funding costs. Since the upfront price offsets the call for variation margin, neither the upfront price nor the variation margin will have any funding costs or benefits. The independent amount, on the other hand, has no offsetting cash flows, and therefore requires financing. In this section it is assumed that the firm finances the independent amount by issuing new debt, and the issue raised by this setup is how the valuation of the derivative is affected.

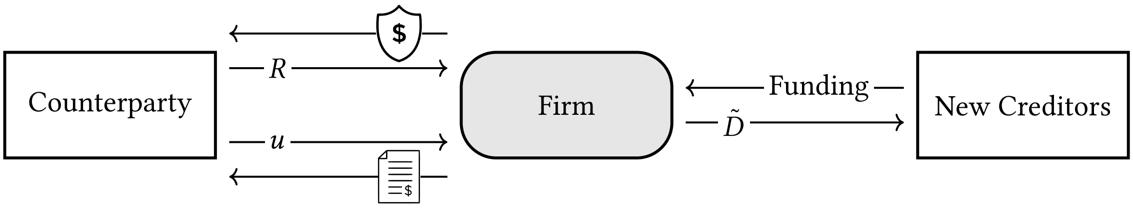

Figure 14: Illustration of the firm funding a derivative traded through a CSA agreement.

As previously, financing with debt issuance is obtained such that the new creditors make a zero net present value investment. The new creditors' debt claim ranks pari passu with the existing debt. At time 0, the asset value increases by the leveraged amount, i.e. the debt obtained for the independent amount. At time 1 the asset value has increased by this amount forward discounted, since the quantity earns the risk-free rate when posted as collateral.

The collateralisation is more than enough to secure the payoff of the derivative; therefore, the counterparty cannot suffer losses on the contract. Assume then that the price of the derivative is given as the credit risk-free value, which is the discounted promised payoff:

$$u = -0.990$$

Since the new debt is unsecured, the firm must pay a credit spread to cover its own credit risk. The deal is still a zero net present value investment for the new creditors. Hence, the present value of the new creditors claim, $\tilde{D}$, must equal the borrowed amount, $-\frac{1}{2}u$, and the face value of the new debt must therefore solve:

$$-\frac{1}{2}u = 0.495 = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[\tilde{D}]$$

Solving this numerically yields a face value of 0.516. The payoff structure at time 1 of the firm and its stakeholders is summarised in table 10.

i	1	2	3	4	5	Present value
$\tilde{A}(\omega_i)$	120.500	110.500	100.500	95.500	60.500	96.895
$D(\omega_i)$	80.000	80.000	80.000	80.000	60.113	76.814
$E(\omega_i)$	39.984	29.984	19.984	14.984	0.000	19.586
$\tilde{D}(\omega_i)$	0.516	0.516	0.516	0.516	0.387	0.495
$Y_C(\omega_i)$	−1.000	−1.000	−1.000	−1.000	−1.000	−0.990

Table 10

The loss rate given default and the new credit spread are respectively computed as:

$$\phi(\omega_5) = 1 - \frac{0.387 + 60.113}{0.516 + 80.000} = 24.859\,\%$$
$$\frac{80.000}{76.814} - R = \frac{0.516}{0.495} - R = 3.138\,\%$$

These have both clearly decreased compared to the pre-project values, which suggests that the legacy creditors' position has improved.

The collateralisation has turned the funding implications around, such that selling the derivative no longer provides funding benefits. In the default state, all creditors share the independent amount, and the payoff of the legacy creditors increases. Since the legacy creditors maintain their payoff in all other states, the total value of their claim increases. The shareholders are unable to capitalise on the upfront price, since that rests with the counterparty earning the risk-free rate. Instead, they bear the funding costs of obtaining unsecured funds for financing the independent amount; hence, the value of the shareholders' claim has decreased. The exact change of wealth for the legacy creditors as well as the shareholders are given respectively by:

$$76.814 - 76.800 = 0.014$$
$$19.586 - 19.600 = -0.014$$

The shareholders pay the funding costs of the project, and their welfare loss is transferred to the legacy creditors who enjoy the increase in asset value.

These results were found numerically, but can be verified by the marginal valuation equations if they are adapted slightly to the collateral postings. If the independent amount for an investment of size q is denoted by $I(q)$, the marginal increase in the value of the firms equity per unit investment is given by:

$$G_{\text{secured}} = \frac{\partial}{\partial q} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{R} \left(A + qY - L - U(q)R - I(q)s(q) \right)^{+} \right] \bigg|_{q=0}$$

The upfront price $U(q)$ earns the risk free rate, while borrowing the independent amount costs $s(q)$. Using the same derivation as earlier this can be rewritten as:

$$G_{\text{secured}} = \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c} \right] \left(\frac{1}{R} \mathbb{E}^{\mathbb{Q}} \left[Y \right] - u \right) - \frac{1}{R} \text{Cov}^{\mathbb{Q}} \left(\mathbb{1}_{\mathcal{D}}, Y \right) - \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c} \right] \frac{1}{R} IS \tag{10.3}$$

The last term might be referred to as the *margin value adjustment* of the derivative; as the name implies, it accounts for the funding implications of the margin postings. In this paper it is simply considered a part of the FVA.

If the independent amount is a share of the upfront price, the wealth increase for shareholders in this setup is lower than if there was no collateralisation. If the independent amount is determined as $I = -\alpha u$ for some positive share α, then, comparing with Φ from equation 9.9:

$$\mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\mathcal{D}^c} \right] \frac{1}{R} (-\alpha u) S > \Phi$$

if $u < 0$. This leads to the, perhaps unsurprising, conclusion that shareholders are worse off when required to invest the upfront price at the risk-free rate and obtaining unsecured funding than if they could retire debt with the upfront price.

Returning to the example at hand; equation 10.3 verifies the results by substituting in the survival probability, the independent amount, and the actual credit spread after obtaining the project:

$$G_{\text{secured}} = -\frac{1}{R} \cdot 87.879\,\% \cdot 0.495 \cdot 3.138\,\% = -0.014$$

In addition, the equation can help estimate the breakeven price. Substituting $I = -\frac{1}{2}u$ into equation 10.3, setting equal to 0, and solving for u yields:

$$0 = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\left(\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[Y\right] - u\right) - \frac{1}{R}\text{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right) + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\mathcal{D}^c}\right]\frac{1}{R}\frac{1}{2}uS$$

$$u^*_{\text{secured}} \equiv \frac{1}{R - \frac{1}{2}S}\left(\mathbb{E}^{\mathbb{Q}}\left[Y\right] - \frac{\text{Cov}^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}}, Y\right)}{p^{\mathbb{Q}}}\right)$$

Substituting the quantities from the example:

$$u^*_{\text{secured}} = -1.006$$

So, for the shareholders to accept entering the project, the counterparty must be willing to donate a price increase of 0.016.

Again, the shareholders are not willing to enter into the project at the credit risk-free price. The shareholders bear the funding costs of financing the independent amount, and so they transfer wealth to the legacy creditors, who reap the benefits of entering the project.

Even if the counterparty donated the amount lost by shareholders, 0.014, the project would still be a losing trade for the shareholders. The donation to breakeven must be more than the amount lost, since the legacy creditors receive a share of the donation.

In conclusion, a strong CSA agreement requiring independent amounts, might quickly turn a derivative with possible funding benefits into one with funding costs, such that shareholders are no longer on board without price adjustments.

Part IV

FVA in the Multi-Period Model

Still missing at this point is to study the effects of a project having cash flows in-between its inception date and maturity date. Such cash flows can also lead to financing requirements, which can have implications for the project's funding costs. Of course, the single-period model is unable to capture this complexity.

Therefore, this part will expand the single-period model to a multi-period model. The first section will provide the framework of a two-period model, which can easily be extended to more periods, and define a firm operating in such an economy. Additionally, the shareholders' financing costs of obtaining unsecured swap contracts will be derived.

The derivations will ultimately be followed by a numerical example with the purpose of quantifying the funding costs and -benefits of obtaining a swap contract. This should give a general comprehension of the effect from financing a derivative cash flows, which could then be carried over to more complex modelling.

First, the setup of the multi-period framework is defined.

11 Extending the Single-Period Model

11.1 The Multi-Period Model

The multi-period model extends the assumptions made in section 9 to include multiple time periods.

An additional time 2 is introduced as the new time of maturity. At the interim date, i.e. time 1, the market economy is defined by a collection of random variables, and at maturity, all the uncertainty is settled. The single-period gross risk-free returns at time 0 and time 1 are R_0 and R_1 respectively, where R_1 is random at time 0. Hence, the risk-free rate is not required to be constant, and the market value of future cash flows is defined as:

$$\pi(C_t) = \mathbb{E}_0^{\mathbb{Q}}\left[\delta_t C_t\right]$$

where $\delta_1 = \frac{1}{R_0}$, and $\delta_2 = \frac{1}{R_0 R_1}$. As time progresses forward through each period, increasingly more random variables will be realised, and the amount of information will grow. Each time period will have an associated information set, formally a σ-algebra, representing the information available at that particular period in time. The expectation operator with a subscript t, $\mathbb{E}_t$, will denote the expected value conditional on the information set at time t.

This setup considers a firm whose assets at time 2 have a payoff given by some random variable A. Additionally, the firm has short-term liabilities, L_1, that expires at time 1 as

well as long-term liabilities, L_2, that expires at time 2.
As always, the firm defaults the first time it fails to meet its financial obligations. At time 1 the firm liquidates a portion of its assets to cover the short-term liabilities. The time 2 value of this portion is denoted by $W = L_1 R_1$. Also, at time 1 the firm pays out a dividend, $\theta_1 \geq 0$, to its shareholders, which is considered a random variable. The default time of the firm is denoted by τ_F, where $\tau_F = \infty$ if the firm survives at time 2. When the firm does not default, its assets are liquidated, and the remaining cash is distributed to the shareholders after the creditors are paid back.
Finally, it is assumed that all liabilities rank pari passu, and the assets' recovery rates given default at time 1 and time 2 are denoted by κ_1 and κ_2 respectively.

As usual, the shareholders receive a payoff of 0 if the firm defaults, and the total value of equity is defined as:

$$\pi(E) = \mathbb{E}_0^{\mathbb{Q}}\left[\delta_1 \mathbb{1}_{\{\tau_F > 1\}} \theta_1\right] + \mathbb{E}_0^{\mathbb{Q}}\left[\delta_2 \mathbb{1}_{\{\tau_F > 2\}} (A - W - L_2)\right]$$

The first term defines the discounted expected value of the dividend payout if the firm does not default at the interim date. If the dividend is non-zero, it benefits the shareholders, in the sense that it guarantees the shareholders to receive some payoff before a possible default event at time 2.
The second term defines the discounted expected residual of the liquidated assets at time 2. If the firm does not have enough assets to cover the long-term liabilities, the creditors receive the remaining asset amount, and the firm defaults. This leaves a payoff of 0 for the shareholders.

On the contrary, the creditors receive a payoff in every scenario. The size of the payoff is fully dependent on if the firm defaults or not. The total value of the creditors' payoff is defined as:

$$\begin{aligned}\pi(D) = \mathbb{E}_0^{\mathbb{Q}}&\left[\delta_1\left(\mathbb{1}_{\{\tau_F > 1\}} L_1 + \mathbb{1}_{\{\tau_F = 1\}} \frac{\kappa_1 \mathbb{E}_1^{\mathbb{Q}}[A]}{R_1}\right)\right] \\ &+ \mathbb{E}_0^{\mathbb{Q}}\left[\delta_2\left(\mathbb{1}_{\{\tau_F > 2\}} L_2 + \mathbb{1}_{\{\tau_F = 2\}} \kappa_2 (A - W)\right)\right]\end{aligned}$$

Both terms basically defines the same thing but in the two different time periods. The first term inside each expectation operator is the expiring liability payoff at the respective point in time given no default. The second term in each expectation is the total asset value if the firm defaults. Note that for $\tau_F > 1$ the firm survives at time 1. Hence, at time 2 for both the shareholders and the creditors, the liquidated value at time 1, W, that was used to cover the short-term liabilities, is subtracted from the asset value.

To ensure that the present value of new creditors' claim equals the amount needed to fund a project, $U(q)$, the idea is fundamentally the same as in the single-period model. To unravel an explicit expression of the credit spread in a two-period model, equation 9.1, which shows the shareholders' breakeven value of a new obtained project, is now extended.

The default time of the firm depends on the size of the new project. To capture this, the default time, after the firm has obtained the project, will be denoted $\tau_F(q)$. The pre-project default time, which corresponds to $\tau_F(0)$ will, for simplicity, still be denoted by τ_F. The firm receives a payoff from its secured position, $Y_{1,t}$, as well as its unsecured position, $Y_{2,t}$, at both time periods. Furthermore, the liabilities are, as mentioned, separated into short-term and long-term.
Denote the no-default value of the new debt in each period by D_0, D_1, and D_2. These quantities represent the value if the firm does not default in the corresponding period, i.e. D_2 is the face value of the debt and D_1 is the time 1 value if the firm does not default at time 1.

The recovery rate of the creditors in each period is the share of liabilities remaining in the bankruptcy estate after distress costs. In period two, the amount paid for short term liabilities must be accounted for, and the recovery rate is given by:

$$\Pi_2(q) = \mathbb{1}_{\{\tau_F(q)=2\}} \frac{\kappa_2 \left(A - W + qY_{1,2} + qY_{2,1}^+\right)}{L_2 + D_2 + qY_{2,2}^-}$$

At time 1, the payoff of everything, but the short term liabilities, are uncertain, so the recovery rate depends on the expected payoffs conditional on the information set at time 1:

$$\Pi_1(q) = \mathbb{1}_{\{\tau_F(q)=1\}} \mathbb{E}_1^{\mathbb{Q}} \left[\frac{\kappa_1 (A + qY_{1,1} + qY_{2,1}^+)}{L_1 + \frac{L_2}{R_1} + D_1 + \frac{qY_{2,1}^-}{R_1}} \right]$$

Since the liabilities have two different terms, it will be useful to also define two different credit spreads; one for the excess yield from time 0 to time 1, and another for the excess yield from time 1 to time 2. The former will be denoted by $s_0(q)$ and the latter by $s_1(q)$. The credit spreads must solve the following forward discounting relations between the debt values:

$$\begin{aligned} D_1 &= D_0(R_0 + s_0(q)) \\ D_2 &= D_1(R_1 + s_1(q)) \end{aligned}$$

including a boundary condition ensuring that the time zero value of the new debt equals the funding needs of the firm, i.e. $D_0 = U(q)$. The debt's value in each period is also related to the following period by the backward discounting relations:

$$\begin{aligned} D_1 &= \frac{1}{R_1} \left(\mathbb{1}_{\{\tau_F(q)>2\}} \mathbb{E}_1^{\mathbb{Q}} [D_2] + \mathbb{E}_1^{\mathbb{Q}} [\Pi_2(q) D_2] \right) \\ D_0 &= \frac{1}{R_0} \left(\mathbb{1}_{\{\tau_F(q)>1\}} \mathbb{E}_0^{\mathbb{Q}} [D_1] + \mathbb{E}_0^{\mathbb{Q}} [\Pi_1(q) D_1] \right) \end{aligned}$$

Recall equation 9.1, which ensured breakeven for the new creditors in the single period model; the two equations just derived are of the exact same form, leave the recovery

rates. With the same motivation as in section 9 and with analogous procedure, the limiting spread for each period can be derived.
At time 0 the limiting spread to time 1 is:

$$S_0 = \frac{\mathbb{E}_0^{\mathbb{Q}}[\phi_1] R_0}{1 - \mathbb{E}_0^{\mathbb{Q}}[\phi_1]}$$

where ϕ_1 is the creditors' loss rate at time 1, given by the complement of the recovery rate for an infinitesimal project:

$$\begin{aligned}\phi_1 &= 1 - \lim_{q \to 0} \Pi_1(q) \\ &= \mathbb{1}_{\{\tau_F = 1\}} \frac{L_1 + \mathbb{E}_1^{\mathbb{Q}}[L_2]/R_1 - \kappa_1 \mathbb{E}_1^{\mathbb{Q}}[A]}{L_1 + \mathbb{E}_1^{\mathbb{Q}}[L_2]/R_1}\end{aligned}$$

Likewise, if the firm survives at time 1, the limiting spread from the interim date to the maturity date is:

$$S_1 = \frac{\mathbb{E}_1^{\mathbb{Q}}[\phi_2] R_1}{1 - \mathbb{E}_1^{\mathbb{Q}}[\phi_2]}$$

where ϕ_2 is the creditors' loss rate at time 2, given by:

$$\begin{aligned}\phi_2 &= 1 - \lim_{q \to 0} \Pi_2(q) \\ &= \mathbb{1}_{\{\tau_F = 2\}} \frac{L_2 - \kappa_2 (A - W)}{L_2}\end{aligned}$$

It should be noted that the framework presented here can be extended with additional periods, by defining the corresponding forward- and backward discounting relations and solving the equations as already shown. For the purposes of this paper, not much would be gained, besides complexity. Therefore, no effort will be funnelled in this direction, and the analysis can move on to quantify the implications of funding costs in the framework presented.

The remainder of this paper will be focused on applying the multi-period model to a swap contract. In this two-period setup, before considering counterparty credit risk, a swap promises a floating payment, X_t, in exchange for a fixed payment, K_t, for $t = 1, 2$, where the fixed payments are not necessarily constant. Hence, for the payer swap, the cash flows stemming from the swap contract are defined as $C_t = X_t - K_t$. Focussing on the payer swap, the positive cash flows will indicate an asset to the firm, whereas the negative cash flows will be a contingent liability.
A position in this swap contract of size q requires the firm to make an upfront payment of $U(q)$ where, as in section 9, $u = \lim_{q \to 0} U(q)/q$ is assumed to exist.

Having in place the basic ideas of a multi-period swap contract, the following section aims to define the impact on the shareholders' welfare by entering such a project.

11.2 Shareholders' Financing Costs of Swaps

This section will derive the marginal shareholder valuation of the firm buying or selling an interest rate swap using debt funding.

For simplicity and for the purpose of marginal valuation, if a swap party defaults at the interim date, the coupon payments will have been paid immediately prior. This assumption allows the contingent liabilities from the swap to be paid along with the rest of the liabilities at the given time of default. And, as all liabilities rank pari passu, the computational part becomes more interpretive.

The default time of the swap counterparty is denoted by τ_C. At this point in time, the firm recovers a fraction of the contractual amount denoted by β_t, where $\beta_t = 1$ if $\tau_C > t$.

Regardless of the method of funding, the apparent marginal market value of the swap is defined by:

$$
\begin{aligned}
V &= V_{rf} + \mathrm{DVA} - \mathrm{CVA} \\
&= \mathbb{E}_0^{\mathbb{Q}}\left[\delta_2\left(C_1 R_1 + C_2\right) - u\right] \\
&\quad + \mathbb{E}_0^{\mathbb{Q}}\left[\sum_{t=1}^{2} \delta_t \mathbb{1}_{\{\tau_F = t\}} \mathbb{1}_{\{\tau_C > t-1\}} \phi_t Y_t^-\right] \\
&\quad - \mathbb{E}_0^{\mathbb{Q}}\left[\sum_{t=1}^{2} \delta_t \mathbb{1}_{\{\tau_C = t\}} \mathbb{1}_{\{\tau_F > t-1\}} (1-\beta_t) Y_t^+\right]
\end{aligned} \tag{11.1}
$$

where $Y_1 = \mathbb{E}_1^{\mathbb{Q}}\left[C_2\right]/R_2$, $Y_2 = C_2$, and V_{rf} is the credit risk-free swap value. The swap value takes into account the credit risks of both the firm and the counterparty. Therefore, two adjustments are performed on the credit risk-free value, V_{rf}. From the firm's point of view, the DVA is the second term in equation 11.1 describing the lost amount of the outstanding liability related to the swap in case the firm defaults. Additionally, the CVA is the third term describing the recovered outstanding positive cash flow in case the counterparty defaults.
As a result of this formulation, the total swap value for the swap counterparty is $-V$.

In this section it is assumed that the firm always uses debt funding for financing financial instruments. Many established companies have a high credit score, hence they have no issues obtaining debt and become leveraged. It is not unusual for large firms to have multiple creditors. This common use of taking on loans constitutes the decision of the above assumption. By financing the upfront with new short-term debt, the marginal valuation of the firm's shareholders of entering the swap contract is now analysed. The net positive cash flows stemming from the swap are perceived as funding benefits as they will be used to retire some of the firm's existing short-term debt.

To finance the upfront price, the firm issues new debt. The face value of the new debt is the upfront price including a spread, such that the loan is a zero net present value investment for the new creditors. With the face value of the new debt now defined as

$D_2(q) = U(q)(R_0 + s_0(q))(R_1 + s_1(q))$, the marginal increase in the value of the firm's equity per unit investment is:

$$\begin{aligned} G_{\text{swap}} = & \frac{\partial}{\partial q}\mathbb{E}_0^{\mathbb{Q}}\left[\delta_2 \mathbb{1}_{\{\tau_F(q)>2\}}\mathbb{1}_{\{\tau_C(q)>1\}}\left(A - W + q\left(C_1R_1 + C_2\right) - L_2 - D_2(q)\right)\right]\Big|_{q=0} \\ & + \frac{\partial}{\partial q}\mathbb{E}_0^{\mathbb{Q}}\left[\delta_2 \mathbb{1}_{\{\tau_F(q)>2\}}\mathbb{1}_{\{\tau_C(q)>1\}}\left(qC_1s_1(q) + \theta_1R_1\right)\right]\Big|_{q=0} \\ & - \frac{\partial}{\partial q}\mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_F(q)>2\}}\left(\sum_{t=1}^{2}\delta_t \mathbb{1}_{\{\tau_C(q)=t\}} q\left(1-\beta_t\right)Y_t^+\right)\right]\Bigg|_{q=0} \end{aligned} \tag{11.2}$$

The first term describes the residual of the total asset value at time 2 if the firm does not default in either of the time periods, and the counterparty does not default at time 1. The residual of the total asset value is the asset value including the swap cash flow less the value of the debt paid at time 1 and the debt to pay at time 2.
At the interim date, the firm is obliged to either pay of receive the contractual coupon payment. If the cash flow is negative for the firm, it issues new debt to finance the payment; if the cash flow is positive for the firm, a portion of the firm's existing debt is retired.
The second term expresses the cost (benefit) of having (retiring) debt from time 1 to time 2. Evidently, if the cash flow at time 1 is positive, the term is a funding benefit; if the cash flow at time 1 is negative, the term is a funding cost. Furthermore, the second term includes the random dividend that the shareholders receive at the interim date.
The third term reflects the cases of the counterparty defaulting. If the counterparty defaults, the contractual outstanding amount, i.e. the potential positive cash flow at time 2, will not be fully paid to the firm.

By expressing the derivative in equation 11.2 as a difference quotient, using $U(0) = 0$, and by the linearity of the expectation operator, the following limit is obtained:

$$\begin{aligned} G_{\text{swap}} = & \lim_{q\to 0}\frac{\mathbb{E}_0^{\mathbb{Q}}\left[\delta_2 \mathbb{1}_{\{\tau_F(q)>2\}}\mathbb{1}_{\{\tau_C(q)>1\}}\left(q\left(C_1R_1 + C_2\right) - D_2(q)\right)\right]}{q} \\ & + \lim_{q\to 0}\frac{\mathbb{E}_0^{\mathbb{Q}}\left[\delta_2 \mathbb{1}_{\{\tau_F(q)>2\}}\mathbb{1}_{\{\tau_C(q)>1\}} qC_1s_1(q)\right]}{q} \\ & - \lim_{q\to 0}\frac{\mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_F(q)>2\}}\left(\sum_{t=1}^{2}\delta_t \mathbb{1}_{\{\tau_C(q)=t\}} q\left(1-\beta_t\right)Y_t^+\right)\right]}{q} \\ & - \lim_{q\to 0}\frac{\mathbb{E}_0^{\mathbb{Q}}\left[\delta_2\left(\mathbb{1}_{\{\tau_F>2\}}\mathbb{1}_{\{\tau_C>1\}} - \mathbb{1}_{\{\tau_F(q)>2\}}\mathbb{1}_{\{\tau_C(q)>1\}}\right)\left(A - W - L_2 + \theta_1R_1\right)\right]}{q} \end{aligned} \tag{11.3}$$

It is assumed that A, W, L_2, C_1, C_2, and θ_1 all have finite expectations, which allows for interchanging the limit and the expectation.
In the single-period model, the probability of the assets being exactly equal to the liabilities is 0. The same argument is assumed in the multi-period model for both the firm and

the counterparty. Then, the derivative of the difference between the post-project default event for an infinitesimal investment and the pre-project default event is well defined. Hence, similarly, L'Hôpital's rule can be applied to show that the fourth term equals 0.

Expanding the limiting face value, D_2, yields:

$$D_2 = u(R_0 + S_0)(R_1 + S_1) = uR_0R_1 + uR_0S_1 + uR_1S_0 + uS_0S_1$$

Focussing on the addend uS_0S_1, recall that the value of each credit spread depends on the loss rate from the respective time period. By substituting the expressions for the credit spread, it can be seen that the term $\mathbb{E}_0^{\mathbb{Q}}[\phi_1]\,\mathbb{E}_1^{\mathbb{Q}}[\phi_2]$ appears in S_0S_1. Using the linearity of expectations and the tower property, the following derivation can then be used:

$$\mathbb{E}_0^{\mathbb{Q}}[\phi_1] \cdot \mathbb{E}_1^{\mathbb{Q}}[\phi_2] = \mathbb{E}_1^{\mathbb{Q}}\left[\phi_2 \mathbb{E}_0^{\mathbb{Q}}[\phi_1]\right] = \mathbb{E}_1^{\mathbb{Q}}\left[\mathbb{E}_0^{\mathbb{Q}}\left[\phi_2 \mathbb{E}_0^{\mathbb{Q}}[\phi_1]\right]\right] = \mathbb{E}_1^{\mathbb{Q}}\left[\mathbb{E}_0^{\mathbb{Q}}[\phi_2\phi_1]\right] = 0$$

where the last equality uses the fact that the loss rate must be zero in at least one of the two periods. This implies $uS_0S_1 = 0$, which can be substituted into equation 11.3.

By dividing q into the expectation of the first limit in equation 11.3 it can be recognised that $U(q)/q$ is the marginal investment cost, u. Rearranging and taking the limits yields:

$$\begin{aligned} G_{\text{swap}} = {} & \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_F>2\}}\left(\delta_2\left(C_1R_1 + C_2\right) - u\right)\right] \\ & - \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_F>2\}}\left(\sum_{t=1}^{2} \delta_t \mathbb{1}_{\{\tau_C=t\}}\left(1-\beta_t\right) Y_t^+\right)\right] \\ & - \mathbb{E}_0^{\mathbb{Q}}\left[\delta_2 \mathbb{1}_{\{\tau_F>2\}} \mathbb{1}_{\{\tau_C>1\}} u\left(R_1S_0 + R_0S_1\right)\right] \\ & + \mathbb{E}_0^{\mathbb{Q}}\left[\delta_2 \mathbb{1}_{\{\tau_F>2\}} \mathbb{1}_{\{\tau_C>1\}} C_1S_1\right] \end{aligned} \tag{11.4}$$

where the third and the fourth terms together constitute the second interpretation of FVA according to section 9.5. They correspond to the quantity Φ, which is the marginal valuation of the swap contract to the firm's legacy creditors.
Since the cash flow at time 2 is 0 if the counterparty defaults at the interim date, and the marginal investment cost is constant regardless of the default times of either counterparties, the indicator of $\{\tau_C > 1\}$ is redundant and has been removed from the first term.

To find the value of the swap contract that makes the shareholders indifferent to entering the project, the marginal shareholder valuation in equation 11.4 can be set equal to 0 and solved for u:

$$G_{\text{debt}} = 0$$

This valuation considers several scenarios, as different counterparties can default at different dates. As a consequence, both the firm and the counterparty have a possibility of losing an otherwise promised income.

This concludes the derivation of the marginal shareholder valuation of a swap contract. These results will be applied in the following section where a quantitative example, will be presented. Just as the marginal shareholder valuations in the single-period model could be extended to include collateralisation and hedges, so can this equation. This will not be pursued in this paper but the section will finish with a few remarks on the topic.

When a firm trades an unsecured swap with a counterparty, it is likely that it combines the position with an appropriate hedge. Practically, it is often the case that the firm would use two hedges; one hedge position to mitigate the counterparty credit risk, e.g. a CDS, and the other hedge position to account for market risk exposure of the floating payments.

For the shareholders to value such a portfolio, an extension to the marginal valuation in equation 11.4 would have to be applied. The cash flows would in this expanded setup, to some extent, offset each other, and most of the funding implications would stem from the collateralisation. As described earlier in this paper, collateralisation agreements can take on many faces. A strict CSA agreement could require the firm to mark-to-market at each time period, whereas partially collateralised projects could also lead to funding issues. Additionally, an independent amount are likely to be established, which would potentially leave the firm with more financial obligations.

Duffie, Andersen, and Song (2018) derives the shareholders' valuation of entering a project including the same swap as above as well as a fully collateralised hedge position. For the purpose of quantifying FVA in a multi-period framework, this paper sticks to the numerical computations of only an unsecured swap, and then draw comparisons to the idea of the hedged position. This is the motive of the following section.

12 Quantifying Funding Costs

This section aims to illustrate how the funding costs and -benefits of a multi-period swap contract can be described by a simple economical setup. The example will be based on the result derived in the previous sections, namely the shareholders' marginal valuation of an unsecured swap. As opposed to the single-period model analysed in section 9, the multi-period model allows for intermediate cash flows, which will necessarily have to be financed.

Before considering the swap contract in question, the following section will introduce a financial institution operating in a multi-period economy.

12.1 A Firm in a Multi-Period Economy

Maintaining the purpose of analysing funding implications, the possible states are now represented by a binomial tree. At the interim date, the two possible states are denoted by ω_u and ω_d. Correspondingly, at the maturity date the two possible states are either $\omega_{u,u}$ and $\omega_{u,d}$, or $\omega_{d,u}$ and $\omega_{d,d}$, depending on the asset value going up or down in the first time period respectively.
In the state $\omega_{d,d}$, the liabilities exceed the asset value, and the firm defaults. In this case

the shareholders receive a payoff of 0.
The asset value structure is illustrated in figure 15. In each period the firm's assets either increase with a factor $u = 1.100$ or decrease with a factor $d = 0.850$. The asset values are determined under the risk-neutral measure, such that at each point in time the value is equal to the expected discounted value of the next period. By that definition, and by a short-term risk-free rate assumed to be $r_{0,1} = 2.000\,\%$ at the inception date, the risk-neutral probability of being in the up-state at the interim date is calculated as:

$$p_u^0 = \frac{1.000 + 2.000\,\% - 0.850}{1.100 - 0.850} = 0.680$$

And consequently, $p_d^0 = 1 - p_u^0 = 0.320$.

The long-term risk-free discount rate from time 0 to time 2 is assumed to be $r_{0,2} = 2.200\,\%$. To determine the expected short-term rate at the interim date, the forward rate is calculated:

$$r_{1,2} = \frac{\left(1.000 + r_{0,2}\right)^2}{1.000 + r_{1,2}} - 1.000 = 2.400\,\%$$

Moving to the up-state from time 1 then has a probability of $p_u^1 = (R_1 - 0.850)/(1.100 - 0.850) = 0.696$.

The risk-neutral probabilities of moving to the three different states at the maturity date are calculated as:

$$\begin{aligned} p_{u,u} &= 0.473 \\ p_{u,d} + p_{d,u} &= 0.429 \\ p_{d,d} &= 0.097 \end{aligned}$$

At the interim date, the firm's creditors have a known claim of $L_1 = 10.000$, meaning $W = 10.000 \cdot R_1 = 10.240$. Meanwhile, the long-term liabilities, which are also known with certainty, are $L_2 = 70.000$. For simplicity, the dividend paid to the shareholders at the interim date is assumed to be $\theta_1 = 0$, and there are no distress costs. As mentioned above, the asset value does not cover the total liabilities at state $\omega_{d,d}$ as $72.250 < L_2 + W$, hence the firm defaults.
The short-term liabilities are, on the other hand, not large enough to trigger a default at the interim date regardless of the state outcome.

Before entering any projects, the firm's loss rate given default is $\phi(\omega_{d,d}) = (L_2 - (d^2 A_0 - W))/L_2 = 11.414\,\%$. The credit spread at the inception date is $0.000\,\%$, as the firm cannot default at time 1, and the credit spread at the interim date is

$$\frac{70}{\frac{70(1-p_{d,d}) + 70(1-\phi)p_{d,d}}{1+r_{1,2}}} - (1 + r_{1,2}) = 1.150\,\%$$

Having defined the setup of the firm's capital structure, a swap contract is now introduced between the firm and a counterparty. For simplicity, the counterparty is assumed

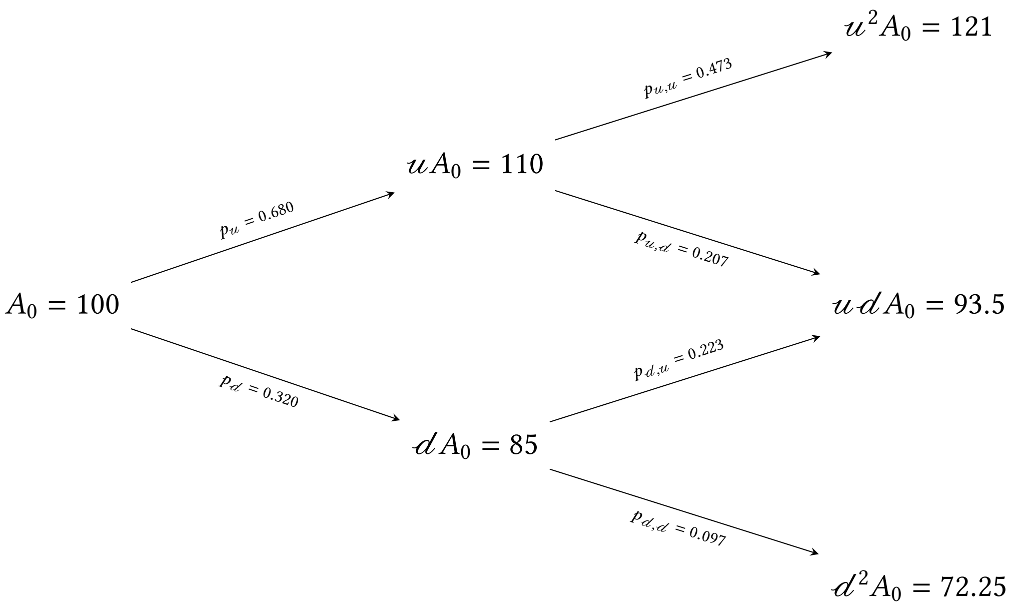

Figure 15: The asset value of the firm in each time period.

to have no credit risk. The swap will have a relatively small notional payoff to the extent that it does not affect the default- and no-default states of the firm.

The numerical example will apply the theory from section 11.2 to find the shareholders' valuation of the project being entered. The purpose of the example is to examine how financing needs in different time periods influence the shareholders' valuation, and consequently how they affect the FVA.

12.2 Obtaining a Swap Contract

Suppose the firm enters a swap contract with a credit risk-free counterparty. The firm pays a constant fixed leg $K = 1.000$ to the counterparty at each time period. Correspondingly, at each time period, the firm receives a random floating leg, X_t, from the counterparty.

Since the counterparty is credit risk-free, the firm can always expect to receive the full contractual amount of the floating leg. Therefore, there is no CVA and the difference between the risk-free value and the credit risk adjusted price is the DVA.

The floating legs are considered random. Their probability distribution will not be specified, but the expected cash flows are known by both the firm and the counterparty. If the firm's swap leg has a higher value than the counterparty's, it will have to pay an upfront price to the counterparty, and vice versa.

The time 0 expected payments of the swap are summarised in table 11. Notice that at the interim date the firm expects a positive cash flow of $\mathbb{E}_0^{\mathbb{Q}}\left[C_1\right] = \mathbb{E}_0^{\mathbb{Q}}\left[X_1 - K\right] = 0.200$, which will be used for retiring existing debt. The expected cash flow at time 2 does not change over time.

t	1	2	Present value
$\mathbb{E}_0^{\mathbb{Q}}[K]$	1.000	1.000	1.938
$\mathbb{E}_0^{\mathbb{Q}}[X_t]$	1.200	0.950	2.086
$\mathbb{E}_0^{\mathbb{Q}}[C_t]$	0.200	−0.050	0.148

Table 11

At the maturity date the firm liquidates all of its assets, which it uses to pay the remaining outstanding amounts. Additionally, the existing creditors' liabilities are assumed to rank pari passu with the counterparty's contingent liabilities.
Since the expected discounted market value of the floating payments is largest, the firm may be obliged to pay the counterparty an upfront price. As per the Modigliani-Miller proposition, a zero net present value investment leaves the aggregate wealth unchanged. For this to be applicable, the price of the swap must also account for the firm's own default risk, i.e. include the aforementioned DVA. Otherwise, it cannot be a zero net present value investment for both the firm and the counterparty.

If the firm defaults at time 2, i.e. if the state $\omega_{d,d}$ is realised, the present value of the total payoff to its creditors is calculated as:

$$\begin{aligned}\mathbb{E}_0^{\mathbb{Q}}\left[(1-\phi(\omega_{d,d}))\,(L_2 - C_1R_1 - C_2)\right] &= (1-\phi(\omega_{d,d}))\,(70.000 - 0.200R_1 + 0.050)\\ &= (1-\phi(\omega_{d,d}))\cdot 69.845\end{aligned}$$

And by definition, this amount is equal to the total remaining asset value. Remember that at time 2 the firm has already paid the liabilities due at the interim date. To find the price of the project as well as the loss rate given default after obtaining the project, the following system of equations must be solved for u and $\phi(\omega_{d,d})$:

$$\begin{cases} d^2A_0 - W = \mathbb{E}_0^{\mathbb{Q}}\left[(1-\phi(\omega_{d,d}))\,(L_2 - C_1R_1 - C_2)\right] \\ u = \mathbb{E}_0^{\mathbb{Q}}\left[\delta_1(X_1 - K) + (\delta_2(X_2 - K(1-\phi)))\right] \end{cases} \tag{12.1}$$

$$\Leftrightarrow \quad \begin{cases} 72.250 - 10.240 = (1-\phi(\omega_{d,d}))\cdot 69.845 \\ u = \delta_1\cdot 0.200 + \delta_2\left(\left(-0.050(1-\phi(\omega_{d,d}))\right)p_{d,d} - 0.050(1-d)\right) \end{cases}$$

The first equation in equation 12.1 ensures that the creditors receive the entire asset base if the firm defaults. The second equation sets the price of the swap as the discounted expected payoff. Solving this system of equations yields a swap price of $u = 0.159$, and a loss rate in the default state of $\phi(\omega_{d,d}) = 11.218\,\%$.
Using the loss rate, the value of the legacy creditors' long term claim can be calculated. At time 2 the payoff in the default state is $L_2\cdot(1-\phi(\omega_{d,d})) = 62.147$. At time 1 the expected value of the claim is therefore 67.613, and at time 0 the value is 66.287.

The credit spread in each period is the debt's no-default value one period ahead divided by its current value less the risk-free rate. Hence, the post-project actual credit spreads for time 0 and time 1 are respectively:

$$\frac{67.618}{66.292} - R_0 = 0.000\,\%$$
$$\frac{70.000}{67.618} - R_1 = 1.122\,\%$$

Since the firm cannot default at time 1, the upfront price can be funded with short-term debt at the risk-free interest rate. If the firm were to pay a spread on the short-term debt starting at time 0, this would be subtracted from the liquidated asset value.

By using the equation for the shareholders' marginal market valuation of the swap contract defined in equation 11.1, the DVA can then be computed as:

$$\begin{aligned} \mathrm{DVA} &= u - V_{rf} \\ &= u - \mathbb{E}_0^{\mathbb{Q}}\left[\sum_{t=1}^{2} \delta_t C_t\right] = 0.010 \end{aligned}$$

Turning to the shareholder valuation of the project. Cash flows from the swap contract may have an impact on the shareholders' wealth. Since the price of the swap is its risk-neutral value adjusted for credit risk, the project is a zero net present value investment for the counterparty. If a wealth transfer has happened, it is bound to be between the shareholders and the long-term creditors.

With $\mathbb{E}_0^{\mathbb{Q}}[C_1] = 0.200$, $\mathbb{E}_0^{\mathbb{Q}}[C_2] = -0.050$, $u = 0.159$, and the assumption that the counterparty is credit risk-free, the total impact on the shareholders' claim is calculated using equation 11.4:

$$G_{\text{swap}} = -0.009$$

This value suggests a wealth transfer of -0.009 between the shareholders and the creditors of the long-term debt, such that the project is a negative net present value investment for the shareholders.

The price that would make this project a zero net present value investment for the shareholders, is found by fixing G_{swap} to 0, and solving for u^*_{swap}. The difference between the credit risk adjusted price and u^*_{swap} is then the FVA:

$$G_{\text{swap}} = 0$$

yielding a funding adjusted price of $u^*_{\text{swap}} = 0.149$. The donation from the counterparty required is then:

$$\mathrm{FVA} = u^*_{\text{swap}} - u = -0.010$$

This example has shown a situation where some of the derivative cash flows have been a source of funding benefits, but where the overall effect were still a loss to the shareholders. The primary effect is clearly from the upfront price, that is the largest outgoing cash flow of the swap. Still, there are smaller effects which are interesting to study.

The payment time of a cash flow has an influence on its funding implications. As the upfront price should be paid at time 0, the firm can obtain funding in the first period at a credit spread of 0.000 %. However, the intermediate incoming cash flow can retire debt that would otherwise have appreciated by the firm's credit spread at time 1 of 1.122 %. Therefore, the time 1 cash flow provides a funding benefit, but the time 0 cash flow has no funding cost. Of course, the setup considered here is incredible simplistic, which exaggerates the actual effects. In reality, creditors would clearly not offer a credit spread of zero on, say, 1 month tenors because they feel certain that the firm can only default in 2 months. Rather, the firm has some default intensity, which fluctuates over time, but not by jumping up from zero when a debt's maturity gets closer.

The general lesson is useful anyway. If institutions have high credit spreads in certain periods, it will be more costly for them to obtain projects that have outgoing cash flows in these periods. This is the case at play in the example. The firm has a large face value maturing at time 2, so its default probability, and therefore credit spread, is high around this time period. Hence funding requirements at time 1 have a high impact.

As has been covered previously, trading unsecured derivatives without hedging market risk has the most extreme funding implications; any outgoing cash flow has to be funded, as there are no offsetting cash flows from hedges.

If the firm was to extend its portfolio by hedging the swap with a fully collateralised position in the opposite trade, the funding implications would mainly be due to the margin requirements. An example of this situation was depicted in figure 4, where differences in the collateral call frequency of the swap and its hedge led to funding shortfalls and surpluses.
Including a collateralised hedge in the current example, would convey some of the same messages. The interest payments from the swap would more or less offset the interest payments from the hedge, which would reduce the amount of funding costs or benefits from the market risk cash flows. The extent of reduction would depend on the accuracy of the hedge; a perfect hedge would have the exact opposite interest payments as the original, but mismatches would provide some funding friction.
Some funding costs would occur as the upfront price paid for the swap would be higher than the upfront price received on the hedge. The hedge would be secured and probably be priced close to its risk-free value, while the swap counterparty would require compensation for taking on the credit risk. Hence, the firm would likely have to provide funding for the differences in upfront payments. Of course, practically speaking, the firm would actually have to provide funding for the entire upfront payment on the original swap; the upfront payment from the hedge would have to be immediately posted as collateral.
Whether to attribute this funding cost to the upfront payment or to the collateralisation

is up to interpretation, but is only a matter of definition.

The asymmetry in the collateral agreements would also be an important contributor, to funding costs or benefits. Since the original swap is assumed completely unsecured, any collateral calls from the hedge counterparty would require funding. Likewise, any collateral postings to the firm would yield funding benefits. Again, trading unsecured derivatives provides the most extreme examples of funding costs.

Extending the portfolio in this example with a hedge, would confirm many of the conclusions already made throughout this paper. The market risk would be reduced and so would the funding implications from the derivative cash flows, but an FVA could still be necessary due to the differences in collateral agreements.

Part V

Conclusion

This paper sat out to study the implications of funding costs in derivatives and the impact of accounting for them with FVAs. This included researching why funding costs appear in the first place, i.e. what mechanisms in financial derivatives lead to funding costs.

Why do funding costs appear in financial derivatives?
When managing a trade position, a derivatives dealer may need to obtain funding to address the financing requirements of the position. Financial institutions cannot obtain external funding for OTC derivatives at risk-free rates; hence, funding comes at a cost due to the dealer's borrowing rate. This cost is *the funding cost.* The needs of financing vary, but the paper has especially considered the cash demands from the hedges of a trade position and from the calls for collateral of a trade position.
Replications of OTC derivatives are often imperfect, and the cash flows from a trade and its hedge might not perfectly offset each other. When the incoming cash flow does not fully cover the outgoing cash flow, the dealer will have to supply additional funding.
In addition to unaligned cash flows, a trade and its hedge might also have asymmetrical collateral agreements, such that collateral calls from one trade cannot be covered by rehypothecated collateral from the other.
In summary, funding costs exist in OTC derivatives because managing them requires financing that has to be obtained at costs above the risk-free rates.

What has been the dispute concerning Funding Value Adjustments?
Increases in banks' borrowing rates after the financial crisis, have led some institutions to adopt valuation adjustments, FVAs, to account for funding costs. Financial institutions and their dealers argue that FVAs are necessary, since the financing requirements of OTC derivatives provide a very tangible cost, which they will necessarily incur. If an FVA is not applied to a derivative that require financing, its valuation will leave out actual costs, which could otherwise have influenced the trading decision.
On the opposite side of the debate are theoreticians. They argue that the valuation of a project should depend on its riskiness and not the riskiness of the institution that undertakes it. This argument implies that dealers should not be using their funding curve to establish valuations, i.e. they should not be applying FVAs.

The paper has not made an attempt to settle the dispute about applying FVAs, but has merely accounted for the arguments of each side. However, in support of this paper's relevance, it has been concluded that FVAs are a necessity when determining a trade's *value to the dealer.* This conclusion is not necessarily in conflict with either side of the debate, but points out a part of the topic where theoreticians and practitioners might not disagree. In addition, it motivates the final research question about applying FVA.

What are the implications of using- and not using Funding Value Adjustments in a simple structural model?
By leveraging the simple setup of a structural model, the paper was able to study how stakeholders were affected by a firm obtaining a new project. A new project creates different incentives for shareholders, creditors and the firm in general.

If a firm obtains a project without considering funding costs, i.e. without applying an FVA, the firm might transfer wealth between its shareholders and creditors. If the project needs financing, the analysis showed that the funding costs of obtaining the project are borne by the shareholders. The financing entity requires a premium for loaning unsecured to a firm that might default; the shareholders are charged with this cost when the firm does not default.
The welfare lost by shareholders is transferred to the legacy creditors, who experience a lower loss rate when the firm defaults, since the project has increased the asset base. Hence, obtaining projects can result in a free rider problem, as shareholders pay the funding costs of financing the project, but do not enjoy the benefits from it. Those benefits are instead enjoyed by the firm's creditors.
This mechanism proved to be a friction for the firm, since it restricted what projects could be obtained if shareholders' wealth should be maintained. Even when a positive net present value investment was offered to the firm, it was not necessarily able to preserve its shareholders' wealth.
That is the implication for a firm of not using FVAs.

Applying an FVA to the valuation of a project is a way for the firm to align its interests with its shareholders'. When a project has funding costs, the firm can use an FVA to evaluate whether a given price is a losing trade for its shareholders or not. By doing so, the firm can ensure that it obtains the project at a high enough discount to offset the funding costs and maintain the shareholders' wealth.

The magnitude of the shareholders' wealth loss depends both on the project's structure but also highly on the way it is financed. The derived pecking order showed that shareholders prefer financing with existing cash over financing with debt issuance, and prefer both over financing with equity issuance. Also, collateralisation agreements can greatly affect the funding costs, as the firm then may need to post collateral, which again needs financing.

The opposite of funding costs, funding benefits, occur when projects provide cash surpluses. In that case, the funds can be used to cover financing needs elsewhere in the organization; then, potentially costly funding does not have to be obtained.
Projects with funding benefits increase the shareholders' wealth, and their financing preferences are reversed, such that they prefer buying back equity, over retiring debt, over investing in risk-free bonds.

In conclusion, if the firm does not apply FVAs, it risks putting the shareholders at a loss corresponding to the funding costs. By using FVAs, the firm can align its interest with its shareholders', such that it avoids trading decisions that deteriorates their wealth.

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Appendices

A Marginal Valuation of Cash Funding

The marginal valuation of stakeholders from funding a risk-free project.