Sandsynlighedsteori og Lineær algebra

Workshop 1 - Linear transformations and system of linear equations

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Polynomial curve fitting

1.1 - The total matrix corresponding to the system of equations

$$\left[\begin{array}{ccccc|c} t_1^0 & t_1^1 & t_1^2 & \cdots & t_1^{n-1} & y_1 \\ t_2^0 & t_2^1 & t_2^2 & \cdots & t_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ t_n^0 & t_n^1 & t_n^2 & \cdots & t_n^{n-1} & y_n \end{array}\right]$$

Polynomial curve fitting

1.3 - Using Gaussian elimination to find the quadratic polynomial passing through the points

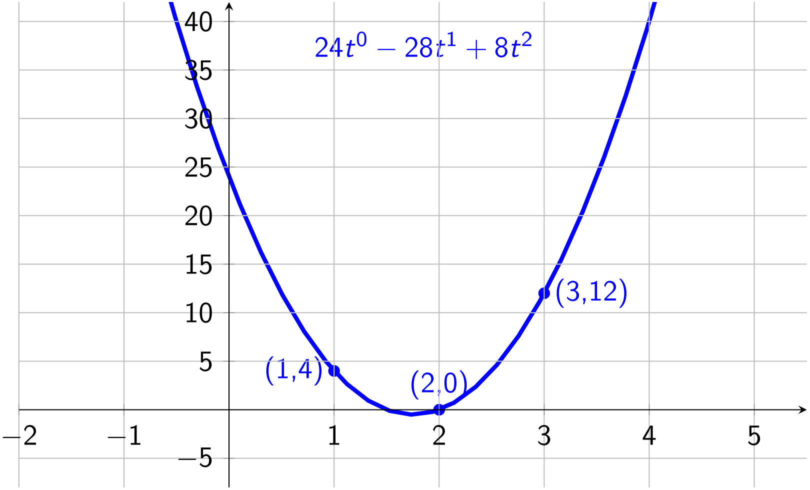
$$\left[\begin{array}{ccc|c} 1^0 & 1^1 & 1^2 & 4 \\ 2^0 & 2^1 & 2^2 & 0 \\ 3^0 & 3^1 & 3^2 & 12 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 0 & 24 \\ 0 & 1 & 0 & -28 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 24 \\ -28 \\ 8 \end{bmatrix}$$

$$p(t) = q_0 + q_1 t + \cdots + q_{n-1} t^{n-1} = 24t^0 - 28t^1 + 8t^2$$

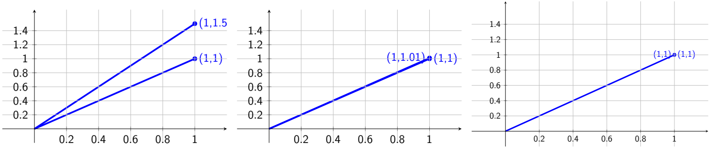
Polynomial curve fitting

1.3 - Using Gaussian elimination to find the quadratic polynomial passing through the points (continuation from last slide)



Linear transformations and digitial image proccesing

3.1 - Discuss in what way the columns of A become "almost linearly dependent" for small *epsilon* ≈ 0



Linear transformations and digitial image proccesing

3.3 - Small changes in δ leads to very large changes in the solution. Comparing pairs of solutions corresponding to $\delta = 0$ and $\delta = 0.01$ for various values of $\epsilon \neq 0$: $\epsilon = 0.1$, $\epsilon = 0.01$, $\epsilon = 0.0001$.

$$\delta = 0, \epsilon = 0.1 : \mathbf{x} = \begin{pmatrix} 2 - \frac{0}{0.1} \\ \frac{0}{0.1} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\delta = 0, \epsilon = 0.01 : \mathbf{x} = \begin{pmatrix} 2 - \frac{0}{0.01} \\ \frac{0}{0.01} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\delta = 0, \epsilon = 0.0001 : \mathbf{x} = \begin{pmatrix} 2 - \frac{0}{0.0001} \\ \frac{0}{0.0001} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\delta = 1, \epsilon = 0.1 : \mathbf{x} = \begin{pmatrix} 2 - \frac{1}{0.1} \\ \frac{1}{0.1} \end{pmatrix} = \begin{pmatrix} -8 \\ 10 \end{pmatrix}$$

$$\delta = 1, \epsilon = 0.01 : \mathbf{x} = \begin{pmatrix} 2 - \frac{0}{0.01} \\ \frac{1}{0.01} \end{pmatrix} = \begin{pmatrix} -98 \\ 100 \end{pmatrix}$$

$$\delta = 1, \epsilon = 0.0001 : \mathbf{x} = \begin{pmatrix} 2 - \frac{1}{0.0001} \\ \frac{1}{0.0001} \end{pmatrix} = \begin{pmatrix} -9998 \\ 10000 \end{pmatrix}$$

Iterative algorithms for solving linear algebraic systems

4.1 - The matrices L and U

$$L = \begin{bmatrix} -12 & 0 & 0 & 0 \\ 6 & 14 & 0 & 0 \\ -5 & -8 & 24 & 0 \\ 1 & -4 & 10 & 16 \end{bmatrix}, U = \begin{bmatrix} 0 & 4 & 0 & -6 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Iterative algorithms for solving linear algebraic systems

4.2 - Compuing the next iterate $x^{(1)}$ in the Gauss-Seidel algorithm

$$x^{(0)} = [1, 2, 3, 4]^T$$

$$x^{(k+1)} = Lx^{(k+1)} = b - Ux^{(k)}$$

$$x^{(1)} = Lx^{(1)} = b - Ux^{(0)}$$

$$x^{(1)} = \begin{bmatrix} -12 & 0 & 0 & 0 \\ 6 & 14 & 0 & 0 \\ -5 & -8 & 24 & 0 \\ 1 & -4 & 10 & 16 \end{bmatrix} x^{(1)} = \begin{bmatrix} -8 \\ 47 \\ -93 \\ -13 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 0 & -6 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$x_1^{(1)} : -12x_1 = -8 - (4(2) - 6(4)) \iff -12x_1 = 8 \iff x_1 = \frac{8}{-12} \iff x_1 = -0.\overline{66}$$

$$x_2^{(2)} : 6x_1 + 14x_2 = 50 \iff x_2 = \frac{-6(-0.\overline{66}) + 50}{14} \iff x_2 = 3.85$$

$$x_3^{(3)} : -5x_1 - 8x_2 + 24x_3 = -125 \iff x_3 = \frac{5(-0.\overline{66}) + 8(3.85) - 125}{24} = x_3 = -4.06$$

$$x_4^{(4)} : 1x_1 - 4x_2 + 10x_3 + 16x_4 = -13 \iff x_4 = \frac{-1(0.\overline{66}) + 4(3.85) - 10(-4.06) - 13}{16} = x_4 = 2.64$$

Solution is therefore:

$$x_1^{(1)} = -0.\overline{66},\ x_2^{(1)} = 3.85,\ x_3^{(1)} = -4.06,\ x_4^{(1)} = 2.64 \iff x^{(1)} = \begin{pmatrix} -0,\overline{66} \\ 3.85 \\ -4.06 \\ 2.64 \end{pmatrix}$$

Iterative algorithms for solving linear algebraic systems

4.3 - Showing the next iterate $x^{(1)}$ in the Gauss-Seidel algorithm equals $x^{(0)} = [1, 4, -3, 2]^T$

$$Lx^{(1)} = b - Ux^{(0)} \text{ We want to show that the next iterate } x^{(1)} \text{ equals } x^{(0)}$$

$$Ux^{(0)} = \begin{bmatrix} 0 & 4(4) & 0 & -6(2) \\ 0 & 0 & 3(-3) & -3(2) \\ 0 & 0 & 0 & 8(2) \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -15 \\ 16 \\ 0 \end{bmatrix}$$

$$Lx^{(1)} = b - Ux^{(0)} \iff \begin{bmatrix} -8 \\ 47 \\ -93 \\ -13 \end{bmatrix} - \begin{bmatrix} 4 \\ -15 \\ 16 \\ 0 \end{bmatrix} = \begin{bmatrix} -12 \\ 62 \\ -109 \\ -13 \end{bmatrix}$$

$$L = \begin{bmatrix} -12 & 0 & 0 & 0 \\ 6 & 14 & 0 & 0 \\ -5 & -8 & 24 & 0 \\ 1 & -4 & 10 & 16 \end{bmatrix} x^{(1)} = \begin{bmatrix} -12 \\ 62 \\ -109 \\ -13 \end{bmatrix} \iff \begin{bmatrix} -12 & 0 & 0 & 0 & -12 \\ 6 & 14 & 0 & 0 & 62 \\ -5 & -8 & 24 & 0 & -109 \\ 1 & -4 & 10 & 16 & -13 \end{bmatrix}$$

$$x_1^{(1)} = \frac{-12}{-12} = 1$$

$$x_2^{(1)} = 62 - 6(x_1) = 62 - 6(1) = \frac{56}{14} = 4$$

$$x_3^{(1)} = \frac{5(x_1) + 8(x_2) - 109}{24} = \frac{5(1) + 8(4) - 109}{24} = \frac{-72}{24} = -3$$

$$x_4^{(1)} = \frac{-1(x_1) + 4(x_2) - 10(x_3) - 13}{16} = \frac{-1 + 4(4) - 10(-3) - 13}{16} = \frac{32}{16} = 2$$

$$x^{(0)} = [1, 4, -3, 2]^T$$

$$x^{(1)} = [1, 4, -3, 2]^T$$

$$x^{(0)} = x^{(1)} \text{ Hence it is a solution}$$

Iterative algorithms for solving linear algebraic systems

4.4 Showing that $x^{(k)}$ solves the system $Ax = b$ by assuming that at some point $x^{(k+1)} = x^{(k)}$

$$
\begin{aligned}
Lx^{(k+1)} &= b - Ux^{(k)} && \text{Line 3 in algorithm} \\
Lx^{(k)} &= b - Ux^{(k)} && \text{Assuming at some iteration } k \text{ that } x^{(k+1)} = x^{(k)} \\
Lx^{(k)} + Ux^{(k)} &= b && \\
(L + U)x^{(k)} &= b && \text{Distributive law} \\
Ax^{(k)} &= b && \text{Given } A = L + U
\end{aligned}
$$