COM1033 FOUNDATIONS OF COMPUTING II

Jim S. Lam

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Contents

1 Vectors

1.1 Vector Definition

Let n $\in \mathbb{N}$ and $n > 0$.

The set of all vectors is the cartesian product of $\mathbb{R}$ by n times, which is a set of ordered n-tuples of real numbers.

For instance, a vector in $\mathbb{R}^3$ is defined as:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

Null Vector

The null vector is a vector with all elements equal to 0.

$$\vec{o} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1.2 Vector Operations

1.2.1 Addition

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$$

1.2.2 Scalar Multiplication

$$\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$$

1.2.3 Dot Product / Scalar Product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

2 Linear Dependence / Independence

2.1 Linear Combination

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be n scalars, and $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ be n vectors.

$$\vec{w} = \lambda_1 \vec{v_1} + \lambda_2 \vec{v_2} + ... + \lambda_n \vec{v_n}$$

$\vec{w}$ is a linear combination of $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ using the scalars $\lambda_1, \lambda_2, ..., \lambda_n$.

2.2 Linear Dependence

Let there be n vectors of the same dimension.

If the null vector $\vec{o}$ can be expressed as linear combination of the n vectors as defined, using non null scalars.

In other words, the n vectors are linearly dependent if:

$$\vec{w} = \lambda_1 \vec{v_1} + \lambda_2 \vec{v_2} + ... + \lambda_n \vec{v_n} \mid \exists \lambda_1, \lambda_2, ..., \lambda_n \neq 0, 0, ..., 0$$

Exercises

Question 1: Sum the following vectors $\in \mathbb{R}^3$:

$$\vec{v_1} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Calculate the product $\lambda \vec{v_1}$ with $\lambda = 2$

$$\lambda \vec{v_1} = \begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}$$

Question 2

$$\vec{u} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \vec{u} \dot{} \vec{v} &= 3 \cdot 2 + 5 \cdot 2 + -4 \cdot 4 \\ &= 6 + 10 - 16 \\ &= 0 \end{aligned}$$

Question 3

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \quad \vec{v_3} = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix} \tag{1}$$

$$\text{when: } \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1 \tag{2}$$

$$\lambda_1 \vec{v_1} + \lambda_2 \vec{v_2} + \lambda_3 \vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

Question 4:

Let $v_1, v_2, ..., v_n$ be n linearly independent vectors. Consider the the set of scalers $\lambda_1, \lambda_2, ..., \lambda_n$ such that $\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = 0$. Find alternative sets of the scalers.

Just multiply all the scalars by a common scaling factor, let's say μ

Question 5:

a_3 is on the same line

3 Matrices

3.1 Matrix Multiplication

Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix.

$$\mathbf{A} \times \mathbf{B} = \mathbf{C} \tag{4}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,n} \\ a_{2,1} & a_{2,2} & ... & a_{2,n} \\ ... & ... & ... & ... \\ a_{m,1} & a_{m,2} & ... & a_{m,n} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & b_{1,2} & ... & b_{1,p} \\ b_{2,1} & b_{2,2} & ... & b_{2,p} \\ ... & ... & ... & ... \\ b_{n,1} & b_{n,2} & ... & b_{n,p} \end{pmatrix} \tag{5}$$

$$= \tag{6}$$

Question 6:

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

$$0 - 2 - 0 + 0 + 12 - 20 = -10$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

$$-1 - (0) - (0) + 0 + 6 - (-12) = 17$$

$$v_1 + 2v_2 - v_3 \text{ i.e.: } \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} - 1 - (0) - (0) + 0 + 6 - (-12)$$

$$= 17$$

Start with a matrix with a determinant of 0.

$$\begin{pmatrix} a_1 1 & a_1 2 & a_1 3 \\ a_2 1 & a_2 2 & a_2 3 \\ a_3 1 & a_3 2 & a_3 3 \end{pmatrix} \tag{7}$$

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{pmatrix} \tag{8}$$

the first laplace theorem is to expand the determinant of a matrix along the first row.

$$a_1 1 \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix} - a_1 2 \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} + a_1 3 \begin{pmatrix} 3 & 5 \\ 2 & 2 \end{pmatrix} \tag{9}$$

for the example matrix $\mathbb{M}$:

$$\det(\mathbb{M}) = 6 - 6 - 0 + 6 + 0 - 12 = -6 \tag{10}$$

$$\tag{11}$$

2. Yes
question 4: 2
Row three is a null row.
has determinant 0
Row three plus a row one multiplied by the some scalar has the same determinant.
Row three plus row two multiplied by the some scalar has the same determinant.
-170 for both det(AB) and det(BA)

4 Linear Equations

4.1 System of Linear Equations

$$\begin{cases} a_{1,1}x + a_{1,2}y = b_1 \\ a_{2,1}x + a_{2,2}y = b_2 \end{cases} \tag{12}$$

the above can be expressed as a matrix equation:

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{13}$$

Let's use a 4 by 4 square matrix as an example:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \tag{14}$$

/subsubsectionKramer's method

For the above 4 by 4 matrix, we can solve for x by using the following formula:

$$x = \frac{\det(\mathbb{M}_x)}{\det(\mathbb{M})} \tag{15}$$

note that $\mathbb{M}_x$ is the matrix $\mathbb{M}$ with the first column replaced by the column vector $\mathbb{B}$.

So x_1 for instance would be:

$$x_1 = \frac{\det(\mathbb{M}_1)}{\det(\mathbb{M})} \tag{16}$$

$$\text{Where } \mathbb{M}_1 = \begin{pmatrix} b_1 & a_{1,2} & a_{1,3} & a_{1,4} \\ b_2 & a_{2,2} & a_{2,3} & a_{2,4} \\ b_3 & a_{3,2} & a_{3,3} & a_{3,4} \\ b_4 & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \tag{17}$$

4.1.1 Mock test

$$\begin{cases} 3x - 2y + z = 2 \\ 2z = 2 \\ x + y = 2 \end{cases} \quad \text{Can be expressed as:}$$

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \text{ the determinant of the matrix is:}$$

$$0 - 6 - 0 - 4 + 0 - 0 = -10$$

It is solvable.

The solution for x is:

$$\det(\mathbb{M}_x) = \det(\begin{pmatrix} 2 & -2 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}) = 0 - 4 + 0 - 8 + 2 - 0 = -10$$

$$= \frac{-10}{-10} = 1$$

Question 2.

$$\begin{pmatrix} 3 & -2 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \text{ the determinant of the matrix is:}$$

$$0 - 6 + 8 - 4 + 2 + 0 = 0$$

It is not solvable.

Question 3.

$$\det(\begin{pmatrix} 1 & 1 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 2 & -1 & 1 & 2 \end{pmatrix}) = \det(\begin{pmatrix} 1 & 1 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix}) = \det(\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -3 & -1 & -4 \\ 0 & -3 & 1 & 0 \end{pmatrix}) = \det(\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & 2 & 4 \end{pmatrix})$$

$$z = 2$$

$$-3y - 1(2) = -4$$

$$-3y = -2$$

$$y = \frac{2}{3}$$

$$x = \frac{1}{3}$$

Question 4.

$$\begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$$

5 Vector Spaces

Fuck you

Let $\mathbf{E}$ be a non-null set ($\mathbf{E} \neq \emptyset$) and $\mathbb{K}$ be a scalar set.

We designate vectors as elements of $\mathbf{E}$

let '+' be an internal compositional law, i.e. $\mathbf{E} \times \mathbf{E} \to \mathbf{E}$

let '.' be an internal compositional law, i.e. $\mathbb{K} \times \mathbf{E} \to \mathbf{E}$