

COM1033 FOUNDATIONS OF COMPUTING

II

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February 27, 2024

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1 Vectors

1.1 Vector Definition

Let $n \in \mathbb{N}$ and $n > 0$.

The set of all vectors is the cartesian product of \mathbb{R} by n times, which is a set of ordered n -tuples of real numbers.

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

1.2 Vector Operations

1.2.1 Addition

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$$

1.2.2 Scalar Multiplication

$$\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$$

1.2.3 Dot Product / Scalar Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = (a \cdot d) + (b \cdot e) + (c \cdot f)$$

1.2.4 Linear Combination

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n scalars, and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be n vectors.

$$\vec{w} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_n \vec{v}_n$$

\vec{w} is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ using the scalars $\lambda_1, \lambda_2, \dots, \lambda_n$.

1.2.5 Linear Dependence

Let there be n vectors of the same dimension.

If the null vector $\vec{0}$ can be expressed as linear combination of the n vectors as defined, using non null scalars.

In other words, the n vectors are linearly dependent if:

$$\vec{w} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_n \vec{v}_n \mid \exists \lambda_1, \lambda_2, \dots, \lambda_n \neq 0, 0, \dots, 0$$

1.2.6 Matrices

Matrices are defined as a table where it's elements have two indicies, limited to the size of the matrix size.

1.2.7 Exercises

Question 1: Sum the following vectors $\in \mathbb{R}^3$:

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Calculate the product $\lambda \vec{v}_1$ with $\lambda = 2$

$$\lambda \vec{v}_1 = \begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}$$

Question 2

$$\vec{u} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 3 \cdot 2 + 5 \cdot 2 + -4 \cdot 4 \\ &= 6 + 10 - 16 \\ &= 0 \end{aligned}$$

Question 3

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix} \tag{1}$$

$$\text{when: } \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1 \tag{2}$$

$$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \lambda_3 \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

Question 4:

Let v_1, v_2, \dots, v_n be n linearly independent vectors. Consider the set of scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$. Find alternative sets of the scalars.

Just multiply all the scalars by a common scaling factor, let's say μ

Question 5:

a_3 is on the same line

2 Matrices

2.1 Matrix Multiplication

Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix.

Question 6:

$$\begin{pmatrix} 1, 0, 2 \\ 3, 5, 1 \\ 2, 2, 0 \end{pmatrix} \quad (4)$$

$$0 - 2 - 0 + 0 + 12 - 20 = -10 \quad (5)$$

$$\begin{pmatrix} 1, 0, 3 \\ 1, -1, 0 \\ 4, 2, 1 \end{pmatrix} \quad (6)$$

$$-1 - (0) - (0) + 0 + 6 - (-12) = 17 \quad (7)$$

$$v_1 + 2v_2 - v_3 \text{ i.e.:} \quad (8)$$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} - 1 - (0) - (0) + 0 + 6 - (-12) = 17 \quad (9)$$

$$(10)$$

Start with a matrix with a determinant of 0.

$$\begin{pmatrix} a_1 1 & a_1 2 & a_1 3 \\ a_2 1 & a_2 2 & a_2 3 \\ a_3 1 & a_3 2 & a_3 3 \end{pmatrix} \quad (11)$$

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{pmatrix} \quad (12)$$

the first laplace theorem is to expand the determinant of a matrix along the first row.

$$a_1 1 \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix} - a_1 2 \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} + a_1 3 \begin{pmatrix} 3 & 5 \\ 2 & 2 \end{pmatrix} \quad (13)$$

for the example matrix \mathbb{M} :

$$\det(\mathbb{M}) = 6 - 6 - 0 + 6 + 0 - 12 = -6 \quad (14)$$

$$(15)$$

2. Yes

question 4: 2

Row three is a null row.

has determinant 0

Row three plus a row one multiplied by the some scalar has the same determinant.

Row three plus row two multiplied by the some scalar has the same determinant.
-170 for both $\det(AB)$ and $\det(BA)$

3 Linear Equations

3.1 System of Linear Equations

$$\begin{cases} a_{1,1}x + a_{1,2}y = b_1 \\ a_{2,1}x + a_{2,2}y = b_2 \end{cases} \quad (16)$$

the above can be expressed as a matrix equation:

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (17)$$

Let's use a 4 by 4 square matrix as an example:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \quad (18)$$

/subsubsectionKramer's method

For the above 4 by 4 matrix, we can solve for x by using the following formula:

$$x = \frac{\det(\mathbb{M}_x)}{\det(\mathbb{M})} \quad (19)$$

note that \mathbb{M}_x is the matrix \mathbb{M} with the first column replaced by the column vector \mathbb{B} .

So x_1 for instance would be:

$$x_1 = \frac{\det(\mathbb{M}_1)}{\det(\mathbb{M})} \quad (20)$$

$$\text{Where } \mathbb{M}_1 = \begin{pmatrix} b_1 & a_{1,2} & a_{1,3} & a_{1,4} \\ b_2 & a_{2,2} & a_{2,3} & a_{2,4} \\ b_3 & a_{3,2} & a_{3,3} & a_{3,4} \\ b_4 & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \quad (21)$$

3.1.1 Mock test

$$\begin{cases} 3x - 2y + z = 2 \\ 2z = 2 \\ x + y = 2 \end{cases} \quad \text{Can be expressed as:}$$

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \text{ the determinant of the matrix is:}$$

$$0 - 6 - 0 - 4 + 0 - 0 = -10$$

It is solvable.

The solution for x is:

$$\det(\mathbb{M}_x) = \det \begin{pmatrix} 2 & -2 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} = 0 - 4 + 0 - 8 + 2 - 0 = -10$$

$$= \frac{-10}{-10} = 1$$

Question 2.

$$\begin{pmatrix} 3 & -2 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \text{ the determinant of the matrix is:}$$

$$0 - 6 + 8 - 4 + 2 + 0 = 0$$

It is not solvable.

Question 3.

$$\det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 2 & -1 & 1 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -3 & -1 & -4 \\ 0 & -3 & 1 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

$$z = 2$$

$$-3y - 1(2) = -4$$

$$-3y = -2$$

$$y = \frac{2}{3}$$

$$x = \frac{1}{3}$$

Question 4.

$$\begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$$

4 Vector Spaces

Fuck you

Let \mathbf{E} be a non-null set ($\mathbf{E} \neq \emptyset$) and \mathbb{K} be a scalar set.

We designate vectors as elements of \mathbf{E}

let '+' be an internal compositional law, i.e. $\mathbf{E} \times \mathbf{E} \rightarrow \mathbf{E}$

let '.' be an internal compositional law, i.e. $\mathbb{K} \times \mathbf{E} \rightarrow \mathbf{E}$