Lecture 17
Radiofrequency Cavities I

Professor Emmanuel Tsesmelis
Principal Physicist, CERN
Department of Physics, University of Oxford

Graduate Accelerator Physics Course
John Adams Institute for Accelerator Science
17 November 2022

Table of Contents I

Introduction

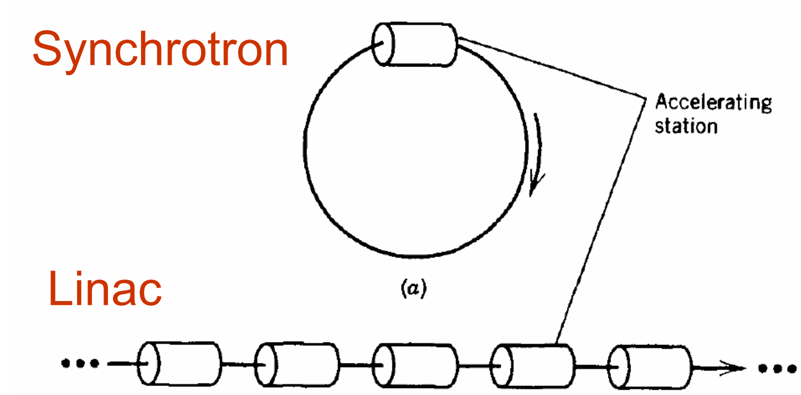
DC versus RF

DC accelerator

RF accelerator

Introduction

- Necessary conditions for acceleration
  - Both linear and circular accelerators use electromagnetic fields oscillating in resonant cavities to apply the accelerating force.



- LINAC – particles follow straight path through series of cavities
- CIRCULAR ACCELERATORS – particles follow circular path in B-field and particles return to same accelerating cavity each time around

Introduction

- Limitations to final beam energy achievable in static accelerators may be overcome by the use of high frequency voltages.
- Virtually all modern accelerators use powerful radiofrequency (RF) systems to produce the required strong electric fields.
  - Frequencies ranging from few MHz to several GHz.

Introduction

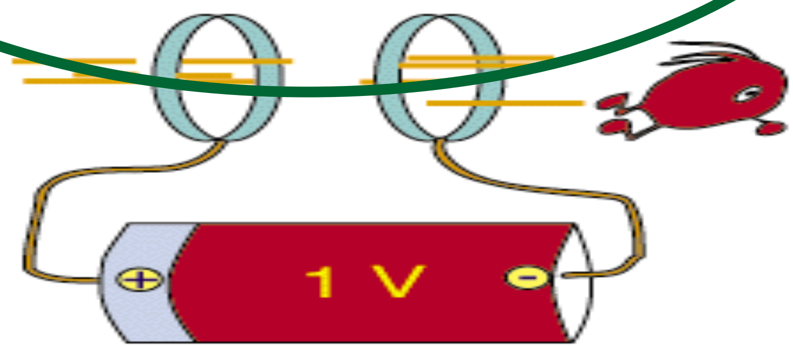
- In classical linac or synchrotron, EM field oscillates in resonant cavity and particles enter and leave by holes in end walls.
- Energy is continuously exchanged between electric and magnetic fields within cavity volume.
- The time-varying fields ensure finite energy increment at each passage through one or a chain of cavities.
- There is no build-up of voltage to ground.
- Equipment which creates and applies field to the charged particles is known as RADIOFREQUENCY (RF), and much of its hardware derived from telecommunications technology.

Introduction – Maxwell’s Equations

$$\nabla \times \mathbf{E} = -\frac{dB}{dt}$$

- Becomes in its integral form

$$\int_{\Gamma} \mathbf{E}.\mathbf{ds} = -\frac{\partial}{\partial t}\int_{S} \mathbf{B}.\mathbf{n}\,da$$



- Hence, there can be no acceleration without time-dependent magnetic field.
- We also see how time-dependent flux may provide particle acceleration.

Wideröe Linear Accelerator

- In order to avoid limitations imposed by corona formation and discharge on electrostatic accelerators, in 1925 Ising suggested using rapidly changing high frequency voltages instead of direct voltages.
- In 1928 Wideröe performed first successful test of linac based on this principle.

- Series of drift tubes arranged along beam axis and connected with alternating polarity to RF supply.
  - Supply delivers high frequency alternating voltage:

$$V(t) = V_{\max} \sin(\omega t)$$

Wideröe Linear Accelerator

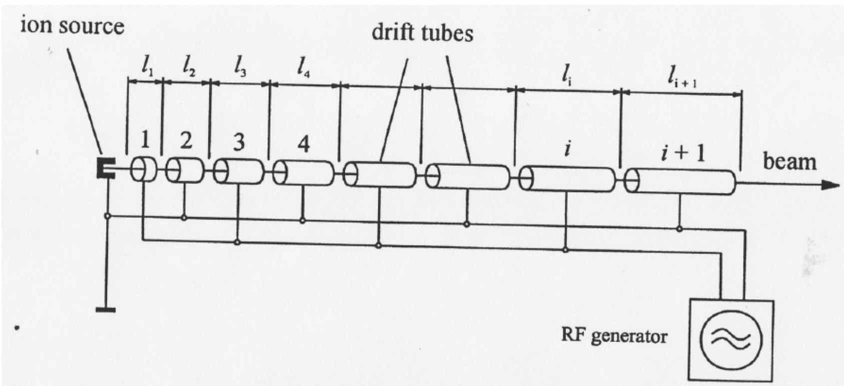
- Acceleration process
  - During first half period, voltage applied to first drift tube acts to accelerate particles leaving ion source.
    - Particles reach first drift tube with velocity v_1.
  - Particles then pass through first drift tube, which acts as a Faraday cage and shields them from external fields.
  - Direction of RF field is reversed without particles feeling any effect.
  - When they reach gap between first and second drift tubes, they again undergo an acceleration.

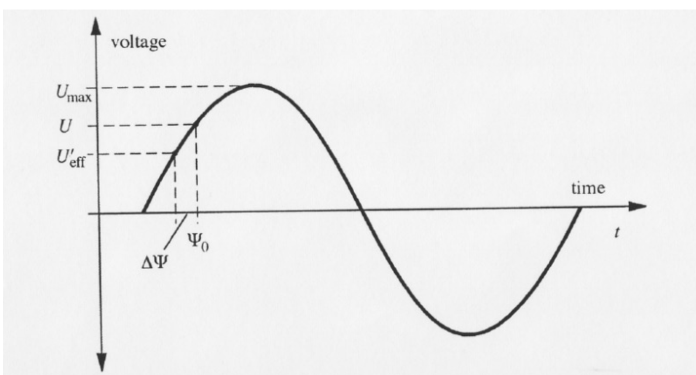
- Acceleration process
  - After the *i-th* drift tube the particles of charge *q* have reached energy

$$E_i = iqV_{\max} \sin \Psi_0$$

where Ψ_0 is average phase of RF voltage that particles see as they cross gaps.

Wideröe Linear Accelerator





- Observations
  - Energy is proportional to number of stages *i* traversed by particle.
  - The largest voltage in entire system is never greater than V_{max}
    - Arbitrary high energies without voltage discharge

Wideröe Linear Accelerator

- Accelerating gaps
  - During acceleration particle velocity increases monotonically but alternating voltage remains constant in order to keep the costs of already expensive RF power supplies reasonable.
    - ➔ Gaps between drift tubes must increase.
- RF voltage moves through exactly half a period $\tau_{RF}/2$ as particle travels through one drift section.
- Fixes distance between *i-th* and *(i+1)-th* gaps

$$l_i = \frac{v_i \tau_{RF}}{2} = \frac{v_i}{2 f_{RF}} = \frac{v_i \lambda_{RF}}{2c} = \beta_i \frac{\lambda_{RF}}{2} = \frac{1}{f_{RF}} \sqrt{\frac{iqV_{\max} \sin \Psi_0}{2m}}$$

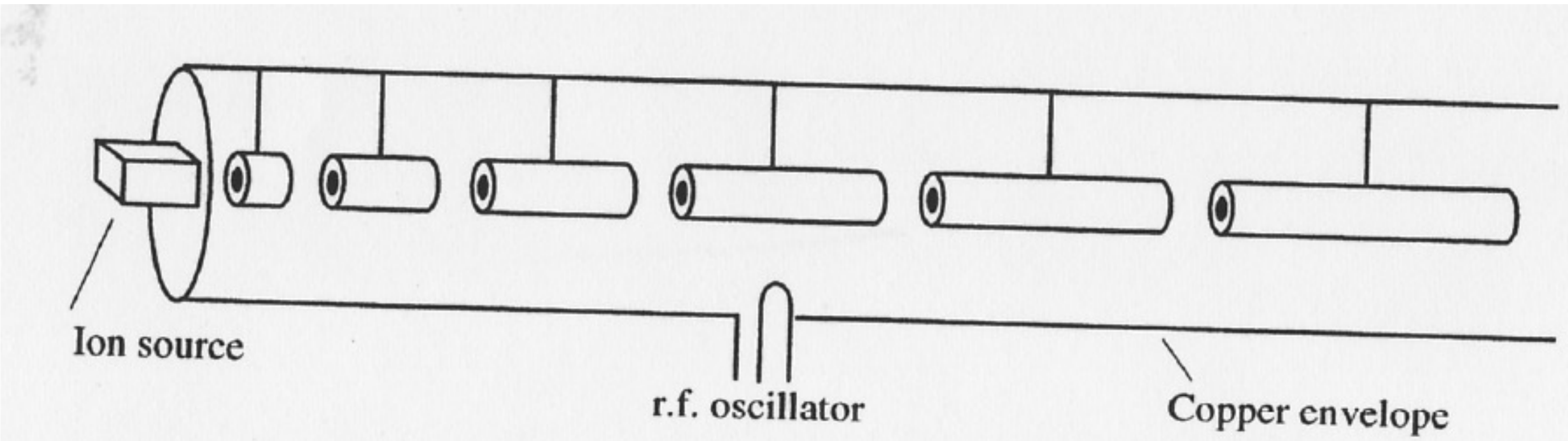
Modern Linear Accelerators

- Drift tubes typically no longer used and have been generally replaced by cavity structures.
- Electron linacs
  - By energies of a few MeV, particles have already reached velocities close to light speed.
  - As they are accelerated electron mass increases with velocity remaining almost constant
    - Allows cavity structures of same size to be situated along whole length of linac.
      - Leading to relatively simple design.

Modern Linear Accelerators

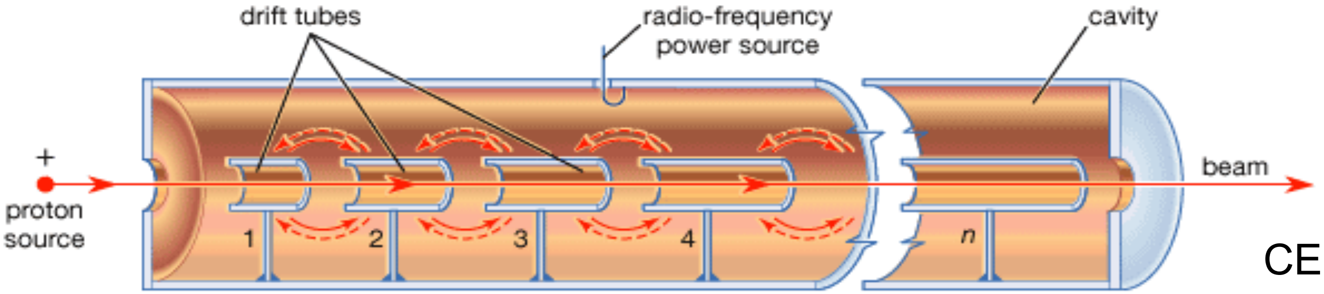
- Hadrons
  - Particles still have non-relativisitc velocities in first few stages and Wideröe-type structure is needed
  - Alvarez structure
    - Drift tubes are today arranged in a tank, made of good conductor (Cu), in which a cavity wave is induced.
    - The drift tubes, which have no field inside them, also contain the magnets to focus the beam.
    - Energy gain from accelerating potential differences between end of drift tubes, but the phase shift between drift tube gaps is 360^o.
    - Alternate tubes need not be earthed and each gap appears to the particle to have identical field gradient which accelerates particle from left to right.

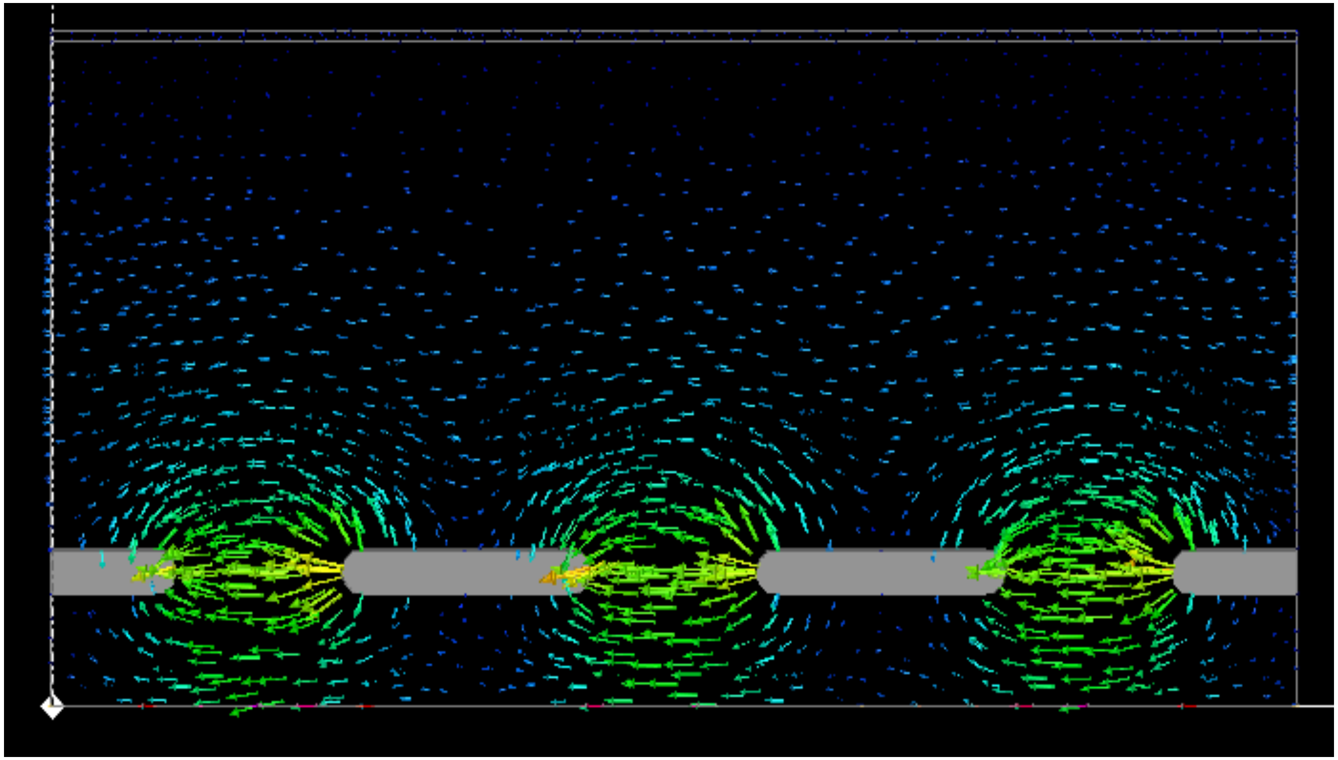
Alvarez Structure



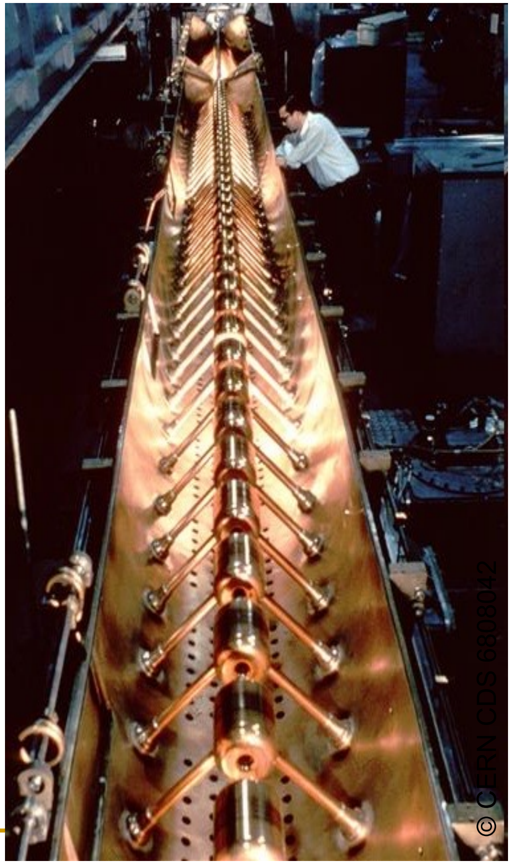
The concept of the Alvarez linear accelerator

Drift Tube Linac: Higher Integrated Field





CERN LINAC1 1982-1992



Courtesy E. Jensen

Phase Focusing

- Energy transferred to particle depends on V_{max} and Ψ_0
  - Small deviation from nominal voltage V_{max} results in particle velocity no longer matching design velocity fixed by length of drift sections.
    - Particles undergo a phase shift relative to RF voltage.
    - Synchronisation of particle motion and RF field is lost.
- Solution based on using $\Psi_0 < \pi/2$ so that the effective accelerating voltage is $V_{eff} < V_{max}$
  - Assume particle gained too much energy in preceding stage and travelling faster than ideal particle and hence arrives earlier.
  - Sees average RF phase $\Psi = \Psi_0 - \Delta\Psi$ and is accelerated by voltage

$$V_{eff}^{'} = V_{\max} \sin(\Psi_0 - \Delta\Psi) < V_{\max} \sin \Psi_0$$

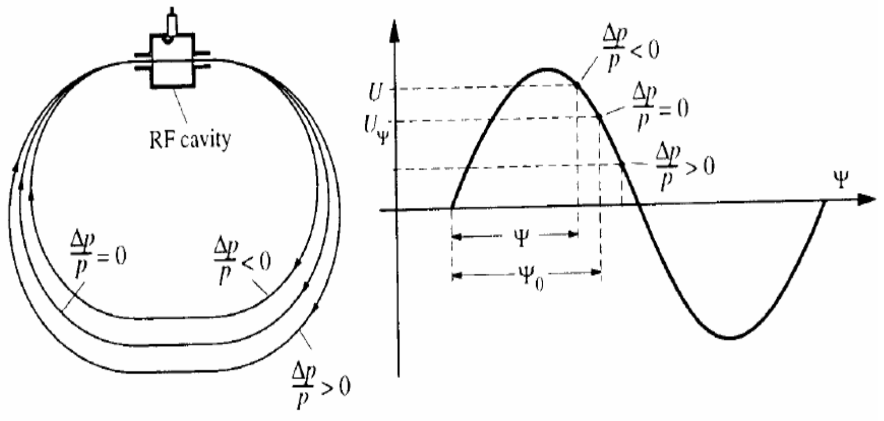
  which is below the ideal voltage.
  - Particle gains less energy & slows down again until it returns to nominal velocity.
  - All particles oscillate about nominal phase Ψ_0

Synchrotron Oscillations

- The periodic longitudinal particle motion about the nominal phase is called synchrotron oscillation.
- As the ideal particle encounters the RF voltage at exactly the nominal phase on each revolution, the RF frequency ω_{RF} must be an integer multiple of the revolution frequency ω_{rev}

$$h = \frac{\omega_{RF}}{\omega_{rev}}$$

where *h* is the harmonic number of the ring.

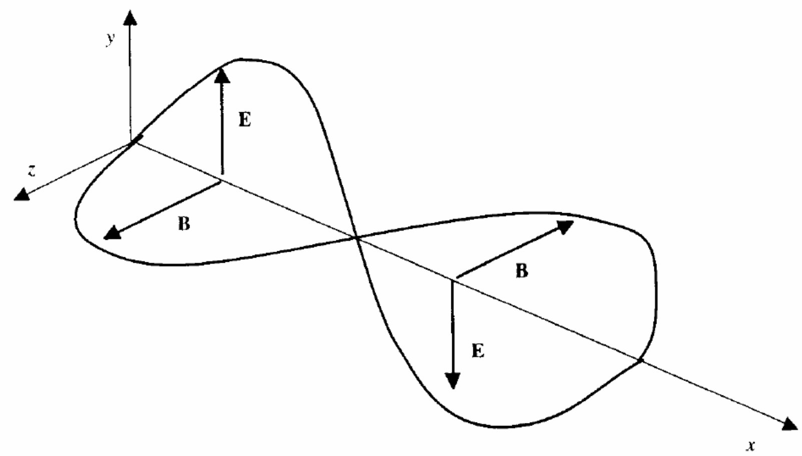


Phase focusing of relativistic particles in circular accelerators

Waves in Free Space

- Wave parameters

Plane transverse electric and magnetic wave (TEM) propagating in free space in x-direction.



$$\text{Velocity in vacuum} \quad v = c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\text{Velocity in medium} \quad v = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r}}$$

With ε_r being the dielectric constant and the magnetic permeability is μ_r

The ratio between the electric and magnetic fields is

$$\frac{E}{H} = 376.6\sqrt{\frac{\mu_r}{\varepsilon_r}} \quad (\Omega)$$

The Poynting flux
(the local power flux) is

$$P = (E \times H) \; W m^{-2}$$

Conducting Surfaces

- Consider waves in metal boxes, recall boundary conditions of a wave at a perfectly conducting metallic surface.
  - $E_{tangential}$ component and H_{normal} component to surface vanish.
- Skin depth – EM wave entering a conductor is damped to 1/e of initial amplitude in distance

$$\delta_s = \frac{1}{\sqrt{\pi f \mu_0 \mu_r \sigma}}$$

- Surface resistance $R_s = \frac{1}{\sigma \delta_s}$

Waveguides

- Propagation of EM wave in waveguide described by general wave equation

$$\nabla^2 E - \frac{1}{c^2}\ddot{E} = 0$$

- As we are interested only in spatial distribution

$$E(r,t) = E(r)e^{i\omega t} \quad with\ r = (x, y, z)$$

- Substituting yields

$$\nabla^2 E + k^2 E(r) = 0$$

$$with\ wavenumber$$

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Waveguides

- Considering only z-component (propagation direction along waveguide) gives

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -k^2 E_z$$

- Which can be solved by using trial equation

$$E_z(x, y, z) = f_x(x) f_y(y) f_z(z)$$

- So that

$$\frac{f_x''}{f_x} + \frac{f_y''}{f_y} + \frac{f_z''}{f_z} = -k^2$$

Waveguides

Defining

$$k_x^2 \equiv -\frac{f_x''}{f_x} \quad k_y^2 \equiv -\frac{f_y''}{f_y} \quad k_z^2 \equiv -\frac{f_z''}{f_z}$$

yields

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

which when setting

$$k_x^2 + k_y^2 = k_c^2$$

gives

$$k_z = \sqrt{k^2 - k_c^2}$$

Waveguides

- The wave propagation along the waveguide is described by

$$f_z'' + k_z^2 f_z = 0$$

from which the differential equation describing the electric field along the waveguide axis is found to be

$$\frac{\partial^2 E_z}{\partial z^2} + k_z^2 E_z = 0$$

whose solution is

$$E_z = E_0 e^{ik_z z}$$

Waveguides

- There are two regimes for waveguide operation:

$$k_z = \begin{cases} complex & if \quad k_c^2 > k^2 \quad (damping) \\ real & if \quad k_c^2 < k^2 \quad (propagation) \end{cases}$$

- The special value of the wavenumber k_c is the cut-off frequency and separates free propagation from damping.

Waveguides

- Corresponding cut-off wavelength is

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_z^2}$$

from which

$$\lambda_z = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}}$$

Waveguides

- In the loss-free wave propagation regime, the wavelength λ_z is always greater than that in free space.
  - → Phase velocity of wave within waveguide is greater than speed of light

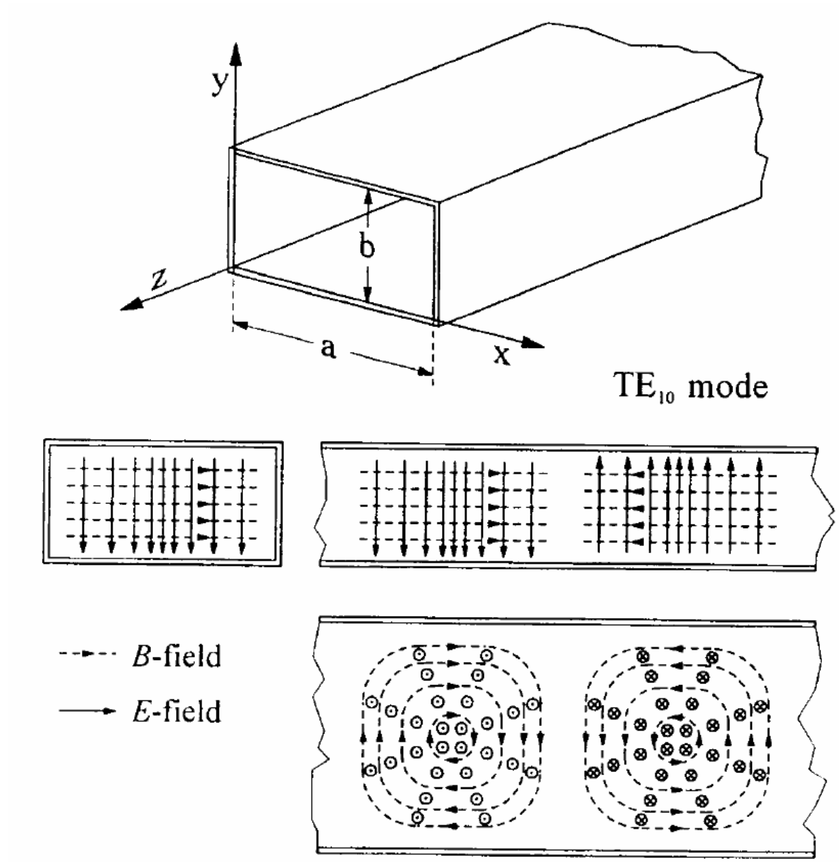
$$v_{\varphi} = \frac{\omega \lambda_z}{2\pi} > c$$

- Dispersion relation for waveguides

$$\omega = c\sqrt{k_z^2 + \left(\frac{2\pi}{\lambda_c}\right)^2}$$

Rectangular Waveguides

- To transport the wave from the transmitter to the accelerator, rectangular waveguides are used.
- Dimensions of waveguide depend on cut-off wavelength.



Rectangular Waveguides

- Cut-off Wavelength

$$f_x'' + k_x^2 f_x = 0$$

$$f_y'' + k_y^2 f_y = 0$$

$$f_x(x) = A\sin(k_x x) + B\cos(k_x x)$$

$$f_y(y) = C\sin(k_y y) + D\cos(k_y y)$$

- Constants A, B, C, D fixed by boundary conditions of wave propagation in waveguide
  - E-field tangential to conducting walls of waveguide vanish at surface of wall. (1)
  - B-field perpendicular to the conducting walls must vanish at the surface due to production of eddy currents. (2)

Rectangular Waveguides

- From boundary conditions

$$f_x(0) = f_y(0) = 0 \quad \Rightarrow \quad B = D = 0$$

$$f_x(a) = f_y(b) = 0 \quad \Rightarrow \quad k_x a = m\pi \quad ; \quad k_y b = n\pi$$

$$\Rightarrow \quad k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\Rightarrow \quad \lambda_c^2 = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

- There are an unlimited number of configurations, called waveguide modes.
  - Only a few are of practical use.

Rectangular Waveguides

- TE_{10}-mode (transverse electric)
  - Electric field lines only run perpendicular to direction of wave motion.
  - Or H_{10} as the magnetic field lines run in the waveguide direction.

$$E_x = 0$$

$$E_y = \hat{E}\sin(\frac{\pi x}{a})e^{-ik_z z}$$

$$E_z = 0$$

$$H_x = \frac{\hat{E}}{Z_0}\frac{\lambda}{\lambda_z}\sin(\frac{\pi x}{a})e^{-ik_z z}$$

$$H_y = 0$$

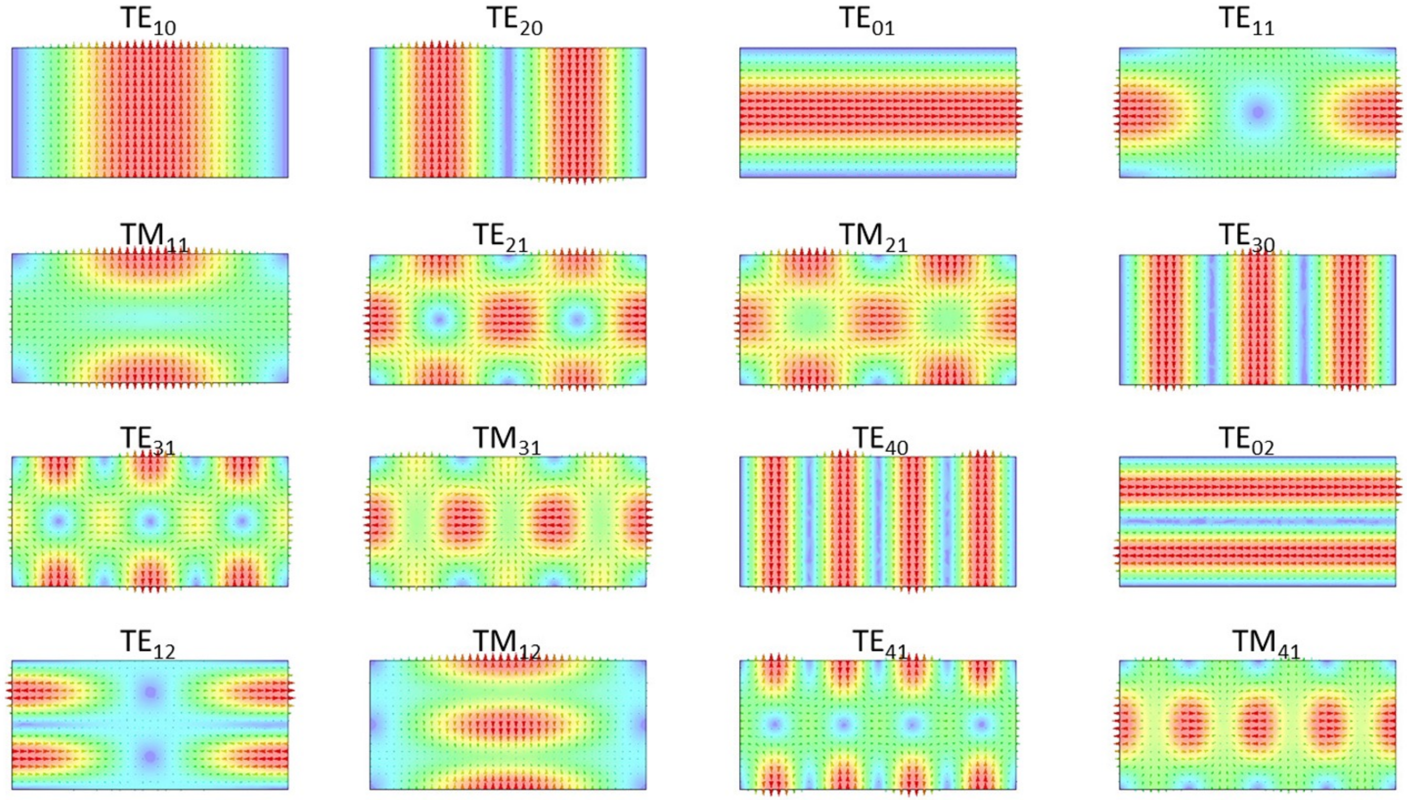
$$H_z = -i\frac{\hat{E}}{Z_0}\frac{\lambda}{2a}\cos(\frac{\pi x}{a})e^{-ik_z z}$$

The individual electric and magnetic field components of TE_{10}-mode.

E^ = arbitrary amplitude
Z_0 = waveguide impedance

Rectangular Waveguides

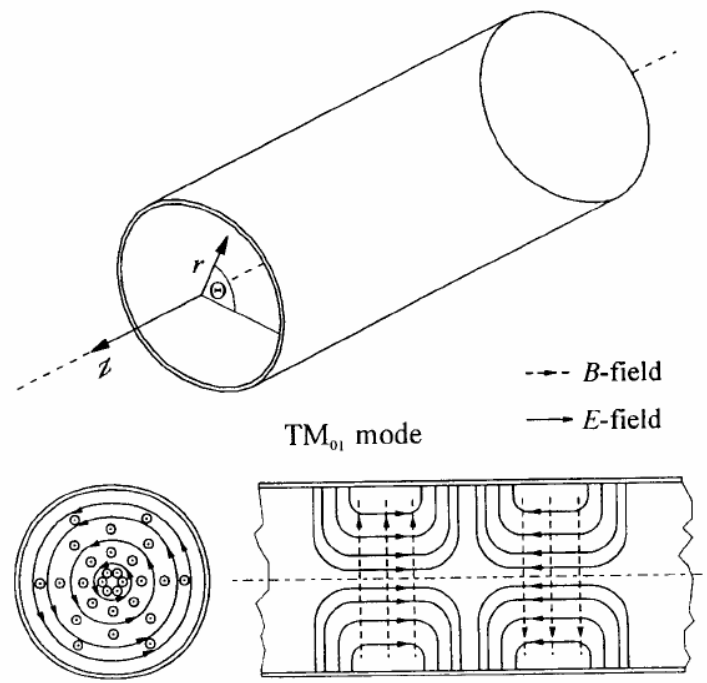
Transverse electric field distributions of different modes in a rectangular waveguide



Courtesy of E. Jensen (CERN)

Cylindrical Waveguides

- The same boundary conditions apply at the surface of the conducting cylinder as for rectangular waveguides.
- The most important mode for acceleration is TM_{01} (or E_{01}).
  - Only transverse magnetic field lines are present.
  - Electrical field lines run parallel to cylinder axis and thus can accelerate charged particles as they travel through waveguide.



Cylindrical Waveguides

- Electromagnetic Field Components

$$E_r = -i\hat{E}\frac{k_z}{k_c}J_0^{'}(k_c r)e^{-ik_z z}$$

$$E_\theta = 0$$

$$E_z = \hat{E}\,J_0(k_c r)e^{-ik_z}$$

$$H_r = 0$$

$$H_\theta = -i\frac{\hat{E}}{Z_0}\frac{k}{k_c}J_0^{'}(k_c r)e^{-ik_z z}$$

$$H_z = 0$$

Cylindrical Waveguides

- Cut-off Wavelength
  - Electrical field components running parallel to the conducting cylinder vanish at surface

$$E_z\left(\frac{D}{2}\right)=0$$

Where D is the cylinder diameter

$$J_0\left(k_c\,\frac{D}{2}\right)=0$$

The above condition is only satisfied if Bessel funtion vanishes

$$k_c=\frac{2x_1}{D}$$

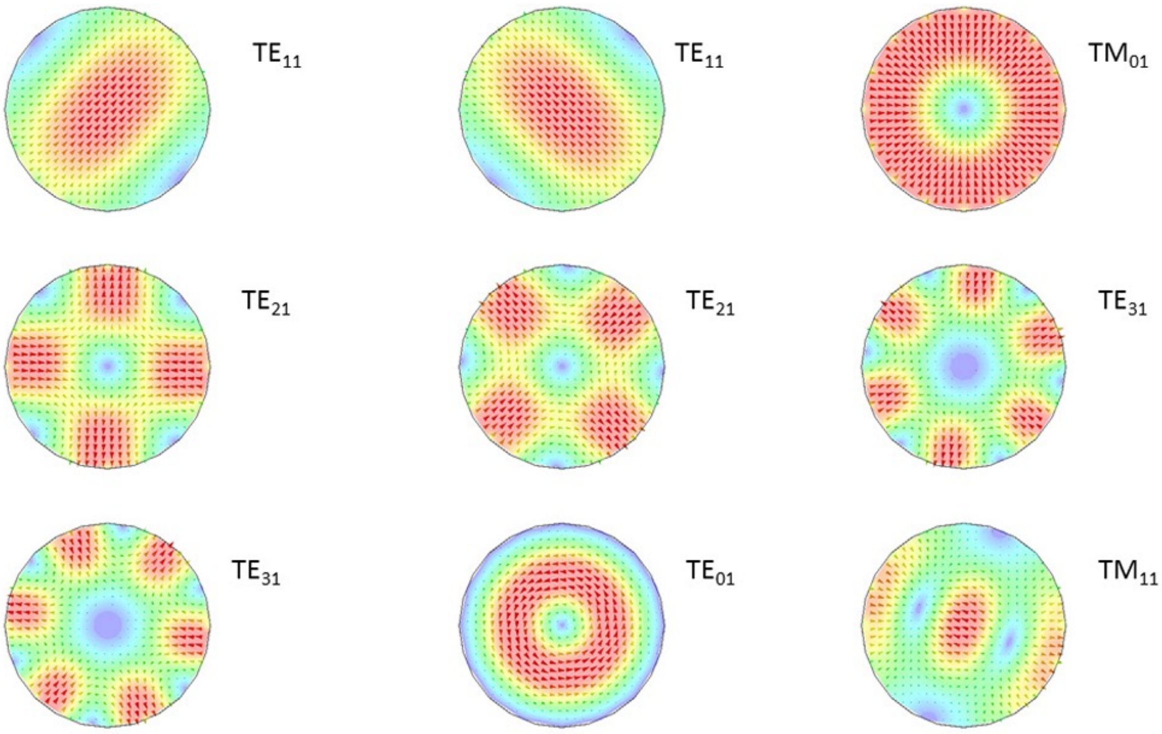
If x_1 (=2.40483) is first zero of Bessel function

$$\lambda_c=\frac{\pi D}{x_1}$$

with the corresponding cut-off wavelength

Cylindrical Waveguides

Transverse electric field distributions of different modes in a round waveguide



Courtesy of E. Jensen (CERN)

General Waveguide Parameters

General cylindrical waveguide	TE modes	TM modes
Boundary conditions	$\vec{n} \cdot \nabla T = 0$	$T = 0$
Longitudinal wave equations	$\frac{\mathrm{d}U(z)}{\mathrm{d}z} + jk_z Z_0 I(z) = 0, \quad \frac{\mathrm{d}I(z)}{\mathrm{d}z} + \frac{jk_z}{Z_0} U(z) = 0$	
Propagation constant	$k_z = k\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$	
Characteristic impedance	$Z_0 = \frac{\omega\mu}{k_z}$	$Z_0 = \frac{k_z}{\omega\varepsilon}$
Ortho-normal eigenvectors	$\vec{e} = \vec{u}_z \times \nabla T$	$\vec{e} = -\nabla T$
Transverse fields	$\vec{E} = U(z)\,\vec{e}, \quad \vec{H} = I(z)\,\vec{u}_z \times \vec{e}$	
Longitudinal fields	$H_z = \left(\frac{\omega_c}{\omega}\right)^2 \frac{T\,U(z)}{j\omega\mu}$	$E_z = \left(\frac{\omega_c}{\omega}\right)^2 \frac{T\,I(z)}{j\omega\varepsilon}$
Rectangular waveguide (width a, height b) (with $\varepsilon_i = \begin{cases} 1 & \text{if} \quad i = 0 \\ 2 & \text{if} \quad i \neq 0 \end{cases}$		
Cutoff	$\frac{\omega_c}{c} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$	
Transverse eigenfunction T_{mn}	$\frac{1}{\pi}\sqrt{\frac{a\,b\,\varepsilon_m\varepsilon_n}{(mb)^2 + (na)^2}} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$	$\frac{2}{\pi}\sqrt{\frac{a\,b}{(mb)^2 + (na)^2}} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$
Round waveguide (radius a)		
Cutoff	$\frac{\omega_c}{c} = \frac{\chi'_{mn}}{a}$	$\frac{\omega_c}{c} = \frac{\chi_{mn}}{a}$
Transverse eigenfunction T_{mn}	$\sqrt{\frac{\varepsilon_m}{\pi(\chi'^2_{mn} - m^2)}} \frac{J_m\left(\chi'_{mn}\frac{\rho}{a}\right)}{J_m(\chi'_{mn})} \begin{Bmatrix} \cos(m\varphi) \\ \sin(m\varphi) \end{Bmatrix}$	$\sqrt{\frac{\varepsilon_m}{\pi}} \frac{J_m\left(\chi_{mn}\frac{\rho}{a}\right)}{J_{m-1}(\chi_{mn})} \begin{Bmatrix} \sin(m\varphi) \\ \cos(m\varphi) \end{Bmatrix}$

Courtesy of E. Jensen (CERN)