

# COM1033 FOUNDATIONS OF COMPUTING

## II

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# 1 Vectors

## 1.1 Vector Definition

Let  $n \in \mathbb{N}$  and  $n > 0$ .

The set of all vectors is the cartesian product of  $\mathbb{R}$  by  $n$  times, which is a set of ordered  $n$ -tuples of real numbers.

For instance, a vector in  $\mathbb{R}^3$  is defined as:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

### Null Vector

The null vector is a vector with all elements equal to 0.

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

## 1.2 Vector Operations

### 1.2.1 Addition

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$$

### 1.2.2 Scalar Multiplication

$$\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$$

### 1.2.3 Dot Product / Scalar Product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

# 2 Linear Dependence / Independence

## 2.1 Linear Combination

Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be  $n$  scalars, and  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  be  $n$  vectors.

$$\vec{w} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_n \vec{v}_n$$

$\vec{w}$  is a linear combination of  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  using the scalars  $\lambda_1, \lambda_2, \dots, \lambda_n$ .

## 2.2 Linear Dependence

Let there be  $n$  vectors of the same dimension.

If the null vector  $\vec{0}$  can be expressed as linear combination of the  $n$  vectors as defined, using non null scalars.

In other words, the  $n$  vectors are linearly dependent if:

$$\vec{w} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_n \vec{v}_n \mid \exists \lambda_1, \lambda_2, \dots, \lambda_n \neq 0, 0, \dots, 0$$

### Exercises

Question 1: Sum the following vectors  $\in \mathbb{R}^3$ :

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Calculate the product  $\lambda \vec{v}_1$  with  $\lambda = 2$

$$\lambda \vec{v}_1 = \begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}$$

Question 2

$$\vec{u} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 3 \cdot 2 + 5 \cdot 2 + (-4) \cdot 4 \\ &= 6 + 10 - 16 \\ &= 0 \end{aligned}$$

Question 3

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix} \tag{1}$$

$$\text{when: } \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1 \tag{2}$$

$$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \lambda_3 \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

Question 4:

Let  $v_1, v_2, \dots, v_n$  be  $n$  linearly independent vectors. Consider the the set of scalars  $\lambda_1, \lambda_2, \dots, \lambda_n$  such that  $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$ . Find alternative sets of the scalars.

Just multiply all the scalars by a common scaling factor, let's say  $\mu$

Question 5:

$a_3$  is on the same line

### 3 Matrices

#### 3.1 Matrix Multiplication

Let  $A$  be a  $m \times n$  matrix, and  $B$  be a  $n \times p$  matrix.

$$\mathbf{A} \times \mathbf{B} = \mathbf{C} \quad (4)$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,p} \\ b_{2,1} & b_{2,2} & \dots & b_{2,p} \\ \dots & \dots & \dots & \dots \\ b_{n,1} & b_{n,2} & \dots & b_{n,p} \end{pmatrix} \quad (5)$$

$$= \quad (6)$$

Question 6:

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

$$0 - 2 - 0 + 0 + 12 - 20 = -10$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

$$-1 - (0) - (0) + 0 + 6 - (-12) = 17$$

$$v_1 + 2v_2 - v_3 \text{ i.e.: } \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} - 1 - (0) - (0) + 0 + 6 - (-12) = 17$$

Start with a matrix with a determinant of 0.

$$\begin{pmatrix} a_1 1 & a_1 2 & a_1 3 \\ a_2 1 & a_2 2 & a_2 3 \\ a_3 1 & a_3 2 & a_3 3 \end{pmatrix} \quad (7)$$

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{pmatrix} \quad (8)$$

the first laplace theorem is to expand the determinant of a matrix along the first row.

$$a_1 1 \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix} - a_1 2 \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} + a_1 3 \begin{pmatrix} 3 & 5 \\ 2 & 2 \end{pmatrix} \quad (9)$$

for the example matrix  $\mathbb{M}$ :

$$\det(\mathbb{M}) = 6 - 6 - 0 + 6 + 0 - 12 = -6 \quad (10)$$

$$(11)$$

2. Yes

question 4: 2

Row three is a null row.

has determinant 0

Row three plus a row one multiplied by the some scalar has the same determinant.

Row three plus row two multiplied by the some scalar has the same determinant.  
-170 for both  $\det(\mathbf{AB})$  and  $\det(\mathbf{BA})$

## 4 Linear Equations

### 4.1 System of Linear Equations

$$\begin{cases} a_{1,1}x + a_{1,2}y = b_1 \\ a_{2,1}x + a_{2,2}y = b_2 \end{cases} \quad (12)$$

the above can be expressed as a matrix equation:

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (13)$$

Let's use a 4 by 4 square matrix as an example:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \quad (14)$$

/subsubsectionKramer's method

For the above 4 by 4 matrix, we can solve for  $x$  by using the following formula:

$$x = \frac{\det(\mathbb{M}_x)}{\det(\mathbb{M})} \quad (15)$$

note that  $\mathbb{M}_x$  is the matrix  $\mathbb{M}$  with the first column replaced by the column vector  $\mathbb{B}$ .

So  $x_1$  for instance would be:

$$x_1 = \frac{\det(\mathbb{M}_1)}{\det(\mathbb{M})} \quad (16)$$

$$\text{Where } \mathbb{M}_1 = \begin{pmatrix} b_1 & a_{1,2} & a_{1,3} & a_{1,4} \\ b_2 & a_{2,2} & a_{2,3} & a_{2,4} \\ b_3 & a_{3,2} & a_{3,3} & a_{3,4} \\ b_4 & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \quad (17)$$

#### 4.1.1 Mock test

$$\begin{cases} 3x - 2y + z = 2 \\ 2z = 2 \\ x + y = 2 \end{cases} \quad \text{Can be expressed as:}$$

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \text{ the determinant of the matrix is:}$$

$$0 - 6 - 0 - 4 + 0 - 0 = -10$$

It is solvable.

The solution for x is:

$$\det(\mathbb{M}_x) = \det \begin{pmatrix} 2 & -2 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} = 0 - 4 + 0 - 8 + 2 - 0 = -10$$

$$= \frac{-10}{-10} = 1$$

Question 2.

$$\begin{pmatrix} 3 & -2 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \text{ the determinant of the matrix is:}$$

$$0 - 6 + 8 - 4 + 2 + 0 = 0$$

It is not solvable.

Question 3.

$$\det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 2 & -1 & 1 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -3 & -1 & -4 \\ 0 & -3 & 1 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

$$z = 2$$

$$-3y - 1(2) = -4$$

$$-3y = -2$$

$$y = \frac{2}{3}$$

$$x = \frac{1}{3}$$

Question 4.

$$\begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$$



## 5 Vector Spaces

Fuck you

Let  $\mathbf{E}$  be a non-null set ( $\mathbf{E} \neq \emptyset$ ) and  $\mathbb{K}$  be a scalar set.

We designate vectors as elements of  $\mathbf{E}$

let '+' be an internal compositional law, i.e.  $\mathbf{E} \times \mathbf{E} \rightarrow \mathbf{E}$

let '.' be an internal compositional law, i.e.  $\mathbb{K} \times \mathbf{E} \rightarrow \mathbf{E}$