Sandsynlighedsteori og Lineær algebra

Workshop 3 - Orthogonal matrices and least-squares problems

Sebastian Livoni Larsen

June 6, 2022

Part 1

Exercise 1 - Finding A and its size.

$$Ax = b \iff A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ x_1 + x_3 \\ x_2 + x_4 \\ x_2 \\ x_1 + x_4 \\ x_3 \\ x_1 \\ x_2 + x_3 \\ x_4 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ x_1 + x_3 \\ x_2 + x_4 \\ x_2 \\ x_1 + x_4 \\ x_3 \\ x_1 \\ x_2 + x_3 \\ x_4 \end{bmatrix}$$

$$A = n \times m, x = m \times p \text{ then } b = n \times p$$

- The size of A must be $\mathbf{10} \times \mathbf{4}$ matrix because if $A = 10 \times 4$ and $x = 4 \times 1$ then $b = 10 \times 1$.

Part 1

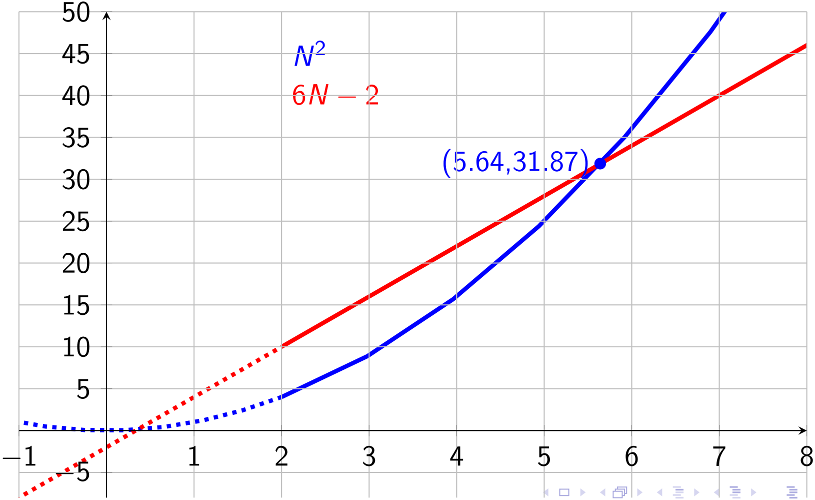
Exercise 3 - What are the dimensions of x, b and A.

$$
\begin{aligned}
A &= 6N - 2 \times N^2 \\
x &= N^2 \times 1 \\
b &= 6N - 2 \times 1
\end{aligned}
$$

Part 1

Exercise 6 - Computing a QR-factorization of the matrix A and use things infromation to compute the orthogonal projection of the vecotr b.

- If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.



Part 1

Exercise 6 - Computing a QR-factorization of the matrix A and use things infromation to compute the orthogonal projection of the vecotr b.

$$Q = \begin{bmatrix} -0.5000 & -0.3873 & 0.2108 & 0.1782 \\ 0 & 0 & -0.5270 & -0.4454 \\ -0.5000 & 0.1291 & -0.4216 & 0.1782 \\ 0 & -0.5164 & 0.1054 & -0.4454 \\ 0 & -0.5164 & 0.1054 & 0.0891 \\ -0.5000 & 0.1291 & 0.1054 & -0.4454 \\ 0 & 0 & -0.5270 & 0.0891 \\ -0.5000 & 0.1291 & 0.1054 & 0.0891 \\ 0 & -0.5164 & -0.4216 & 0.1782 \\ 0 & 0 & 0 & -0.5345 \end{bmatrix}$$

$$R = \begin{bmatrix} -2 & -0.5 & -0.5 & -0.5 \\ 0 & -1.9365 & -0.3873 & -0.3873 \\ 0 & 0 & -1.8974 & -0.3162 \\ 0 & 0 & 0 & -1.8708 \end{bmatrix}$$

$$proj_Q\mathbf{b} = Q(Q^T\mathbf{b}) = \begin{bmatrix} 3.5714 \\ 5.5714 \\ 4.2381 \\ 4.9048 \\ 1.9524 \\ 4.5714 \\ 2.6190 \\ 1.6190 \\ 4.5714 \\ 2.9524 \end{bmatrix}$$

Part 1

Exercise 7 - Finding the solution to the least-squares problem by solving the normal equations and utilizing the QR-factorization.

Method 1 (normal equations):

$$\hat{\mathbf{x}} \iff A^T A\mathbf{x} = A^T\mathbf{b} \iff \begin{bmatrix} 1.6190 \\ 1.9524 \\ 2.6190 \\ 2.9524 \end{bmatrix}$$

Method 2 (Utilizing QR-factorization):

$$\hat{\mathbf{x}} \iff R\mathbf{x} = Q^T\mathbf{b} \begin{bmatrix} 1.6190 \\ 1.9524 \\ 2.6190 \\ 2.9524 \end{bmatrix}$$

Part 2

Exercise 2 - Verifying G is an orthogonal matrix that maps $[a, b]$ to $[d, 0]$.

$$G^T G = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} cc + ss & cs - sc \\ sc - cs & ss + cc \end{bmatrix} = \begin{bmatrix} cc + ss & 0 \\ 0 & ss + cc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a/\sqrt{a^2 + b^2} & b/\sqrt{a^2 + b^2} \\ -(b/\sqrt{a^2 + b^2}) & a/\sqrt{a^2 + b^2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a/\sqrt{a^2 + b^2} * a + b/\sqrt{a^2 + b^2} * b \\ -(b/\sqrt{a^2 + b^2}) * a + a/\sqrt{a^2 + b^2} * b \end{bmatrix}$$
$$= \begin{bmatrix} a^2/\sqrt{a^2 + b^2} + b^2/\sqrt{a^2 + b^2} \\ 0 \end{bmatrix}$$

- We can see that in fact, G does map $[a, b]$ to a vector $[d, 0]$ in $\mathbb{R}^2$.

Part 2

Exercise 3 - Verifying that $G(i,j,a,b)$ is an orthogonal matrix and that is maps to [d,0] in $\mathbb{R}^m$.

$$GG^T = \begin{bmatrix} I & & & & \\ & c & & s & \\ & & I & & \\ & -s & & c & \\ & & & & I \end{bmatrix} \begin{bmatrix} I & & & & \\ & c & & -s & \\ & & I & & \\ & s & & c & \\ & & & & I \end{bmatrix} = \begin{bmatrix} I & & & & \\ & cc+ss & & sc-sc & \\ & & I & & \\ & sc-cs & & ss+cc & \\ & & & & I \end{bmatrix} = \begin{bmatrix} I & & & & \\ & 1 & & & \\ & & I & & \\ & & & 1 & \\ & & & & I \end{bmatrix}$$

$$G(i,j,a,b)x = \begin{bmatrix} I & & & & & & \\ & \ddots & & & & & \\ & & c_{ii} & & s_{ij} & & \\ & & & I & & & \\ & & -s_{ji} & & c_{jj} & & \\ & & & & & \ddots & \\ & & & & & & I \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{i-1} \\ a \\ x_{i+1} \\ \vdots \\ x_{j-1} \\ b \\ x_{j+1} \\ \vdots \\ x_m \end{bmatrix}$$

Part 2

Exercise 4 - Explain why a product of square orthogonal matrices is itself an orthogonal matrix. Explain why $Q_1 Q_2 \cdots Q_k$ is orthogonal.

Let A and B be orthogonal matrices.

$$AA^T = A^T A = I$$

and

$$BB^T = B^T B = I$$

$$(AB)^T(AB) = (B^T A^T)AB = \mathbf{B^T}(\mathbf{A^T A})\mathbf{B} = B^T(I)B = B^T B = I$$