Sandsynlighedsteori og Lineær algebra

Workshop 4 - Linear optimization

Sebastian Livoni Larsen

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Sparse solutions to linear algebraic systems

Exercise 1 - Explain why the objective function does not satisfy the definition of a linear tranformation.

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{n} |x_i| \\
-1 \cdot f(x) &\neq f(-1 \cdot x) \\
-1 \cdot f(1) &= -1 \\
f(-1 \cdot 1) &= 1 \\
\text{Hence, } -1 \cdot f(1) &\neq f(-1 \cdot 1)
\end{aligned}
$$

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Exercise 3 - Determining all 10 basic solutions and select out of them 5 feasible basic solutions

$$\tilde{A}\tilde{x} = b$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & -1 & -2 & -3 & -4 & -5 \end{bmatrix} \tilde{x} = \begin{bmatrix} 10 \end{bmatrix}$$

The 10 basic solutions are therefore:

$$\tilde{A}\begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_1 = \begin{bmatrix} \frac{10}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & \frac{10}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 0 & 0 & \frac{10}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_4 = \begin{bmatrix} 0 & 0 & 0 & \frac{10}{4} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{10}{5} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{10}{-1} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & x_7 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{-2} & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_8 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{-3} & 0 & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_9 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{-4} & 0 \end{bmatrix}$$

$$\tilde{A}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{10} \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix}, \quad s_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{-5} \end{bmatrix}$$

s_6, s_7, s_8, s_9, s_{10} are not feasible solutions because they contains negative entries. Therefore the remaining s_1, s_2, s_3, s_4, s_5 are feasible solutions with the best solution being s_5 because:

$$\frac{10}{\ } < \frac{10}{\ } < \frac{10}{\ } < \frac{10}{\ } < \frac{10}{\ }$$

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Exercise 4 - Converting the problem (1) to a linear optimization problem in canonical form

$$
\begin{aligned}
\text{minimize} \quad & \tilde{c} \cdot \tilde{x}, \\
\text{subject to} \quad & \\
& \tilde{A}\tilde{x} = b, \\
& \tilde{x} \geq 0.
\end{aligned}
\tag{1}
$$

Converted to canonical form:

$$
\begin{aligned}
\text{maximize} \quad & -\tilde{c} \cdot \tilde{x}, \\
\text{subject to} \quad & \\
& \tilde{A}\tilde{x} \leq b, \\
& -\tilde{A}\tilde{x} \leq -b, \\
& \tilde{x} \geq 0.
\end{aligned}
$$

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Exercise 5 - Writing down the dual problem for the linear optimization problem in canonical form.

Converted to canonical form:

$$\begin{aligned} \text{maximize} \quad & -\tilde{c}\cdot\tilde{x}, \\ \text{subject to} \quad & \\ & \tilde{A}\tilde{x} \le b, \\ & -\tilde{A}\tilde{x} \le -b, \\ & \tilde{x} \ge 0. \end{aligned}$$

Converting it into a dual problem:

$$\begin{aligned} \text{minimize} \quad & b\cdot y_1 - b\cdot y_2, \\ \text{subject to} \quad & \\ & \begin{bmatrix} \tilde{A} \\ -\tilde{A} \end{bmatrix}^T y \ge -\tilde{c} \\ & y \ge 0. \end{aligned}$$

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Exercise 5 - Sketching the feasible set for the dual problem and solve it graphically (continuation).

The feasible set sketched is:

$$\text{minimize} \quad [10] \cdot y_1 - [10] \cdot y_2,$$

$$\text{subject to}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \\ 4 & -4 \\ 5 & -5 \\ -1 & 1 \\ -2 & 2 \\ -3 & 3 \\ -4 & 4 \\ -5 & 5 \end{bmatrix} y \geq \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$y \geq 0.$$

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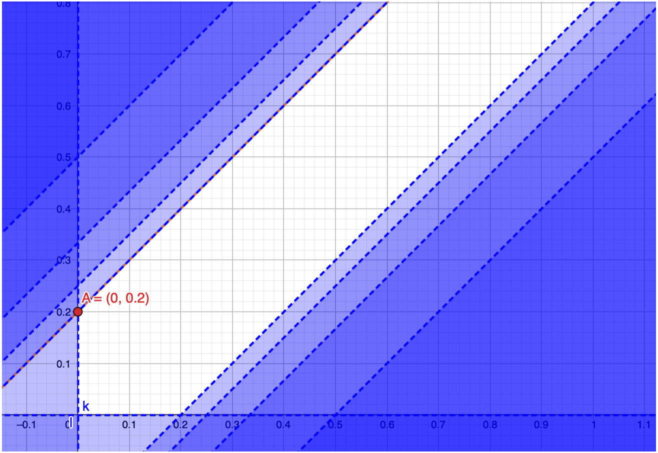
Exercise 5 - Sketching the feasible set for the dual problem and solve it graphically (continuation). Do we obtain the same optimal value as in question 3?



- We obtain the value $10 \cdot 0 - 10 \cdot 0.2 = -2$.
- Yes, we obtain the same value: $-(-2) = 2$.