Name, DoB: Answer Key

IT University of Copenhagen

Foundations of Computing – Discrete Mathematics MSc

Exam

3 January, 2018

Instructions (Read Carefully)

Contents: The exam contains 13 questions for a total of 100 points. The exam is divided into two parts: The first part has 9 multiple choice questions and the second part has 4 open ended questions.

What to check: In the multiple-choice questions, there is one and only one correct answer. You should only check 1 box.

Definitions and theorems: At the end of this document (page 11) you can find some definitions and theorems that could be useful for answering some of the questions.

Info about you: Write *clearly* your full name and your date of birth (DoB) on every page (top-right).

—IMPORTANT—
Only information written on the pages 1–10 will be evaluated.
Anything else that you hand-in will NOT be considered for the final evaluation!

Part I. Answer the following multiple choice questions.

S **1.** (6 pts) Which of the following statements is **true**?

[A] $\{1, 2\} \in \{1, 2, 3\}$

[B] $\{1, 2\} \times \{3, 4\} = \{(1, 3), (2, 4)\}$

[✔] $\{1, 2, 3\} \cap \{0, 1\} \subseteq \{1, 2\}$

[D] $\{1, 2\} \cap \mathcal{P}(\{1, 2, 3\}) = \{1, 2\}$

Solution: Because $\{1, 2, 3\} \cap \{0, 1\} = \{1\}$ and $\{1\} \subseteq \{1, 2\}$.

N **2.** (6 pts) Which of the following is **true** for all integers n?

[A] $(2 \cdot n) \bmod 2 = n$

[✔] $(n^2 + 1) \bmod 2 = (n + 1)^2 \bmod 2$

[C] $(2 \cdot n) \bmod 2 = n \bmod 2$

[D] $(n + 2) \bmod 2 = n$

Solution: By definition

$$(n^2 + 1) \bmod 2 = (n + 1)^2 \bmod 2$$

is equivalent to

$$(n^2 + 1) \equiv (n + 1)^2 \pmod 2$$

Using Theorem 3, we need to show that

$$2 \mid ((n^2 + 1) - (n + 1)^2)$$

This follows from the fact that

$$((n^2 + 1) - (n + 1)^2) = (n^2 + 1) - (n^2 + 2n + 1) = -2n$$

because $2 \mid (-2n)$.

F **3.** (6 pts) Let $f : \mathbb{N} \to \mathbb{N}$ be the function from the set of natural numbers to the set of natural numbers defined as follows:

$$f(n) = n \bmod 3$$

How many elements are there in the range of f?

[A] 6

[B] Infinite number of elements

[✔] 3

[D] None; The range of f is the empty set

Solution: The range of f is the set $\{0, 1, 2\}$.

F **4.** (6 pts) How many one-to-one functions are there from the set $\{1, 2, 3\}$ to the set $\{a, b, c, d\}$?

[A] 12

[✔] 24

[C] 36

[D] 48

Solution: A one-to-one function from $\{1, 2, 3\}$ to $\{a, b, c, d\}$ is a 3-permutation of the set $\{a, b, c, d\}$. Hence, there are $P(n, r)$ such functions, with $n = 4$ and $r = 3$:

$$P(4, 3) = \frac{4!}{(4-3)!} = \frac{24}{1} = 24$$

CP **5.** (6 pts) You throw two fair six-sided dice without looking. Given that one of them came up 6 what is the probability that the other one also came up 6?

[A] $\frac{1}{9}$

[B] $\frac{1}{10}$

[✔] $\frac{1}{11}$

[D] $\frac{1}{12}$

Solution: Let A be the event of getting two sixes when rolling two dice and let B be the event of getting at least one six when rolling two dice. We need to calculate the conditional probability $P(A|B)$. By the definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Moreover, $A \cap B = A$ since $A \subseteq B$. To calculate $P(A \cap B)$ and $P(B)$ we use the probability of rolling a six with a single die. Let S be the event of rolling a six with a single die. $P(S) = \frac{1}{6}$. By the multiplication rules we have that

$$P(A \cap B) = P(A) = P(S) \cdot P(S) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

In order to get at least one six with two dice, one either needs to roll a six with the first die, or roll something other than a six with the first and then a six with the second. Hence,

$$\begin{aligned} P(B) &= P(S) + P(S^C) \cdot P(S) \\ &= P(S) + (1 - P(S)) \cdot P(S) \\ &= \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} \\ &= \frac{11}{36} \end{aligned}$$

Hence,

$$P(A|B) = \frac{\frac{1}{36}}{\frac{11}{36}} = \frac{1 \cdot 36}{36 \cdot 11} = \frac{1}{11}$$

Alternative solution: Let A be the event that the first die shows a six and B the event that the second die shows a six, i.e. $A = \{(6, n) \mid 1 \le n \le 6\}$ and $B = \{(n, 6) \mid 1 \le n \le 6\}$. Our sample space is $A \cup B$ and we need to calculate the probability of the event $A \cap B$. Since all outcomes are equally likely, we can calculate the probability as follows:

$$P(A \cap B) = \frac{N(A \cap B)}{N(A \cup B)}$$

Clearly, $A \cap B = \{(6, 6)\}$. Moreover we may use the fact that

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

Since $N(A) = N(B) = 6$ and $N(A \cap B) = 1$, we have that

$$P(A \cap B) = \frac{1}{6 + 6 - 1} = \frac{1}{11}$$

L **6.** (6 pts) One of the compound propositions below is logically equivalent to the compound proposition $\sim p \vee q$. Which one?

[A] $\sim(\sim p \wedge \sim q)$.

[B] ✔ $(p \wedge q) \vee \sim p$

[C] $p \wedge \sim(\sim q \wedge p)$

[D] $(p \vee q) \vee \sim p$

Solution:

$$
\begin{aligned}
&(p \wedge q) \vee \sim p \\
\equiv\ &(p \vee \sim p) \wedge (q \vee \sim p) \\
\equiv\ &\mathbf{t} \wedge (q \vee \sim p) \\
\equiv\ &q \vee \sim p \\
\equiv\ &\sim p \vee q
\end{aligned}
$$

R **7.** (6 pts) Let $R = \{(0,0),(0,1),(0,2),(2,2),(3,1),(3,3),(4,1),(4,3)\}$ be a relation on the set $S = \{0,1,2,3,4\}$. Which of the following statements is **true**?

[A] ✔ R is transitive and antisymmetric, but not reflexive.

[B] R is antisymmetric, but neither reflexive nor transitive.

[C] R is reflexive and transitive, but not antisymmetric.

[D] R is reflexive, but neither transitive nor antisymmetric.

A **8.** (6 pts) Consider the following six strings:

$$
\begin{aligned}
w_1 &= \varepsilon & w_4 &= baaba \\
w_2 &= baba & w_5 &= bbababaa \\
w_3 &= aaa & w_6 &= bbb
\end{aligned}
$$

Which of those strings belong to the language defined by the regular expression $(a^*b)^*(\varepsilon \,|\, aa)$?

[A] w_2, w_4, w_6

[B] ✔ w_1, w_5, w_6

C w_2, w_3, w_4

D w_1, w_3, w_5

A **9.** (6 pts) Let G be the grammar with vocabulary $V = \{a, b, S, A, B\}$, terminal symbols $T = \{a, b\}$, starting symbol S, and productions

$$\begin{aligned} S &\to AB \\ A &\to aBb \\ A &\to \varepsilon \\ B &\to bAa \\ B &\to ba \end{aligned}$$

Which one of the following strings can be generated by G?

A $aabbba$

B ✔ $bababa$

C ab

D $ababab$

Solution: $bababa$ can be generated as follows :

$$S \Rightarrow AB \Rightarrow B \Rightarrow bAa \Rightarrow baBba \Rightarrow bababa$$

Part II. Answer the following questions. Be brief but precise. Your correct use of mathematical notation is an important aspect of your answer.

L

1. (12 pts) Show that $\sim(q \to p) \leftrightarrow (\sim p \wedge q)$ is a tautology by constructing a truth table for all subexpressions.

Solution:

p	q	$\sim p$	$q \to p$	$\sim p \wedge q$	$\sim(q \to p)$	$\sim(q \to p) \leftrightarrow (\sim p \wedge q)$
T	**T**	**F**	**T**	**F**	**F**	**T**
T	**F**	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**F**	**F**	**T**

L **2.** (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$\begin{aligned} a_k &= a_{k-1} + 2k + 1 \qquad \text{for all } k \geq 1 \\ a_0 &= 0 \end{aligned}$$

Prove by mathematical induction that $a_n = n(n+2)$ for all $n \geq 0$.

Solution: We want to prove the statement

$$a_n = n(n+2) \tag{$P(n)$}$$

for all $n \geq 0$.

Basis step: Let $n = 0$. We have

$$0(0+2) = 0 \cdot 2 = 0$$

and, by definition $a_0 = 0$. Hence, the basis step is verified.

Inductive step:

Suppose that $k \geq 0$ and that $P(k)$ holds, that is,

$$a_k = k(k+2) \tag{inductive hypothesis}$$

We must show that $P(k+1)$ holds, that is,

$$a_{k+1} = (k+1)(k+3) \tag{$P(k+1)$}$$

We will show that the left-hand side of $P(k+1)$ equals the right-hand side. We start with the right-hand side:

$$(k+1)(k+3) = k^2 + 3k + 1k + 3 = k^2 + 4k + 3$$

For the left-hand side of $P(k+1)$ we have:

$$\begin{aligned} a_{k+1} &= a_k + 2(k+1) + 1 && \text{(by definition of the sequence)} \\ &= k(k+2) + 2(k+1) + 1 && \text{(inductive hypothesis)} \\ &= k^2 + 2k + 2k + 2 + 1 \\ &= k^2 + 4k + 3 \end{aligned}$$

That shows that both sides of $P(k+1)$ are equal.

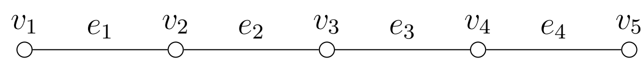
G **3.** (15 pts) There are three conditions for a graph listed below. For each of these conditions give *either* an example of a graph that satisfies the condition, *or* a reason why no such graph exists. In order to give an example, either draw the corresponding graph or give the triple (V, E, f) of vertices, edges and edge-endpoint function.
Hint: Use the definitions and theorems about graphs and trees on pages 16–17.

(a) A tree with 5 vertices and a total degree of 10.

Solution: Such a graph does not exist. According to the Handshake Theorem, a graph with total degree 10 must have exactly 5 edges. However, according to Theorem 13 on page 17, a tree with 5 vertices must have exactly 4 edges.

(b) A tree with 5 vertices and an Euler trail.

Solution: Such a graph does exist:



It has the Euler trail $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_4$.

(c) A simple graph with 6 vertices: one vertex of degree 0, one of degree 1, one of degree 2, one of degree 3, one of degree 4 and one of degree 5.

Solution: Such a graph does not exist. Such a graph would have total degree of $0 + 1 + 2 + 3 + 4 + 5 = 15$. However, according to the Handshake Theorem, every graph has an even total degree.
Alternative Solution: The vertex with degree 5 must have an edge to each other vertex, because a simple graph may not have loops or parallel edges. But that means that there cannot be a vertex with degree 0.

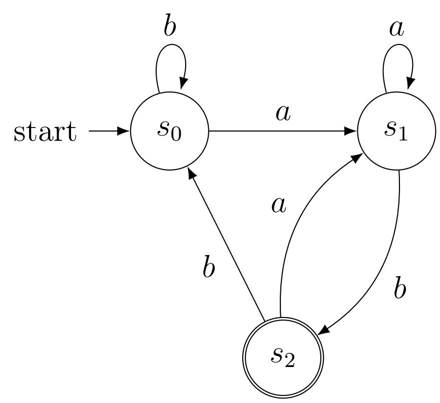
A

4. (7 pts) Construct a finite-state automaton A with input alphabet $\{a, b\}$ that accepts the set of all strings that end with ab. That is, A must satisfy

$$L(A) = \{wab \mid w \in \{a, b\}^*\}$$

Describe the automaton A using a next-state table *or* a transition diagram.

Solution:



Definitions and theorems

Logic

The truth table for a number of logical operators is given below.

p	q	$\sim p$	$p \vee q$	$p \wedge q$	$p \to q$	$p \leftrightarrow q$
T	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**T**	**F**
F	**F**	**T**	**F**	**F**	**T**	**T**

A compound proposition is called a *tautology* if it is always true no matter what the truth values of the propositional variables are. A compound proposition that is always false is called a *contradiction*.

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

Sets

A *set* is an (unordered) collection of objects, called *elements* or *members*. We write $a \in A$ to indicate that a is an *element of* A

A set A is a *subset* of a set B, written $A \subseteq B$, if for all a, $a \in A$ implies $a \in B$.

Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.

The *power set* of a set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A. That is, $\mathcal{P}(A) = \{B \mid B \subseteq A\}$.

The *union* of two sets A and B is the set $A \cup B = \{x \mid x \in A \vee x \in B\}$.

The *intersection* of A and B is the set $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

The *difference* of two sets A and B is the set $A - B = \{x \mid x \in A \wedge x \notin B\}$.

The *complement* of a set A is the set $A^c = \{x \in \mathcal{U} \mid x \notin A\}$, where $\mathcal{U}$ is the *universal set.*

The *Cartesian product* of A and B is the set $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. That is, $A \times B$ is the set of all pairs (a, b) with $a \in A$ and $b \in B$.

Functions

Given two sets A and B, a *function* f from A to B (written $f : A \to B$) is an assignment of exactly one element of B to each element of A.

A function $f : A \to B$ is *onto* if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

A function $f : A \to B$ is *one-to-one* if $f(a) = f(b)$ implies $a = b$ for all a and b in the domain of f.

A function f is a *one-to-one correspondence* if it is both one-to-one and onto.

Given two functions $g : A \to B$ and $f : B \to C$, the *composition* of f and g (written $f \circ g : A \to C$) is given by

$$(f \circ g)(a) = f(g(a)) \qquad \text{for all } a \in A$$

Let $f : X \to Y$. If $f(x) = y$ we say that y is the *image* of x and x is an *inverse image* (or *preimage*) of y. The *range* (or *image*) of f is the set of all values of f, i.e.

$$\mathsf{range}\,(f) = \{y \in Y \mid \exists x \in X. f(x) = y\}$$

The *inverse image* of an element $y \in Y$ is the set of all inverse images of y, i.e.

$$\text{inverse image of } y = \{x \in X \mid f(x) = y\}$$

Relations

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

A *relation on a set* A is a binary relation from A to A.

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

A relation R on a set A is called *antisymmetric* if whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$, for all $a, b \in A$,.

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

The *transitive closure* of a relation R is the smallest transitive relation T such that $R \subseteq T$.

A relation is an *equivalence relation* if it is reflexive, symmetric and transitive.

A relation is a *partial order relation* if it is reflexive, antisymmetric and transitive.

A *partially ordered set* is a pair (A, R) consisting of a set A and a partial order relation R on A.

Elements a and b of a partially ordered set $(A, \preceq)$ are said to be *comparable* if $a \preceq b$ or $b \preceq a$. Otherwise, a and b are said to be *incomparable*.

Let $(A, \preceq)$ be a partially ordered set. An element $a \in A$ is called

- a *greatest element* of A, if $x \preceq a$ for all $x \in A$
- a *least element* of A, if $a \preceq x$ for all $x \in A$
- a *maximal element* of A if, for all $x \in A$, either $x \preceq a$ or a and x are incomparable
- a *minimal element* of A if, for all $x \in A$, either $a \preceq x$ or a and x are incomparable

Number Theory

Given two integers a and b, with $a \neq 0$, we say that a *divides* b (written $a \mid b$) if there exist an integer c such that $b = ac$, or equivalently, if $\frac{b}{a}$ is an integer.

Theorem 1. *Let a, b, and c be any integers.*

1. *If $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.*
2. *If $a \mid b$ then $a \mid bc$.*
3. *If $a \mid b$ and $b \mid c$ then $a \mid c$.*

Theorem 2 (The Quotient/Remainder Theorem)**.** *Given any integer a and a positive integer d, there exist unique integers q and r such that*

$$a = dq + r \qquad \text{and } 0 \leq r < d$$

In Theorem 2 the value d is called the *divisor*, a is the *dividend*, q is the *quotient*, and r is the *remainder*. Then div and mod are defined as $a \text{ div } d = q$, $a \bmod d = r$. Remember that the remainder cannot be negative.

Let n and m be any integers and d a positive integer. We write $n \equiv m \pmod{d}$ if

$$(n \bmod d) = (m \bmod d)$$

Theorem 3. *Let n and m be any integers and d a positive integer then*

$$n \equiv m \pmod{d} \textit{ if and only if } d \mid (n - m)$$

Let n be a nonnegative integer and $b_r b_{r-1} ... b_0$ a finite sequence of binary digits, i.e. $b_i \in \{0, 1\}$ for $0 \leq i \leq r$. The sequence $b_r b_{r-1} ... b_0$ is a binary representation of n if

$$n = b_r \cdot 2^r + b_{r-1} 2^{r-1} + ... + b_1 \cdot 2 + b_0$$

For example 101101 is the binary representation of $1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 1 = 45$

The *greatest common divisor* of two integers a and b, not both zero, is denoted by $\gcd(a, b)$ and is the largest integer that both divides a and divides b.

The *least common multiple* of two positive integers a and b is denoted by $\text{lcm}(a, b)$ and is the smallest positive integer c such that $a \mid c$ and $b \mid c$.

Theorem 4. *If a and b integer, not both zero, then*

$$\gcd(a, b) = \gcd(a, a \bmod b)$$

Counting

The table below states how many different ways there are to order n distinct objects, and how many ways there are to choose r objects out of n distinct objects (depending on whether the order matters or repetition is allowed).

Order n objects (permutation)	$P(n) = n! = n \cdot (n-1) \ldots 2 \cdot 1$	
Choose r objects from n	without repetition	with repetition
- order matters (r-permutation)	$P(n, r) = \frac{n!}{(n-r)!}$	n^r
- order doesn't matter (r-combination)	$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$	$\binom{n+r-1}{r}$

Theorem 5 (Binomial Theorem)**.** *Given real numbers a and b, and non-negative integer n,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Let A be a finite set. We write $N(A)$ to denote the number of elements in A. We say A has $N(A)$ elements or A is a set of size $N(A)$

Theorem 6. *If A and B are finite sets then*

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

Probability

A *sample space* is the set of all possible outcomes of a random process or experiment. An *event* is a subset of a sample space. The probability of an event E is denoted $P(E)$. If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the probability of E is

$$P(E) = \frac{N(E)}{N(S)}$$

Let S be a sample space. A *probability function* P is a function from the set of all events in S to the set of real numbers satisfying

1. $0 \leq P(A) \leq 1$ for all events A in S.
2. $P(\emptyset) = 0$ and $P(S) = 1$.
3. If A and B are disjoint events in S, i.e. $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

Theorem 7. *If A is any event in a sample space S then the probability of the complement event $A^c = S - A$ is*

$$P(A^c) = 1 - P(A)$$

Theorem 8. *If A and B are events in a sample space S then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Suppose the sample space of an experiment or a random process is given by $S = \{a_1, ..., a_n\}$ where a_i is a real number for all $1 \leq i \leq n$. Suppose that each a_i occur with probability p_i for $1 \leq i \leq n$. The *expected value* of the process is

$$\sum_{i=1}^{i=n} a_i p_i = a_1 p_1 + a_2 p_2 + ... + a_n p_n$$

Let A and B be events in a sample space S. If $P(A) \neq 0$, then the *conditional probability of B given A*, denoted $P(B|A)$, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Theorem 9 (Bayes' Theorem)**.** *Suppose that a sample space S is the union of mutually disjoint events $B_1, ..., B_n$ where $P(B_i) \neq 0$ for all $1 \leq i \leq n$. Suppose that A is an event in S and that $P(A) \neq 0$. Then*

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + ... + P(A|B_n)P(B_n)}$$

Graphs & Trees

A *graph* is a triple (V, E, f) consisting of: a finite, nonempty set V of vertices; a finite set E of edges; and a function $f : E \rightarrow \{\{x, y\} \mid x, y \in V\}$.

An edge with only one endpoint is called a *loop*. Two edges with the same set of endpoints are called *parallel*. Vertices are called *adjacent* if they are endpoints of the same edge (if the edge is a loop, its endpoint is called *adjacent to itself*). We say that an edge is *incident on* its endpoints. Two edges with a common endpoint are called *adjacent*. A vertex on which no edge is incident is called *isolated*.

A graph is called *simple* if it has no loops or parallel edges.

Let G be a graph and v a vertex of G. The *degree of v*, denoted $\deg(v)$, is the number of edges that are incident on v, with an edge that is a loop counted twice. The *total degree of G* is the sum of the degrees of all vertices in G.

Theorem 10 (Handshake Theorem)**.** *A graph with m edges has total degree $2m$.*

The table below gives an overview over the different kinds of walks in a graph:

	repeated edge	**repeated vertex**	**same start & end vertex**	**must contain at least 1 edge**
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple Circuit	no	first and last only	yes	yes

A *subgraph* of G is a graph (V', E', f') with $V' \subseteq V$, $E' \subseteq E$ and $f'(e) = f(e)$ for all $e \in E'$.

A graph (V, E, f) is *connected* if there exists a walk from u to v for all $u, v \in V$. A *connected component* of a graph G is a maximal connected subgraph C of G (i.e. every connected subgraph of G is either a subgraph of C or has no common vertices with C).

An *Euler circuit* of a graph G is a circuit that contains every edge and every vertex of G.

An *Euler trail* of a graph G is a trail that contains every edge and every vertex of G.

Theorem 11. *A graph has an* Euler circuit *if and only if it is connected and every vertex has positive even degree.*

Theorem 12. *Given two distinct vertices u and v in a graph G, there exists an Euler trail from u to v if and only if G is connected, u and v have odd degree, and all other vertices have positive even degree.*

A graph is called *circuit-free* if it has no circuits. A *tree* is a connected and circuit-free graph. In a tree, a vertex is called a *leaf* if it has degree 1, and an *internal vertex* if it has degree 2 or greater. A tree which consists of only one vertex is called *trivial.*

Theorem 13. *Let n be a positive integer. A tree with n vertices has $n - 1$ edges.*

Automata, Regular Expressions, Grammars

An *alphabet* Σ is a finite set of *symbols.*

A *string* (or *word*) w over Σ is a finite sequence of symbols from Σ.

A *language* A is a set of strings over some Σ.

ε is the *empty string.*

The *length* of w, written $|w|$, is the number of symbols that w contains. We write $|w|_a$ for the number of times the symbol a occurs in w.

Given words $w_1 = a_1a_2 \dots a_m$ and $w_2 = b_1b_2 \dots b_n$, we write w_1w_2 to mean their *concatenation*: $a_1a_2 \dots a_mb_1b_2 \dots b_n$.

The concatenation of two languages L_1 and L_2 is defined as the language

$$L_1L_2 = \{w_1w_2 \mid w_1 \in L_1 \text{ and } w_2 \in L_2\}$$

The *Kleene closure* of a language L is the language $L^* = \{\varepsilon\} \cup L \cup LL \cup LLL \cup \dots$

A *finite-state automaton* is a 5-tuple (S, I, N, s_0, F) consisting of: a finite set of states S; a finite set I (the input alphabet); a transition function $N : S \times I \to S$; an initial state s_0; and a set $F \subseteq S$ of accepting states.

The language accepted by an automaton A is denoted by $L(A)$:

$$L(A) = \{w \mid w \in I^* \text{ and } N^*(s_0, w) \in F\}$$

where $N^* : S \times I^* \to S$ is defined by

$$\begin{aligned} N^*(s, \varepsilon) &= s \\ N^*(s, aw) &= N^*(s', w) \qquad \text{where } s' = N(s, a) \end{aligned}$$

A *regular expression* r over an alphabet Σ can be built using: $\emptyset$; ε; a symbol $a \in \Sigma$; concatenation $r\,s$ and union $r \mid s$ of regular expressions r, s; and Kleene closure r^* of a regular expression r.

The language $L(r)$ defined by a regular expression r is

$$\begin{array}{lll} L(\emptyset) = \emptyset & L(a) = \{a\} & L(r \mid t) = L(r) \cup L(r) \\ L(\varepsilon) = \{\varepsilon\} & L(r\,t) = L(r)\,L(r) & L(r^*) = L(r)^* \end{array}$$

A *grammar* $G = (V, T, S, P)$ consists of: A set of symbols V called *vocabulary*; a set $T \subset V$ of terminal symbols (symbols in $N = V - T$ are called *non-terminal*); a starting symbol $S \in V$; and a set P of productions of the form $z_0 \to z_1$, where $z_0, z_1 \in V^*$ and z_0 must contain at least one non-terminal symbol.

Given a grammar G and two strings $s, t \in V^*$, we write $s \Rightarrow t$, if s can be rewritten to t by applying one production from P. We write $s \overset{*}{\Rightarrow} t$, and say that t is derivable from s, if s can be rewritten to t by applying several productions from P. The language defined by the grammar G is the set of strings over T derivable from S:

$$L(G) = \left\{w \mid w \in T^* \text{ and } S \overset{*}{\Rightarrow} w\right\}$$