JAI Transverse Beam Optics Tutorial

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1 Preliminary Exercises

1.1

Watch this Iron Man clip and discuss the main accelerator physics concepts involved either if they are properly represented or not in the movie.
Well, Tony Stark single-handedly bulding an accelerator in one afternoon is obviously super realistic, yes. Creating new elements in accelerators is not science fiction, however, the system obviously lacks some kind of robustness, vacuum, superconductive magnets with proper cryogenic systems, etc. Also, when the beam is scattered from the crystal (...?), it doesn't seem to damage the wall of the accelerator in any way. Also, where is the beam coming from..? It is deflected but there needs to be some source that would inject new particles within that one turn...

1.2

What is faster? An electron/positron at LEP (E = 100 GeV), or a proton in the LHC (E = 7000 GeV)
We can compare the relativistic $\gamma = \frac{E}{E_0}$ factors of each particle. The rest energy of an electron is 0.511 MeV, and the rest mass of proton is 938 MeV. The ratios are then $\gamma_e = \frac{100000}{0.511} = 195695$, and $\gamma_p = \frac{7000000}{938} = 7462$. From this, we can see that LEP electrons are significantly faster than the LHC protons.

1.3

Why do we use magnets for bending the trajectory of the beam?
The magnetic force scales with the second power of the particle speed, rendering itself much more practical for the bending.

1.4

Given current technology (B_{max} ∼10 T), what is the maximum energy of a particle accelerator around the Earth equator, and of an accelerator around the Solar System?
Given the expression for magnetic rigidity, $B\rho = 3.33p[GeV/c]$, and the two (im)possible radii, $\rho_{Earth} = 6.378 * 10^6$m and $\rho_{SS} = 4.545 * 10^{12}$m (aphelion of Neptune, the first google search result...), we get $p_E = 19.15$MeV/c, and

1.5

Derive the limits for the thin lens approximation and find the new matrices for quadrupoles in thin lens approximation.
The focusing quadrupole transport matrix is

$$M_f = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \tag{1}$$

For the thin lens approximation, we can use the result of Taylor expansion of sines and cosines, namely that $\cos(x) \approx 1$ if x is small, and $\sin(x) \approx x$, with the same assumption. Therefore, assuming that $L \to 0$ (argument of the sines and cosines is small) we can say that

$$M_f \approx \begin{pmatrix} 1 & \frac{1}{\sqrt{K}}\sqrt{K}L \\ -\sqrt{K}\sqrt{K}L & 1 \end{pmatrix} = \begin{pmatrix} 1 & L \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \tag{2}$$

since $K = \frac{1}{Lf}$ and $L \to 0$. Similarly for the defocusing quadrupole matrix

$$M_d = \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh\sqrt{K}L \end{pmatrix} \tag{3}$$

we can use the result of the Taylor expansions $\cosh(x) \approx 1$ and $\sinh(x) \approx x$ if x is small. Following the same logic as in the previous case, this leads to

$$M_d \approx \begin{pmatrix} 1 & \frac{1}{\sqrt{K}}\sqrt{K}L \\ \sqrt{K}\sqrt{K}L & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \tag{4}$$

1.6

The LHC contains 1232 dipole magnets. Each is 15 m long. What is the length of the full circumference?
the dipole length is $1232 * 15 = 18480$m, however, the full circumference is well over 27km. The difference is caused by straight elements rather than dipoles (quadrupoles, sextupoles, etc...), and by injection points, experiments, etc.

2 To Think About

2.1

β^* cannot be measured directly at the interaction point but surely can be extrapolated from the measurements done at the last BPM. The function is directly related to the transverse size of the beam through the emittance, so measurements of the amplitude at BPMs around the interaction point should give enough info to calculate it.

2.2

Ground motions (like earthquakes, I guess..?) can pretty much cause misalignments in the lattice, possibly having us lose the beam.

2.3

When something is blocking the aperture, we can either heat up the accelerator, open it and clean it, or we can try to shift the beam's trajectory using a par of kickers before and after the object in order to avoid it. Alternatively, we can just blast through it with the beam.

3 Exercise: Understanding the phase space concept

4 Exercise: Stability Condition

Consider a lattice composed by a single 2 meters long quadrupole, with f = 1 m

4.1 Prove that if the quadrupole is defocusing, then a lattice is not stable

A defocusing quadrupole has the transfer matrix

$$M_d = \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh\sqrt{K}L \end{pmatrix} \tag{5}$$

The stability condition essentially states that $Tr(M_d) \leq 2$, which translates into

$$2\cosh\sqrt{K}L = 2\frac{1}{2}(e^{\sqrt{K}L} + e^{-\sqrt{K}L}) \tag{6}$$

Also, $K = \sqrt{1/\rho + k}$, where $k = \frac{1}{fL}$, and $\rho = \inf$ as this is a quadrupole with no bending. This means that $\sqrt{K}L = \frac{1}{\sqrt{fL}}L = \sqrt{\frac{L}{f}} = \sqrt{2}$ after plugging in the quadrupole length L and focal length f. Thus, we have

$$e^{\sqrt{2}} + e^{-\sqrt{2}} \geq 4 \tag{7}$$

which violates the stability condition. Therefore, this defocusing quadrupole is unstable.

4.2 Prove that if the quadrupole is focusing, then the lattice is stable

This exercise does not require plugging in any numbers. Taking the transport matrix for a focusing quadrupole

$$M_f = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos\sqrt{K}L \end{pmatrix} \tag{8}$$

and taking its trace, we have $2\cos\sqrt{K}L \leq 2$ always, as cosine only takes up values from -1 to 1. Thus, a focusing quadrupole is always stable.

5 Twiss Function Evolution

Which of the optics parameters can be constant:

- **In a drift.**
- **In a quadrupole with constant strength K**

We can use the differential equation representing the evolution of the β function $\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 K = 1$.

In a drift space, $K = 0$ and $2\beta\beta'' - \beta'^2 = 4$. In a drift space, however, β cannot be constant; it is directly related to the beam transverse width, which will necessarily change due to the natural diverging of the beam. $\alpha(s) = -\frac{1}{2}\frac{d\beta}{ds} = -\frac{1}{2}\beta'$ also cannot be constant; if it were, then $\beta'' = \frac{d(-2\alpha)}{ds} = 0$, and $-\beta'^2 = -(2\alpha)^2 = -4\alpha^2 \neq 4$. The sign difference prevents the equation from holding true.

In a quadrupole with constant K and $\beta =$ const., we have $\alpha = 0$ and all $\beta' = \beta'' = 0$. But again, this would mean that the beam width is not changing inside the quadrupole, which goes against the very purpose of a quadrupole... So β cannot be constant there. Theoretically, one could have $-1/4\beta'^2 + \beta^2 K = -\frac{1}{4}(2\alpha)^2 + \beta^2 K = 1$, and thus $K = \frac{1+\alpha^2}{\beta^2}$ when $\beta'' = 0$ (α being constant). Since β changes withing the length of the quadrupole, however, keeping α constant would thus require K changing as a function of s, which is practically never the case, I suppose.
In other words, I think we simply cannot have α or β constant in either of these cases.

6 Exercise: Bump and Orbit Control

Given two kickers located at the two ends of a FODO cell with phase advance 45 degrees (the two kickers are located at L_{cell} distance from each other), compute the strengths of such kickers (in radians) in order to give the beam, initially at $(x, x')_i = (0, 0)$, an arbitrary set at the end of the cell while preserving its angle, $(x, x')_f = (x_{arb}, 0)$

7 Exercise: Low-Beta Insertion

Consider the following low-beta insertion around an interaction point (IP). The quadrupoles are placed with mirror-symmetry with respect to the IP. The beam enters the quadrupole with Twiss parameters $\beta_0 = 20$m and $\alpha_0 = 0$. The drift space has length L = 10 m.

- **Determine the focal length of the quadrupole in order to locate the waist at the IP.**
- **What is the value of β^*?**
- **What is the phase advance between the quadrupole and the IP?**

We can use of the general transfer matrix for the optic parameters α, β, γ:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2CS & ^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0 \tag{9}$$

where the C, S, C' and S' expressions need to be found from the constructed trasport matrix representing the elements between s_0 and s. This is a combination of a focusing quadrupole, and a drift space. Using the thin element approximation, we have

$$M_{transfer} = M_{drift} M_f = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{l}{f} & l \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \tag{10}$$

plugging these into eq. 9, we get an equation for β_s

$$\beta_s = C^2\beta_0 - 2CS\alpha_0 + S^2\gamma_0 = C^2\beta_0 + \frac{S^2}{\beta_0} = \left(1 - \frac{l}{f}\right)^2 + \frac{l}{\beta_0} = \beta_0 - 2\beta_0\frac{l}{f} + \beta_0\frac{l^2}{f^2} + \frac{l^2}{\beta_0} \tag{11}$$

where we used the fact that $\alpha_0 = 0$ (given) and $\gamma_0 = \frac{1+\alpha_0}{\beta_0} = \frac{1}{\beta_0}$. Now, we want to find the focal length of the quadrupole at which the waist ($\beta^* = \beta_{min}$) is located at the interaction point ($l = L$). This can be done by setting $\frac{d\beta_s}{dl}|_{f_o, L} = 0$

$$\frac{d\beta_s}{dl}|_{f_o} = -\frac{2\beta_0}{f_0} + \frac{2\beta_0 L}{f_0^2} + \frac{2L}{\beta_0} = 0 \tag{12}$$

solving for f_0 and plugging in $\beta_0 = 20$m and $L = 10$m gives

$$f_0 = \frac{\beta_0 \pm \sqrt{\beta_0^2 - 4L^2}}{\frac{2L}{\beta_0}} = \frac{\beta_0^2}{2L} = 20m \tag{13}$$

The value of β^* can be found by plugging the calculated value for f_0 and $l = L$ into eq. 11. This gives β^*=10m.
The phase advance can be calculated as

$$\phi(s|s_0) = \int_{s_0}^{s} \frac{ds'}{\beta_s} = \int_0^L \frac{dl}{\beta_0 - 2\beta_0\frac{l}{f_0} + \beta_0\frac{l^2}{f_0^2} + \frac{l^2}{\beta_0}} = \int_0^{10} \frac{dl}{20 - 2l + \frac{l^2}{10}} = \frac{\pi}{4} \tag{14}$$

I simply put the last integral into Symbolab.