Advanced Mathematical Reference

Contents

1 Set Theory

1.1 Basic Definitions

- Set: Collection of distinct objects
- $\emptyset$: Empty set
- $\{x : P(x)\}$: Set of all x satisfying property P
- $A \subseteq B$: A is a subset of B

1.2 Set Operations

- $A \cup B$: Union
- $A \cap B$: Intersection
- $A \setminus B$: Set difference
- A^c: Complement of A
- $\mathcal{P}(A)$: Power set of A

2 Linear Algebra

2.1 Vector Spaces

Definition 2.1. *A vector space V over a field F is a set with operations + and $\cdot$ satisfying:*

1. *Closure under addition and scalar multiplication*
2. *Associativity and commutativity of addition*
3. *Existence of zero vector and additive inverses*
4. *Distributivity of scalar multiplication over vector addition*
5. *Distributivity of scalar multiplication over field addition*
6. *Associativity of scalar multiplication*
7. *Existence of multiplicative identity for scalars*

2.2 Linear Independence

Definition 2.2. *Vectors $\{v_1, \ldots, v_n\}$ are linearly independent if $\sum_{i=1}^{n} c_i v_i = 0$ implies $c_i = 0$ for all i.*

2.3 Basis and Dimension

Definition 2.3. *A basis for a vector space V is a linearly independent set that spans V. The dimension of V is the cardinality of its basis.*

2.4 Linear Transformations

Definition 2.4. *A linear transformation T: V → W is a function satisfying:*

1. *T(u + v) = T(u) + T(v)*
2. *T(cv) = cT(v)*

2.5 Matrices

- Matrix multiplication: $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$
- Determinant (2x2): $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
- Trace: $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$
- Eigenvalues: $\det(A - \lambda I) = 0$
- Eigenvectors: $(A - \lambda I)v = 0$

3 Analysis

3.1 Limits

Definition 3.1. $\lim_{x\to a} f(x) = L$ *if* $\forall\epsilon > 0, \exists\delta > 0$ *such that* $0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$

3.2 Continuity

Definition 3.2. *f is continuous at a if* $\lim_{x\to a} f(x) = f(a)$

3.3 Differentiation

- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
- Product rule: $(fg)' = f'g + fg'$
- Chain rule: $(f \circ g)' = (f' \circ g) \cdot g'$

3.4 Integration

- Definite integral: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$
- Fundamental Theorem of Calculus:
  1. If $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$
  2. $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

4 Topology

4.1 Metric Spaces

Definition 4.1. *A metric space (X, d) is a set X with a function d: X × X → R satisfying:*

1. *d(x, y) $\geq$ 0 and d(x, y) = 0 iff x = y*
2. *d(x, y) = d(y, x)*
3. *d(x, z) $\leq$ d(x, y) + d(y, z)*

4.2 Open and Closed Sets

- Open ball: $B_r(x) = \{y \in X : d(x,y) < r\}$
- Open set: A $\subseteq$ X is open if $\forall x \in A, \exists r > 0$ such that $B_r(x) \subseteq A$
- Closed set: A is closed if A^c is open

5 Complex Analysis

5.1 Complex Numbers

- $z = a + bi$, where $i^2 = -1$
- $|z| = \sqrt{a^2 + b^2}$
- $\bar{z} = a - bi$
- $e^{i\theta} = \cos\theta + i\sin\theta$

5.2 Analytic Functions

Definition 5.1. *f is analytic at z_0 if it is differentiable in a neighborhood of z_0.*

5.3 Cauchy-Riemann Equations

For $f(x + yi) = u(x,y) + v(x,y)i$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

5.4 Contour Integration

- $\oint_C f(z)dz = \int_a^b f(z(t))z'(t)dt$
- Cauchy's Integral Formula: $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$

6 Probability and Statistics

6.1 Probability Basics

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A|B) = \frac{P(A\cap B)}{P(B)}$
- Bayes' Theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

6.2 Random Variables

- Expected value: $E[X] = \sum_x xP(X = x)$ or $\int_{-\infty}^{\infty} xf(x)dx$
- Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$

6.3 Common Distributions

- Binomial: $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$
- Poisson: $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$
- Normal: $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$