The Physics of the Plasma Modulator

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1 Introduction

This document is the summary of the underlying physics behind the modulator stage of the P-MoPA experiment. The theory has been developed by Oscar Jakobsson, Simon Hooker, and Roman Walczak and is neatly summarized in [1]. This document aims to provide the complete derivation of the theory in the paper's supplemental material. It is meant to properly establish the understanding of the topic by the author of this document while hopefully providing additional help to other researchers.

2 Deriving the Wave Equation

We want to see how EM radiation is affected while propagating through plasma of changing refractive index. This section loosely follows Chapter 3 of W.L. Kruer's classic book "The Physics of Laser Plasma Interactions" [2]. Kruer uses Gaussian units and replaces the operator $\frac{\partial}{\partial t}$ by $-i\omega$. SI units are used here. We can start from the very beginning with Maxwell's equations. Namely, with Faraday's law and Ampere's laws, respectively:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \tag{2}$$

Taking the curl of Faraday's law, we have

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla \times \left(-\frac{\partial \boldsymbol{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \boldsymbol{B}) \tag{3}$$

Now, we can use the common vector identity for the curl of a curl for the LHS, and plug Ampere's law (eq. 2) into the RHS. We then have

$$\nabla \cdot (\nabla \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t} \left(\mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) = -\mu_0 \frac{\partial \boldsymbol{J}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{4}$$

Now, if we were dealing with a vacuum, we could disregard the current density $\boldsymbol{J}$ since there are no charges in free space. That would be the classic way of deriving the wave equation of light in a vacuum. Since we are dealing with plasmas, however, we have current arising from the quiver motion of electrons.

$$\boldsymbol{J} = -n_e(\boldsymbol{r}) e \boldsymbol{v}_e \tag{5}$$

where $n_e(\boldsymbol{r})$ is the local electron density, and $\boldsymbol{v_e}$ the local electron velocity. The (Lorentz) force acting on an electron can be generally written as

$$m_e \frac{\partial \boldsymbol{v}}{\partial t} = -e(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{B}) \tag{6}$$

Taking the time derivative of $\boldsymbol{J}$ and using this result, we have

$$\frac{\partial \boldsymbol{J}}{\partial t} = -n_e(\boldsymbol{r}) e \frac{\partial \boldsymbol{v}_e}{\partial t} = n_e(\boldsymbol{r}) \frac{e^2}{m_e} \boldsymbol{E}(\boldsymbol{r}, t) = \omega_p^2 \epsilon_0 \boldsymbol{E}(\boldsymbol{r}, t) \tag{7}$$

Here we used the characteristic plasma frequency $\omega_p^2 = \frac{n_e e^2}{\epsilon_0 m_e}$, and the assumption that the vector product $\boldsymbol{v} \times \boldsymbol{B}$ in equation 6 is small and can be neglected. The latter comes from the assumption that we are dealing with non-magnetized plasmas and that self-generated magnetic fields are very small. Also, we plug in the result given by Gauss' law in quasineutral plasma, namely that the divergence of the electric field $\nabla \cdot \boldsymbol{E}$ in the absence of charge (or charge separation $\Delta\rho$ in plasma) is zero:

$$\nabla \cdot \boldsymbol{E} = \frac{\Delta \rho}{\epsilon_0} = 0 \tag{8}$$

Together with this result, we substitute eq. 7 back in equation 4, getting

$$-\nabla^2 \boldsymbol{E} = -\mu_0 \epsilon_0 \omega_p^2 \boldsymbol{E} - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = -\frac{1}{c^2} \left(\omega_p^2 \boldsymbol{E} + \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \right) \implies \nabla^2 \boldsymbol{E} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{\omega_p^2}{c^2} \boldsymbol{E} \tag{9}$$

this can be rewritten as

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \boldsymbol{E} = k_p^2 \boldsymbol{E} \tag{10}$$

where $k_p = \frac{\omega_p}{c}$. k_p conveniently contains information about relativistic effects should the plasma enter such a regime. It is important to note a difference between k_p and k_{p0} in the rest of this document. The wavenumber k_{p0} relates to the situation where the plasma electron density is unperturbed, denoted as n_0, and k_p is valid for general (perturbed) electron density $n_e = n_0 + \delta n$. Plugging the plasma frequency ω_p and using the relativistic electron mass $m_e = \gamma m_{e0}$ (where m_{e0} is the electron rest mass), we get the following relationship between k_p and k_{p0}:

$$\left. \begin{aligned} k_p^2 &= \tfrac{n_e e^2}{c^2 \epsilon_0 \gamma m_{e0}} \\ k_{p0}^2 &= \tfrac{n_{e0} e^2}{c^2 \epsilon_0 m_{e0}} \end{aligned} \right\} k_p^2 = k_{p0}^2 \frac{n_e}{\gamma n_{e0}}$$

The relativistic factor γ equals 1 in the unperturbed case and is therefore omitted in the k_{p0} equation. It is customary to express the wave equation not in terms of the electric field $\boldsymbol{E}$ but instead using the normalized vector potential of the laser pulse. The relationship between the electric field $\boldsymbol{E}$ and the vector field corresponding to linear polarization $\boldsymbol{A} = A_0 \cos(kz - \omega t) \boldsymbol{e}_\perp$ is as follows:

$$\boldsymbol{E} = -\frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial t} = \frac{\omega}{c} \boldsymbol{A} \tag{11}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{12}$$

The normalized vector potential is written as

$$\boldsymbol{a} = \frac{e \boldsymbol{A}}{m_e c^2} \tag{13}$$

and thus

$$\boldsymbol{E} = \frac{\omega}{c} \frac{m_e c^2}{e} \boldsymbol{a} = \frac{\omega m_e c}{e} \boldsymbol{a} \tag{14}$$

Hence, we can replace $\boldsymbol{E}$ by $\boldsymbol{a}$ in equation 10 without changing its validity:

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \boldsymbol{a} = k_p^2 \boldsymbol{a} \tag{15}$$

This is equation (1) in [3].

3 Solving the Wave Equation

We'll deal with a laser pulse propagating through a cylindrical plasma channel along direction z. Hence, it makes sense to write the periodically changing vector potential $a(z,t)$ as:

$$a(z,t) = b(z,t)e^{i(k_0 z - \omega_0 t)} \tag{16}$$

Plugging this into eq. 15, we get:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] b(z,t)e^{i(k_0 z - \omega_0 t)} = k_p^2 b(z,t)e^{i(k_0 z - \omega_0 t)} \tag{17}$$

We can separate the left-hand side into individual derivatives:

$$\frac{\partial}{\partial z}\left[b(z,t)e^{i(k_0 z - \omega_0 t)}\right] = \frac{\partial b}{\partial z}e^{i(k_0 z - \omega_0 t)} + ik_0 b(z,t)e^{i(k_0 z - \omega_0 t)} \tag{18}$$

$$\frac{\partial^2}{\partial z^2}\left[b(z,t)e^{i(k_0 z - \omega_0 t)}\right] = e^{i(k_0 z - \omega_0 t)}\left[\frac{\partial^2}{\partial z^2} + 2ik_0\frac{\partial}{\partial z} - k_0^2\right] b(z,t) \tag{19}$$

$$\frac{\partial}{\partial t}\left[b(z,t)e^{i(k_0 z - \omega_0 t)}\right] = \frac{\partial b}{\partial t}e^{i(k_0 z - \omega_0 t)} - i\omega_0 b(z,t)e^{i(k_0 z - \omega_0 t)} \tag{20}$$

$$\frac{\partial^2}{\partial t^2}\left[b(z,t)e^{i(k_0 z - \omega_0 t)}\right] = e^{i(k_0 z - \omega_0 t)}\left[\frac{\partial^2}{\partial t^2} - 2i\omega_0\frac{\partial}{\partial t} - \omega_0^2\right] b(z,t) \tag{21}$$

After plugging equations 19 and 21 into 17, we get

$$\left[\frac{\partial^2}{\partial z^2} + 2ik_0\frac{\partial}{\partial z} - ik_0 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2i\omega_0}{c^2}\frac{\partial}{\partial t} + \frac{\omega_0^2}{c^2}\right] b(z,t) = k_p^2 b(z,t) \tag{22}$$

We can define

$$\delta k_p^2 = k_p^2 - k_{p0}^2 = k_p^2 + k_0^2 - \frac{\omega_0^2}{c^2} \tag{23}$$

and simplify equation 22 further:

$$\left[\frac{\partial^2}{\partial z^2} + 2ik_0\frac{\partial}{\partial z} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2i\omega_0}{c^2}\frac{\partial}{\partial t}\right] b(z,t) = \delta k_p^2 b(z,t) \tag{24}$$

Now we are ready to make a coordinate transformation from the lab frame into the reference frame co-propagating with the laser pulse (assuming the base electron density n_{e0}):

$$\zeta = z - v_g t = z - \frac{c^2 k_0}{\omega_0}t \tag{25}$$

$$\tau = t \tag{26}$$

where $v_g = c^2 k_0/\omega_0$ is the group velocity of the laser pulse in plasma of electron density n_{e0}. We need to replace the derivatives in equation 24 using the following transformations:

$$\frac{\partial}{\partial z} = \cancelto{0}{\frac{\partial \tau}{\partial z}}\frac{\partial}{\partial \tau} + \cancelto{1}{\frac{\partial \zeta}{\partial z}}\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial \zeta} \implies \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \zeta^2} \tag{27}$$

$$\frac{\partial}{\partial t} = \cancelto{1}{\frac{\partial \tau}{\partial t}}\frac{\partial}{\partial \tau} + \cancelto{-v_g}{\frac{\partial \zeta}{\partial t}}\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial \tau} - v_g\frac{\partial}{\partial \zeta} \tag{28}$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \tau^2} - 2v_g\frac{\partial^2}{\partial \zeta \partial \tau} + v_g^2\frac{\partial^2}{\partial \zeta^2} \tag{29}$$

After plugging these derivatives into 24, we get

$$\left[\left(1-\frac{v_g^2}{c^2}\right)\frac{\partial^2}{\partial\zeta^2}+2\frac{v_g}{c^2}\frac{\partial^2}{\partial\zeta\partial\tau}+2i\cancelto{0}{\left(k_0-\frac{\omega_0}{c^2}v_g\right)}\frac{\partial}{\partial\zeta}+\frac{2i\omega_0}{c^2}\frac{\partial}{\partial\tau}-\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right]b(\zeta,\tau)=\delta k_p^2(\zeta,\tau)b(\zeta,\tau) \tag{30}$$

At this moment, it is important to stress that δk_p is a function of ζ and τ. We can also define $1/\gamma_g^2 = 1-v_g^2/c^2$, and eq. 30 becomes

$$\left[2i\frac{\omega_0}{c^2}\frac{\partial}{\partial\tau}+2\frac{v_g}{c^2}\frac{\partial^2}{\partial\zeta\partial\tau}+\frac{1}{\gamma_g^2}\frac{\partial^2}{\partial\zeta^2}-\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right]b(\zeta,\tau)=\delta k_p^2(\zeta,\tau)b(\zeta,\tau) \tag{31}$$

This is equation (2) as given by [3].

We can neglect the higher-order derivatives in the slow-moving envelope approximation (meaning that $b(\zeta,\tau)$ is slowly varying compared to the radiation frequency ω_0 [3]). Equation 31 then simplifies into

$$2i\frac{\omega_0}{c^2}\frac{\partial b(\zeta,\tau)}{\partial\tau}\simeq\delta k_p^2(\zeta,\tau)b(\zeta,\tau) \tag{32}$$

This is a separable differential equation and is easy to solve:

$$\int_{b_0}^{b}\frac{1}{b(\zeta,\tau)}db=-\frac{ic^2}{2\omega_0}\int_0^{\tau}\delta k_p^2(\zeta,\tau')d\tau' \tag{33}$$

$$b(\zeta,\tau)=|b_0(\zeta)=b(\zeta,0)|\exp\left[-\frac{ic^2}{2\omega_0}\int_0^{\tau}\delta k_p^2(\zeta,\tau')d\tau'\right] \tag{34}$$

This is equation (3) in [3] and (2) in the supplemental material of [1]. This equation neatly describes the pulse and is written in a form allowing for the investigation of what effect a varying index of refraction has on the beam.

4 Frequency Modulation

We want to study the effect of a periodically changing plasma density on the laser pulse propagating through a plasma channel. We can write this longitudinal density variation (in the lab frame) as

$$n_e(z,t)=n_{e0}+\delta n\cos(k_{p0}z-\omega_{p0}t+\Delta\phi) \tag{35}$$

Moving into the co-moving reference frame by simply plugging in the transformations for z and t, we have

$$n_e(\zeta,\tau)=n_{e0}+\delta n\cos\left[\frac{\omega_{p0}}{v_p}(\zeta+v_g\tau)-\omega_{p0}\tau+\Delta\phi\right]=n_{e0}+\delta n\cos\left[\frac{\omega_{p0}}{v_p}(\zeta+\Delta v\tau)+\Delta\phi\right] \tag{36}$$

where we defined $\Delta v = v_g - v_p$, the difference between the speed of the laser pulse (its group velocity v_g) and the propagation speed of the plasma wake (wake phase velocity v_p). We now recall that

$$\delta k_p^2(\zeta,\tau)=k_p^2-k_{p0}^2=k_{p0}^2\left(\frac{n_e(\zeta,\tau)}{\gamma n_{e0}}-1\right) \tag{37}$$

Plugging equation 36 for $n_e(\zeta,\tau)$ and ignoring relativistic effects ($\gamma\simeq 1$) yields

$$\delta k_p^2(\zeta,\tau)=k_{p0}^2\frac{\delta n_e}{n_{e0}}\cos\left[\frac{\omega_{p0}}{v_p}(\zeta+\Delta v\tau)+\Delta\phi\right] \tag{38}$$

We can now plug this expression for $\delta k_p^2(\zeta,\tau)$ into the general solution of the wave equation 34:

$$b(\zeta,\tau)=|b_0(\zeta)|\exp\left[-\frac{ic^2}{2\omega_0}k_{p0}^2\frac{\delta n_e}{{}_{e0}}\int_0^{\tau}\cos\left(\frac{\omega_{p0}}{v_p}\zeta+\frac{\Delta v}{v_p}\omega_{p0}\tau'+\Delta\phi\right)d\tau'\right] \tag{39}$$

Generally, we have the solution to the definitive integral

$$\int_0^\tau \cos(\alpha\tau' + \beta)d\tau' = \frac{1}{\alpha}\left[\sin(\alpha\tau + \beta) - \sin(\beta)\right] \tag{40}$$

We can massage this expression a little further using the trig identities for sine and cosine of a double angle, and cosine of a sum of angles, to get the following:

$$\begin{aligned}\int_0^\tau \cos(\alpha\tau' + \beta)d\tau' &= \frac{1}{\alpha}\left[\sin(\alpha\tau)\cos(\beta) + \cos(\alpha\tau)\sin(\beta) - \sin(\beta)\right] \\ &= \frac{1}{\alpha}\left[2\sin\left(\frac{\alpha\tau}{2}\right)\cos\left(\frac{\alpha\tau}{2}\right)\cos(\beta) + \left(1 - 2\sin^2\left(\frac{\alpha\tau}{2}\right)\right)\sin(\beta) - \sin(\beta)\right] \\ &= \frac{1}{\alpha}\left[2\sin\left(\frac{\alpha\tau}{2}\right)\cos\left(\frac{\alpha\tau}{2}\right)\cos(\beta) + \cancel{\sin(\beta)} - 2\sin^2\left(\frac{\alpha\tau}{2}\right)\sin(\beta) - \cancel{\sin(\beta)}\right] \\ &= \frac{2}{\alpha}\sin\left(\frac{\alpha\tau}{2}\right)\left[\cos\left(\frac{\alpha\tau}{2}\right)\cos(\beta) - \sin\left(\frac{\alpha\tau}{2}\right)\sin(\beta)\right] \\ &= \tau\frac{\sin\left(\frac{\alpha\tau}{2}\right)}{\left(\frac{\alpha\tau}{2}\right)}\cos\left(\frac{\alpha\tau}{2} + \beta\right) = \tau\,\mathrm{sinc}\left(\frac{\alpha\tau}{2}\right)\cos\left(\frac{\alpha\tau}{2} + \beta\right)\end{aligned} \tag{41}$$

We can then simply plug $\alpha = \frac{\Delta v}{v_p}\omega_{p0}$ and $\beta = \frac{\omega_{p0}}{v_p}\zeta + \Delta\phi$, and plug it all back into equation 39, yielding

$$b(\zeta,\tau) = |b_0(\zeta)|\exp\left[-\frac{i}{2}\frac{\omega_{p0}^2}{\omega_0}\frac{c^2}{v_p^2}\frac{\delta n_e}{n_{e0}}\tau\,\mathrm{sinc}\left(\frac{1}{2}\frac{\omega_{p0}}{v_p}\Delta v\tau\right)\cos\left(\frac{\omega_{p0}}{v_p}\zeta + \frac{1}{2}\frac{\omega_{p0}}{v_p}\Delta v\tau + \Delta\phi\right)\right] \tag{42}$$

which is equation (5) in the supplemental material of [1].

Now, we can make the approximation that $\Delta v \to 0$, meaning that the wake propagates at nearly the speed of light in plasma, and also $v_p \simeq c$ Then, we have the common result $\lim_{x\to 0}\mathrm{sinc}(x) = 1$ and the above equation simplifies to

$$b(\zeta,\tau) \simeq |b_o(\zeta)|\exp\left[\frac{i}{2}\frac{\omega_{p0}^2}{\omega_0}\frac{\delta n_e}{n_{e0}}\tau\cos\left(\frac{\omega_{p0}}{v_p}\zeta + \Delta\phi\right)\right] \tag{43}$$

Let's group parameters into a common constant β:

$$\beta = \frac{1}{2}\frac{\omega_{p0}^2}{\omega_0}\frac{\delta n_e}{n_{e0}}\frac{z}{v_g} \tag{44}$$

Also, the argument of the cosine in equation 43 can be re-written using a new definition $\Delta\phi' = -\left(\Delta\phi - \omega_{p0}\frac{z}{v_p}\right)$ as

$$\frac{\omega_{p0}}{v_p}\zeta + \Delta\phi = \frac{\omega_{p0}}{v_p}(z - v_g\tau)_\delta\phi = \cancel{\omega_{p0}\frac{z}{v_p}} - \omega_{p0}\frac{v_g}{v_p}\frac{z}{v_g} - \Delta\phi' - \cancel{\omega_{p0}\frac{z}{v_p}} = -\left(\omega_{p0}\frac{z}{v_p} + \Delta\phi'\right) \simeq -(\omega_{p0}\tau + \Delta\phi) \tag{45}$$

Using the collective parameter β and the fact that cosine is an even function ($\cos(-\alpha) = \cos(\alpha)$), we can rewrite equation 43 as

$$b(\zeta,\tau) \simeq |b_0(\zeta)|\exp\left[-i\beta\cos(\omega_{p0}\tau + \Delta\phi')\right] \tag{46}$$

where we used $\tau \simeq z/v_g$, arising from "the times of interest, i.e. times corresponding to the arrival of the pulse". For a general complex number z, the Jacobi-Anger expansion is written as

$$\boxed{e^{iz\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(z)e^{im\theta}} \tag{47}$$

where J_m are bessel functions of the first kind. Using this, we can write eq. 46 as

$$b(\zeta,\tau) \simeq |b_0(\zeta)|\sum_{m=-\infty}^{\infty} i^m J_m(-\beta)e^{im(\omega_{p0}\tau + \Delta\phi')} \tag{48}$$

which is equation (1) in [1] and eq. (7) in the supplemental material.

References

[1] O Jakobsson, SM Hooker, and R Walczak. “GeV-scale accelerators driven by plasma-modulated pulses from kilohertz lasers”. In: *Physical Review Letters* 127.18 (2021), p. 184801.

[2] William Kruer. *The physics of laser plasma interactions.* crc Press, 2019.

[3] E Esarey, A Ting, and P Sprangle. “Frequency shifts induced in laser pulses by plasma waves”. In: *Physical Review A* 42.6 (1990), p. 3526.