JAI Longitudinal Problem Set

Sebastian Kalos, sebastian.kalos@physics.ac.uk

November 2022

1

1.1 ISIS is a proton synchrotron that operates from 70 to 800 MeV on a 50 Hz sinusoidal main magnet field. Given the main dipole field at 70 MeV is 0.17639 T calculate: the magnetic rigidity at that energy, the bending radius and the magnetic rigidity and dipole field at top energy (800 MeV).

Magnetic rigidity can be calculated as $B\rho = 3.3356p$ [T/m] where p is given in GeV/c. Considering that the rest mass of a proton is 938 MeV, I assume that the energy given (70 MeV and 800 MeV) refers to kinetic energy E_k only, where $E = E_k + E_0$ and $E_0 = m_0c^2 = 938$MeV for protons. Then, the relativistic energy-momentum relation

$$E^2 = (pc)^2 + (m_0c^2)^2 \tag{1}$$

turns into

$$(E_k + E_0)^2 = (pc)^2 + E_0^2 \tag{2}$$

$$p_0 = \frac{\sqrt{2E_0E_k + E_k^2}}{c} = \frac{\sqrt{2 \cdot 938 \cdot 70 + 70^2}\ \text{MeV}}{c} \simeq 0.369\ \text{GeV/c} \tag{3}$$

for the beginning energy 70 MeV, and

$$p_{max} = \frac{\sqrt{2 \cdot 938 \cdot 800 + 800^2}\ \text{MeV}}{c} \simeq 1.463\ \text{GeV/c} \tag{4}$$

for the final energy 800 MeV. Plugging these into the formula for magnetic rigidity, we get $(B\rho)_0 \simeq 3.3365 * 0.369$ [Tm] $\simeq 1.23$ [Tm], and $(B\rho)_{max} \simeq 3.3365 * 1.463$ [Tm] $\simeq 4.88$ [Tm]. To get the bending radius ρ_0, we just divide this by the magn. field at that energy: $\rho_0 = (B\rho)_0/B_0 = 1.23/0.17639$ m $= 6.97$ m. Now, as the protons are being accelerated, the bending radius needs to remain constant so that the beam can be stored in the ring with fixed geometry. This means

$$\rho_0 = \frac{(B\rho)_0}{B_0} = \frac{(B\rho)_{max}}{B_{max}} \implies B_{max} = \frac{(B\rho)_{max}}{\rho_0} = \frac{4.88}{6.97}\text{T} = 0.700\ \text{T} \tag{5}$$

1.2 If 26.8% of the circumference is taken up with dipoles what is the mean radius R of the synchrotron? Calculate the revolution frequency at 0 ms (70 MeV) and 10 ms (extraction at 800 MeV).

We can make use for the bending angle in a dipole: $\theta = \frac{BL}{p/e} \implies L = \frac{\theta p}{Be}$. Acknowledging that the full cycle nees to add up to $\theta_{total} = 2\pi$, and setting the real accelerator circumference $S = 2\pi R$, and the hypothetical circumference of an accelerator only composed of dipoles as $C = 2\pi\rho$, where ρ is the bending radius, we have

$$\frac{C}{S} = 0.268 = \frac{2\pi p}{Be2\pi R} = \frac{p}{eBR} = \frac{\rho}{R} \implies R = \frac{\rho}{0.268} = 6.97/0.268\ \text{m} \simeq 26.01\ \text{m} \tag{6}$$

This radius also seems reasonable to me, considering the picture of the synchrotron given in the assignment. We can get the velocity of the protons from the relativistic γ factor

$$\gamma = \frac{E}{E_0} = 1 + \frac{E_k}{E_0} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \implies v = c\sqrt{\frac{E_k}{E_0+E_k}} \tag{7}$$

This gives $v_0 = 3\cdot 10^8 \cdot \sqrt{\frac{70}{70+938}} \simeq 7.91\cdot 10^7$m/s and $v_{max} = 3\cdot 10^8 \cdot \sqrt{\frac{800}{800+938}} \simeq 2.04\cdot 10^8$m/s. The revolution frequency is given simply as

$$f = \frac{1}{T} = \frac{v}{2\pi R} = \frac{c}{2\pi R}\sqrt{\frac{E_k}{E_0+E_k}} \tag{8}$$

This gives $f_0 \simeq 0.484$ MHz at 70 MeV (0 ms), and $f_{max} \simeq 1.245$ MHz at 800 MeV (10 ms)

1.3 Calculate the following parameters at 0, 5 and 10 ms

1.3.1 Momentum

I already calculated p_0 (0 ms) and p_{max} (10 ms) in part 1.1. The ramp-up is not linear, however, I suppose that at t=5ms, virtually all quantities will be midway through their values at t=0 and t=10ms, hence $p_{t=5ms} = 0.369 + \frac{1}{2}(1.463 - 0.369)$ GeV/c $= 0.916$ GeV/c

1.3.2 Kinetic Energy

I assumed that the energy given is kinetic energy only. Hence, the starting (t=0ms) KE is 70MeV, the final (t=10ms) KE is 800MeV, and KE at 1=5ms will be $70 + 1/2(800 - 70)$ MeV $= 435$MeV

1.3.3 Relativistic parameters β, γ

$$\gamma = 1 + \frac{E_k}{E_0} \tag{9}$$

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} = \frac{\sqrt{\gamma^2-1}}{\gamma} \tag{10}$$

This gives $\gamma_{t=0} \simeq 1.0746$, $\beta_{t=0} \simeq 0.3662$, $\gamma_{t=5ns} \simeq 1.464$, $\beta_{t=5ms} \simeq 0.7303$, and $\gamma_{t=10ns} \simeq 1.8529$, $\beta_{t=10ms} \simeq 0.8419$

1.3.4 What does γ_t have to be for ISIS to remain below transition throughout acceleration?

"Below transition" means that a particle with momentum higher than is the one of a synchronous particle also has a higher revolution frequency (the slip factor η is positive):

$$\frac{df}{f} = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}\right)\frac{dp}{p} = \eta\frac{dp}{p} \implies \gamma_t > \gamma \tag{11}$$

This means that γ_t needs to be greater than the highest value of γ, which is obtained at $\gamma_{t=10ns} \simeq 1.8529$.

1.4 What is the minimum RF voltage required as a function of time (0-10 ms) to accelerate a proton at ISIS? Why do we need more?

We can use

$$\Delta E_t = 2\pi R\rho\dot{B} = eV\sin\phi_s \tag{12}$$

Assuming $\sin\phi_s = 1$, we can get $V(t) = 2\pi R\rho\dot{B}(t)$. Now, based on the image and other information given in the assignment, I assume that $B = -\frac{1}{2}(B_{max} - B_{min})\cos(2\pi ft) \implies \dot{B} = 2\pi f\frac{1}{2}(B_{max} - B_{min})\sin(2\pi ft)$ this gives the final answer

$$V(t) = 2\pi R\rho\dot{B} = \pi(B_{max} - B_{min})\sin(2\pi ft) \tag{13}$$

This is the minimum voltage required so as not to get the particle bunches decelerated by the ahead phase. To actually accelerate them, we need a bit more than this.

1.5 Given a mean dispersion of 1 m, what is the γ_t? What transition kinetic energy does that correspond to? Calculate the slip factor η at 0, 5 and 10 ms.

We can use the fact that $\gamma = \gamma_t = \alpha_c^{-\frac{1}{2}}$ for slip factor $\eta = 0$

$$\alpha_c = \frac{dL/L}{dp/p} = \frac{1}{L}\oint \frac{D_x(s)}{\rho(s)}ds = \frac{1}{2\pi R}\frac{< D_x >}{\rho} \cdot 2\pi R \cdot \frac{S}{C} = \frac{< D_x >}{\rho} \cdot 0.268 = \frac{1 \text{ m} \cdot 0.268}{6.97 \text{ m}} \simeq 0.03845 \tag{14}$$

this corresponds to $\gamma_t \simeq 5.1$. This can be used to find the corresponding kinetic energy of the protons using $\gamma = 1 + \frac{E_k}{E_0} \implies E_k = (\gamma - 1)E_0 = (5.1 - 1) \cdot 938$ MeV $\simeq 3.85$ GeV. The slip factor can be calculated as $\eta = \gamma^{-2} - \gamma_t^{-2}$. We can use previous results to calculate this. For $t = 0$ms, this means $\eta_{t=0} \simeq 0.735$, $\eta_{t=5ms} = 0.487$, $\eta_{t=10ms} = 0.344$

2

My solutions to this problem can be found in a Jupyter notebook on my GitHub under this link:
`https://github.com/sebastiankalos/JAI_accelerator_course/blob/master/JAI_longitudinal.ipynb`
Please let me know in case of any difficulties with accessing the webpage (sebastian.kalos@physics.ox.ac.uk).