Nonlinear Systems and Control

Quadcopter Project - Task 1: Linear Control

Sebastiano Oliani (22-952-683)

Spring Semester 2023

1 Definitions

A set of generalized coordinates that fully describes the state of the quadcopter is given by:

$$X = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & \phi & \theta & \psi & p & q & r \end{bmatrix}^T$$

The control input can be written as:

$$U = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T$$

2 Task 1a

Write the equations of motion of the system in the form:

$$\dot{X} = f(X, t) \tag{1}$$

and show that the points:

$$X_{SS} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi & 0 & 0 & 0 \end{bmatrix}^T$$

$\forall \psi \in [0; 2\pi]$ are equilibrium points of the non-linear system for the input:

$$U_{SS} = \begin{bmatrix} mg & 0 & 0 & 0 \end{bmatrix}^T$$

Solution: The state vector can be rewritten as:

$$X = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & \phi & \theta & \psi & p & q & r \end{bmatrix}^T$$
$$= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{bmatrix}^T$$

then, the state dynamics is:

$$\dot{X} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \frac{1}{m}\{[\cos(x_7)\sin(x_8)\cos(x_9) + \sin(x_7)\sin(x_9)]u_1 - k_x x_4\} \\ \frac{1}{m}\{[\cos(x_7)\cos(x_8)\sin(x_9) + \sin(x_7)\cos(x_9)]u_1 - k_y x_5\} \\ \frac{1}{m}\{[\cos(x_7)\cos(x_8)]u_1 - mg - k_z x_5\} \\ x_{10} + x_{11}\sin(x_7)\tan(x_8) + x_{12}\cos(x_7)\tan(x_8) \\ x_{11}\cos(x_7) + x_{12}\sin(x_7) \\ x_{11}\frac{\sin(x_7)}{\cos(x_8)} + x_{12}\frac{\cos(x_7)}{\cos(x_8)} \\ \frac{1}{I_x}[(I_y - I_z)x_{11}x_{12} + u_2 - k_p x_{10}] \\ \frac{1}{I_y}[(I_z - I_x)x_{10}x_{12} + u_3 - k_q x_{11}] \\ \frac{1}{I_z}[(I_x - I_y)x_{10}x_{111} + u_4 - k_r x_{12}] \end{bmatrix} \tag{2}$$

The equilibrium points can be found by setting (2) equal to 0 and substituting X_{SS} into it:

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m}\{0u_1 - 0\} \\ \frac{1}{m}\{0u_1 - 0\} \\ \frac{1}{m}\{u_1 - mg - 0\} \\ 0 \\ 0 \\ 0 \\ \frac{1}{I_x}[0 + u_2 - 0] \\ \frac{1}{I_y}[0 + u_3 - 0] \\ \frac{1}{I_z}[0 + u_4 - 0] \end{bmatrix}$$

Therefore, it is easy to obtain $u_2 = u_3 = u_4 = 0$ and $u_1 = mg$, which correspond to the values of U_{SS}. To sum up, the points:

$$X_{SS} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi & 0 & 0 & 0 \end{bmatrix}^T$$

$\forall \psi \in [0; 2\pi]$ are equilibrium points of the non-linear system.

3 Task 1b

Find the linearized system matrices A and B around $X_{SS}(\psi)$ and U_{SS}, as a function of ψ.

Solution: Matrix $A \in \mathbb{R}^{12\times 12}$ can be computed by taking the derivative of the state dynamics with respect to the state vector X and substituting the values of $X_{SS}(\psi)$ and U_{SS}:

$$A = \left.\frac{df}{dX}\right|_{X=X_{SS}(\psi), U=U_{SS}} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{k_x}{m} & 0 & 0 & \sin(\psi)g & \cos(\psi)g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{k_y}{m} & 0 & -\cos(\psi)g & \sin(\psi)g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{k_z}{m} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k_p}{I_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k_q}{I_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k_r}{I_z} \end{bmatrix} \tag{3}$$

Matrix $B \in \mathbb{R}^{12\times 4}$ can be computed by taking the derivative of the state dynamics with respect to the input vector U and substituting the values of $X_{SS}(\psi)$ and U_{SS}:

$$B = \left.\frac{df}{dU}\right|_{X=X_{SS}(\psi),U=U_{SS}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{I_x} & 0 & 0 \\ 0 & 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & 0 & \frac{1}{I_z} \end{bmatrix} \tag{4}$$

Finally, we can write the linearized state space equations of the system in the form:

$$\Delta\dot{X} = A\Delta X + B\Delta U \tag{5}$$

where $\Delta X = X - X_{SS}$ and $\Delta U = U - U_{SS}$.

4 Task 1e

Design an LQR controller for the operating point $X_{SS}(\psi)$ and U_{SS} with $\psi = 0$, taking the cost matrices Q and R to be identity matrices of appropriate size. Simulate the time evolution of the system from different initial conditions and reference values. In particular show the results for:

$$X_{init} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi_0 & 0 & 0 & 0 \end{bmatrix}^T$$

with $\psi_0 \in \{\frac{2\pi}{16}, \frac{2\pi}{8}, \frac{2\pi}{4}, \frac{2\pi}{2}\}$, regulating the system to the origin. Plot the position x, y, z and orientation ψ, θ, ϕ in two separate figures for each value of ψ_0. Briefly explain the observed behavior.

Solution: By running the simulations on Matlab, we can see that the system converges to the value:

$$X_{ref} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

irrespective of the value of ψ_0 from the vector X_{init}. It is important to notice that the more the value of ψ_0 increases, the more the oscillations in the response. This is due to the fact that the simulation is starting far away from the operating point $X_{SS}(0)$ and U_{SS}.

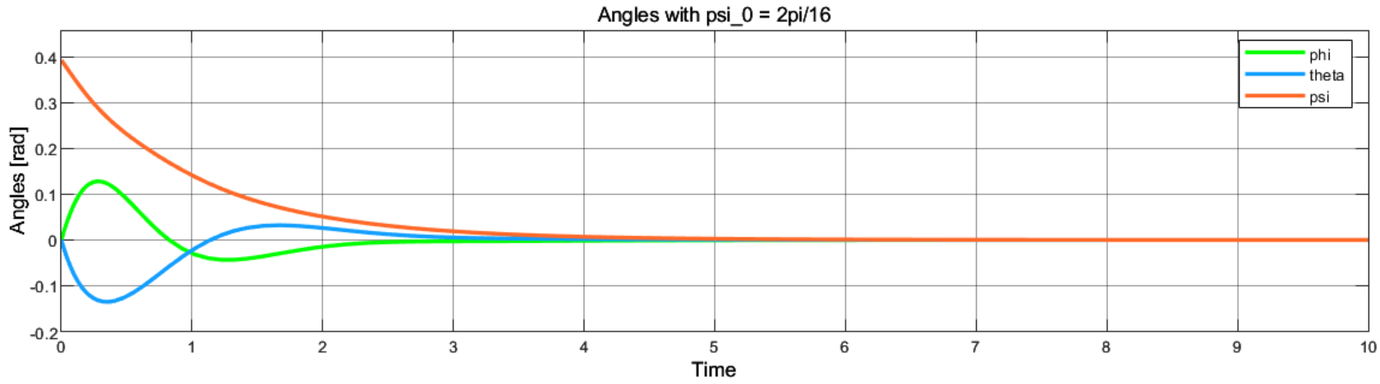

Figure 1: Angles with $\psi_0 = \frac{2\pi}{16}$

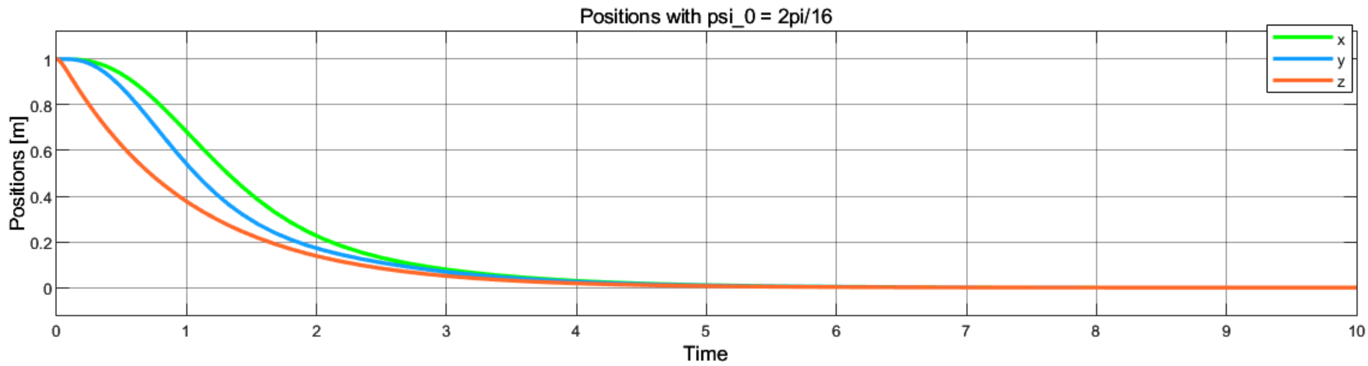

Figure 2: Positions with $\psi_0 = \frac{2\pi}{16}$

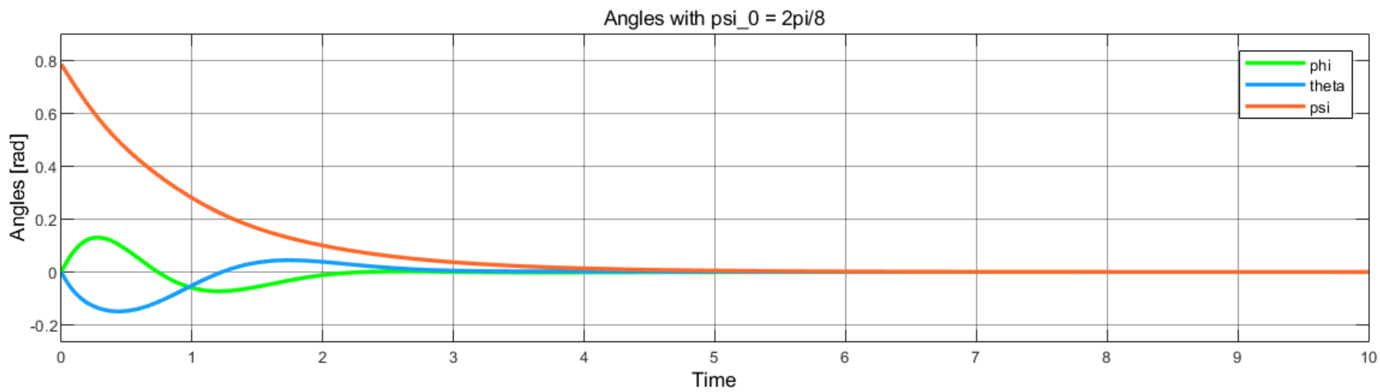

Figure 3: Angles with $\psi_0 = \frac{2\pi}{8}$

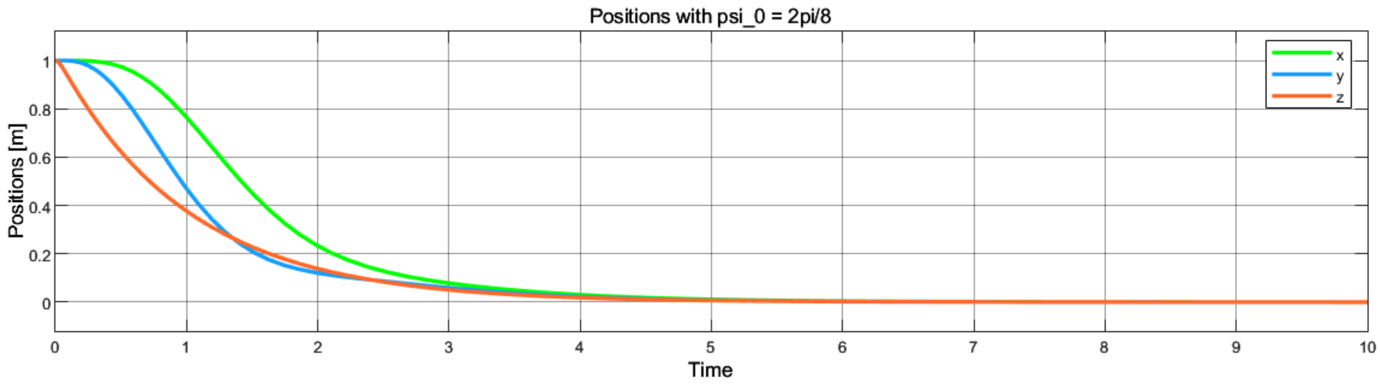

Figure 4: Positions with $\psi_0 = \frac{2\pi}{8}$

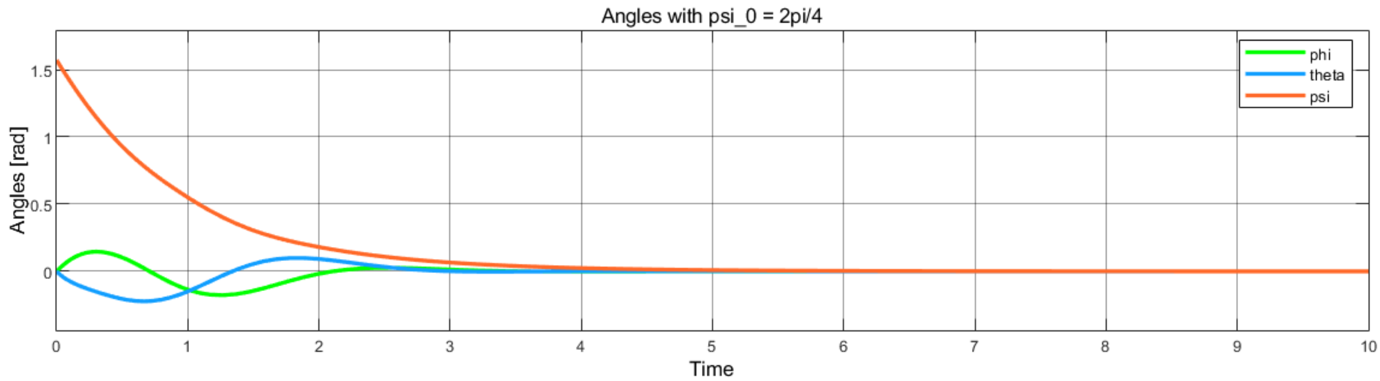

Figure 5: Angles with $\psi_0 = \frac{2\pi}{4}$

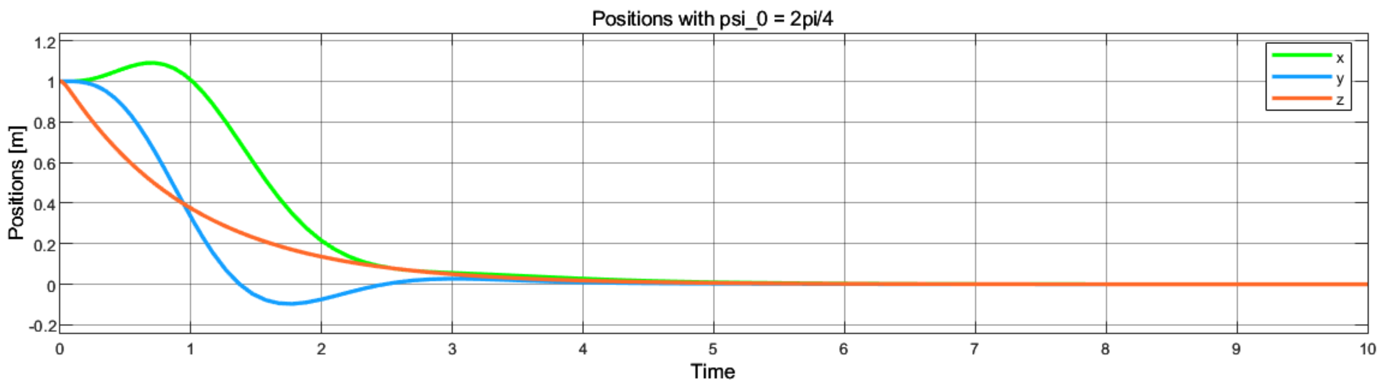

Figure 6: Positions with $\psi_0 = \frac{2\pi}{4}$

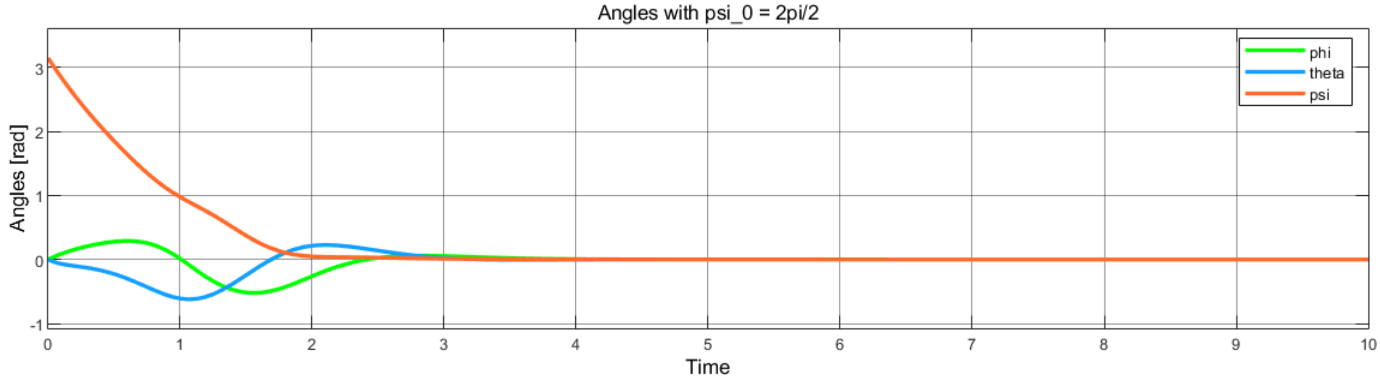

Figure 7: Angles with $\psi_0 = \frac{2\pi}{2}$

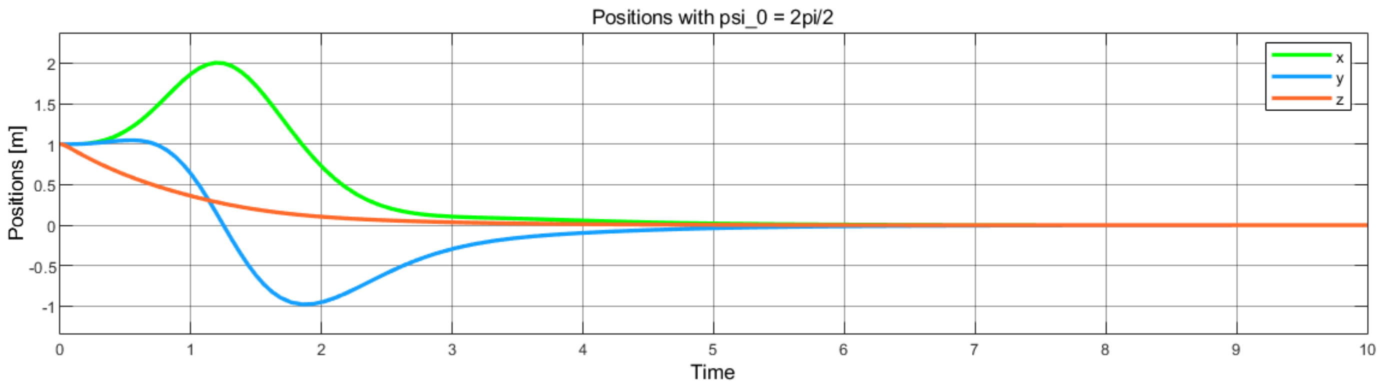

Figure 8: Positions with $\psi_0 = \frac{2\pi}{2}$

5 Task 1f

What happens to the system response if you set the reference x, y, z coordinates to a non-zero value? Explain the observed behavior. What happens if the initial condition is X_{init} defined above with $\psi_0 = \frac{\pi}{4}$ and the reference is

$$X_{ref} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\pi}{2} & 0 & 0 & 0 \end{bmatrix}^T ?$$

What goes wrong? How can one fix it?

Solution: When simulating the system response with the position references x, y, z having values different from zero and very large, the system becomes unstable. This behaviour comes from the fact that the system is driven far away from the linearization point X_{SS} and therefore the controller fails in stabilizing the system.

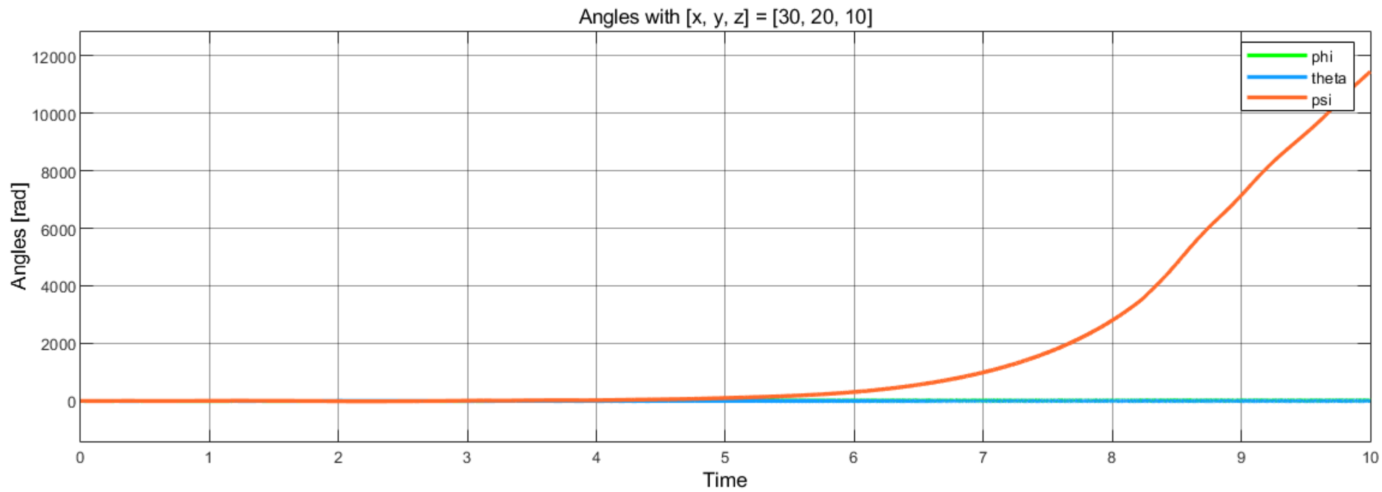

Figure 9: Angles with $\psi_0 = \frac{2\pi}{16}$ and $[x, y, z] = [30, 20, 10]$

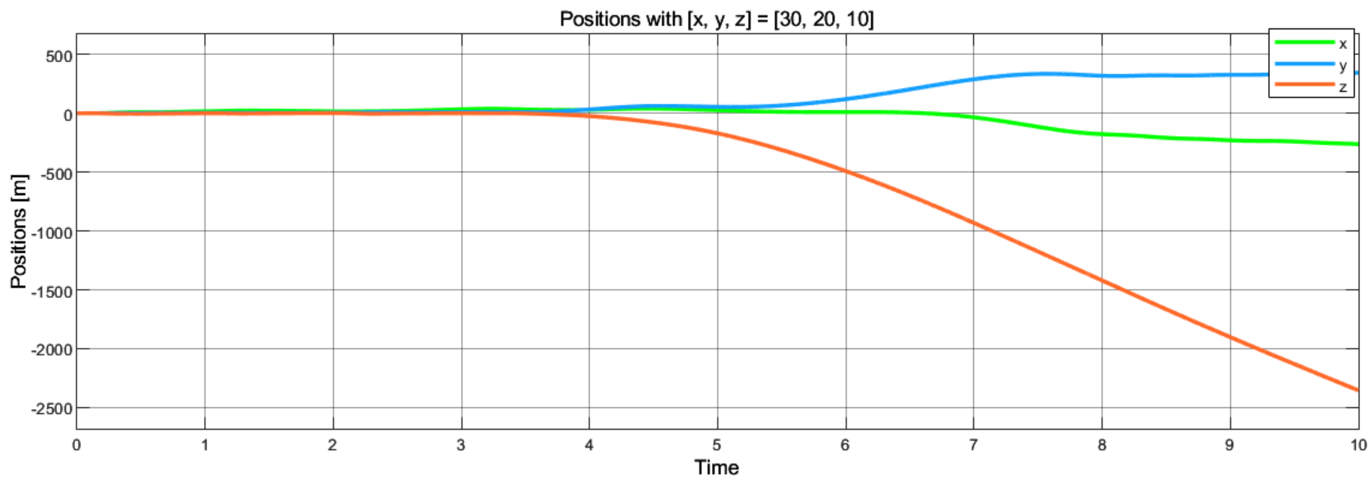

Figure 10: Positions with $\psi_0 = \frac{2\pi}{16}$ and $[x, y, z] = [30, 20, 10]$

When the initial condition is X_{init} defined above with $\psi_0 = \frac{\pi}{4}$ and the reference is set to:

$$X_{ref} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\pi}{2} & 0 & 0 & 0 \end{bmatrix}^T$$

the system's position and orientation coordinates show an unstable behaviour. This is due to the fact that the system is linearized around $\psi = 0$, but here $\psi = \frac{\pi}{2}$. This means that the linearized model with matrices A and B computed in Task 1e is not valid anymore.

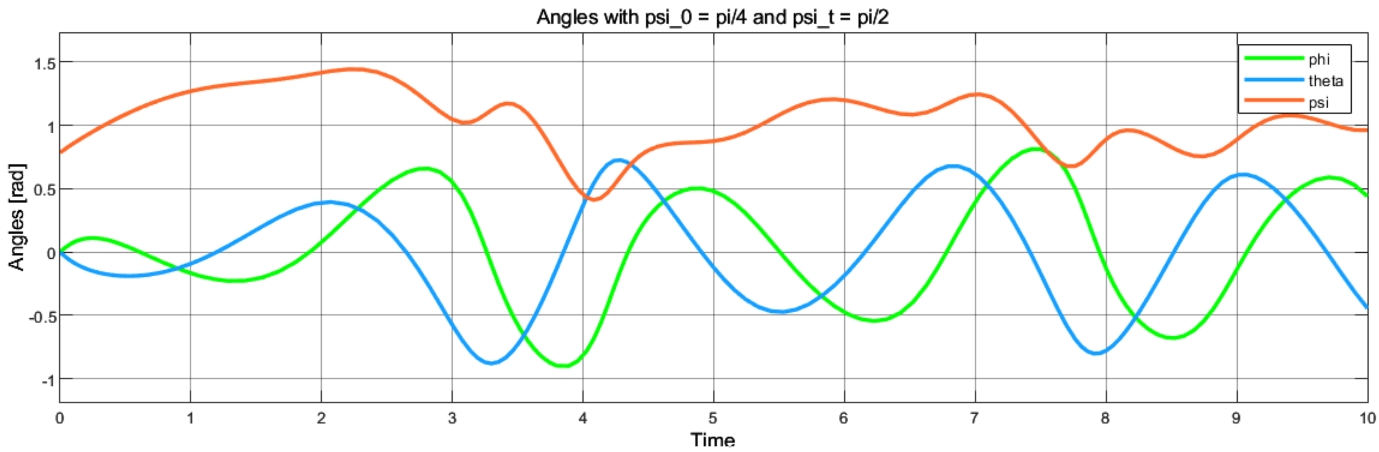

Figure 11: Angles with $\psi_0 = \frac{\pi}{4}$ and $\psi_t = \frac{\pi}{2}$, linearization around $\psi = 0$

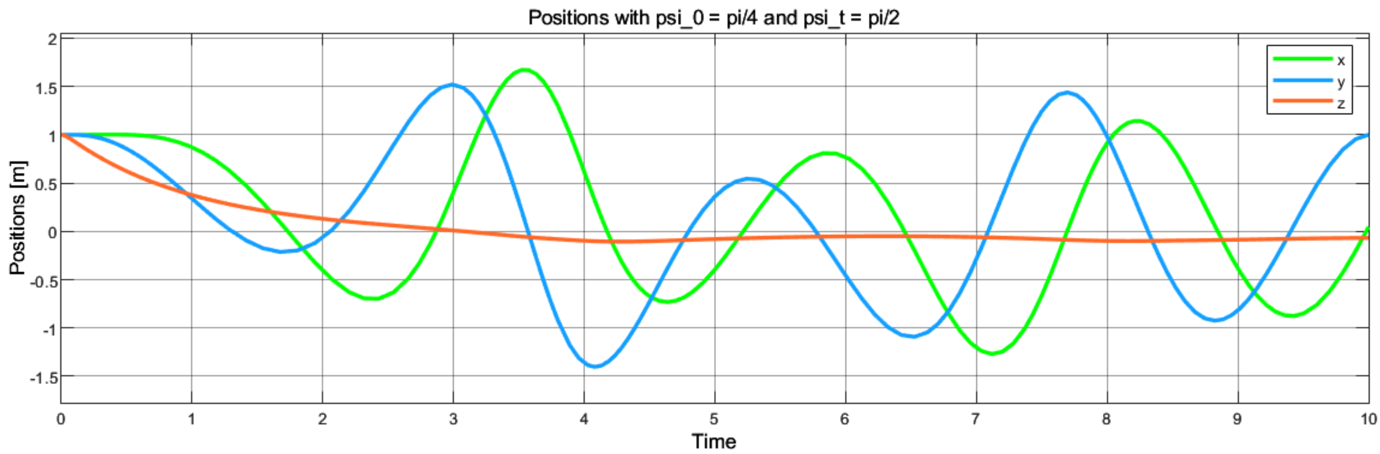

Figure 12: Positions with $\psi_0 = \frac{\pi}{4}$ and $\psi_t = \frac{\pi}{2}$, linearization around $\psi = 0$

In order to fix this behaviour, one can linearize the system around the new operating point, i.e.:

$$X_{ref} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\pi}{2} & 0 & 0 & 0 \end{bmatrix}^T$$

obtaining a new matrix A for the linearized system, a new LQR controller, and the following responses:

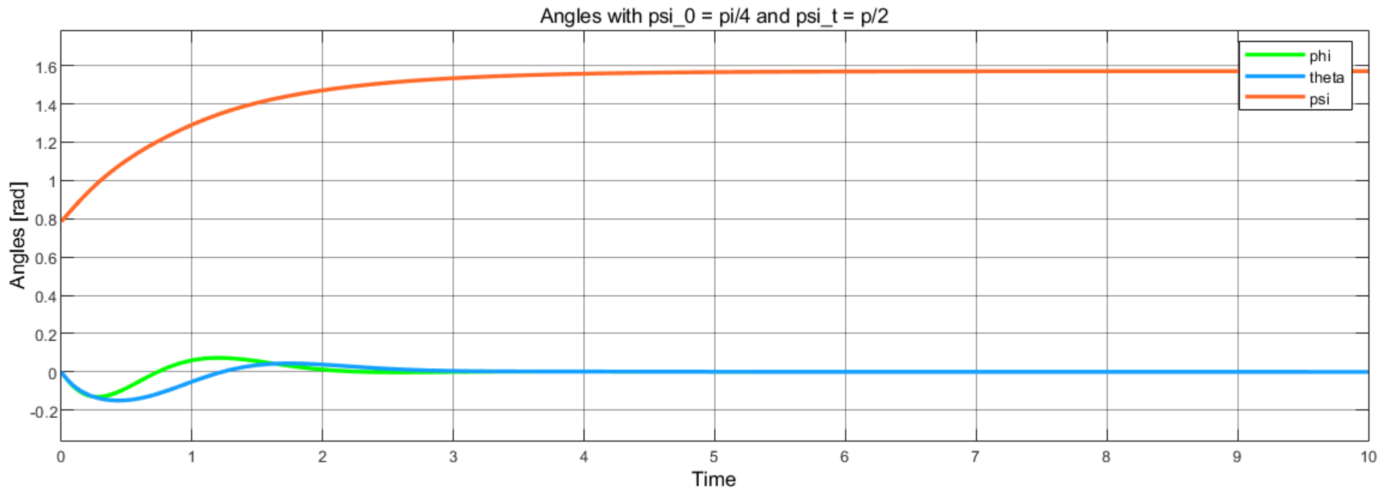

Figure 13: Angles with $\psi_0 = \frac{\pi}{4}$ and $\psi_t = \frac{\pi}{2}$, linearization around $\psi = \frac{\pi}{2}$

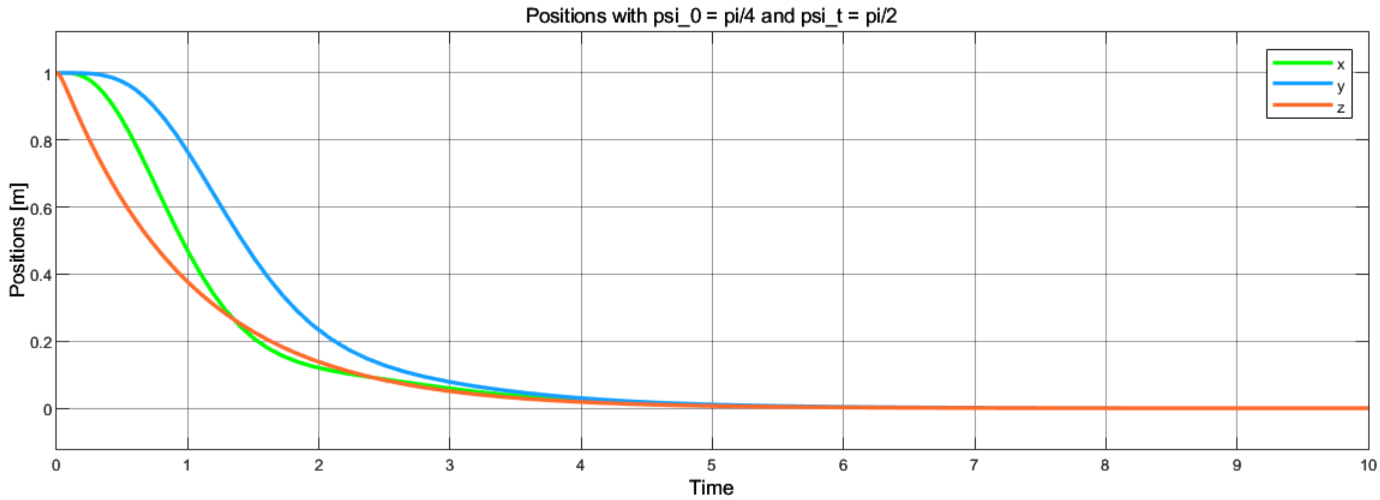

Figure 14: Positions with $\psi_0 = \frac{\pi}{4}$ and $\psi_t = \frac{\pi}{2}$, linearization around $\psi = \frac{\pi}{2}$