Advanced Mathematical Reference

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Contents

1 Vectors

Explanation

Vectors are fundamental objects in linear algebra, representing quantities with both magnitude and direction.

1.1 Vector Definition

$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$

1.1.1 Null Vector

$\vec{0} = (0, 0, \ldots, 0)$

1.2 Vector Operations

1.2.1 Addition

$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$

1.2.2 Scalar Multiplication

$\lambda(a_1, \ldots, a_n) = (\lambda a_1, \ldots, \lambda a_n)$

1.2.3 Dot Product

$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$

2 Linear Dependence / Independence

Explanation

Linear dependence and independence are crucial concepts in understanding the relationships between vectors.

2.1 Linear Combination

$\vec{w} = \sum_{i=1}^{n} \lambda_i \vec{v_i}$

2.2 Linear Dependence

Vectors are linearly dependent if $\vec{0} = \sum_{i=1}^{n} \lambda_i \vec{v_i}$, with at least one $\lambda_i \neq 0$

3 Matrices

Explanation

Matrices are rectangular arrays of numbers used to represent linear transformations and solve systems of linear equations.

3.1 Matrix Multiplication

$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

3.2 Determinant

For 3x3 matrix:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

3.3 Properties of Determinants

1. $\det(AB) = \det(A)\det(B)$
2. Row/column swap changes sign of determinant
3. $\det(A) = 0$ if A has a row/column of zeros or two identical rows/columns

4 Linear Equations

Explanation

Linear equations form the backbone of many mathematical models in science and engineering.

Ax = b

4.1 Cramer's Rule

For a system Ax = b, $x_i = \frac{\det(A_i)}{\det(A)}$, where A_i is A with i-th column replaced by b

5 Vector Spaces

Explanation

Vector spaces are abstract mathematical structures that generalize the properties of vectors in Euclidean space.

A set V with vector addition and scalar multiplication satisfying:

1. Commutativity and associativity of addition
2. Existence of zero vector and additive inverses
3. Distributivity of scalar multiplication over vector and field addition
4. Associativity of scalar multiplication
5. Existence of multiplicative identity for scalars

6 Linear Transformations

Explanation

Linear transformations are functions between vector spaces that preserve vector addition and scalar multiplication.

A function T: V → W between vector spaces satisfying:

1. T(u + v) = T(u) + T(v)
2. T(αv) = αT(v)

7 Multi-variate Calculus

Explanation

Multi-variate calculus extends the concepts of single-variable calculus to functions of several variables.

7.1 Limits and Continuity

Similar to single-variable calculus, but with multiple variables

7.2 Partial Derivatives

$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$

7.3 Gradient

$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$

7.4 Directional Derivative

$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$

7.5 Chain Rule

$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$

7.6 Multiple Integrals

$\int \int_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$