Lecture 18
Radiofrequency Cavities II

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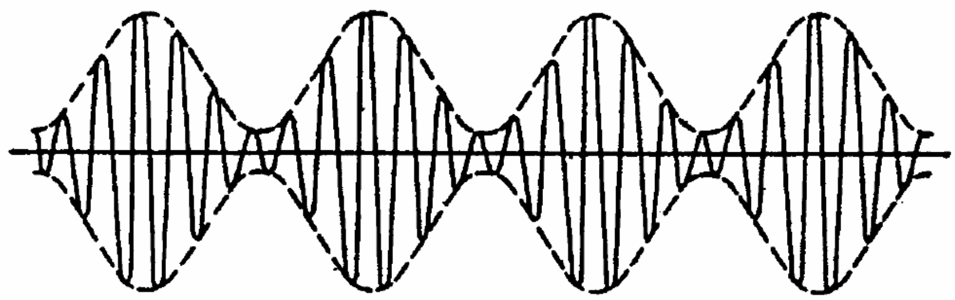
Group Velocity

- Energy (and information) travel with wave group velocity.
- Interference of two continuous waves of slightly different frequencies described by:



$$
\begin{aligned}
E &= E_0 \sin\left[(k+dk)x-(\omega+d\omega)t\right]+E_0 \sin\left[(k-dk)x-(\omega-d\omega)t\right] \\
&= E_0 \sin\left[kx-\omega t\right]\cos\left[dk\,x-d\omega\,t\right] \\
&= 2E_0 f_1(x,t) f_2(x,t)
\end{aligned}
$$

Group Velocity

- Mean wavenumber & frequency represented by continuous wave

$$f_1(x,t) = \sin\left[kx - \omega t\right]$$

  - Any given phase in this wave is propagated such that $kx - \omega t$ remains constant.
  - Phase velocity of wave is thus

$$v_p = -\frac{\partial f_1(x,t)/\partial t}{\partial f_1(x,t)/\partial x} = \frac{\omega}{k}$$

- Envelope of pattern described by

$$f_2(x,t) = \cos[dkx - d\omega t]$$

  - Any point in the envelope propagates such that $x\,dt - t\,d\omega$ remains constant and its velocity, i.e. group velocity, is

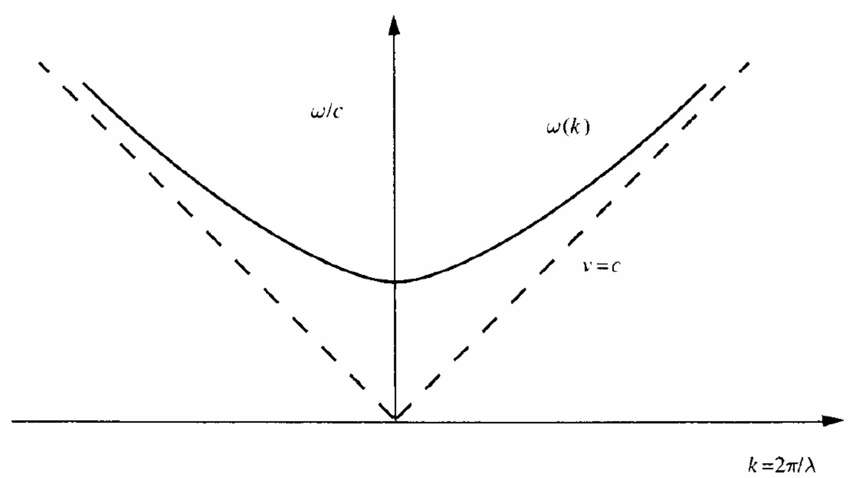
$$v_g = -\frac{\partial f_2(x,t)/\partial t}{\partial f_2(x,t)/\partial x} = \frac{d\omega}{dk}$$

Dispersion Diagramme for Waveguide

- Description of wave propagation down a waveguide by plotting graph of frequency, ω, against wavenumber, $k = 2\pi/\lambda$
  - Imagine experiment in which signals of different frequencies are injected down a waveguide and the wavelength of the modes transmitted are measured.
- Measurables
  - Phase velocity for given frequency: ω/k
  - Group velocity: slope of tangent

Dispersion Diagramme for Waveguide

- Observations
  - However small the *k*, the frequency is always greater than the cut-off frequency.
  - The longer the wavelength or lower the frequency, the slower is the group velocity.
  - At cut-off frequency, no energy flows along the waveguide.
  - Also $v_{ph} v_g = c^2$



Dispersion diagramme for waveguide is the hyperbola

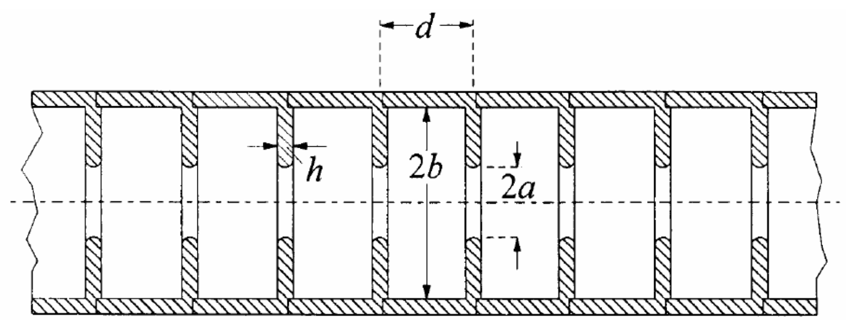
$$\left(\frac{\omega}{c}\right)^2 = k^2 + \left(\frac{\omega_c}{c}\right)^2$$

Iris-loaded Structures

- Acceleration in a waveguide is not possible as the phase velocity of the wave exceeds that of light.
  - Particles, which are travelling slower, undergo acceleration from the passing wave for half the period but then experience an equal deceleration.
  - Averaged over long time interval results in no net transfer of energy to the particles.

•Need to modify waveguide to reduce phase velocity to match that of the particle (less than speed of light).

•Install iris-shaped screens with a constant separation in the waveguide.

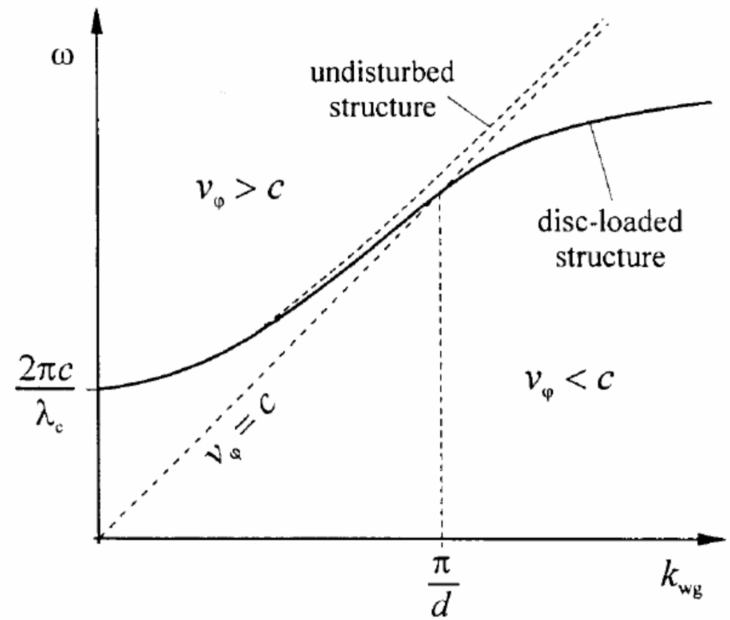


Iris-loaded Structures

- Recall that the dispersion relation in a waveguide is

$$\omega = c\sqrt{k_z^2 + \left(\frac{2\pi}{\lambda_c}\right)^2}$$

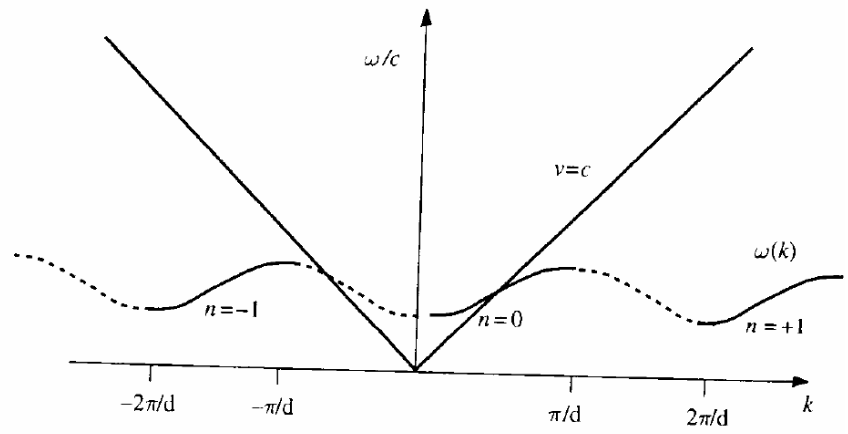
- With the installation of irises, curve flattens off and crosses boundary at $v_\varphi = c$ at $k_z = \pi/2$



With suitable choice of iris separation *d* the phase velocity can be set to any value

Iris-loaded Structures

- Waveguides cannot be used for sustained acceleration as all points on dispersion curve lie above diagonal in dispersion diagramme.
  - Phase velocity > *c*
- An iris-loaded structure slows down the phase velocity.



Dispersion diagramme for a loaded waveguide

The *k*-value for each space harmonic is

$$k_n = k_0 + \frac{2n\pi}{d}$$

By choosing any frequency in dispersion diagramme it will intercept dispersion curve at k values spaced by 2nπ/d

First rising slope used for acceleration.

Resonant Cavities

- General solution of wave equation

$$W(r,t) = Ae^{i(\omega t + k \bullet r)} + Be^{i(\omega t - k \bullet r)}$$

  - Describes sum of two waves – one moving in one direction and another in opposite direction
- If wave is totally reflected at surface then both amplitudes are the same, *A=B,* and

$$W(r,t) = Ae^{i\omega t}(e^{ik \bullet r} + e^{-ik \bullet r})$$
$$= 2A\cos(k \bullet r)e^{i\omega t}$$

  - Describes field configuration which has a static amplitude *2Acos(k·r),* i.e. a standing wave.

Resonant Cavities

- Resonant Wavelengths
  - Stable standing wave forms in fully-closed cavity if

$$l = q\frac{\lambda_z}{2} \quad with \quad q = 0, 1, 2, \ldots$$

    - where l = distance between entrance and exit of waveguide after being closed off by two perpendicular sheets.
    - → only certain well-defined wavelengths λ_r are present in the cavity.
- General resonant condition $\frac{1}{\lambda_r^2} = \frac{1}{\lambda_c^2} + \frac{1}{4}\left(\frac{q}{l}\right)^2$
- Near the resonant wavelength, resonant cavity behaves like electrical oscillator but with much higher Q-value and corresponding lower losses of resonators made of individual coils and capacitors.
  - Exploited to generate high-accelerating voltages

Rectangular Resonant Cavities

- Inserting

$$\left(\frac{2}{\lambda_c}\right)^2 = \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2$$

  into the resonance condition yields

$$\lambda_r = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{q}{l}\right)^2}} \quad with \quad m,n,q = \text{integers}$$

- Integers *m*,*n*,and *q* define modes in resonant cavity.
  - Number of modes is unlimited but only a few of them used in practical situations.
    - *m*,*n*,and *q* between 0 and 2

Cylindrical Resonant Cavities

- Inserting the expression for cut-off frequency into general resonance condition yields

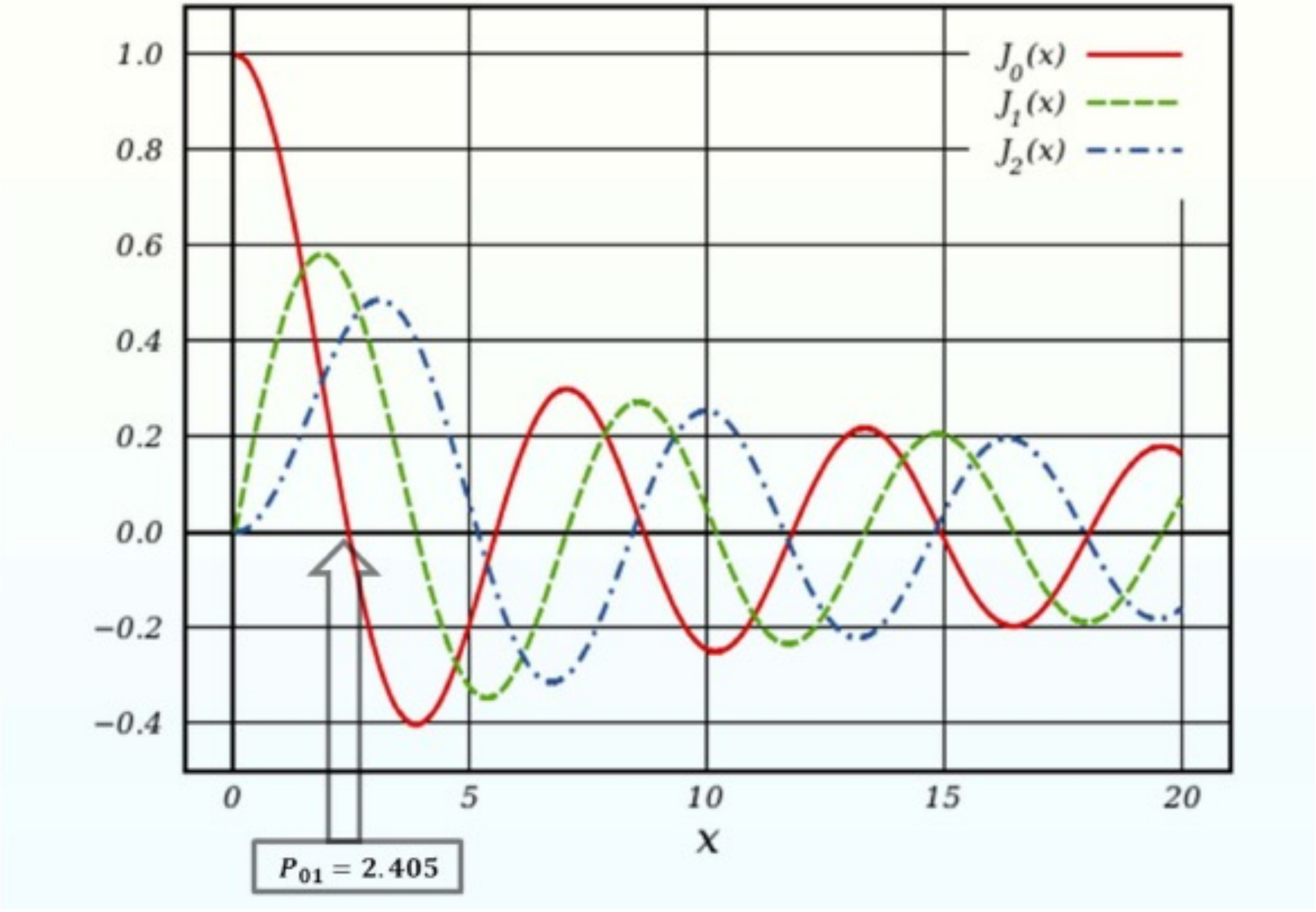
$$\frac{1}{\lambda_r^2} = \left(\frac{x_1}{\pi D}\right)^2 + \frac{1}{4}\left(\frac{q}{l}\right)^2 \quad with \quad q = 0,1,2,\ldots$$

  - where x_1=2.0483 is the first zero of the Bessel function.

- For the case of $q=0$, termed the TM_{010} mode, the resonant wavelength reduces to

$$\lambda_r = \frac{\pi D}{x_1}$$

Bessel Functions



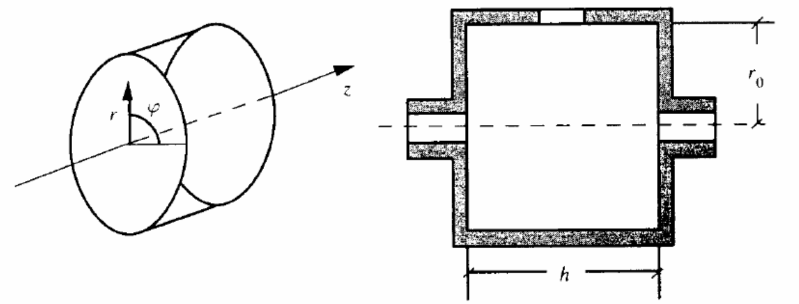
Pill-box Cylindrical Cavity

- The simplest RF cavity type

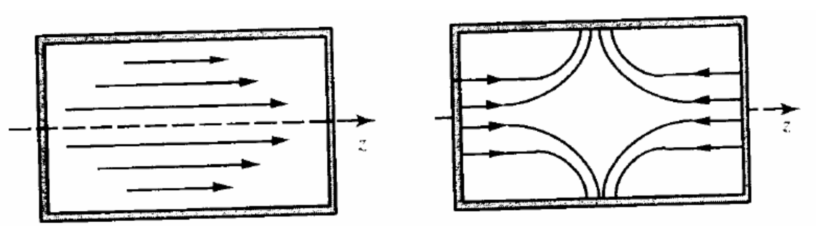
- The accelerating modes of this cavity are TM_{0lm}

  - Indices refer to the polar co-ordinates φ, r and z

Cylindrical pill-box cavity with holes for beam and coupler.



TM_{010} TM_{011}



Lines of force for the electrical field.

Pill-box Cylindrical Cavity

- The modes with no φ variation are:

$$\nabla^2 E + \Lambda^2 E = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right) + \frac{1}{r^2}\frac{\partial E}{\partial r} + \frac{\partial^2 E}{\partial z^2} + \Lambda^2 E = 0$$

$$E_z = E_0 J_0\left(\frac{P_{0l}}{r_0} r\right)\cos\left(\frac{m\pi}{h} z\right)$$

$$E_r = E_0 \frac{m\pi}{P_{0l}}\frac{r_0}{h} J_1\left(\frac{P_{0l}}{r_0} r\right)\sin\left(\frac{m\pi}{h} z\right)$$

$$\Lambda_{0lm}^2 = \left(\frac{P_{0l}}{r_0}\right)^2 + \left(\frac{m\pi}{h}\right)^2$$

- *l* indicates the radial variation while *m* controls the number of wavelengths in the z-direction.
- P_{0l} is the argument of the Bessel function when it crosses zero for the *lth* time.
  - $J_0(P_{0l}) = 0$ for $P_{0l} = 2.405$

Pill-box Cylindrical Cavity

- TM_{010} Mode

$$E = E_0 J_0\left(\frac{2.405}{r_0} r\right); \quad \Lambda_{010} = \frac{2.405}{r_0}; \quad \omega_{010} = \frac{\Lambda_{010}}{\sqrt{\varepsilon\mu}}$$

$$\nu_{010} = \frac{\omega_{010}}{2\pi}; \quad \lambda_{010} = \frac{1}{\nu_{010}\sqrt{\varepsilon\mu}}$$

Quality Factor of Resonator, Q

- Ratio of stored energy to energy dissipated per cycle divided by 2π

$$Q = \frac{W_s}{W_d} = \omega \frac{W_s}{P_d}$$

W_s = stored energy in cavity

W_d = energy dissipated per cycle divided by 2π

P_d = power dissipated in cavity walls

ω = frequency

Quality Factor of Resonator, Q

- Stored energy over cavity volume is

$$W_s = \frac{\varepsilon_0}{2} \int |E|^2 dv \qquad W_s = \frac{\mu_0}{2} \int |H|^2 dv$$

where the first integral applies to the time the energy is stored in the *E*-field and the second integral as it oscillates back into the *H*-field.

Quality Factor of Resonator, Q

- Losses on cavity walls are introduced by taking into account the finite conductivity σ of the walls.
- Since, for a perfect conductor, the linear density of the current j along walls of structure is

$$j = n \times H$$

we can write

$$P_d = \frac{R_{surf}}{2} \int_s |H|^2 ds$$

with s = inner surface of conductor

Quality Factor of Resonator, Q

R_{surf} = surface resistance

δ = skin depth

$$R_{surf} = \sqrt{\pi f \mu_0 \mu_\tau / \sigma} = \frac{1}{\sigma\delta}$$

For Cu, $R_{surf} = 2.61 \times 10^{-7} \sqrt{\omega}\ \Omega$

Shunt Impedance - R_s

- Figure of merit for an accelerating cavity
  - Relates accelerating voltage to the power P_d to be provided to balance the dissipation in the walls.
- Voltage along path followed by beam in electric field E_z is

$$V = \int_{path} \left| E_z(x,y,z) \right| \, dl$$

from which (peak-to-peak)

$$R_s = \frac{V^2}{2P_d}$$

Shunt Impedance - R_s

$$R_s = 5.12 \times 10^8 \frac{\beta_z (1-\eta)^2}{p + 2.61\beta_z (1-\eta)} \left(\frac{\sin D/2}{D/2} \right)^2$$

with

$$\beta_z \equiv \frac{\upsilon_\varphi}{c} \text{ (phase velocity)}$$

$$\eta \equiv \frac{h}{d} \text{ (}h = \text{thickness, } d = \text{iris separation)}$$

$$p \equiv \text{number of irises per wavelength (equal to mode number)}$$

$$D \equiv \frac{2\pi}{p}(1-\eta)$$

Energy Gain

- Energy gain of particle as it travels a distance through linac structure depends only on potential difference crossed by particle:

$$U = K\sqrt{P_{RF} l R_s}$$

$$where$$

$$P_{RF} \equiv \text{supplied RF power}$$

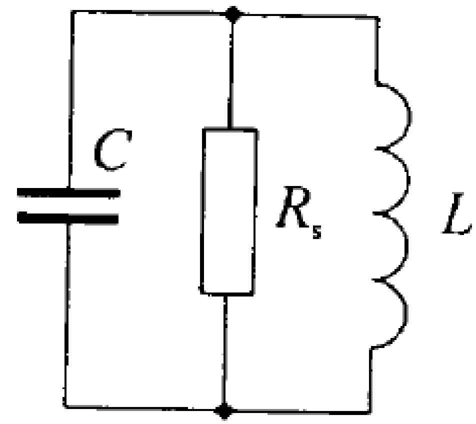
$$l \equiv \text{length of linac structure}$$

$$R_s \equiv \text{shunt impedance}$$

$$K \equiv \text{correction factor } (\approx 0.8)$$

Analogous to Electrical Oscillator

- Cavity behaves as an electrical oscillator but with very high quality factor (sharp resonance)



Electrical response of cavity described by parallel circuit containing C, L, and R_s

$$Q = \frac{\omega_r}{\Delta\omega} = \frac{R_s}{Z}$$

On resonance the impedance is

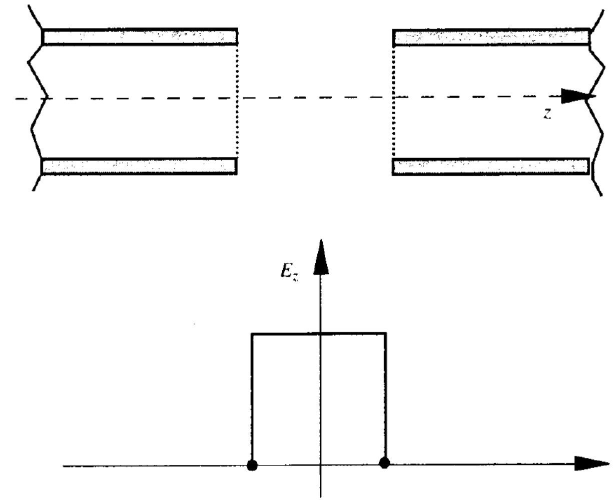
ω_r resonant frequency
$\Delta\omega$ = frequency shift at which amplitude is reduced by -3 dB relative to resonance peak

$$Z = \omega L = \frac{1}{\omega C}$$

Transit-Time Factor

- Accelerating gap
  - Space between drift tubes in linac structure
  - Space between entrance and exit orifices of cavity resonator
- Field is varying as the particle traverses the gap
  - Makes cavity less efficient and resultant energy gain which is only a fraction of the peak voltage

The RF Gap



$$E_z = E_0 \cos(\omega t + \varphi)$$

Field is uniform along gap axis and depends sinusoidally on time

Phase ϕ refers to particle in middle of gap z=0 at t=0

Transit-Time Factor

- Transit-Time Factor is ratio of energy actually given to a particle passing the cavity centre at peak field to the energy that would be received if the field were constant with time at its peak value
- The energy gained over the gap G is:

$$V = \int_{-G/2}^{+G/2} E_0 \cos(\omega t + \varphi) dz = \frac{\sin(\omega G/2\beta c)}{\omega G / 2\beta c} \left(E_0 G \cos\varphi \right)$$

Transit-Time Factor

The Transit Gap Factor is defined as

$$\textit{Transit Gap Factor}\ \ \Gamma \equiv \frac{\sin(\omega G/2\beta c)}{\omega G/2\beta c}$$

Defining a transit angle

$$\textit{Transit Angle} \equiv \theta = \omega G/\beta c = 2\pi G/\beta\lambda$$

the Transit Gap Factor becomes

$$\Gamma = \frac{\sin\theta/2}{\theta/2}$$

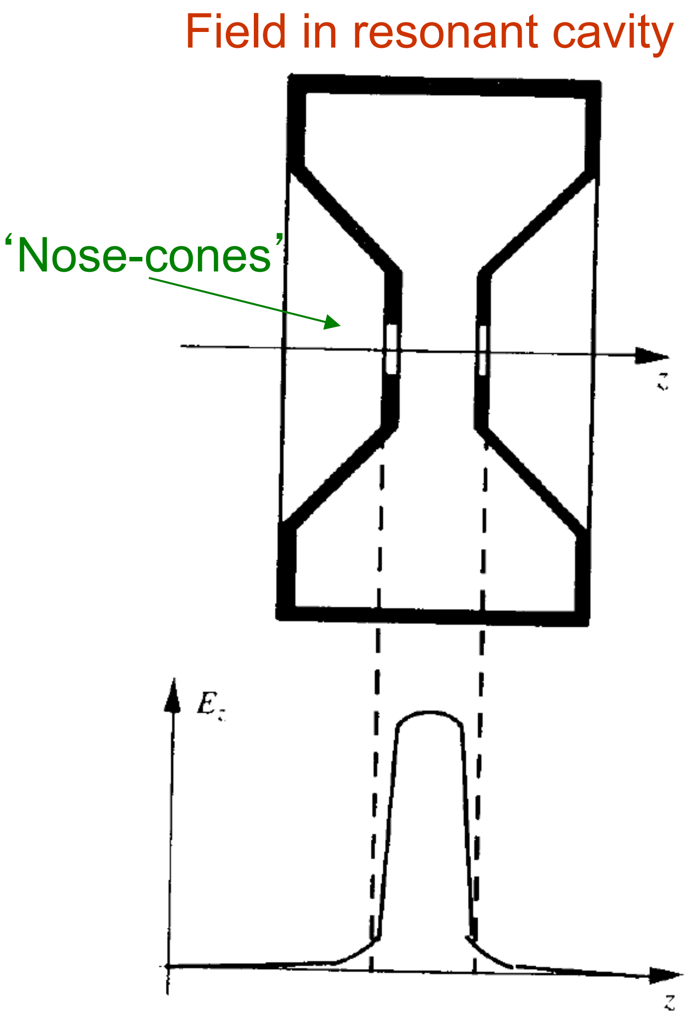
with 0 < θ < 1

The Transit-Time Factor

- Observations
  - At relativistic energies, cavity dimensions are comparable with $\lambda/2$
    - Reduction in efficiency due to transit-time factor is acceptable.
- At low energies, this is not the case
  - Cavities have strange re-entrant configuration to keep G short compared to dimensions of its resonant volume.

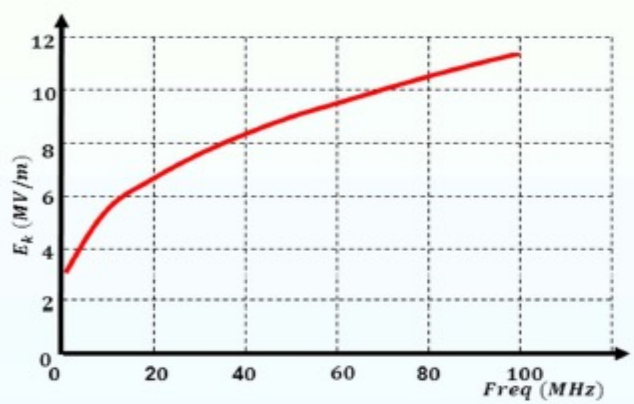
The Transit-Time Factor

- Compromise cavity design
  - Increasing ratio of volume/surface area
    - Reduces ohmic losses
    - Increases Q factor
  - Minimise gap factor



Kilpatrick Limit

- RF breakdown observed at very high fields.
- Kilpatrick Limit expresses empirical relation between accelerating frequency and E-field
  - $f = 1.64 E_k^2 e^{-8.5/E_k}$



Software for Cavity Design

- Poisson and Superfish are the main solver programs in a collection of programs from LANL for calculating static magnetic and electric fields and radio-frequency electromagnetic fields in either 2-D Cartesian coordinates or axially symmetric cylindrical coordinates.
- Finite Element Method

Solvers:
- Automesh – generates the mesh (always the first program to run)
- Fish – RF solver
- Cfish – version of Fish that uses complex variables for the rf fields, permittivity, and permeability.
- Poisson – magnetostatic and electrostatic field solver
- Pandira – another static field solver (can handle permanent magnets)
- SFO, SF7 – postprocessing
- Autofish – combines Automesh, Fish and SFO
- DTLfish, DTLCells, CCLfish, CCLcells, CDTfish, ELLfish, ELLCAV, MDTfish, RFQfish, SCCfish – for tuning specific cavity types.
- Kilpat, Force, WSFPlot, etc.

- http://laacg1.lanl.gov/laacg/services/download_sf.phtml

Structures usually solved by Finite Element Analysis