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FVA: Funding Value Adjustment

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Preface

This master thesis within the area of financial mathematics concerns the study and performing numerical simulation of funding value adjustment concept. The research is performed in Quantitative Research group at Swedbank, Stockholm, under the supervision of Dr. Ola Hammarlid.

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Abstract

Department of Mathematics

Master of Science

FVA: Funding Value Adjustment

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This thesis is intended to provide an overview of funding value adjustment (FVA) as one of the XVA's. We discuss the controversial subject of FVA and whether or not a bank should include it in pricing process. In addition we present the results of expanding the traditional pricing framework to the new one where the funding costs and counterparty risk is taken into account. Also we focus on the situation where there are multiple discounting curves involving in the valuation formulas. Finally we propose a new methodology for FVA calculation which is designed based on the concept of Net Stable Funding Ration (NSFR). Numerical experiments show the performance of the calculation frameworks.

Acknowledgements

Reaching to the end of my master thesis I would like to formally thank people who helped me during this research. During this project I granted to be present at the Quantitative Research group in Swedbank where I learned enormously.

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Finally I wish to show my sincerest gratitude to my family: my parents Hajar and Saied, my brother Mohsen and my husband Azad for supporting me spiritually throughout my life. I would also like to thank all my kinfolk in Sweden who keep outstanding me with their benevolence.

Contents

List of Abbreviations

CSA	Credit Support Annex
CVA	Credit Value Adjustment
DVA	Debt Value Adjustment
FBA	Funding Benefit Adjustment
FCA	Funding Cost Adjustment
FDA	Funding Debt Adjustment
FVA	Funding Value Adjustment
ISDA	International Swaps and Derivative Association
LIBOR	London Interbank Offered Rate
NSFR	Net Stable Funding Ratio
OIS	Overnight Indexed Swap
OTC	Over The Counter
NDV	No Default Value

To Azad

Chapter 1

Introduction

In 2007 a credit crisis has started that was the worst financial crisis since 1930 and caused many dealers to critically review their approach. The crisis started from financial institutions in USA and became a global crisis for real economy.

Before financial crisis LIBOR was the benchmark that banks used to charge for funding and for posted collateral, as well. Hence the charge of funding was almost offset by the interest rate received on the posted collateral. As a result no cost for funding were taken into account in portfolio pricing. However starting the financial crisis made the market very unstable such that the spread between 3 month LIBOR and OIS reached a peak of 365bp [1] which was already 10bp in average, so the banks were very reluctant to use the discount rate based on three-month LIBOR rate and they managed to change it into OIS instead. Figure 1.1 shows the trend of spread between 3-month LIBOR and 3-month OIS from January 2007 to October 2011.

In the aftermath of the global crisis, financial institutions found out that one area that needed special attention is counterparty credit risk for OTC trades since, for example, the high profile investment bank Lehman Brothers had notional amount of $800 billion of OTC derivatives at the point of bankruptcy. Financial market Analysts end up with the idea that absence of appropriate assessment of credit exposure and ignoring the default probability of both parties were the key reason of having the credit crisis of 2007. Some companies were given zero or close to zero default probability which turn out to be inaccurate in many cases. The financial crash revealed that all institutions need to improve their performance in terms of understanding, assessment and proper management of their counterparty risk as well as their own credit risk.

Post crisis derivative trading is much more complex as it requires careful and explicit pricing that reflects all aspects of credit risk. Nowadays pricing derivatives needs multiple discounting curves and large scale simulations to calculate all elements of counterparty credit risk. It is now a standard practice to adjust the derivative prices for the risk of the counterparty (CVA) and one's own default (DVA). Also banks know that considering funding collateral has a significant role in pricing the trading portfolio (a common example of FVA). More such adjustments are used in practice, for example, the cost of regulatory capital (KVA) and initial margin costs (MVA). All these adjustments are commonly referred to as XVA.

The massive post crisis changes in the accounting setting of banks makes it inevitable to include FVA in pricing process. Banks began looking for a

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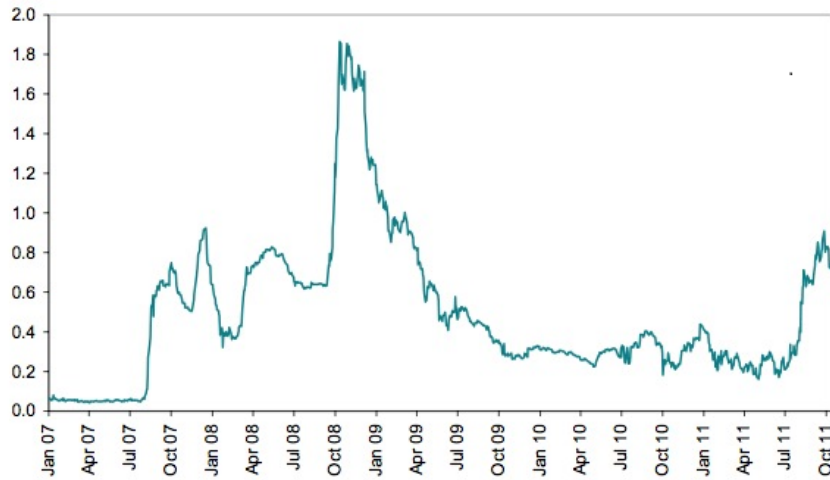[1]Basis Point which is equal to 0.01%

FIGURE 1.1: Spread between 3-month LIBOR and 3-month
OIS

way to measure and account for the funding cost that could change over time due to the bank's own credit quality and the notion of a funding value adjustment emerged. Generally speaking FVA can be defined as the difference between the value of a portfolio with risk-free rate (which is typically LIBOR or OIS) and valuing it using the bank's actual average funding cost.

We give a simplistic model to show that how funding costs arise in practice (see Figure 1.2). Consider a bank that enters into a derivative transaction with a client. Bank needs cash (to post as collateral) in order to hedge this derivative. The required cash is usually obtained by borrowing from the market (debt) with interest rate r_B. On the other hand banks receives an interest rate on the posted collateral which usually is $r_{\textbf{OIS}}$. As mentioned earlier, before the crisis $r_B \simeq r_{\textbf{OIS}}$ meaning that the bank was able to use the earned interest rate on the posted collateral to offset the interest rate of the money borrowed from the market but this is not the case any more. The spread between the bank's funding rate and the OIS rate creates a funding cost for the bank that makes it completely sensible to include funding value adjustment in derivative pricing.

FIGURE 1.2: FVA mechanism in a simple model

As will be discussed in the following, FVA is still a controversial issue. Most practitioners strongly believe that it must be included in pricing while some theoreticians do not accept it as an element of derivative pricing.In this thesis we try to discuss the ideas of both groups and go into the technical details of calculating funding value adjustment.

1.1 Preliminaries

In this section we provide the basic relevant definitions for some terms that are widely used in the rest of the thesis.

Over-The-Counter (OTC) Trading. Financial activities do not always take place on an organized exchange market. One important alternative which is much larger than the exchange-traded market is to use over-the-counter market where no public report is usually made on these transactions. In fact, OTC trading is a telephone and computer-linked network of dealers and trades that are usually done over the phone. For example, bonds are mostly traded over the counter.

An exchange trading has the benefit of mitigating almost all the risk concerning the default of one of the parties (credit risk) while there is usually a small risk that one party defaults on its obligations in an OTC trade. In other words the probability of default in exchange trading is smaller than the one in OTC trading. However, a key advantage of the OTC market is that there are few limitations on the contract and therefore more flexibility in the terms of a contract.

Credit Risk. Credit risk is the risk that one party cause a financial loss for the other party by failing to make required payments. Banks or other financial institutions calculate the credit risk of a client based on his/her ability to repay which in turn includes client's collateral assets, taxing authority, etc. So the higher the risk is, the higher rate of interest will be demanded for lending capital.

Credit value adjustment (CVA) and debt value adjustment (DVA). CVA is the most straight forward case among the XVA s. CVA is designed to take the possibility of default of parties into account in pricing a derivative. simply speaking,CVA says if party A enters into a contract with party B, each party may default. In the time of default one party, say A, has an exposure meaning that the value of the contract is positive to A and negative to B. So one needs to take this into account which is called credit value adjustment (CVA). Basically the value of the transaction is calculated without considering the possibility of default and then it is subtracted by CVA to reflect the possibility that one party may default. In other words it is an adjustment to the measurement of derivative assets to reflect the credit risk of the counterparty.

A trend that has become more relevant and popular, particularly since the global credit crisis, has been to integrate the bilateral nature of counterparty credit risk. A bilateral CVA takes into account the probability of defaults of both parties. The bilateral nature of CVA has brought a measure called debt valuation adjustment. DVA is simply CVA from the counterparty's perspective. So if company A inters into a contract with company B then

$$CVA_A = DVA_B, \qquad CVA_B = DVA_A$$

In practice we might not see this symmetry since two parties use different CVA models.

Swap. A swap is an OTC contract negotiated in 1980s for the first time. It is an agreement between two parties to exchange financial instruments (usually cash flows) in the future.

The most common type of swap is an *interest rate swap (plain vanilla)* where a party agrees to pay fixed cash flows equal to interest at a predetermined fixed rate on the *notional principal* amount and the other party pays a floating interest on the same notional principle. The payments are done on predetermined future times for the predetermined number of years.

One of the common floating rates for interest rate swap agreements is the London Interbank Offered Rate (LIBOR). It is a reference rate produced once a day by British Banker's Association and is designed to reflect the rate of interest at which a bank is prepared to deposit money with other banks. (Swedish market rate is called *STIBOR: Stockholm Interbank Offered Rate*) One-month, three-month, six-month, and 12-month LIBOR are quoted in all major currencies. For example, 1-month LIBOR is the rate at which 1-month deposits are offered. For more details regarding the mechanism of swaps see [1] and [2].

Overnight Indexed Swap (OIS). Overnight indexed swaps have been introduced in 1990s and become very popular between banks and other financial institutions. An OIS is an interest rate swap where a fixed rate for a period of time (e.g 1 month, 3 month,1 year or 2 years) is exchanged for the floating payment based on geometric average of the overnight rates during the period. this rate is often a rate targeted by the central bank to influence monetary policy. In the United States the rate is called the Fed Funds rate.

REPO market. A repurchase agreement (repo) is a form of transaction between two parties where one party sells an asset at a specified price and commits to repurchase the asset from the other party at a different price at a future date. Consider a company that issues bonds then if the company defaults during the life of the repo, the buyer (as the new owner) can sell the asset to a third party to offset his loss. The asset therefore acts as collateral and mitigates the credit risk that the buyer has on the seller.

For the party selling the security (and agreeing to repurchase it in the future) it is a repo; for the counterparty, (buying the security and agreeing to sell in the future) it is called a reverse repo.

Margin. Organizing the trades in such a way that the contract defaults are avoided is very important in all types of transactions. Margins are introduced to decrease this type of risk (credit risk) of a trade.

The amount that must be deposited at the time the contract is started is known as *initial margin*. If the balance in the margin account falls below the maintenance margin, the investor receives a margin call and is expected to top up the margin account to the initial margin level by the end of the next day. The extra funds deposited are known as a variation margin.

Zero coupon bonds. A zero coupon bond with maturity date T, also called a T-bond, is a contract which guarantees the holder 1 dollar (face value or notional value) to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t, T)$. The convention that face value, equals one is made for computational convenience while in reality different face values are used , e.g $\$1000$ is a common face value for zero coupon bonds.

Outline of the thesis

The thesis is structured as follows. Chapter 2 gives an overview of the papers [3] and [4] and we try to discuss the subject of whether or not FVA calculation should be considered in derivative pricing. chapters 3 and 4 discuss the new framework for pricing derivatives in the presence of funding costs. We also present a numerical example to illustrate that how taking funding costs into account affects the pricing process. In chapter 5 we provide an overview of the available methodology and we propose a formulation for FVA calculation, as well. Besides, we build up a simple model that shows how the new proposed FVA formula is applied in practice. The relevant fundamental mathematical theory of derivative pricing can be found in Appendix.

Chapter 2

The FVA Debate

One of the most controversial questions of post crisis is whether or not a bank should make a funding value adjustment (FVA) to value derivatives. On one hand theoreticians think that accounting FVA leads to arbitrage opportunities, on the other hand practitioners say that it is inevitable to account for it since they need to use it to reflect their costs in the pricing process.

Among many various ideas in this context John Hull and Alan White have a challenging debate: The FVA debate regarding should a derivative dealer make a funding value adjustment(*Risk 25th anniversary issue, July 2012*). In this section we provide a summary of [3] containing their arguments that leads to the idea that it is not relevant to consider the actual funding rate paid by the banks .

Lets start by the basic definition of FVA stated in [3]. An FVA is an adjustment to the value of a derivative or a portfolio of derivatives which is designed to make sure that a dealer recovers its average funding costs when it trades and hedges derivatives. The arguments in FVA can be stated for different situations:

- *The trader's* point of view is that the derivatives desk of bank must make an FVA for uncollateralized trades. Simply because the funding desk charges the derivatives desk by the bank's average funding cost hence

- *The accountant* on the other hand believes that all the derivatives should be valued at the exit price which strongly depends on the market price not on the funding costs. As it is clearly mentioned in the IFRS 13, Fair Value Measurement: fair value is a market-based measurement, not an entity-specific measurement . The accountants are concerned that the FVA leads to a conflict since there might be different prices for one transaction.

- *The theoreticians* claim that there is no theoretical basis for considering funding value adjustment since it is mentioned in many introductory finance books that the discount rate used by a company for the project's expected cash flow is determined by the risk of the project not the funding cost.

As can be seen, although the arguments of the theoretician and the accountants are different but they lead to similar results while the trader's argument is completely different which leads to markedly different valuation of the derivatives. So the issue of involving FVA in valuing the derivatives is basically to make the decision of choosing the discount rate. In other words,

the question is whether the products should be valued at cost (the trader's view point) or at the market prices.

The paper by J. Hull and A. White ([3]) tries to show that taking FVA into account in derivative pricing is not relevant. As Prof. John Hull says: "*FVA is an adjustment that is applicable on transactions that are not collateralized because in the transactions that are collateralized each party receives the federal fund's rate on the posted collateral so the value of the contract is discounted on the OIS rate which is linked with the federal fund's rate. In the case that one party receives more than OIS, for example, OIS+50bp, then an adjustment is needed since that is a real cash flow. But this is different story from when the transaction is not collateralized. The question is which rate one should use as the discount rate. Then we come back to the argument that is stated in many finance books and says you should keep the funding of your business separate from the risk of the transaction*".

2.1   Pricing Model

If we name the standard pricing model of Black-Scholes-Merton as the no-default value (NDV) then we have the following

$$\text{Portfolio value} = NDV - CVA + DVA$$

Where CVA (credit value adjustment) reflects the possibility that the counterparty will default and DVA (debt value adjustment) reflects the possibility that the dealer will default. So considering FVA as an adjustment is designed to incorporate the dealer's average funding costs for uncollateralized transactions. We can update the previous equation as:

$$\text{Portfolio value} = NDV - CVA + DVA - FVA$$

FVA is defined as the difference between the NDV obtained when the risk-free rate is used for discounting and the NDV based on discounting at the dealer's cost of funds (FVA can be defined in so many different ways and this is one of the confusing aspects of FVA).

2.2   Funding costs and performance measurement

We start this section by the simple example from [3] that shows that there are two interest rates that affect the valuation of the derivative: the rate at which a position in the underlying asset is funded and the rate at which a position in the derivative can be funded.

Suppose that a trader wants to enter into a forward contract to buy a non-dividend-paying stock in one year's time. Consider how a trader might view this transaction in an FVA world. If she enters into a contract to sell forward, she will hedge the forward contract by buying the stock now so that she will have it available to deliver one year from now. Her profit at the end of the year will be the delivery price minus the value of the current stock price when it is compounded forward at the funding rate for a stock purchase. If the current stock price is 100 and her funding rate for equity purchases is 4%, the delivery price must be higher than 104 for the trader to earn a profit. The delivery price at which the trader is willing to sell in

one year's time reflects the rate at which a position in the underlying asset can be funded.

The trader's funding costs also affect the discount rate that she uses. Suppose that the delivery price is set at 106 and the trader pays the counterparty X to enter into this forward contract. This expense is another cash outflow that must be funded by borrowing. If the funding rate for this payment is 5%, the year-end profit is $106 - 104 - 1.05X$. Thus, the trader will pay no more than $\dfrac{2}{1.05} = 1.905$ to enter into this contract. The present value of the forward contract –the maximum amount the trader will pay– is determined by discounting the payoff on the contract at the rate at which a position in the derivative can be funded.

Also the authors of **all** extend the Black-Scholes-Merton arguments in the case that the interest rate at which the derivative is funded to be different from the rate at which the stock is funded. In fact this consideration makes sense because assets such as stocks are often funded using repo agreement while there is no repo market for derivatives thus they are funded at a higher rate in the uncollateralized market. In the rest we consider r_s as the rate at which a position in the underlying asset is funded and r_d as the rate at which a position in the derivative is funded (discounting payoff). Let S shows the stock price with the dynamics:

$$dS = \mu S dt + \sigma S dW \tag{2.1}$$

where μ is the expected return on the stock, σ is the volatility and dW is a srandard Briwnian motion. We know that the price of the dericative, $f(t, s)$ is a function of S so we can apply Ito lemma to get the following

$$
\begin{aligned}
df &= f_t dt + f_s ds + 1/2 f_{ss}(dS)^2 \tag{2.2}\\
&= \left(f_t + \mu S f_s + \frac{1}{2}\sigma^2 S^2 f_{ss}\right)dt + \sigma S f_s dW \tag{2.3}
\end{aligned}
$$

Consider a portfolio consisting of a short position in the derivative and a position of $\dfrac{\partial f}{\partial S}$ in the underlying asset. The value of this portfolio is:

$$\Pi = -f + \frac{\partial f}{\partial S}S$$

Since the portfolio is risk-free, the changes of its value in time period Δt is

$$\Delta\Pi = -r_d f + r_s \frac{\partial f}{\partial S}S \tag{2.4}$$

Substituting equations (2.1) and (2.2) we get the following differential equation

$$f_t + r_s S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r_d f$$

Today's price of the derivative is

$$f_0 = \exp(-r_d T)\hat{E}(\phi(S_T))$$

where ϕ is the pay off function, T is the maturity time and $\hat{E}$ denotes expectations in a world where the expected growth rate of the stock price is r_s.

Using these conventions, value of a European call option on a stock with strike price K and maturity T is

$$S_0 N(d_1)e^{(r_s - r_d)T} - Ke^{-r_d T}N(d_2)$$

where

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r_s + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

(see [2] for more details).

2.3   FVA, DVA and double counting

In this section the FVA and DVA definitions are checked and the relationship between them is discussed. FVA, as it is clear from its title concerns funding and DVA concerns participants own credit risk. In this paper the authors divide the DVA into two separate components: DVA1 and DVA2:

> DVA1: The value to the bank that arises because it might default on its derivative obligations.

> DVA2: The value to the bank that arises because it might default on the funding required for the derivative portfolio.

From the accounting perspective both DVA1 and DVA2 are approved as DVA. Now we have the following equation for the portfolio value:

$$\text{Portfolio value} = NDV - CVA + DVA1 + DVA2 - FVA \qquad (2.5)$$

Consider a situation where a dealer has m uncollateralzied transactions where T is the life of the longest transaction. Let $v_j(t), \quad (j = 1, \cdots, m)$ be the value of jth transaction to the end user and $q(t)$ is the dealer's default rate at time t. Then

$$\text{DVA1} = \int_{t=0}^{T} w(t)q(t)[1 - R(t)]E\Big[max(\sum_{j=1}^{m} v_j(t), 0)\Big]dt \qquad (2.6)$$

where $R(t)$ is the recovery rate at time t, $w(t)$ is the value of \$1 recieved at time t (discount factor) and E denotes the risk neutral expectation. We see that whenever the net value of the transactions is positive to the counterparty (hence negative to the dealer), the dealer makes a profit as DVA1 from his own default.

Considering $q(t)[1 - R(t)]$ as the instantaneous forward credit spread at time t, the FVA can be defined as

$$\text{FVA} = -\int_{t=0}^{T} w(t)q(t)[1 - R(t)]E\Big[\sum_{j=1}^{m} v_j(t)\Big]dt \qquad (2.7)$$

If $\sum_{j=1}^{m} v_j > 0$ then the net value of portfolio is negative to the dealer. So if the dealer defaults then $FVA < 0$ hence it interpreted as a benefit for the dealer provided by FVA (since it is added to the other elements in equation (2.5)).

As can be seen the difference between DVA1 and FVA (apart from the sign) is that the DVA only considers the expected value of the positive net value of the transactions while in FVA formula the expectation is over the sum of the transaction values regardless it is negative or positive. When the v_i's are always positive (e.g., when the end user buys options from the dealer), DVA1 = -FVA and DVA1 has the same effect as FVA. In the case that there are there exists $v_j(t) < 0$ for some $j = 1 \cdots m$ then using the fact that

$$\max \Big(\sum_{j=1}^{m} v_j(t), 0 \Big) \geq \sum_{j=1}^{m} v_j(t)$$

we have

$$\text{DVA1} \geq -\text{FVA}.$$

The later case (having negative transaction values) is more realistic because if the transaction never has a positive value to the end user (negative value to the dealer) then it wont be attractive to the end users.

So if a dealer sells an option ($v > 0$) then DVA1 $= -$FVA. So in the case of dealer's default DVA1 is a benefit to the dealer (since the dealer doesn't pay the counterparty anymore). In fact in the case of a sold option both DVA1 and FVA are benefit for the dealer. This phenomena is referred to as double counting. So some practitioners choose to include FVA but not DVA1 in their pricing. The reality is that DVA2 cancels FVA and thus only DVA1 should be included in pricing. Except in the case that all derivatives have a negative value to the dealer, in this case $DVA1 = 0$ so it is excluded and FVA is included in equation (2.5).

2.4 Yes, FVA is a cost for derivatives desk

J. Hull and A.White state in [4] that the correct discounting rate for pricing a derivative is the risk-free rate which is consistent with risk–neutral valuation. However, It is well-known that the situation is totally different in practice. Papers like [5], [6] carefully discuss the arguments of Hull and White and explain how and why some of them can not be true in real world. Here we provide a summary of the FVA discussion presented in [5], [6].

Based on the FVA debate discussed in previous section, using the risk-free rate as the discount rate is necessary because it is required by the risk neutral valuation which is the correct economic valuation for derivative pricing. It is completely true that the risk-free rate is the correct rate to use in the risk neutral valuation. However, the risk-neutral valuation is designed under some terms and conditions which is completely different from the real world model.

It is shown in risk–neutral pricing that any option claim can be replicated by a portfolio of stocks and bank account (risk-free debt) while the value of the portfolio is equal to the value of the contract. But all the calculations and results are based on some special assumptions about the economic environment and the distribution of the stock price ([7]):

- The risk-free rate is known and constant.

- There is no transaction costs or taxes.

- It is possible to short-sell with no charge and to borrow at the risk-free rate.

- The volatility of continuously compounded increase in the stock price is known and constant.

- Follows a lognormal distribution; that is, returns on the underlying are normally distributed.

So the risk-free discounting rate is required in the risk-neutral approach but this outcome is possible only within the Black-Scholes economy. For example, in a risk-neutral economy the credit spreads (and hence the funding spreads) do not exist which is obviously in contrast with the real world situation. If r shows the risk-free rate and r_F shows the funding rate that the bank has to pay on its debt, then in the Black-Scholes economy $r = r_F$ while in practice we may have an economy at which $r_F > r$. Then if the bank borrows money (sells bonds) with the rate r_F and on the other hand purchase bonds at the risk-free rates, the resulting PDE depends on both rates which is not consistent with the pricing in risk-neutral world.

So using the risk-free rate for discounting does not necessarily produce the correct result in pricing framework simply because it does not consider the actual costs related to the replication strategy.

The other statement of [4] which is not really sensible is about the funding of hedges. They argue that *trades on hedging instruments involve buying or selling assets for their markets prices and are therefore, zero net present value instruments. As a result, the decision to hedge does not effect valuation.* The answer is since banks need to borrow money on funding rate (not risk-free rate) to hedge the derivative, the replicating strategy involves unsecured funding that would require paying funding spread. For example consider a bank that sells a call option so it need to hedge it (replication with the opposite sign) hence in order to set up the replication portfolio the bank needs to borrow money (to buy the underlying asset) and pay its funding rate so it is likely for the pricing PDE to involve a a rate other than the risk-free rate (see [5] for more details).

The other controversial issue regarding FVA a the well-known principal from finance theory which says that *pricing should be kept separate from funding*. It says that *the rate used to discount the value of a derivative should depend on the risk of the project rather than the riskiness of the firm that undertakes it.* The authors refer to Fisher-Hirshleifer's separation principle ([8], [9]) that shows that it is possible to separate a firm's investment decisions from the firm's financial decisions. This theory says in an economy with the following conditions, a firm's value is not affected by how its investments are financed,

- there are no taxes.

- there is no separation between stockholders and bondholders.

- firms know their future financing needs.

- there are no bankruptcy costs.

So again the argument is not accepted because of unrealistic assumptions that do not correspond to the real world model.

The other argument that they have is concerned with the relationship between FVA and DVA. They argue that since DVA can be viewed as a funding component,to account FVA is some sort of double-counting. On the other hand, practitioners say that they cannot price a deal without accounting for the costs which is not covered by Black-Scholes risk-neutral derivative pricing framework. Hence, FVA is needed.They agree that FVA and DVA may overlap in some cases but they are different in their core ideas. *DVA shows the cost of the money we need to borrow today to make sure that we will be able to meet my cash liabilities with the counterparty during the life of the trade. However the FVA goes beyond that, and it represents the cost of the money we need to borrow globally, which is driven by many factors, including the asymmetry between the collateral agreements with the counterparty and the exchange. These are two very different issues.*

Despite the fact that academics argue that the risk-free rate is the correct rate to use in pricing derivatives and FVA breaks financial principals, the story is different in the real incomplete market, where dealers live. Nowadays quants (quantitative analysts) are working on mathematical structure of pricing process in the presence of FVA ([10]–[13]) and derivatives in many banks are valued at the banks funding rate which makes it quiet relevant to consider the funding cost.

In the following we take a closer look at [10] and [12] to see how the Black-Scholes pricing framework is extended to include a rate other than the risk-free rate.

Chapter 3

Funding Beyond Discounting: Collateral Agreements and Derivatives Pricing

After global crisis 2008, the traditional stable relationships between bank's funding rate, government rate, etc, have been broken. So it is obvious that standard derivative pricing theory which relies on the assumption that one can borrow and lend at the same rate (risk free rate) is not valid any more. Vladimir Piterbarg discusses this subject in his article [12]. He establishes the derivative pricing theory in presence of such complications. In this section we will provide a brief summary of his findings regarding how the new raised challenges like stochastic funding and collateral posting affect the price of derivative.

In a very simplified manner, a derivative desk of a financial institution, e.g. a bank, sells derivative securities to clients while hedging them with other dealers. So the derivative desk needs money to use in its operations, this money is provided by different sources e.g. collateral posted by counterparties or borrowing from the market. In the case of the bank's default, the clients will join the queue of the bank's creditors but the situation is more complicated among the dealers of the hedging process since in order to reduce the credit risk, agreements have been put to collateralize mutual exposures. The agreements about the collaterals is done based on the *credit support annex (CSA)* which provides credit protection by setting forth the rules governing the mutual posting of collateral. The trade is documented under a standard contract called a *master agreement*, developed by the *International Swaps and Derivatives Association (ISDA)*. Since in the case of default the collateral can be used to offset the liabilities, the CSA rate usually referred to as risk-free rate, because purchased assets can be used as collateral so in the case of default of one party the other side can take it, such as in the *repo* market for shares used in delta hedging. The task of specifying the funding rates is done by treasury desk. The unsecured rates that treasury desk provides are generally linked to the unsecured funding rate at which bank itself can borrow or lend. This rate typically depends on bank's credit rating.

So one problem is how to take all the different rates, posted collaterals, etc, into account in pricing a derivative. Here the author avoids different rates for lending and borrowing (to reduce the complication of the task). This article extends the standard no-arbitrage pricing framework for pricing derivatives and takes a closer look at the difference of the CSA (collateralized) and the non-CSA (not collateralized) versions of the same derivative

security. This is important because in reality the dealers refer to mark-to-market prices in their calculations which is very close to CSA-based valuation, however, they still have a large amount of OTC transactions which are non-CSA hence looking carefully at both of them could be very helpful in understanding the whole situation.

In this chapter we take a closer look at the problem of using different discounting rates for CSA and non-CSA prices of the same derivative. This gives us a sense of the importance of FVA in practice.

3.1 Preliminaries

Suppose that the risk free rate for lending is shown by $r_C(t)$, "C" here stands for collateral (the safest available collateral is cash). Also this is agreed that the overnight rate is paid on collateral among dealers under CSA. The discount factor is denoted by $P_C(t,T)$, for $0 \le t \le T \le \infty$ with the following dynamics:

$$dP_C(t,T) = P_C(t,T)\Big(r_C(t)dt - \sigma_C(t,T)^T dW_C(t)\Big) \tag{3.1}$$

where $dW_C(t)$ is the d-dimensional Brownian motion and σ_C is a vector valued, d-dimensional, stochastic process. We consider a derivative on the underlying asset whose price process is $S(t)$ and we indicate the secured funding rate as $r_R(t)$ (R stands for "repo") . The difference $r_C(t) - r_R(t)$ is sometimes called the stock lending fee. Finally we define the short rate for unsecured funding by $r_F(t)$.

We expect $r_C(t) \le r_R(t) \le r_F(t)$ which means that we have the highest rate for unsecured assets and the lowest short rate for the collateralized asset (which is secured by collateral), in other words the riskier the asset is the higher rate it has. Having different rates, which leads to non-zero spreads, can be interpreted as credit risk by introducing joint defaults between banks and various assets used as collateral for funding , e.g funding spread $s_F(t) = r_F(t) - r_C(t)$.

3.2 Black–Scholes with collateral

In this section we check that how the presence of a collateral affects the Black-Sholes pricing formulas. Let $S(t)$ be the asset price under the following dynamics:

$$dS(t) = S(t)\mu_S(t)dt + \sigma_S(t)dW(t) \tag{3.2}$$

Let $V(t)$ be the value of the derivative defined on the underlying asset $S(t)$, then by Ito formula we have:

$$\begin{aligned} dV(t) &= \frac{\partial V(t)}{\partial t}dt + \frac{\partial V(t)}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V(t)}{\partial S^2}(dS)^2 \\ &= \Big(\frac{\partial V(t)}{\partial t} + \frac{1}{2}S^2(t)\sigma_S^2\frac{\partial^2 V(t)}{\partial S^2}\Big)dt + \frac{\partial V(t)}{\partial S}dS \end{aligned} \tag{3.3}$$

So if we define operator $\mathcal{L}$ as the standard pricing operator and Δ as the option's delta :

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{S^2(t)\sigma_S^2}{2}\frac{\partial^2}{\partial S^2}$$

$$\Delta(t) = \frac{\partial V(t)}{\partial S}$$

then the equation (3.3) becomes

$$dV(t) = (\mathcal{L}V(t))dt + \Delta(t)dS(t) \tag{3.4}$$

Let $C(t)$ be the collateral (cash in the collateral account) held at time t against the derivative, for flexibility we allow $C(t)$ be different from $V(t)$. Suppose that in order to replicate the portfolio $V(t)$, we use an adjusted portfolio consisting x units of stock S and cash $\gamma(t)$:

$$V(t) = xS(t) + \gamma(t)$$

Then from the delta hedging argument (see Appendix C), in order to make this portfolio delta neutral we have to choose x such that $\dfrac{\partial V}{\partial S} = 0$ and this gives us the equation:

$$x = \frac{\partial V}{\partial S} = \Delta(t)$$

Then the value of the replication portfolio which is denoted by $\Pi(t)$ is given by:

$$V(t) = \Pi(t) = \Delta(t)S(t) + \gamma(t) \tag{3.5}$$

The cash amount $\gamma(t)$ is divided among a number of accounts:

- Amount $C(t)$ is in collateral.

- Amount $V(t) - C(t)$ needs to be borrowed/lent secured from the treasury.

- Amount $\Delta(t)S(t)$ is borrowed to finance the purchases of $\Delta(t)$ stocks.

Considering the fact that the stock is paying dividend at rate r_D, we can compute the growth of all cash accounts(collateral, unsecured, stock secured, dividends):

$$d\gamma(t) = [r_C(t)C(t) + r_F(t)(V(t) - C(t)) - r_R(t)\Delta(t)S(t) + r_D(t)\Delta(t)S(t)]dt \tag{3.6}$$

On the other hand, from (3.5) by the self financing condition we get:

$$d\gamma(t) = dV(t) - \Delta(t)dS(t) \tag{3.7}$$

Substituting (3.4) in the above equation we have:

$$\begin{aligned} d\gamma(t) &= (\mathcal{L}V(t))dt \\ &= \left(\frac{\partial}{\partial t} + \frac{\sigma_S^2(t)}{2}\frac{\partial^2}{\partial S^2}\right)V(t)dt \end{aligned} \tag{3.8}$$

Thus

$$(\frac{\partial}{\partial t} + \frac{\sigma_S^2(t)}{2}\frac{\partial^2}{\partial S^2})V(t) = r_C(t)C(t) + r_F(t)(V(t) - C(t)) + (r_D(t) - r_R(t))\frac{\partial V}{\partial S}S(t)$$

Rearranging the above equation we get

$$\frac{\partial V}{\partial t} + (r_D(t) - r_R(t))\frac{\partial V}{\partial S}S(t) + \frac{\sigma_S^2(t)}{2}\frac{\partial V^2}{\partial S^2}S^2 = r_F(t)V(t) + (r_F(t) - r_C(t))C(t)$$
(3.9)

Now the Feynman-Kac (see Appendix A) can be applied to solve the above PDE which yields:

$$V(t) = E_t\Big[e^{-\int_t^T r_F(u)du}V(T) + \int_t^T e^{-\int_t^u r_F(v)dv}(r_F(u) - r_C(u))C(u)du\Big]$$
(3.10)

It can be seen from equation (3.9) that the stock grows at rate $r_R(t) - r_D(t)$, that is:

$$\frac{dS(t)}{S(t)} = (r_R(t) - r_D(t))dt + \sigma_S(t)dW_s(t)$$
(3.11)

By rearranging terms in (3.9), we get

$$\frac{\partial V}{\partial t} + (r_D(t) - r_R(t))\frac{\partial V}{\partial S}S(t) + \frac{\sigma_S^2(t)}{2}\frac{\partial V^2}{\partial S^2}S^2 = r_F(t)(V(t) - (t)) + r_C(t)C(t)$$
(3.12)

Hence another useful formula can be obtained for the value of derivative:

$$V(t) = E_t\Big[e^{-\int_t^T r_C(u)du}V(T)\Big] - E_t\Big[\int_t^T e^{-\int_t^u r_C(v)dv}(r_F(u) - r_C(u))(V(u) - C(u))du\Big]$$
(3.13)

Substituting equations (3.6) in (3.7) and using (3.11) we get

$$dV(t) = [r_F(t)V(t) - (r_F(t) - r_C(t)C(t)]dt + \Delta(t)S(t)dW_S(t)$$

hence

$$\begin{aligned} E_t\Big[dV(t)\Big] &= [r_F(t)V(t) - (r_F(t) - r_C(t))C(t)]dt \\ &= (r_F(t)V(t) - s_F(t)C(t))dt \end{aligned}$$

So from the above equation we can observe that the growth of the derivative's value equals r_F applied on the value of the derivative minus the funding spread s_F applied to the collateral. In particular we look at two cases:

- when the collateral is equal to the value of the derivative (or the derivative can be used as collateral) then $C(t) = V(t)$ so we have

$$E_t\Big[dV(t)\Big] = r_C(t)V(t)dt, \qquad V(t) = E_t\Big[e^{-\int_t^T r_C(u)du}V(T)\Big] \quad (3.14)$$

and the derivative grows at risk free rate (since we assumed r_C as the rate corresponds to the safest available collateral).

- When the collateral is zero then $C(t) = 0$ and

$$E_t\Big[dV(t)\Big] = r_F(t)V(t)dt, \qquad V(t) = E_t\Big[e^{-\int_t^T r_F(u)du}V(T)\Big] \quad (3.15)$$

in this case, as it is expected, the rate of growth is the bank's unsecured rate of funding (r_F).

Note that the collateral position on the whole portfolio depends on each individual trade. The reason is when two dealers are trading , the collateral

is applied to the overall value of the portfolio of derivatives between them, with positive exposures on some trades offsetting negative exposures on other trades (called netting). In the simple cases mentioned above: $C(t) = 0$ and $C(t) = V(t)$, the collateral is a linear function of the exact value of the portfolio hence from (3.10) we see that the value of the portfolio is just the sum of values of individual trades .

We saw that the standard Black-Scholes framework for pricing derivatives is significantly changed in the presence of collateral agreements. Also it is shown that the pricing of non-collateralised derivatives needs to be adjusted and the required adjustment comes from the correlation between market factors for the derivative and the funding spread.

Chapter 4

PDE representation of derivatives with bilateral counterparty risk and funding costs

During the last years the risk embedded in counterparty default has become increasingly important. This risk describes the possibility that a counterparty defaults while owing money. In other words the counterpaty does not live up to its contractual obligations when the mark-to-market value of the derivative is positive to the seller. The most common techniques that have been established to mitigate this risk in OTC products are netting agreements and collateral mechanism. However, the counterparty faces a similar situation when the seller defaults. So the credit risk of both party influences the value of a derivative. There are plenty of books and papers that develop techniques for valuation of derivatives under counterparty risk, see .e.g. [11], [14]–[19]. Also we discussed the paper from Piterbarg where it discusses the effect of funding costs on derivative valuation in the presence of collateral (chapter 3).

   In this chapter we go into the details of the paper by C. Burgard and M.Kjaer, [10], which provides a partial differential equation representation for the value of a financial derivative considering the effects of bilateral counterparty risk and the funding costs. The designed model is very flexible in the sense that the funding rate may be different for lending and borrowing, the mark-to-market value can be specified exogenously and the delta hedging is done based on the buying back of a party's own bonds. The paper analyses the obtained PDE in different situations where the mark-to-market value of the derivative is given by either the risky value of the derivative or the counterparty-riskless value of the derivative (the value of the derivative without considering the counterparty risk which is obtained using the standard pricing formula). The extension of the Black-Scholes PDE is driven using hedging arguments. The strategy described in this paper includes repurchase by the seller of its own bonds to hedge out its own credit risk. It is shown in the paper that in the first case where the default-risky derivative price is used as the mark-to-market value, the pricing PDE is in general non-linear. Under certain conditions on the payoff, the non-linear terms vanish and then Feynmen–Kac can be used to solve the PDE. In the second case where the counteparty-riskless derivative price is used as the mark-to-market value the resulting pricing PDE is linear and the solution is obtained by Feynmen–Kac.

Rate	Definition
r	risk-free rate
r_B	yield on recovery-less bond of B
r_C	yield on recovery-less bond of C
λ_B	$\lambda_B = r_B - r$
λ_C	$\lambda_C = r_C - r$
r_F	seller's funding rate
s_F	$s_F = r_f - r$
R_B	recovery rate on mark-to-market value for B
R_C	recovery rate on mark-to-market value for C

TABLE 4.1: Definition of the rates used through this chapter

Let $\hat{V}$ be the risky value of a derivative's contract on asset S between a seller B and a counterparty C, the asset S is not affected by a default of either B or C and it is assumed to follow a Markov process with generator $\mathcal{A}_t$. Also let V be the same derivative contract with assumption that neither B nor C defaults (counterparty risk-less value). In table 4.1 you can see all the symbols used in the rest of this chapter.

Notice that when the derivative can be used as collateral then we have $r_F = r$ otherwise $r_F = r + (1 - R_B)\lambda_B$.

In the following a general PDE for price of the derivative will be driven while the counterparty default is taken into account.

4.1 Model set up and deviation of a bilateral risky PDE

Consider a portfolio consisting of four traded assets:

P_r : default risk-free zero coupon bond
P_B : default risky, zero-recovery, zero-coupon bond of party B
r_C : default risky, zero-recovery, zero-coupon bond of party C
λ_B : spot asset with no default risk

Note that both P_B and P_C are simple bonds that are very useful for simplifying complicated models. If B (seller) does not default then it should pay 1 at maturity T and it pays nothing otherwise. The same situation is defined for counterparty, as well. So we have the following dynamics for above traded assets:

$$\frac{dP_R}{P_R} = r(t)dt \tag{4.1}$$

$$\frac{dP_B}{P_B} = r_B(t)dt - dJ_B$$

$$\frac{dP_C}{P_C} = r_C(t)dt - dJ_C$$

$$\frac{dS}{S} = \mu(t)dt + \sigma(t)dW(t)$$

where $W(t)$ is a standard Brownian motion, $r(t) > 0$, $r_B(t) > 0$, $r_C(t) > 0$, $\sigma(t) > 0$ are deterministic functions of t. dJ_B and dJ_C are processes that jump between 0 and 1 in the case of default of either B or C.

Suppose that the parties B and C enter a derivative on the spot asset S with the payoff $H(S)$ at maturity T. When $H(S) > 0$ to the seller it means that B receives cash or asset from counterparty. we denote the price of this derivative to the seller at time t by $\hat{V}(t, S, J_B, J_C)$. It can be seen than not only it depends on the underlying asset but also depends on the default states J_B and J_C of the seller and counterparty, so it is clearly a risky price. On other hand the counterpary-riskless value of the derivative to the seller is shown by $V(t, S)$ meaning that the parties are assumed to be default free.

Let $M(t, S)$ be the mark-to-market value of the derivative . When one of the parties defaults the close-out (the end of a trading session in financial markets) or claim on the position is determined by $M(t, S)$. Generally speaking, there are two scenarios to determine the mark-to-market value of a derivative:

1. $M(t, S) = \hat{V}(t, S, J_B, J_C)$

2. $M(t, S) = V(t, S)$

Let $R_B \in [0, 1]$ and $R_C \in [0, 1]$ the recovery rates (the value of a security when it emerges from default) on the derivative positions of parties B and C, respectively. So we have the following boundary conditions:

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S) \tag{4.2}$$

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S) \tag{4.3}$$

Equation (4.2) discusses the situation where the seller (B) defaults first, so if the mark-to-market value is positive to the seller then $M^-(t, S) = 0$ and we have $\hat{V}(t, S, 1, 0) = M(t, S)$ which means that the counterparty should pay the full mark-to-market value to the seller. If the market-to-market value is negative to the seller then $M^+(t, S) = 0$ hence $\hat{V}(t, S, 1, 0) = R_B M(t, S)$ which means that the recovery value of mark-to-market value $(R_B M(t, S))$ should be paid to the counterparty. Equation (4.3) can be analysed in the same manner.

Similar to the standard Black-Scholes model for pricing derivatives, we start by replicating the derivative with a self-financing portfolio Π that covers all the underlying risk factors of the model:

$$\Pi(t) = \delta(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t) \tag{4.4}$$

So the portfolio consists of $\delta(t)$ units of asset S, $\alpha_B(t)$ units of P_B, $\delta_C(t)$ units of P_C and $\beta(t)$ units of cash. By the assumption of self-financing portfolio we have

$$\hat{V}(t) + \Pi(t) = 0.$$

As it is already mentioned, the bonds P_B and P_C are used for hedging. For example suppose $\hat{V} > 0$ to the seller. Then if the counterparty defaults, the seller incur a loss. The bond P_C is borrowed by the seller (through a repurchase agreement) to hedge this loss, so the counterparty takes a short

position in P_C hence $\alpha_C < 0$. If the borrowing rate of P_C be r_C then the spread $\lambda_C = r_C - r$ corresponds to the default intensity of C.

Now suppose that $\hat{V} < 0$ to the seller, then if the seller defaults he will gain at his own default. Then $\alpha_B > 0$ since he can replicate this profit by buying back P_B bonds. For this to work, we need to ensure that enough cash is generated and that any remaining cash (after purchase of P_B) is invested in a way that does not generate additional credit risk to the seller: i.e. any remaining positive cash generates a yield at the risk free rate r.

The change in the cash account is:

$$d\beta = \delta(\gamma - q_S)Sdt + \{r(-\hat{V} - \alpha_B P_B) + s_F(-\hat{V} - \alpha_B P_B)^-\}dt - r\alpha_C P_C dt \quad (4.5)$$

(see [10] for more details about the mechanism of cash account)

Imposing that the portfolio $\Pi(t)$ is self-financing we have:

$$-d\hat{V} = \delta(t)dS(t) + \alpha_B(t)dP_B(t) + \alpha_C(t)dP_C(t) + d\beta(t) \quad (4.6)$$

Replacing (4.5) in (4.6) we get:

$$\begin{aligned} -d\hat{V} &= \{-r\hat{V} + s_F(-\hat{V} - \alpha_B P_B) + (\gamma_S - q_S)\delta S + (r_B - r)\alpha_B P_B \\ &\quad + (r_C - r)\alpha_C P_C\}dt - \alpha_B P_B dJ_B - \alpha_C P_C dJ_C + \delta dS \end{aligned} \quad (4.7)$$

On the other hand from Ito lemma we have

$$d\hat{V} = \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_c \quad (4.8)$$

where

$$\Delta \hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0) \quad (4.9)$$
$$\Delta \hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0) \quad (4.10)$$

Replacing $d\hat{V}$ from (4.8) in (4.7) we can find out the coefficients of equation (4.4) and hence eliminate all risks in the portfolio by choosing them as follows:

$$\delta = -\partial_S \hat{V} \quad (4.11)$$

$$\alpha_B = \frac{\Delta \hat{V}_B}{P_B} = -\frac{\hat{V} - (M^+ + R_B M^-)}{P_B} \quad (4.12)$$

$$\alpha_C = \frac{\Delta \hat{V}_C}{P_C} = -\frac{\hat{V} - (M^- + R_C M^+)}{P_C} \quad (4.13)$$

Let us introduce the operator $\mathcal{A}_t$ as:

$$\mathcal{A}_t V = \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (q_S - \gamma_S)S\partial_S V \quad (4.14)$$

So using equation (4.8) one can say that $\hat{V}$ is the solution of the following PDE:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = s_F(\hat{V} + \Delta \hat{V}_B)^+ - \lambda_B \Delta \hat{V}_B - \lambda_C \Delta \hat{V}_C \\ \hat{V}(T, C) = H(S) \end{cases} \quad (4.15)$$

Replacing (4.9) and (4.10) and using boundary conditions introduced in (4.2) and (4.3) we finally have:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-) \\ \hat{V}(T,C) = H(S) \end{cases}$$

(4.16)

In the above replacement we used the fact that $(\hat{V} + \Delta\hat{V}_B)^+ = (R_B M^- + M^+)^+ = M^+$

However, the risk-free value V satisfies in the standard Black-Scholes PDE:

$$\begin{cases} \partial_t V + \mathcal{A}V - rV = 0 \\ \hat{V}(T,C) = H(S) \end{cases}$$

(4.17)

Paying attention to the differences of (4.16) and (4.17) is very important since it reveals the effects of considering the probability of default of parties in pricing the derivative.

Looking at the (4.16) you can see that terms one, three and four are related to counterparty risk while the second term ($s_F M^+$) shows the funding cost.

In the following we check the PDE obtained in (4.16) in four different cases:

- $M(t, S) = \hat{V}(t, S, 0, 0)$ and $r_F = r$

- $M(t, S) = \hat{V}(t, S, 0, 0)$ and $r_F = r + s_F$

- $M(t, S) = V(t, S)$ and $r_F = r$

- $M(t, S) = V(t, S)$ and $r_F = r + s_F$

4.2 Using $\hat{V}$ as mark-to-market value at default

In this case the mark-to-market value at default is equal to the risky value of the derivative:

$$M(t, S) = \hat{V}(t, S, 0, 0)$$

This case is conceptually simpler than the others. Lets analyse the situation by a simple example. Assume that company B enters into a contract with company C and suppose B defaults first. If at the time of default B is in the money then there is no additional effect on the profit and loss at the point of default. If at the time of default C is in the money then it losses $(1 - R)\hat{V}$. So the PDE (4.16) becomes:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+ \\ \hat{V}(T,C) = H(S) \end{cases}$$

(4.18)

The above PDE can be checked further in two different cases:

- $s_F = 0$ which happens when $r_F = r$ and means that the derivative can be used as collateral.

- $s_F = (1 - R_B)\lambda_B$ when the derivative can not be posted as collateral.

Then assuming that the value of the derivative is negative to the dealer (hence positive to the counterparty), we have the hedge ratios α_B and α_C as follows:

$$\begin{aligned}
\alpha_B &= -\frac{\hat{V} - (M^+ + R_B M^-)}{P_B} \\
&= -\frac{\hat{V} - (\hat{V}^+ + R_B \hat{V}^-)}{P_B} = -\frac{(1 - R_B)\hat{V}^-}{P_B}
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\alpha_C &= -\frac{\hat{V} - (M^- + R_C M^+)}{P_C} \\
&= -\frac{\hat{V} - (\hat{V}^- + R_C \hat{V}^+)}{P_C} = -\frac{(1 - R_C)\hat{V}^+}{P_B}
\end{aligned} \tag{4.20}$$

In order to see the mechanism of the above equations we consider two simple example namely when $\hat{V}$ is the bonds of the counterparty.

1. The seller sells P_B to the counterparty

In this example we suppose that the risky, recovery-less bond sold by the seller B to the counterparty C, so we have $\hat{V} = \hat{V}^- = -P_B$ and since the bond is recovery-less $R_B = 0$. Since we consider the rates to be deterministic and we don not have any risk regarding the underlying market factors so the variation of the derivative's value (bond price) is zero and hence the term $\mathcal{A}_t V$ vanishes. So we have the PDE (4.18) as:

$$\begin{cases} \partial_t \hat{V} = r\hat{V} + \lambda_B \hat{V}^- = r_B \hat{V} \\ \hat{V}(T, C) = -1 \end{cases} \tag{4.21}$$

The solution of the above PDE is :

$$\hat{V}(t) = -\exp\left(-\int_t^T r_B(s)ds\right) \tag{4.22}$$

which clearly matches with our assumption that $\hat{V}(t) = -P_B(t)$.

If we suppose the bond P_B has the recovery R_B, then the PDE (4.18) becomes:

$$\begin{cases} \partial_t \hat{V} = r\hat{V} + (1 - R_B)\lambda_B \hat{V}^- \\ \hat{V}(T, C) = -1 \end{cases} \tag{4.23}$$

with the solution

$$\hat{V}(t) = -\exp\left(-\int_t^T \{r(s) + (1 - R_B)\lambda_B(s)\}ds\right) \tag{4.24}$$

From the above equation we see that the rate payable on the bond with recovery is $r(s) + (1 - R_B)\lambda_B(s)$ which in fact is unsecured funding rate $r_F = r + s_F$ that the seller should pay on non-negative cash balances when the derivative can not be posted as collateral.

2. The seller buys P_c from the counterparty In this example, first we suppose that the seller B purchases the recovery-less ($R_C = 0$) bond P_C so

$\hat{V} = \hat{V}^+ = P_C$ and equation (4.18) becomes

$$\begin{cases} \partial_t \hat{V} = s_F \hat{V} + r\hat{V} + \lambda_C \hat{V} = (r_F + (r_C - r))\hat{V} \\ \hat{V}(T, C) = 1 \end{cases} \qquad (4.25)$$

The assumption of being able to use the derivative as collateral, $r_F = r$, makes the above PDE even simpler and the solution is

$$\hat{V}(t) = -\exp\left(-\int_t^T r_C(s)ds\right) \qquad (4.26)$$

as expected for $\hat{V}(t) = P_C(t)$.

Now lets assume that the purchased bond P_C has recovery rate R_C. in this case the solution is:

$$\hat{V}(t) = -\exp\left(-\int_t^T \{r(s) + (1 - R_C)\lambda_C(s)\}ds\right) \qquad (4.27)$$

Similar to the previous example, the above solution shows that $r(s) + (1 - R_C)\lambda_C(s)$ is the rate at which the seller receives on the bond P_C.

4.2.1 The case $r_F = r$

In this part we continue working on the general equation (4.18) in the case that the derivative can be used as collateral, namely $s_F = r_F - r = 0$. So the PDE (4.18) becomes:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ \\ \hat{V}(T, C) = H(S) \end{cases} \qquad (4.28)$$

The above PDE is non-linear so needs numerical methods to be solved but if we suppose that either $\hat{V} < 0$ or $\hat{V} > 0$ then it becomes linear and the well-known Feynman-Kac theorem can be applied to solve it explicitly.

Suppose for example, that B sells an option to C then we have

$$\hat{V} < 0 \quad \text{and} \quad H(S) < 0$$

and using Feynman-Kac we have the solution of PDE (4.28) as :

$$\hat{V}(t, S) = E_t[D_{r+(1-R_B)\lambda_B}(t, T)H(S(T))] \qquad (4.29)$$

where D is the discount factor defined as $D_k(t, S) = \exp\left(-\int_t^T k(s)ds\right)$ given the rate k.

It is customary in counterparty risk literature to write $\hat{V} = V + U$ where U is the *credit valuation adjustment (CVA)*. Inserting this into the equation (4.29) when $\hat{V} < 0$ and using the fact that V satisfies PDE (4.17) we have for

$$\begin{cases} \partial_t U_0 + \mathcal{A}_t U_0 - rU_0 = (1 - R_B)\lambda_B U_0 \\ U(T, C) = 0 \end{cases} \qquad (4.30)$$

Where U_0 refers to $s_F = 0$.

Applying Feynman-Kac on equation (4.30) and using the fact that $V(t, S) = D_r(t, u)E_t[V(u, S(u))]$ we get

$$
\begin{aligned}
U_0(t, S) &= E_t[D_{r+(1-R_B)\lambda_B}(t, S)(-(1 - R_B)\lambda_B V(u, S(u)))du] \\
&= -V(t, S)\{(1 - R_B)\lambda_B D_{(1-R_B)\lambda_B}(t, u)du\} \qquad (4.31)
\end{aligned}
$$

Notice that $D_{r+(1-R_B)\lambda_B}(t, S) = D_r(t, S)D_{(1-R_B)\lambda_B}(t, S)$.

Similarly, for the case that $\hat{V} \geq 0$, when the seller buys an option, symmetry yields:

$$
U_0(t, S) = -V(t, S)\{(1 - R_B)\lambda_B D_{(1-R_C)\lambda_C}(t, u)du\} \qquad (4.32)
$$

We see from (4.31) and (4.32) that when B sells an option then CVA only depends on the credit of the seller (R_B and λ_B) and when B buys an option then CVA only depends on the credit of the counterparty.

4.2.2   The case $r_F = r + (1 - R_B)\lambda_B$

Now we check the situation where the derivative can not be posted as collateral so we have $s_F = r_F - r = (1 - R_B)\lambda_B$. The PDE (4.18) becomes:

$$
\begin{cases}
\partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + \{(1 - R_C)\lambda_C \hat{V}^+ + (1 - R_B)\lambda_B\}\hat{V}^+ \\
\hat{V}(T, C) = H(S)
\end{cases} \qquad (4.33)
$$

Similar to the previous case the obtained PDE is non-linear unless we make some additional assumptions.

So first suppose that $\hat{V} \leq 0$ (so $\hat{V}^+ = 0$) and the situation is exactly the same as (4.28) (where s_F was zero). So considering the fact that $\hat{V} = V + U$, we see that $U = U_0$ in (4.31).

If $\hat{V} \geq 0$ then by applying Feynman-Kac on the obtained PDE we have

$$
\hat{V}(t, S) = E_t\left[D_{r+(1-R_B)\lambda_B+(1-R_C)\lambda_C}(t, T)H(S(T))\right] \qquad (4.34)
$$

Substituting $\hat{V} = V + U$ in the obtained PDE and using the same argument as the case $r_F = r$ we have

$$
U(t, S) = -V(t, S)\{\int_t^T k(u)D_k(t, u)du\} \qquad (4.35)
$$

where $k = (1 - R_B)\lambda_B + (1 - R_C)\lambda_C$.

Comparing the two discussed cases $r_F = r$ and $r_F = r + (1 - R_B)\lambda_B$, we see that when B sells an option the CVA is the same for both cases. On other hand when B buys an option then by comparing (4.35) and (4.32) we see that when $s_F \neq 0$ then an additional funding spread $s_F = (1 - R_B)\lambda_B$ is encountered.

4.3   Using V as mark-to-market value at default

We will now consider the case that the payments at default are based on V instead of $\hat{V}$:

$$
M(t, S) = V(t, S)
$$

So $M(t, S) = V(t, S)$ are replaced in the boundary conditions (4.2) and (4.3). So the PDE (4.16) will change into

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^- + s_F V^+ \\ \hat{V}(T, C) = H(S) \end{cases} \quad (4.36)$$

Writing $\hat{V} = V + U$ also gives the following linear PDE

$$\begin{cases} \partial_t U + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = (1 - R_B)\lambda_B V^- + (1 - \lambda_C)R_C\lambda_C V^+ + s_F V^+ \\ U(T, C) = 0 \end{cases} \quad (4.37)$$

Note that $V = V^+ + V^-$ where $V^- = \min(V, 0)$. Applying Feynman-Kac on the above PDE yields the CVA as follows:

$$\begin{aligned} U(t, S) &= -(1 - R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t, u)E_t\Big[V^-(u, S(u))\Big]du \\ &\quad - (1 - R_C)\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t, u)E_t\Big[V^+(u, S(u))\Big]du \\ &\quad + \int_t^T s_F(u)D_{r+\lambda_B+\lambda_C}(t, u)E_t\Big[V^+(u, S(u))\Big]du \end{aligned} \quad (4.38)$$

When $s_F = 0$, the last term vanishes and the above equation is reduced to the common bilateral CVA equation discussed in many books and papers (e.g. [15]). However, using the derivative as collateral is not so common in practice, so the equation (4.38) gives a consistent adjustment of the derivatives price for bilateral counterparty risk and funding costs. As a special case, consider the situation where the funding spread corresponds to that of unsecured bond of B with recovery R_B, then $s_F = (1 - R_B)\lambda_B$ and by combining the first and third term in (4.38) we get CVA as:

$$\begin{aligned} U(t, S) &= -(1 - R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t, u)E_t\Big[V(u, S(u))\Big]du \\ &= -(1 - R_C)\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t, u)E_t\Big[V^+(u, S(u))\Big]du \end{aligned} \quad (4.39)$$

4.4 Numerical experiments

In this section we suppose that the seller B buys a call option and we calculate the derivative value $\hat{V} = V + U$, where V is the risk-less derivative price which is obtained from standard Black-Scholes price and U is calculated in four different cases:

Case 1: $M = \hat{V}$ and $s_F = 0$

$$U_0 = -(1 - \exp\{-(1 - R_C)\lambda_C(T - t)\})V(t, S)$$

Case 2: $M = \hat{V}$ and $s_F = (1 - R_B)\lambda_B$

$$U = -(1 - \exp\{-((1 - R_B)\lambda_B + (1 - R_C)\lambda_C)(T - t)\})V(t, S)$$

Case 3: $M = V$ and $s_F = 0$

$$U_0 = -\frac{(1 - R_C)\lambda_C(1 - \exp\{-(\lambda_B + \lambda_C)(T - t)\})}{\lambda_B + \lambda_C}V(t, S)$$

Case 4: $M = V$ and $s_F = (1 - R_B)\lambda_B$

$$U = -\frac{\{(1 - R_B)\lambda_B + (1 - R_C)\lambda_C\}(1 - \exp\{-(\lambda_B + \lambda_C)(T - t)\})}{\lambda_B + \lambda_C}V(t, S)$$

Example 1 *We suppose that B buys a call option on the stock S with the initial price $S_0 = 200$, maturity $T = 5$, strike price $K = 200$, volatility $\sigma = 0.25$, risk-free interest rate $r = 60\%$ and recovery rates $R_B = R_C = 40\%$. Since the option is bought by B we have $V \geq 0$ and $\hat{V} \geq 0$.*

We used the above formulas to calculate the CVA in each case. We checked the changes in CVA for different values of λ_B. You can see the result in Figure 4.1 where λ_C is fixed at 0% (top), 2.5% (middle) and 5% (bottom). The results show that the effect of funding cost is spectacular such that in all three cases the graphs of $s_F > 0$ (case 1 and case 3) are close to each other but very different from the cases that $s_F = 0$ (case 2 and case 4). So taking funding into account is completely relevant since it has a great impact on the CVA.

The standard pricing framework of Black-Scholes for the value of financial derivatives with bilateral counterparty risk and funding costs has been extended in this chapter. Especially we saw in the numerical results that in liquid markets like swap pricing and vanilla option pricing paying attention to the funding cost can change the computations significantly.
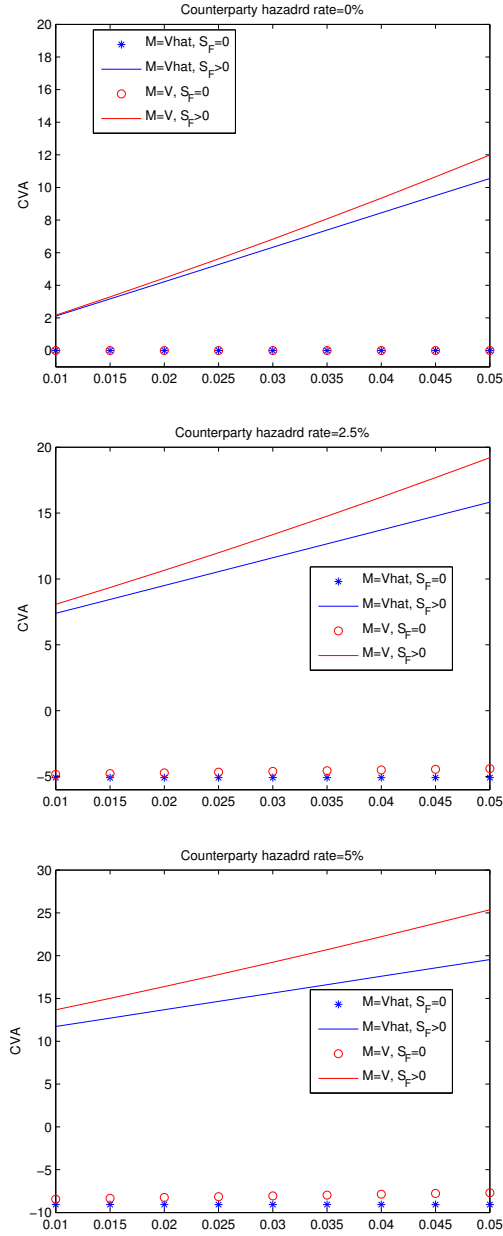
FIGURE 4.1: CVA changes for different values of λ_B. λ_C is considered to be 0% (top), 2.5% (middle) and 5% (bottom). The mark-to-market value is considered in four different cases.

Chapter 5

FVA Calculation

As mentioned earlier, FVA mostly reflects the costs of collaterals that banks post to hedge uncollaterallized derivatives. this cost increased rapidly after financial crisis. This raise encouraged banks to pay attention to funding valuation adjustment in addition to CVA and DVA.

Quants like Claudio Albanese have proposed formulations for calculating FVA in different scenarios: FCA/FBA and FVA/FDA, ([20]–[23]). Since the proposed methods of funding costs are closely related to accounting concepts, we can not go into the details of them in this dissertation. However, we try to focus on their mathematical structure instead and present some numerical experiments to show how the FVA looks like in practice.

5.1 FCA/FBA method

The *symmetric funding value adjustment* (SFVA) is a metric that is formed based on the assumption that we can price derivatives as their cost of replication while ignoring the possibility of bank's default. The details can be found in [20] but the formula is as follows:

$$SFVA = E\Big[\int_0^\infty e^{-\int_0^t r_B(s)ds} s_B(t)\Big(\sum_i V_i(t)\Big)dt\Big] \tag{5.1}$$

where E is the expectation under the risk neutral measure, $V_i(t)$ is the riskless (default free) value of the i'th derivative, $r_B(t)$ is the bank's funding rate and $s_B(t) = r_B(t) - r_{OIS}$ is the bank's funding spread over the OIS. As can be seen in the formula, the bounds of integral are from 0 to ∞ which refers to the assumption of ignoring the bank's default.

In equation (5.1) ,$V_i(t)$ has no sign so it can be split into positive and negative part, which in turn gives us *funding benefit adjustment* (FBA) and *funding cost adjustment* (FCA):

$$FBA = E\Big[\int_0^\infty e^{-\int_0^t r_B(s)ds} s_B(t) \sum_i V_i^-(t)dt\Big] \tag{5.2}$$

$$FCA = E\Big[\int_0^\infty e^{-\int_0^t r_B(s)ds} s_B(t) \sum_i V_i^+(t)dt\Big] \tag{5.3}$$

5.2   FVA/FDA method

In this section the funding costs are calculated when the bank or counter-prty defaults. Lets τ_B and τ_i show default time of bank and i'th counter-party , respectively. Then the cost of uncollateralized trading, FVA, is:

$$FVA = E\Big[\int_0^{\tau_B} e^{-\int_0^t r_{OIS}(s)ds} s_B(t) \Big(\sum_i V_i(t)\mathbb{I}_{t<\tau_i} \Big)^+ dt \Big] \qquad (5.4)$$

We see from $(\sum_i V_i(t)\mathbb{I}_{t<\tau_i})+$ that the positive value of portfolio before the default of counterparty, incorporates the funding cost. The exterior integral is from zero to τ_B meaning that if at the time of bank's default, it has an exposure then the funding cost is obtained from (5.4) which in fact is a benefit to bank.

FVA has a "twin" denoted by FDA,*Funding Debt Adjustment*, raised from the fact that if funding is a cost for bank then the counterparty must receive a benefit of equal size. So from the accounting point of view FVA belongs to asset account while FDA belongs to liability account.

Roughly speaking, FVA is the result of not having a repo market for unsecured OTC derivatives which causes the banks to borrow the required variational margin for the hedging process. The other side of this story are the senior creditors of the bank that make profit of the unsecured borrowing receivable after bank's default. The value of this recovery is FDA and

$$FDA = FVA.$$

5.3   Numerical Experiment

Example 2 *We checked the proposed model for funding cost adjustment in* (5.3) *for an imaginary model (Figure 5.1) where bank inters into an uncollateralized swap with a client. To hedge this derivative bank enters into another swap with exactly the same floating rate and fixed rate but in the reverse direction (back to back hedge) while the hedging side is collateralized. Bank borrows money from market to fund the initial margin and the posted collateral in the hedging side, so*

$$Debt = initial\ margin + posted\ collateral \qquad (5.5)$$

In this example we supposed for simplicity that the initial margin is zero. Since we have a single derivative (swap with client) then $\sum_i V_i^+(t) = V^+(t)$ where $V^+(t)$ is the value of the swap calculated in its life time with maturity time $T = 18$ months. Swap price calculation is done based on the traditional approach explained in [1] (chapter 7).

We supposed $r_B = 3\%$ and r_{OIS} is obtained by simulating the interest rate. We used Vasicek model as our interest rate model that specifies the instantaneous interest rate by following the stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma dW_t \qquad (5.6)$$

where a,b and σ are the model parameters and W_t is the standard Brownian motion under the Q measure. In Figure 5.3 you can see 100 sample paths of the stochastic model in (5.6).
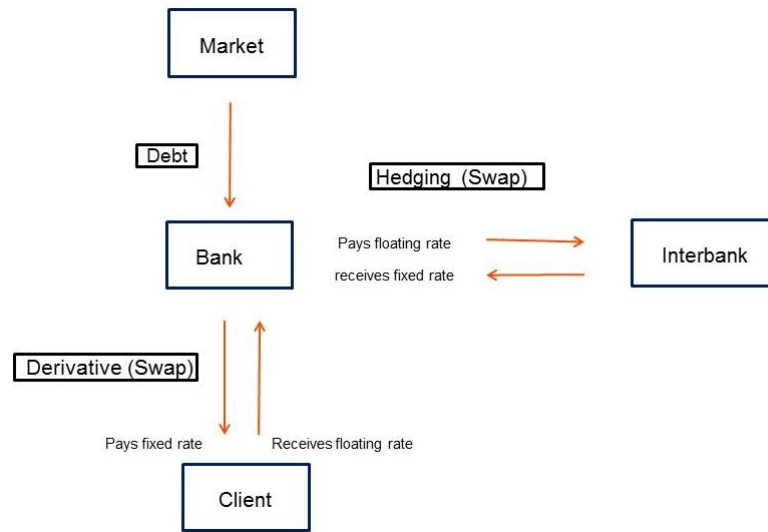
FIGURE 5.1: A single transaction model along with the hedging strategy

Figure 5.4 shows that how the FCA changes by choosing different starting rates in simulating Vasicek model.

In the following we explain a measure called *Net Stable Funding Ratio (NSFR)* that requires banks to maintain a stable funding profile in relation to the composition of their assets and activities. Then we propose a new formulation to calculate FVA which is intertwined with the concept of NSFR (for more details regarding NSFR see [24], [25]).

5.4 NSFR

The Bassel Committee on Banking Supervision (BCBS) introduced the Net Stable Funding Ratio (NSFR) in 2010 and the final standards were published in October 2014. NSFR is a measure designed to compare a firm's available stable funding (ASF) to its required stable funding (RSF). NSFR is defined based on the liquidity characteristics of the firm's assets and activities and the aim is to ensure that the firm holds a minimum amount of stable funding over a one year horizon. By taking NSFR into account banks and other financial institutions can reduce funding risks originating from lack of liquidity. In fact NSFR is a complementary for Liquidity Coverage Ratio (LCR) which is a similar short term measure that requires banks to hold enough high quality liquid assets in order to survive under stressed conditions within 30 days time horizon.

The NSFR standards set out the framework for calculating NSFR as follows:

$$NSFR = \frac{ASF}{RSF}$$

The ratio should always be equal or greater than 1 and it is mentioned by BCBS that the NSFR will become a minimum standard by 1st January 2018.

As mentioned earlier, ASF refers to the those types of equity and liability that are supposed to provide stable sources of funding over one year
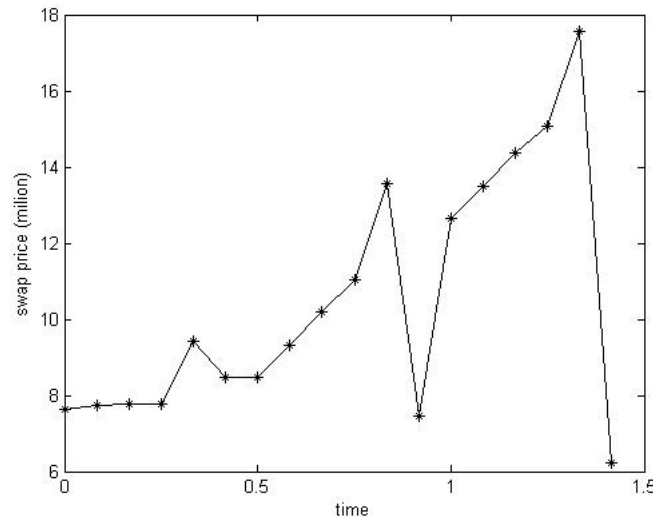
FIGURE 5.2: Price of an interest rate swap in its life time when the short rate simulation is started from $r_0 = 3\%$. Fixed leg rate of the swap is $R = 4\%$ and floating leg rate is $r = 5.5\%$

time horizon. ASF is calculated by first assigning the carrying value of a firm's equity and liability to one of the specified categories and then multiplying each assigned value by a weight which is already defined in NSFR standards (see Figure 5.5). The total ASF is the sum of weighted amounts.

The RSF, on the other hand, measures different assets in terms of the proportion of stable funding required to support them. Similar to the ASF case, the total RSF is calculated as the sum of the value of assets multiplied by a specific weigh assigned to each particular asset type. Figure 5.5 shows ASF and RSF weights for different categories of assets and liabilities. The idea behind introducing NSFR is to restrict over-reliance on short-term funding during times and to promote banks to have better assessment of liquidity risk across all on- and off-balance sheet items.

5.5   A new FVA calculation methodology

Generally speaking, a derivative desk of a financial institution, e.g. a bank, sells derivative securities to clients while hedging them with other dealers. So the derivative desk needs money to use in its operations, this money is provided by different sources e.g. collateral posted by counterparties or borrowing from the market, etc, however any of these transactions directly effects the value of NSFR. This refers us to the potential for NSFR to be linked with FVA.

The idea is to define the FVA in such a way that it also contains the information of net stable funding ratio. As it is stated by BCBS, banks are required to keep $NSFR \geq 1$ which obviously brings some costs to the bank. We use this fact to define funding costs of the contract such that we have $NSFR = 1$. We combine the concepts of NSFR and FVA such that we end up with a new methodology to calculate FVA. The proposed method not
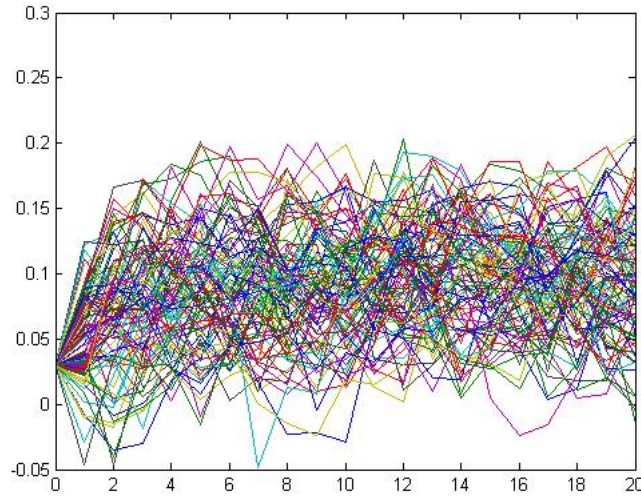
FIGURE 5.3: 100 trajectories of the short rate corresponding
to the Vasicek model where $a = 0.3$, $b = 0.1$, $\sigma = 0.03$ and
$r_0 = 3\%$.

only contains the funding costs but also guarantees the bank to stay in an
acceptable level of NSFR.

Consider the model described in Figure 5.1, the NSFR calculation for
this model can be easily done using the table in Figure 5.5. The ASF calcu-
lation includes multiplying regulatory capital (Reg.Cap) and the borrowed
amount from the market (D) by their associated weights from Figure 5.5, so

$$ASF = 100\%\text{Reg.Cap} + \alpha D$$

where

$$\alpha = \begin{cases} 50\% & \text{Debt with 6 months–12 months maturity} \\ 100\% & \text{Debt with more than 1 year maturity} \end{cases}$$

Also the RSF can be calculated in a same way:

$$RSF = 100\%(\text{Net Derivatives - Net Collateral}) + 20\%(\text{Derivative Liabilities})$$

And the value of NSFR is obtained by dividing ASF over RSF.

Now the objective is to manage NSFR to stay at the desired level, which
is supposed to be 1 in this example. It can be seen from the ASF and RSF
calculations that the only viable modification that one can do is to change
D (debt from the market) both in terms of amount and the maturity time.
At this stage we assume α fixed and try to find D such that $NSFR = 1$, so
we have:

$$1 = \frac{ASF}{RSF} = \frac{100\%\text{Reg.Cap} + \alpha D}{100\%(\text{Net Derivatives - Net Collateral}) + 20\%(\text{Derivative Liabilities})}$$
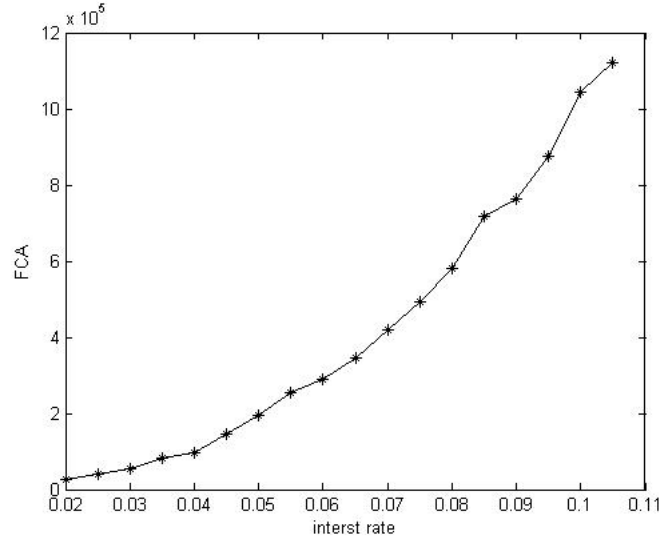$$(5.7)$$

FIGURE 5.4: Variations of FCA when the initial point is
changing in Vasicek model simulation

So we can find D in the above equation as :

$$D = \frac{100\%(\text{Net Derivatives - Net Collateral}) + 20\%(\text{Derivative Liabilities}) - 100\%\text{Reg.Cap}}{\alpha}$$
(5.8)

Where D is the amount of debt needed to have NSFR $= 1$. Let C^+ shows the posted collateral then two scenarios might happen:

- $D \geq C^+$ which means that D is enough to provide the posted collateral in hedging side,

- $D < C^+$, meaning that bank needs more money to post collateral.

Now we can define a new formula based on the above discussion and similar to the ones defined by Albanese in 5.1 and 5.2 ,

$$FVA = FVA_1 + FVA_2 \tag{5.9}$$

where

$$FVA_1 = E\Big[\int_0^{\tau_B} e^{-\int_0^t r_B(s)ds} s_B(t)D(t)dt \Big] \tag{5.10}$$

$$FVA_2 = E\Big[\int_0^{\tau_B} e^{-\int_0^t r_B(s)ds} s_B(t)\Big(C^+(t) - D(t)\Big)^+ dt \Big] \tag{5.11}$$

It can be seen that the proposed formulation is very general and flexible as it includes both funding costs generated by posting collateral and the required debt to have NSFR $= 1$, as well.

Coming back to example 2, we consider an interest rate swap which is traded with a client while it is back-to-back hedged in the other side with interbank. The funding is done through a debt from market with short rate $r_B = 3\%$. The fixed leg rate of the swap is $R = 4\%$ and the floating rate is $r = 5.5\%$. The ASF and RSF are calculated from Table 5.5 for each time step t_i where $0 = t_1 < t_2 \cdots < t_m = T$.
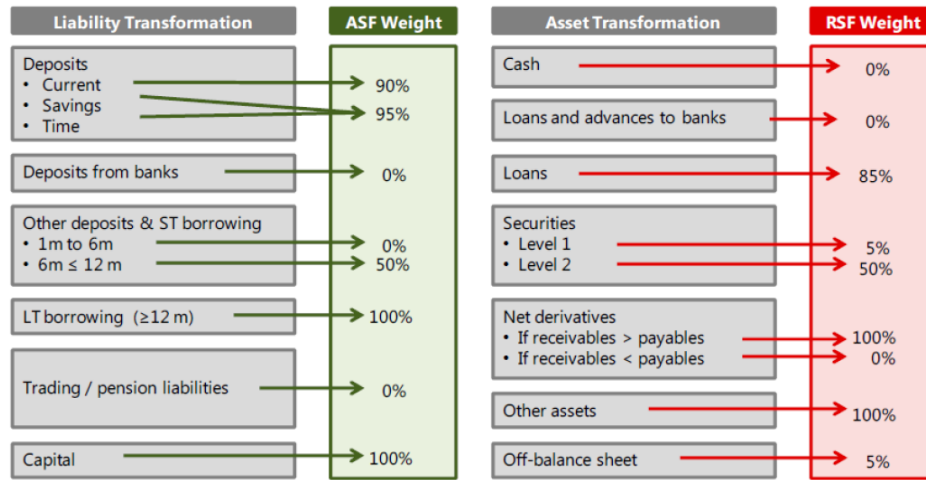
FIGURE 5.5: Summary of transformation of assets and liabilities to NSFR. Source: BCBS (2014).

The required debt for each time step is calculated from (5.8) and r_{OIS} is obtained by simulating the Vasicek model. As it is mentioned in previous example the swap price is calculated by the standard method explained in [1]. Having all ingredients of the integrals (5.10) and (5.11), we can easily use the Monte-Carlo method to calculate FVA from (5.9). The calculation of regulatory capital (Reg.Cap) for ASF valuation is done according to standards presented in [26].

As can be seen in Figure 5.6, the level of ordinary NSFR is below 1 (left plot), however applying (5.8) to compute the "Debt" increases the level of NSFR to above 1 which can be seen in the right plot of Figure 5.6. Also we repeat the experiment of FCA where we calculated the swap price with different initial interest rates (see Figure 5.4). Assuming that $\alpha = 100\%$ is fixed, we use the same model parameters as Example 2 to implement the new FVA formulation and compare the resulted graph with FCA case. If we use (5.8) to obtain D for each initial interest rate then we will be able to calculate the FVA in (5.9) for different initial interest rates. The result is presented in Figure 5.7.

The trends of the FCA plot (5.4) and the FVA plot (5.9) looks very similar, however if we take a closer look at the numerical results we see that the value of FVA is slightly higher than FCA which is quite predictable. The reason is maintaining the level of NSFR above 1 causes some additional costs which is included in our FVA calculation framework so we get a higher overall funding cost which makes our propose method more expensive than the other ones. For example suppose that we start the interest rate simulation from $r_0 = 3\%$, also assume that the funding is done over one year horizon, namely $\alpha = 100\%$, and assume that all the parameters are the same for both models FVA and FCA, then we will get $\text{FCA} = 1.58 \times 10^5$ and $\text{FVA} = 2,87 \times 10^5$.
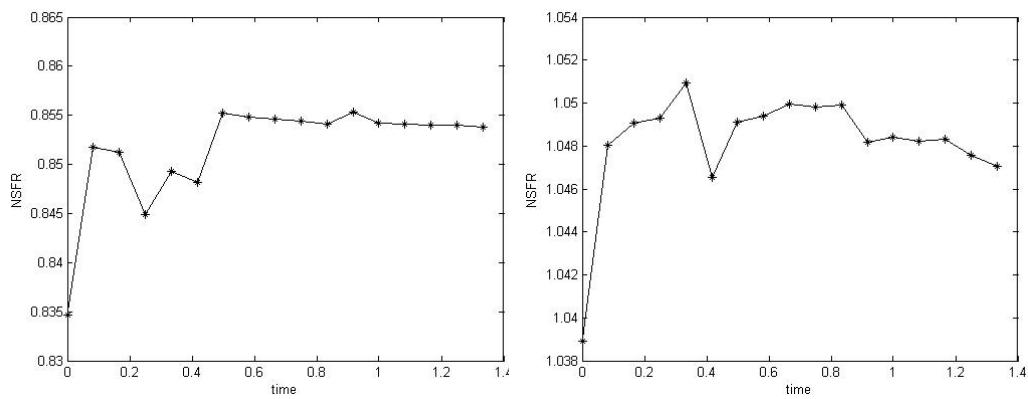
FIGURE 5.6: Left: NSFR measure in the life time of the derivative when debt (*D*) in ASF calculation is obtained from the standard approach of (5.5). Right: NSFR measure in the life time of the derivative when debt (*D*) in ASF calculation is obtained from the proposed approach of (5.8).
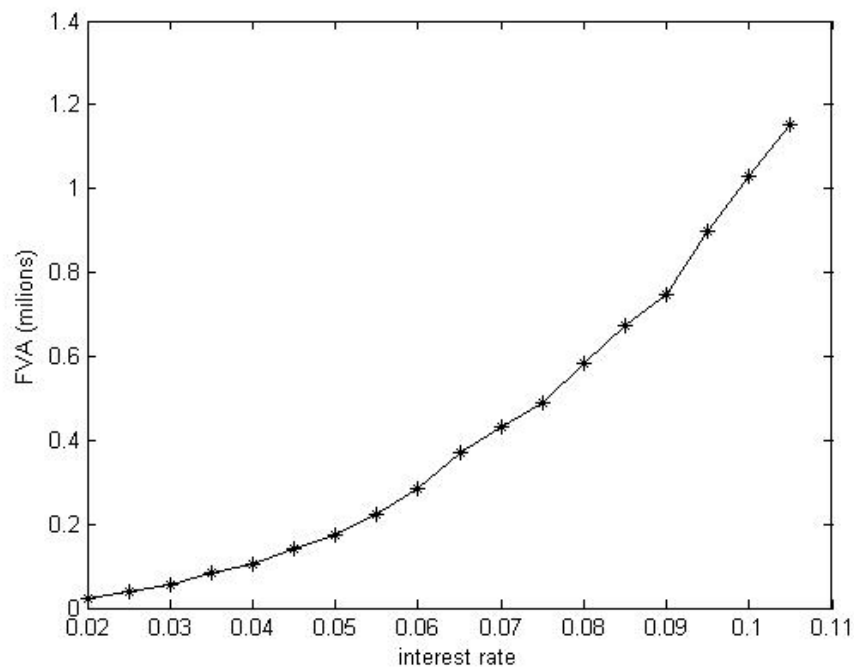

FIGURE 5.7: Variations of FVA with different interest rates as the initial point of Vasicek model simulation. The FVA is calculated from the proposed methodology of (5.9)

5.5.1 Conclusion

The funding value adjustment (FVA) as one of the XVAs is discussed in this thesis. The concept of FVA which refers to the funding costs of a transaction, is reviewed from different viewpoints. On one hand some theoreticians believe that FVA is in contrast with financial principles and the correct interest rate to be involved in the pricing process is the risk free rate which is also consistent with the risk-neutral valuation. On the other hand practitioners argue that the standard Black–Scholes pricing is based on some unrealistic assumptions. They believe that funding cost is a real cost that should be considered in pricing process therefore it is unavoidable to have multiple discounting curves to reflect the actual funding costs of a trade.

We presented the extended partial differential equation for derivative contracts in the presence of funding costs and collateral agreements. Also we used the examples of a call option to illustrate that taking funding into account is completely relevant since it has a significant impact on the CVA.

Finally we discussed the FVA formulation as a metric for funding costs which followed by a numerical example that showed how does FCA look like for a single transaction model.

An overview of the NSFR measure is given which shows the firm's liquidity risk profile. NSFR is designed to reduce the likelihood of liquidity risk which is a result of over-reliance on short term funding. We discussed how the NSFR and FVA are linked then we used this connection to introduce a new calculation methodology for FVA. The new proposed approach not only reflects the funding costs of the transaction but also guarantees an acceptable level of NSFR which contributes to enhancing the stability of banks. The implementation results show that funding costs obtained by our proposed method are higher than classic FVA formulations as the new methodology includes the additional cost of keeping NSFR above 1, as well.

Appendix A

Black–sholes framework of pricing

In this section we briefly study the theoretical pricing model for financial derivatives summarized from [2]. Generally speaking, a derivative is a contract between two or more parties based on one or more underlying assets. The value of a derivative is derived from fluctuations of the underlying asset(s) which can be stocks, bonds, interest rates, etc. Derivatives can be traded either on an exchange or OTC. In this section we provide an answer to the crucial a question:"what is the fair price for a derivative?" To this end we start by the formal mathematical definition of a derivative:

Definition 1 *A contingent claim (financial derivative) is any stochastic variable X of the form $X = \Phi(Z)$ where Z is the stochastic variable driving the stock price process.*

Consider a fixed portfolio h with the value process V_t^h at time t. A self-financing portfolio is characterised by that all trades are financed by selling or buying assets in the portfolio. No money is withdrawn or inserted after the initial forming of the portfolio. So if we have

$$V_t^h = \sum_i \lambda_i S_i(t)$$

then for a self financing portfolio we should have:

$$V_{t+\Delta t}^h - V_t^h = \sum_i \lambda_i (S_i(t + \Delta t) S_i(t)).$$

Definition 2 *The portfolio h is called an* arbitrage portfolio *if it has the following properties:*

$$
\begin{aligned}
V_0^h &= 0 \\
V_1^h &> 0 \qquad \text{with probability 1}
\end{aligned}
$$

So basically having an arbitrage portfolio means to make a positive amount of money out of nothing, and the existence of arbitrage is interpreted as a serious case of mispricing. We can now say that a given market model is arbitrage free if there is no arbitrage portfolios.

The original approach to derivative pricing, going back to Black, Scholes, and Merton[7], is to use a replication argument together with the assumption of having an arbitrage free market. Having the concept of no arbitrage portfolio in mind we now state the definition of a replicating portfolio:

Definition 3 *A given contingent claim X is said to be reachable if there exists a portfolio h such that*

$$V_1^h = X$$

with probability 1. In that case we say that the portfolio h is a hedging portfolio *or a* replicating portfolio. *If all claims can be replicated we say that the market is* complete.

In order to model asset prices we need concepts like stochastic processes, Brownian motion and some other preliminaries from probability theory which are skipped here but it can be found e.g. in [2]. Here we state the most important theories to reach the Black-Scholes standard pricing model.

Theorem 1 *Ito's formula*

Assume that the process X has a stochastic differential given by

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \tag{A.1}$$

where μ and σ are drift and diffusion coefficients and let f be a $C^{1,2}$ function. Define process $Z(t) = f(t, X(t))$. Then Z has a stochastic differential given by

$$df(t, X(t)) = \{\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}\}dt + \sigma \frac{\partial f}{\partial x}dW(t) \tag{A.2}$$

Now we start to build the pricing framework by considering a financial market consisting of only two assets:

- A risk-free asset with the price process $B(t)$ where being risk-free means that there is no stochastic term (dW) in its dynamics

$$dB(t) = r(t)B(t)dt \tag{A.3}$$

  Hence we have deterministic knowledge of the return at time t by looking at r. A risk-free asset usually refers to a bank account with short rate of interest r.

- A stock with price process $S(t)$ with the dynamics:

$$dS(t) = \alpha\Big(t, S(t)\Big)S(t)dt + \sigma\Big(t, S(t)\Big)S(t)dW(t) \tag{A.4}$$

  The function σ is known as the volatility of S, while α is the local mean rate of return of S.

The Black-Scholes model is based on having a portfolio consisting the assets introduced in (A.3) and (A.4), the only difference is α, σ and r are considered as deterministic constants:

$$dB(t) = rB(t)dt \tag{A.5}$$
$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \tag{A.6}$$

The most important derivative which is frequently used in pricing theory are European call option and European put option:

Definition 4 *A European call (put) option with strike price K and time of maturity T on the underlying asset S is a contract defined as:*
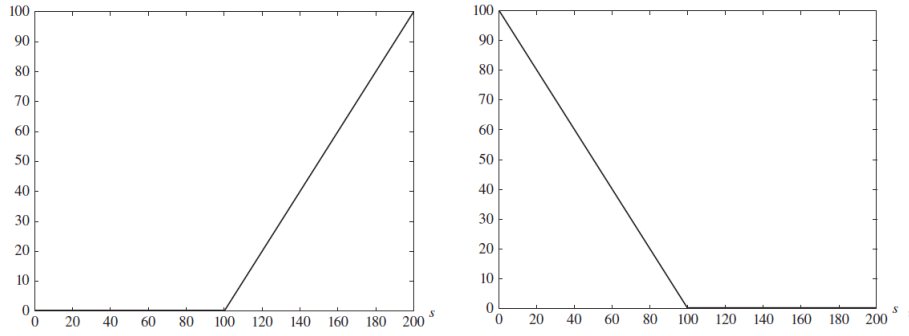
FIGURE A.1: Left: Contract function of European call with
$K = 100$. Right: Contract function of European put with
$K = 100$.

- *The holder of the option has, at time T, the right but not the obligation to buy (sell) one share of the underlying stock at the price K from the underwriter of the option.*

- *The right to buy the underlying stock at the price K can only be exercised at the precise time T.*

A contingent claim X is called a simple claim if it is of the form

$$\mathcal{X} = \Phi(S(T)) \tag{A.7}$$

The function Φ then is called a contract function. So a European call is a simple T–claim for which the contract function is given by

$$\Phi(x) = \max[x - K, 0]$$

the European put option's contract can be formulated similarly:

$$\Phi(x) = \max[K - x, 0]$$

The graphs of European options are shown in Figure **??** and Figure **??**.

The main goal of this section is to calculate the price of a contingent claim $\mathcal{X}$ shown by $\Pi(t; \mathcal{X})$. We assume that the price process $\Pi(t)$ is such that there are no arbitrage possibilities on the market made of $(B(t), S(t), \Pi(t))$.

Proposition 1 *Suppose that there exists a self-financed portfolio h, such that the value process V^h has the dynamics*

$$dV^h(t) = k(t)V^h(t)dt,$$

where k is an adapted process. Then it must hold that $k(t) = r(t)$ for all t, or there exists an arbitrage possibility.

The above proposition says that for a locally risk-less portfolio the rate of return should be equal to the bank's short rate. In other words, on an arbitrage free market there can be only one short rate of interest.

The Black-Scholes Equation

We assume that given a market consists of two assets with the following dynamics

$$dB(t) = rB(t)dt \tag{A.8}$$

$$dS(t) = \alpha\Big(t, S(t)\Big)S(t)dt + \sigma\Big(t, S(t)\Big)S(t)d\overline{W}(t) \tag{A.9}$$

where $\overline{W}_t$ is Brownian motion under the P measure (will be discussed further in the following). Also we consider a simple contingent claim of the form

$$\mathcal{X} = \Phi(S(T)) \tag{A.10}$$

The price process for this derivative asset is of the form

$$\Pi(t; \mathcal{X}) = F(t, S(t))) \tag{A.11}$$

where F is some smooth function. Our task is to determine what F might look like if the market consisting of $S(t)$, $B(t)$ and $\Pi(t; \mathcal{X})$ is arbitrage free.

By applying Ito formula on equation (A.11) and using (A.4)we have:

$$d\Pi(t) = \alpha_\pi(t)\Pi(t)dt + \sigma_\pi(t)\Pi(t)d\overline{W}_t \tag{A.12}$$

where α_π and σ_π are defined as:

$$\alpha_\pi(t) = \frac{F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{F} \tag{A.13}$$

$$\sigma_\pi(t) = \frac{\sigma S F_s}{F} \tag{A.14}$$

Now we build a portfolio with vale V consisting two assets: the underlying stock and the derivative asset. We show the formed portfolio by (u_s, u_π). From the definition of relative portfolio and its price formula (see section 6.2 of [2]) we have:

$$dV = V[u_s\alpha + u_\pi\alpha_\pi]dt + V[u_s\sigma + u_\pi\sigma_\pi]d\overline{W}. \tag{A.15}$$

In order to find u_s and u_π which are the ingredients of the replication portfolio, we use the following linear systems at which the first equation comes from the definition of relative portfolios and the second one is designed to vanish the stochastic $d\overline{W}$ term:

$$u_s + u_\pi = 1 \tag{A.16}$$

$$u_s\sigma + u_\pi\sigma_\pi = 0 \tag{A.17}$$

Solving the above linear system gives us:

$$u_s = \frac{\sigma_\pi}{\sigma_\pi - \sigma}, \tag{A.18}$$

$$u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma} \tag{A.19}$$

Replacing (A.13) and (A.14) gives us the explicit form of the portfolio:

$$u_s = \frac{S(t)F_s(t, S(t))}{S(t)F_s(t, S(t)) - F(t, S(t))} \tag{A.20}$$

$$u_\pi = \frac{-F(t, S(t))}{S(t)F_s(t, S(t)) - F(t, S(t))} \tag{A.21}$$

On the other hand by vanishing the stochastic term in equation (A.15) we get:

$$dV = V[u_s\alpha + u_\pi\alpha_\pi]dt,$$

from proposition 1 we must have the following relation which guarantees the absence of arbitrage:

$$u_s\alpha_+ u_\pi\alpha_\pi = r. \tag{A.22}$$

Now substituting (A.13),(A.20) and (A.21) into (A.22) we obtain the equation

$$F_t(t, S(t)) + rS(t)F_s(t, S(t)) + \frac{1}{2}\sigma^2(t, S(t))S^2(t)F_{ss}(t, S(t)) - rF(t, S(t)) = 0 \tag{A.23}$$

Also we already know that the the price of the derivative asset at the time of maturity equals the value of the contract:

$$\Pi(T) = \Phi(S(T)) \tag{A.24}$$

We summarize the above findings in the *Black Scholes equation* which is the most important result of this section.

Theorem 2 *Assume that the market is specified by equations (A.8)and (A.9) and that we want to price a contingent claim of the form (A.7). Then the only pricing function of the form (A.11) which is consistent with the absence of arbitrage is when F is the solution of the following boundary value problem in the domain $[0, T] \times \mathbb{R}^+$.*

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}\sigma^2(t, s)s^2F_{ss}(t, s) - rF(t, s) = 0 \tag{A.25}$$

$$F(T, s) = \Phi(s) \tag{A.26}$$

It can be seen from the Black-Scholes equation that the price of the claim is given as a function of the price of the underlying asset ($\Pi(t; \mathcal{X}) = F(t, S(t))$) which means that the derivative pricing is a relative pricing in terms of the price of the underlying asset. The other interesting point about the obtained PDE in (A.25) is that it does not contain the local mean rate of return $\alpha(t, s)$ of the underlying asset. However, the pricing PDE obtained here has some weak points, specially when it comes to real world problems. For example we will be in trouble with pricing an OTC instrument (which is not traded on a regular basis) using the above theorem since we assumed that there exists a price process for the derivative.

The equation (A.25) with the boundary condition (A.26) can be explicitly solved by *Feynman–Kac* formula.

Proposition 2 *(Feynman–Kac)*

Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) - r(t,x)F(t,x) = 0$$
$$F(T,x) = \Phi(x)$$

Assume furthermore that the process $\sigma(s,X_s)\dfrac{\partial F}{\partial x}(s,X_s) \in \mathcal{L}^2$ *with X defined as*

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s$$
$$X(t) = x$$

Then F has the representation

$$F(t,x) = E_{t,x}[e^{-\int_t^T r(s,x)ds}\Phi(X_T)]. \tag{A.27}$$

In our case the discounting rate is considered to be constant, $r(t,x) = r$, so we get

$$F(t,s) = E_{t,s}e^{-r(T-t)}[\Phi(S(T))]. \tag{A.28}$$

If we call the dynamics introduced in (A.8) and (A.9) as the P–dynamics then we can define a new probability measure Q under which the S-process has a different probability distribution. This is done by defining the Q–dynamics of S as

$$dS(t) = rS(t)dt + S(t)\sigma(t,S(t))dW(t) \tag{A.29}$$

where W is the Brownian motion under the Q measure.

In the following we see that by defining the Q probability which is known as *risk neutral measure* we will have a risk neutral valuation of the derivative's price given absence of arbitrage.The model derives a risk neutral valuation formula for derivatives price in the sense that it gives today's price as the discounted expected value of future's price. The following proposition states the condition of no arbitrage:

Proposition 3 *The market model is arbitrage free if and only if there exists a risk neutral measure Q.*

Considering the new dynamics of S and using the Feynman–Kac theorem we can state the following important result which gives us the risk neutral valuation of the derivative's price.

Theorem 3 *Risk Neutral Valuation*
The arbitrage free price of the claim $\Phi(S(T))$ *is given by* $\Pi(t;\Phi) = F(t,S(t))$, *where F is given by the formula*

$$F(t,s) = e^{-r(T-t)}E_{t,s}^Q\Big[\Phi(S(T))\Big] \tag{A.30}$$

where E^Q *denotes expectations taken under the Q–measure given in* (A.29) .

Appendix B

Numeraire and change of measure

In this section we will provide the definition of numeraire and the theory of changing the numeraire as a useful tool that helps to drastically reduce the computational cost.

A numeraire is a positively priced asset which denominate other assets. A typical example of a numeraire is the currency of a country, since people usually measure other asset's price in terms of the unit of currency. One can also use other numeraire when it is more convenient. For example consider a financial market with the locally risk free asset B and a risk neutral measure Q. It is well-known that a measure is a martingale measure only relative to a specific numaraire. Also we know that when the numeraire is chosen as the money account B, the risk neutral martingale measure has the property of martingalizing all processes of the form $\dfrac{S(t)}{B(t)}$ where S is the arbitrage free price process of any (non-dividend paying) traded process. In the following we use a simple example to show that how changing the numeraire can be helpful in decreasing the computational cost required for calculating the arbitrage free price of a derivative.

Suppose that we have a contingent claim $\mathcal{X}$ with a stochastic short rate $r(t)$. We know that the price of $\mathcal{X}$ at $t = 0$ is given by:

$$\Pi(0; \mathcal{X}) = E^Q \left[e^{-\int_0^T r(s)ds} . \mathcal{X} \right] \tag{B.1}$$

From the computational point of view the above problem is expensive since there are two stochastic process $\int_0^T r(s)$ and $\mathcal{X}$ and in order to compute the expectation we need compute a double integral which is hard work.

For a moment suppose that the two mentioned stochastic variables are independent then we can rewrite (B.1) as :

$$\Pi(0; \mathcal{X}) = E^Q \left[e^{-\int_0^T r(s)ds} \right] . E^Q [\mathcal{X}]$$

Where $E^Q \left[e^{-\int_0^T r(s)ds} \right] = p(0, T)$ is the bond price (discounting factor).

The above formula is much easier to compute than (B.1) since the bond price $p(0, T)$ can be directly observed instead of being computed. The main problem with the above argument is that in most cases $\int_0^T r(s)$ and $\mathcal{X}$ are not independent under Q (think about the case that $\mathcal{X}$ is a contingent claim on the underlying bond.)

The above problem can be rectified by defining a new measure in which the two stochastic variables are independent.

Let $\{N_t\}$ be a numerair that pays no dividend, and $\{S_t\}$ any price process, then there exists a measure $\tilde{Q}$ which is equivalent to the risk neutral measure Q and under the measure $\tilde{Q}$ the process

$$S_t^B = \frac{S_t}{N_t}$$

is a martingale. So we replicate the measure Q with $\tilde{Q}$ as the new risk neutral measure. For example the price of the derivative $\mathcal{X}$ at time t is:

$$\Pi(t;\mathcal{X}) = N_t E^{\tilde{Q}}\left[\frac{\mathcal{X}}{N_T}|\mathcal{F}_t\right]$$

where the expectation is under the new measure $\tilde{Q}$. One example that changing the numerair can facilitates the computations is when dealing with derivatives defined in terms of several underlying assets (see [2], chapter 26 for more details).

Appendix C

Parity relation and Delta hedging

Parity Relation

We saw in Appendix A that any contingent claim can be replicated by a portfolio compounded of the underlying asset $S(t)$ and the bank account $B(t)$ (risk-free asset). But using only S and B leads to some limitations in replicating some types of claims. So we need to add some other ingredients to out replication portfolio, like European call options. In the following which is a summary of section 9 of [2], we will provide important properties of the replication portfolios and the relevant theories. See [2] for the proofs and more details.

Lets start with the following proposition which shows the linearity property for the contract's price:

Proposition 4 *Let Φ and Ψ be contract functions for the T-claims $\mathcal{X} = \Phi(S(T))$ and $\mathcal{Y} = \Psi(S(T))$. Then for any real number α and β we have the following price relation.*

$$\Pi(t; \alpha\Phi + \beta\Psi) = \alpha\Pi(t; \Phi) + \beta\Pi(t; \Psi) \tag{C.1}$$

Let $C(t, s; K, T)$ and $P(t, s; K, T)$ show the price of a European call option and a European put option at time t given $S(t) = s$ with T as the time to maturity and strike price K. Also we need to make the following basic assumption which state that the corresponding contracts provide the holder, one share of the stock, 1\$ and a European call with strike price K.

$$\begin{aligned}
\Phi_S(x) &= x, & \text{(C.2)} \\
\Phi_B(x) &= 1, & \text{(C.3)} \\
\Phi_{C,K}(x) &= \max[x - K, 0]. & \text{(C.4)}
\end{aligned}$$

We show the prices of the above contracts as follows:

$$\begin{aligned}
\Pi(t; \Phi_S) &= S(t) & \text{(C.5)} \\
\Pi(t; \Phi_B) &= e^{-r(T-t)}, & \text{(C.6)} \\
\Pi(t; \Phi_{C,K}) &= C(t, s; K, T). & \text{(C.7)}
\end{aligned}$$

Now fix T and suppose that Φ is a linear combination of the basic contracts mentioned in (C.1), so :

$$\Phi = \alpha\Phi_S + \beta\Phi_B + \sum_{i=1}^{n} \gamma_i \Phi_{C,K}, \tag{C.8}$$

So according to proposition 4 the price of Φ can be written as:

$$\Pi(t;\Phi) = \alpha\Pi(t;\Phi_S) + \beta\Pi(t;\Phi_B) + \sum_{i=1}^{n}\gamma_i\Pi(t;\Phi_{C,K}), \qquad \text{(C.9)}$$

We can construct the replicating portfolio such that it contains :

- α shares of the underlying stock,

- β zero coupon T-bonds with face value \$1,

- γ_i European call option with strike price K_i, all maturing at T.

Now we consider a put option $\Phi_{P,K}(x) = \max[K - x, 0]$ and try to replicate it with the mentioned contacts in (C.2), (C.3) and (C.4). Drawing the a simple figure for each contract we can see that:

$$\Phi_{P,K} = K\Phi_B(x) + \Phi_{C,K} - \Phi_S \qquad \text{(C.10)}$$

From the above relation we state the so called put-call parity relation:

Proposition 5 *Consider a European call and a European put option, both with strike price K and the time to maturity T. Denoting the corresponding pricing functions by $C(t,s)$ and $P(t,s)$, we have the following relation:*

$$P(t,s) = Ke^{-r(T-t)} + C(t,s) - S \qquad \text{(C.11)}$$

In other words, a put option can be replicated by a portfolio consisting of a long position in a zero coupon T-bond with face value K ($K\Phi_B$), a long position in a European call ($\Phi_{C,K}$) and a short position in one share of the underlying asset (Φ_S).

The following proposition answers to the important question that which contracts can be replicated by such a portfolio.

Proposition 6 *Fix an arbitrary continuous contract function Φ with compact support. Then the corresponding contract can be replicated with arbitrary precision using a constant portfolio consisting only bonds, call options and underlying stock.*

Delta Hedging

Consider a portfolio constructed based on a single asset with price process S_t. So the portfolio may consist of the positions in the underlying asset itself or positions in various options written on the underlying asset. In the following we want to check the sensitivity of the price of portfolio, $P(t,s)$ with respect to

1. Price changes of the underlying asset.

2. Changes in the model parameters.

The first case tells us about how much the price of the portfolio will change with the changes of the underlying asset which in fact is to measure the exposure risk. The second case,does not contribute into risk exposure but shows the sensitivity of the portfolio value with respect to one of the model parameters.
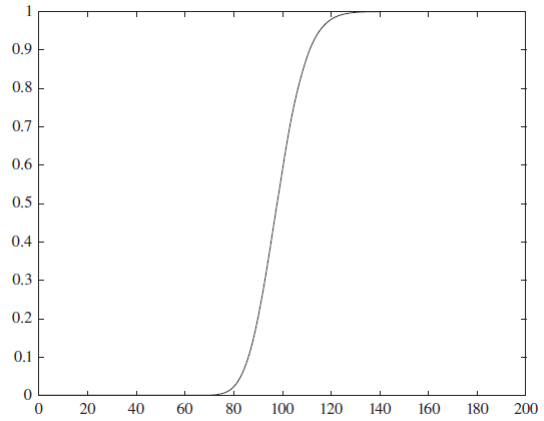
FIGURE C.1: Delta for a European call

Sensitivity measures are introduced under some standard notations, while all of the following measures are called *the greeks*.

$$\Delta = \frac{\partial P}{\partial S}$$
$$\Gamma = \frac{\partial^2 P}{\partial S^2}$$
$$\rho = \frac{\partial P}{\partial r}$$
$$\Theta = \frac{\partial P}{\partial t}$$
$$\nu = \frac{\partial P}{\partial \sigma}$$

A portfolio which is insensitive to the above parameters is said to be *neutral*. For example when the value of the portfolio is not sensitive with respect to the changes of the underlying stock then Δ equals zero and the portfolio is called delta neutral. You can see the graph of delta in figure C.1.

Among all the greeks, delta is of great importance since it can be effectively used for hedging purposes. Delta hedging of a portfolio means to immunize it against small changes in the underlying asset price, in other words, to reduce the risk associated with price movements in the underlying asset. If we have $\Delta = o$ then the portfolio is delta neutral and we are already done otherwise we need to have a strategy to hedge the portfolio. The simplest idea to mitigate the risk is to sell the whole portfolio and invest the obtained money in the bank (risk-free account) but this method is not practically preferable. The other option is to add a derivative to the portfolio such that the adjusted portfolio becomes delta neural. So if we show the price of the added derivative by $F(t, s)$ and x shows the number of units of the added derivative then the value of the new portfolio, V, is given by

$$V(t, s) = P(t, s) + x \cdot F(t, s) \tag{C.12}$$

So we just need to choose x such that we obtain $\Delta = 0$:

$$\Delta = \frac{\partial V}{\partial S} = \frac{\partial P}{\partial S} + x \cdot \frac{\partial F}{\partial S} = 0$$

As a result we obtain $x = -\dfrac{\Delta_P}{\Delta_F}$ which gives us the required units of the added derivative in order to have a delta neutral portfolio.

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