

Elementary Logic (PrepCamp)

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- 1 Statements
- 2 Logical operations
- 3 Implication
- 4 Quantifiers
- 5 Proofs

Slides and exercises:

<https://github.com/sebastianotronto/preplogic>

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Statements

- Unambiguous

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Example (A bad joke)

Q: How many months have 30 days?

- Unambiguous

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A: 11, some of them have even more!

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Statements

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- Objective

Example

Good: 3 is greater than 4

Bad: 3 is nicer than 4

- Mathematical: “*Three is greater than four*” (or “ $3 > 4$ ”)
- ...or not: “*I am 26 years old*”
- **Key point:** statements can be **true** or **false**

Logical operations

- We can combine statements to make new ones
- Negation (**not**), conjunction (**and**), disjunction (**or**)

Negation (**not**)

If A is a statement, the statement “not A ” (in symbols: $\neg A$) is **true** when A is **false**, and it is **false** when A is **true**.

Negation (**not**)

If A is a statement, the statement “not A ” (in symbols: $\neg A$) is **true** when A is **false**, and it is **false** when A is **true**.

Example

$\neg(3 > 4)$ is equivalent to $3 \leq 4$

“3 is **not** greater than 4” is equivalent to “3 is less or equal than 4”

Conjunction (**and**)

The statement “ A and B ” (in symbols: $A \wedge B$) is **true** when both A and B are **true**, and it is **false** if at *at least* one of them is **false**.

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Example

“ $(3 < 4) \wedge (5 \text{ is an odd number})$ ” is **true**

Example

“(Today is Monday) \wedge (we are in France)” is **false**

Disjunction (**or**)

The statement “ A or B ” (in symbols: $A \vee B$) is **true** when at least one of A and B is **true**, and it is **false** if both of them are **false**.

Disjunction (**or**)

The statement “ A or B ” (in symbols: $A \vee B$) is **true** when at least one of A and B is **true**, and it is **false** if both of them are **false**.

Example

“(3 = 4) \vee (5 is an even number)” is **false**

Example

“(Today is Monday) \vee (we are in Luxembourg)” is **true**

- **Important:** \vee is always *inclusive*:

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Example (Another bad joke)

Waiter: "Would you like cheese or dessert?"

Mathematician: "Yes."

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- \neg has precedence over \wedge and \vee :

$$\neg A \wedge B \text{ means } (\neg A) \wedge B, \quad \neg A \vee B \text{ means } (\neg A) \vee B$$

(or just use parenthesis)

If A , B and C are statements:

| | | |
|--|---|-------------------------|
| $A \wedge B = B \wedge A$ | $A \vee B = B \vee A$ | commutativity |
| $A \wedge (B \wedge C) = (A \wedge B) \wedge C$ | $A \vee (B \vee C) = (A \vee B) \vee C$ | associativity |
| $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ | | distributivity |
| $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ | | distributivity* |
| $\neg(\neg A) = A$ | | double negation |
| $A \wedge \text{true} = A$ | $A \wedge \text{false} = \text{false}$ | |
| $A \vee \text{true} = \text{true}$ | $A \vee \text{false} = A$ | |
| $(\neg A) \wedge A = \text{false}$ | $(\neg A) \vee A = \text{true}$ | |
| $\neg(A \wedge B) = (\neg A) \vee (\neg B)$ | $\neg(A \vee B) = (\neg A) \wedge (\neg B)$ | De Morgan's laws |

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Boolean algebra

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- This is called a **Boolean algebra**

Truth tables

A compact way of describing an operator, or a composition of operators

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Example:

| A | B | $\neg A$ | $A \wedge B$ | $A \vee B$ | $(A \vee B) \wedge (\neg A)$ |
|-----|-----|----------|--------------|------------|------------------------------|
| 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 |

Truth tables

We can check that two statements are equivalent with truth tables

| A | B | $\neg(A \wedge B)$ | $(\neg A) \vee (\neg B)$ |
|-----|-----|--------------------|--------------------------|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |

Implication

- “ $A \implies B$ ” means “*If A (is true), then B (is true)*”

- “ $A \implies B$ ” means “If A (is true), then B (is true)”

Example

“If it rains, I will bring an umbrella”
 $(\text{It rains}) \implies (\text{I will bring an umbrella})$

Implication

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Example

“If it rains, I will bring an umbrella”
(It rains) \implies (I will bring an umbrella)

Example

“If my grandpa had wheels, he would be a bike”
(My grandpa has wheels) \implies (My grandpa is a bike)

Implication

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| A | B | $A \implies B$ | |
|-----|-----|----------------|-------------------------------------|
| 0 | 0 | 1 | No rain, I don't bring an umbrella |
| 0 | 1 | 1 | No rain, I bring an umbrella anyway |
| 1 | 0 | 0 | It rains, I don't bring an umbrella |
| 1 | 1 | 1 | It rains, I bring an umbrella |

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Remark

"**false** $\implies A$ " is always **true**, whatever A is (*ex falso quodlibet*)

" $A \implies$ **true**" is always true, whatever A is

Sometimes we use the following symbols:

- “ $A \Leftarrow B$ ” is the same as “ $B \Rightarrow A$ ”
- “ $A \iff B$ ” is the same as “ $(A \Rightarrow B) \wedge (B \Rightarrow A)$ ”.
It is read “ A is equivalent to B ” or “ A if and only if B ”.

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- It is equivalent to “ $A \implies B$ ”
- Two proofs:
 - 1 Properties of logical operations
 - 2 Truth tables

See you tomorrow!

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Welcome Back!

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Let S be a set and let $A(x)$ be a “variable statement” that depends on $x \in S$ (for example $S = \mathbb{N}$ and $A(x) = “x \text{ is an even number}”$).

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- **Universal quantifier** (\forall or “for all”): “ $\forall x \in S, A(x)$ ” means that if we replace “ x ” with any element of S , $A(x)$ is always **true**.
- **Existential quantifier** (\exists or “there exists”): “ $\exists x \in S, A(x)$ ” means that $A(x)$ is **true** for at least one value of x in S .

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You always need a set S

Quantifiers - examples

Example

$S = \text{"the set of all cars"} , A(x) = \text{"x is red"}$

$\forall x \in S, A(x)$ is **false**.

$\exists x \in S, A(x)$ is **true**.

Example

$S = \mathbb{N}, A(x) = x > 5$

$\forall x \in S, A(x)$ is **false**.

$\exists x \in S, A(x)$ is **true**.

Today's most important fact:

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Example

\neg “ $\exists x \in \mathbb{N}, x + 3 = 9$ ” = “ $\forall x \in \mathbb{N}, x + 3 \neq 9$ ”

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- Proofs are used to derive new statements from statements that are known to be true.
- If A is known to be true and the implication $A \implies B$ is logically clear, then also B must be true.
- Every mathematical theorem must be justified with a proof.

Example: direct proof

Theorem

The sum of two even numbers is even.

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- 2 If x and y are even numbers, then there are natural numbers n and m such that $x = 2n$ and $y = 2m$.

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- 2 If x and y are even numbers, then there are natural numbers n and m such that $x = 2n$ and $y = 2m$.
- 3 Then $x + y = 2n + 2m = 2(n + m)$.

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- ② If x and y are even numbers, then there are natural numbers n and m such that $x = 2n$ and $y = 2m$.
- ③ Then $x + y = 2n + 2m = 2(n + m)$.
- ④ Then $x + y$ is even.



Proofs by contradiction

Idea: I want to show $A = \mathbf{true}$. I show that the implication
“ $(\neg A) \implies \mathbf{false}$ ” is **true**. Then $\neg A = \mathbf{false}$, so $A = \mathbf{true}$.

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Definition

A natural number is called *prime* if it is different from 1 and it is only divisible by 1 and itself.

Theorem

There are infinitely many prime numbers.

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Proof.

- 1 Assume that there are only finitely many prime numbers.

Proofs by contradiction

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- ① Assume that there are only finitely many prime numbers.
- ② $\exists n \in \mathbb{N}$, there are n prime numbers. Call them p_1, p_2, \dots, p_n .

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There are infinitely many prime numbers.

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- 1 Assume that there are only finitely many prime numbers.
- 2 $\exists n \in \mathbb{N}$, there are n prime numbers. Call them p_1, p_2, \dots, p_n .
- 3 Let $u = p_1 \times p_2 \times \dots \times p_n + 1$.

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- ⑤ Therefore u is only divisible by 1 and itself. So u is prime.

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- ③ Let $u = p_1 \times p_2 \times \dots \times p_n + 1$.
- ④ u is not divisible by any of the prime numbers p_1, \dots, p_n .
- ⑤ Therefore u is only divisible by 1 and itself. So u is prime.
- ⑥ So p_1, \dots, p_n are not the only prime numbers.



Proofs by induction

If I want to prove $\forall n \in \mathbb{N}, A(n)$:

- 1 Prove $A(0)$ (*base step*)
- 2 Prove $\forall n \in \mathbb{N}, (A(n) \implies A(n+1))$ (*inductive step*)

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Theorem (Sum of natural numbers)

$$\forall n \in \mathbb{N}, \quad 0 + 1 + \cdots + n = \frac{n(n+1)}{2}$$

Proof.

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- 2 Let n be any natural number.
If $A(n) = \mathbf{false}$, then $A(n) \implies A(n + 1)$ is **true**.

Proof.

① Base case: $0 = 0$.

② Let n be any natural number.

If $A(n) = \mathbf{false}$, then $A(n) \implies A(n+1)$ is **true**.

If $A(n) = \mathbf{true}$, we have to show that $A(n+1) = \mathbf{true}$.

Proof.

① Base case: $0 = 0$.

② Let n be any natural number.

If $A(n) = \mathbf{false}$, then $A(n) \implies A(n+1)$ is **true**.

If $A(n) = \mathbf{true}$, we have to show that $A(n+1) = \mathbf{true}$.

$$\begin{aligned} 0 + \cdots + (n+1) &= (0 + \cdots + n) + (n+1) = \\ &= \frac{n(n+1)}{2} + (n+1) = && \text{since } A(n) = \mathbf{true} \\ &= \frac{n^2 + n + 2n + 2}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

so $A(n+1) = \mathbf{true}$



The end

What you have learned in this course:

- Basic logic operations and logic implication
- Quantifiers and **their negation**
- Some basic mathematical proofs

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