Nonlinear Systems and Control

Quadcopter Project - Task 2: Feedback Linearization

Sebastiano Oliani (22-952-683)

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1 Definitions

A set of generalized coordinates that fully describes the state of the quadcopter is given by:

$$X = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & \phi & \theta & \psi & p & q & r \end{bmatrix}^T$$

The control input can be written as:

$$U = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T$$

2 Task 2a

Using ψ as the output and u_4 as the input, compute the relative degree of the system. Is the system input-output linearizable from the input u_4 to the output $y = \psi$?

Solution: The state vector can be rewritten as:

$$X = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & \phi & \theta & \psi & p & q & r \end{bmatrix}^T$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{bmatrix}^T$$

then, the state dynamics is:

$$\dot{X} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \frac{1}{m}\{[\cos(x_7)\sin(x_8)\cos(x_9) + \sin(x_7)\sin(x_9)]u_1 - k_x x_4\} \\ \frac{1}{m}\{[\cos(x_7)\cos(x_8)\sin(x_9) + \sin(x_7)\cos(x_9)]u_1 - k_y x_5\} \\ \frac{1}{m}\{[\cos(x_7)\cos(x_8)]u_1 - mg - k_z x_5\} \\ x_{10} + x_{11}\sin(x_7)\tan(x_8) + x_{12}\cos(x_7)\tan(x_8) \\ x_{11}\cos(x_7) + x_{12}\sin(x_7) \\ x_{11}\frac{\sin(x_7)}{\cos(x_8)} + x_{12}\frac{\cos(x_7)}{\cos(x_8)} \\ \frac{1}{I_x}[(I_y - I_z)x_{11}x_{12} + u_2 - k_p x_{10}] \\ \frac{1}{I_y}[(I_z - I_x)x_{10}x_{12} + u_3 - k_q x_{11}] \\ \frac{1}{I_z}[(I_x - I_y)x_{10}x_{111} + u_4 - k_r x_{12}] \end{bmatrix} \tag{1}$$

Considering the output map:

$$y = h(X) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} X = \psi \tag{2}$$

Recall that the system (1)-(2) is said to have a relative degree ρ if:

$$\begin{cases} L_g L_f^i h(X) = 0, & i = 0, ..., \rho - 2 \\ L_g L_f^{\rho-1} h(X) \neq 0 \end{cases} \tag{3}$$

where $L_f h(X) = \frac{\partial h}{\partial X} f(X)$ is called Lie Derivative. The system is then called input-output linearizable, and the state feedback control:

$$u = \frac{1}{L_g L_f^{\rho-1} h(X)} \left[-L_f^{\rho} h(X) + v \right]$$

reduces the input-output map to:

$$y^{(\rho)} = v$$

By taking the first order derivative of the output (2):

$$\dot{y} = \frac{\partial h(X)}{\partial X} \dot{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \dot{X}$$

$$\rightarrow \dot{y} = x_{11} \frac{\sin(x_7)}{\cos(x_8)} + x_{12} \frac{\cos(x_7)}{\cos(x_8)}$$

By taking the second order derivative:

$$\ddot{y} = \frac{\partial L_f h(X)}{\partial X} \dot{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_{11} \frac{\cos(x_7)}{\cos(x_8)} - x_{12} \frac{\sin(x_7)}{\cos(x_8)} \\ x_{11} \frac{\sin(x_7)\sin(x_8)}{(\cos(x_8))^2} + x_{12} \frac{\cos(x_7)\sin(x_8)}{(\cos(x_8))^2} \\ 0 \\ 0 \\ \frac{\sin(x_7)}{\cos(x_8)} \\ \frac{\cos(x_7)}{\cos(x_8)} \end{bmatrix}^T \dot{X}$$

$$= \left[x_{11} \frac{\cos(x_7)}{\cos(x_8)} - x_{12} \frac{\sin(x_7)}{\cos(x_8)} \right] \dot{x}_7 + \left[x_{11} \frac{\sin(x_7)\sin(x_8)}{(\cos(x_8))^2} + x_{12} \frac{\cos(x_7)\sin(x_8)}{(\cos(x_8))^2} \right] \dot{x}_8 + \frac{\sin(x_7)}{\cos(x_8)} \dot{x}_{11} + \frac{\cos(x_7)}{\cos(x_8)} \dot{x}_{12}$$

$$\rightarrow \ddot{y} = \left[x_{11} \frac{\cos(x_7)}{\cos(x_8)} - x_{12} \frac{\sin(x_7)}{\cos(x_8)} \right] [x_{10} + x_{11} \sin(x_7) \tan(x_8) + x_{12} \cos(x_7) \tan(x_8)] +$$
$$+ \left[x_{11} \frac{\sin(x_7)\sin(x_8)}{(\cos(x_8))^2} + x_{12} \frac{\cos(x_7)\sin(x_8)}{(\cos(x_8))^2} \right] [x_{11} \cos(x_7) + x_{12} \sin(x_7)] +$$
$$+ \frac{\sin(x_7)}{\cos(x_8)} \frac{1}{I_y} [(I_z - I_x) x_{10} x_{12} + u_3 - k_q x_{11}] + \frac{\cos(x_7)}{\cos(x_8)} \frac{1}{I_z} [(I_x - I_y) x_{10} x_{111} + u_4 - k_r x_{12}]$$

From the second order derivative of the output, we obtain that:

$$L_f^2 h(X) = \left[x_{11} \frac{\cos(x_7)}{\cos(x_8)} - x_{12} \frac{\sin(x_7)}{\cos(x_8)} \right] [x_{10} + x_{11} \sin(x_7) \tan(x_8) + x_{12} \cos(x_7) \tan(x_8)] +$$

$$+\left[x_{11}\frac{\sin(x_7)\sin(x_8)}{(\cos(x_8))^2}+x_{12}\frac{\cos(x_7)\sin(x_8)}{(\cos(x_8))^2}\right][x_{11}\cos(x_7)+x_{12}\sin(x_7)]+$$

$$+\frac{\sin(x_7)}{\cos(x_8)}\frac{1}{I_y}[(I_z-I_x)x_{10}x_{12}+u_3-k_q x_{11}]+\frac{\cos(x_7)}{\cos(x_8)}\frac{1}{I_z}[(I_x-I_y)x_{10}x_{111}+-k_r x_{12}]$$

$$L_g L_f h(X)=\frac{\cos(x_7)}{I_z\cos(x_8)}$$

From (3), it follows that:

$$L_g L_f h(X)=\frac{\cos(x_7)}{I_z\cos(x_8)}\neq 0$$

if $x_7 \neq \frac{\pi}{2}+k\pi$, $\forall k\in R$. Then, the relative degree ρ of the system is equal to 2, and the control input is:

$$u_4=\frac{1}{\frac{\cos(x_7)}{I_z\cos(x_8)}}\left[-L_f^2 h(X)+v\right]$$

Finally, the system is input-output linearizable from the input u_4 to the output $y=\psi$.

3 Task 2b

Given a reference signal $\psi_{ref}(t)$, design an asymptotically stable tracking controller for yaw control using input-output linearization from the input u_4 to output $y=\psi$. For the design, you can assume that the derivatives $\dot{\psi}_{ref}(t)$, $\ddot{\psi}_{ref}(t)$ are available to you online, and are bounded. Explain the design procedure.

Solution: In order to obtain an asymptotically stable tracking controller, it is possible to define the error signal as:

$$e=\psi-\psi_{ref}$$

Then:

$$\dot{e}=\dot{\psi}-\dot{\psi}_{ref}$$
$$\ddot{e}=\ddot{\psi}-\ddot{\psi}_{ref}$$

Therefore, the signal v can be chosen as:

$$v=-K_1 e-K_2\dot{e}$$

$$\rightarrow u_4=\frac{1}{\frac{\cos(x_7)}{I_z\cos(x_8)}}\left[-L_f^2 h(X)-K_1 e-K_2\dot{e}\right]$$

In this way, it is possible to cancel the non-linearities and the closed loop dynamics becomes:

$$\ddot{e}=-K_1 e-K_2\dot{e}$$

$$\ddot{\psi}-\ddot{\psi}_{ref}=-K_1(\psi-\psi_{ref})-K_2(\dot{\psi}-\dot{\psi}_{ref})$$

which is asymptotically stable for any $K_1>0$ and $K_2>0$. Hence, the tracking error tends to zero when time tends to infinity.

4 Task 2d

Tune your controller such that the maximum overshoot is less than 5% and settling time (to within 4% of a step change) is less than 5 seconds. Simulate the time evolution of the system for the given initial condition and reference set-point, and plot the orientation ψ, θ, ϕ as a function of time. In this plot, indicate the overshoot and settling time achieved by your controller, highlighting the percentage values using annotations.

Solution: By choosing:

$$K_1 = 4$$

$$K_2 = 3$$

the controller satisfies both the performances. In particular, the maximum overshoot is around 3% and the settling time with 4% of a step reference is around 1.5 seconds.

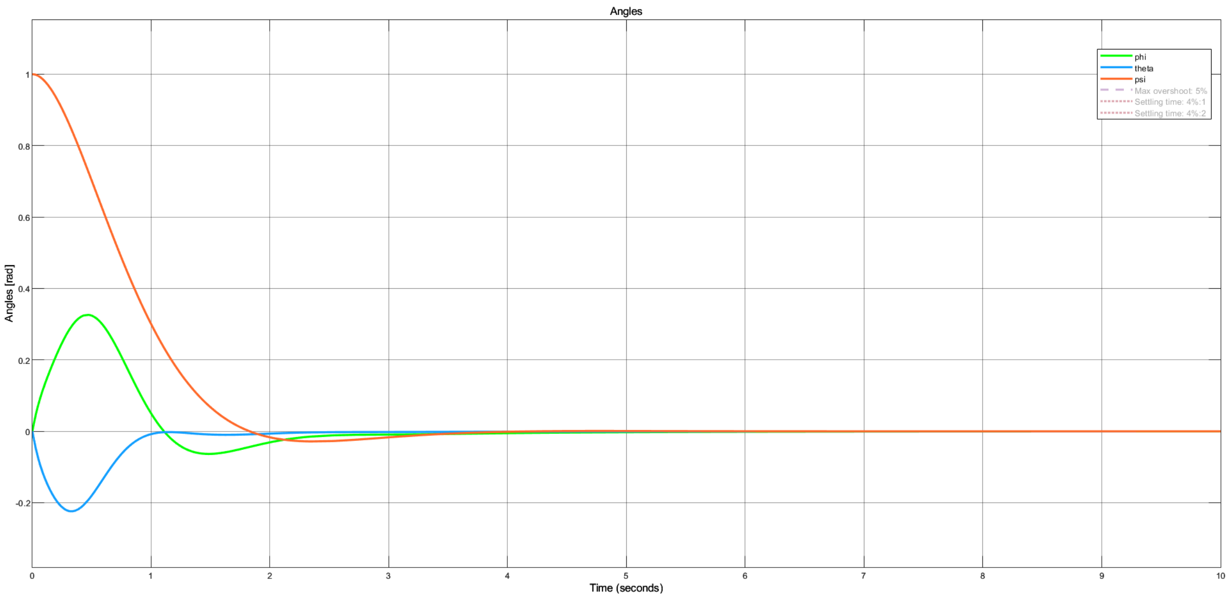

Figure 1: Orientation response

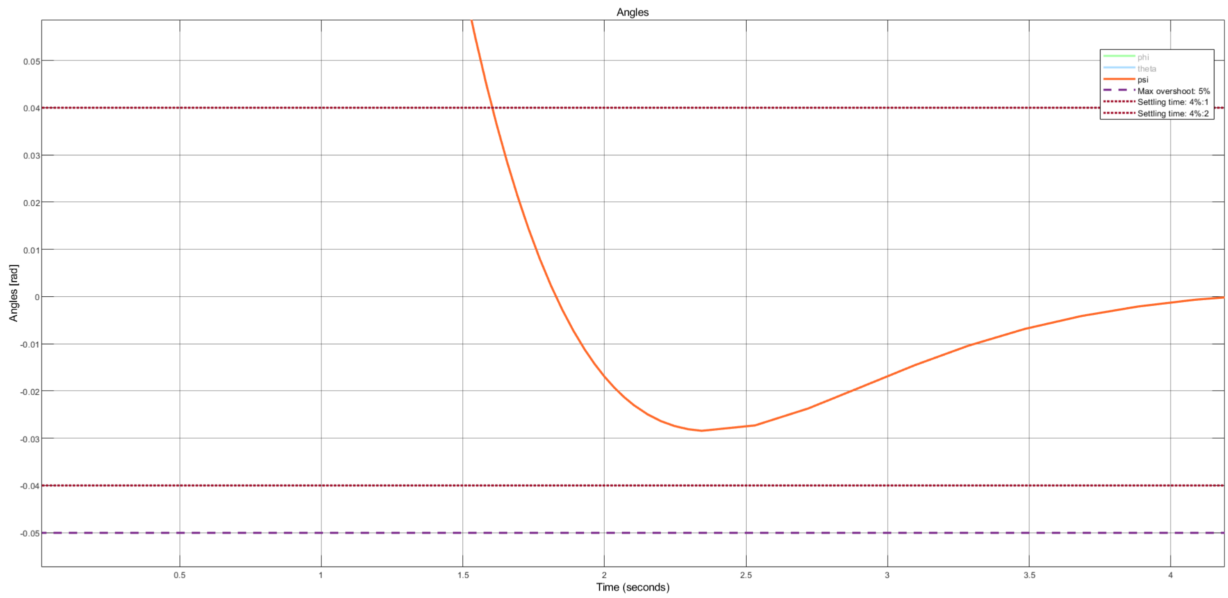

Figure 2: Detailed view - ψ response with performances