Relations (R)

	Chapter	Lecture	Assignment
Relations	8	8	4

6. (6 pts) Consider the binary relation R on $\mathbb{N}$ defined as follows for all $n, m \in \mathbb{N}$:

$$m\,R\,n \quad \text{iff} \quad n = 5 \cdot m$$

Which of the following statements is **true**?

☐ R is an equivalence relation.

☐ R is a partial order relation.

☐ R is antisymmetric, but not transitive.

☐ R is symmetric, but not reflexive.

3. (6 pts) Let A be an arbitrary set. Which of the following statements is **true** for all binary relations R and S on A?

☐ If R and S are transitive, then also $R \cup S$ is transitive.

☐ If R and S are symmetric, then also $R \cup S$ is symmetric.

☐ If R is transitive and $S \subseteq R$, then also S is transitive.

☐ If R is symmetric and $S \subseteq R$, then also S is symmetric.

7. (6 pts) Let $R = \{(0,0),(0,1),(1,3),(2,1)(2,2),(2,3),(3,0)\}$ be a relation on the set $S = \{0,1,2,3\}$. Which of the following statements is **true**?

☐ R is reflexive and transitive, but not antisymmetric.

☐ R is antisymmetric, but neither reflexive nor transitive.

☐ R is transitive and antisymmetric, but not reflexive.

☐ R is reflexive, but neither transitive nor antisymmetric.

6. (6 pts) Consider the binary relation R on $\mathbb{N}$ defined as follows for all $n, m \in \mathbb{N}$:

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[B] R is a partial order relation.

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[D] R is symmetric, but not reflexive.

3. (6 pts) Let A be an arbitrary set. Which of the following statements is **true** for all binary relations R and S on A?

[A] If R and S are transitive, then also $R \cup S$ is transitive.

☑ If R and S are symmetric, then also $R \cup S$ is symmetric.

[C] If R is transitive and $S \subseteq R$, then also S is transitive.

[D] If R is symmetric and $S \subseteq R$, then also S is symmetric.

Solution: Let R, S be symmetric, and let $(x,y) \in R \cup S$. Then $(x,y) \in R$ or $(x,y) \in S$. Since both relations are symmetric, we obtain $(y,x) \in R$ or $(y,x) \in S$. Consequently, $(y,x) \in R \cup S$.

7. (6 pts) Let $R = \{(0,0),(0,1),(1,3),(2,1)(2,2),(2,3),(3,0)\}$ be a relation on the set $S = \{0,1,2,3\}$. Which of the following statements is **true**?

[A] R is reflexive and transitive, but not antisymmetric.

☑ R is antisymmetric, but neither reflexive nor transitive.

[C] R is transitive and antisymmetric, but not reflexive.

[D] R is reflexive, but neither transitive nor antisymmetric.

7. (6 pts) Let $R = \{(0,0),(0,1),(0,2),(2,2),(3,1),(3,3),(4,1),(4,3)\}$ be a relation on the set $S = \{0,1,2,3,4\}$. Which of the following statements is **true**?

☐ R is transitive and antisymmetric, but not reflexive.

☐ R is antisymmetric, but neither reflexive nor transitive.

☐ R is reflexive and transitive, but not antisymmetric.

☐ R is reflexive, but neither transitive nor antisymmetric.

7. (6 pts) Let R be a relation on the set $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ defined as follows: For all $m,n \in \mathbb{Z}$, $m\,R\,n$ if and only if $m-n$ is odd. Which of the following statements is **true**?

☐ R is reflexive and transitive.

☐ R is reflexive but not transitive.

☐ R is not reflexive but transitive.

☐ R is neither reflexive nor transitive.

7. (6 pts) Let $R = \{(0,0),(0,1),(0,3),(1,1),(2,2),(3,0),(3,1),(3,3)\}$ be a relation on the set $S = \{0,1,2,3\}$. Which of the following statements is **true**?

☐ R is symmetric and transitive.

☐ R is symmetric but not transitive.

☐ R is not symmetric but transitive.

☐ R is neither symmetric nor transitive.

7. (6 pts) Let $R = \{(0,0),(0,1),(0,2),(2,2),(3,1),(3,3),(4,1),(4,3)\}$ be a relation on the set $S = \{0,1,2,3,4\}$. Which of the following statements is **true**?

☒ R is transitive and antisymmetric, but not reflexive.

B R is antisymmetric, but neither reflexive nor transitive.

C R is reflexive and transitive, but not antisymmetric.

D R is reflexive, but neither transitive nor antisymmetric.

7. (6 pts) Let R be a relation on the set $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ defined as follows: For all $m,n \in \mathbb{Z}$, $m\,R\,n$ if and only if $m-n$ is odd. Which of the following statements is **true**?

A R is reflexive and transitive.

B R is reflexive but not transitive.

C R is not reflexive but transitive.

☒ R is neither reflexive nor transitive.

Solution: For example we don't have $1\,R\,1$ since $1-1=0$ is not odd. And we have $1\,R\,2$ and $2\,R\,3$ but not $1\,R\,3$.

7. (6 pts) Let $R = \{(0,0),(0,1),(0,3),(1,1),(2,2),(3,0),(3,1),(3,3)\}$ be a relation on the set $S = \{0,1,2,3\}$. Which of the following statements is **true**?

A R is symmetric and transitive.

B R is symmetric but not transitive.

☒ R is not symmetric but transitive.

D R is neither symmetric nor transitive.

Solution: R is not symmetric because $(0,1) \in R$ but $(1,0) \notin R$. To check that R is transitive, we look at all cases where $(a,b),(b,c) \in R$ and show that $(a,c) \in R$. If $a=b$ or $b=c$, then $(a,c) \in R$ follows immediately. Let's look at all other cases:

- $(0,3),(3,0) \in R$: $(0,0) \in R$
- $(3,0),(0,3) \in R$: $(3,3) \in R$
- $(0,3),(3,1) \in R$: $(0,1) \in R$
- $(3,0),(0,1) \in R$: $(3,1) \in R$

12. (12 pts) Which of the following statements are true for all relations R and S on $\mathbb{N}$? Justify your answer by giving a proof or a counterexample.

(a) If R and S are antisymmetric, then $R \cup S$ is antisymmetric.

(b) If R and S are antisymmetric, then $R \cap S$ is antisymmetric.

3. (6 pts) Consider the binary relation R on $\mathbb{N}$ defined as follows

$$m \, R \, n \quad \text{iff} \quad m \equiv n \pmod 4$$

Which of the following statements is **true**?

☐ Each equivalence class of R has exactly four elements.

☐ R has exactly four distinct equivalence classes.

☐ R is antisymmetric.

☐ $[1]_R = [4]_R$.

12. (12 pts) Which of the following statements are true for all relations R and S on $\mathbb{N}$? Justify your answer by giving a proof or a counterexample.

(a) If R and S are antisymmetric, then $R \cup S$ is antisymmetric.

Solution: $R \cup S$ is not necessarily antisymmetric.
For example, if $R = \{(0,1)\}$ and $S = \{(1,0)\}$, then R and S are antisymmetric, but the relation $R \cup S = \{(0,1),(1,0)\}$ is not.

(b) If R and S are antisymmetric, then $R \cap S$ is antisymmetric.

Solution: This statement is true.
Suppose that R and S are antisymmetric. To show that $R \cap S$ is antisymmetric, we assume some $x, y \in \mathbb{N}$ such that $(x,y) \in R \cap S$ and $(y,x) \in R \cap S$. We must then show that also $x = y$. By definition of intersection, we know that $(x,y) \in R$ and $(y,x) \in R$. Because R is antisymmetric, it follows that $x = y$.

3. (6 pts) Consider the binary relation R on $\mathbb{N}$ defined as follows

$$m \, R \, n \quad \text{iff} \quad m \equiv n \pmod 4$$

Which of the following statements is **true**?

$\boxed{\text{A}}$ Each equivalence class of R has exactly four elements.

☑ R has exactly four distinct equivalence classes.

$\boxed{\text{C}}$ R is antisymmetric.

$\boxed{\text{D}}$ $[1]_R = [4]_R$.

12. (12 pts) Consider the “divides” relation on the set $A = \{1, 2, 3, 5, 10, 15\}$. That is, consider the relation R defined as follows for all $n, m \in A$:

$$m\,R\,n \iff m \mid n$$

The relation R is a partial order relation.

(a) Draw the Hasse diagram for R.

(b) Find all minimal elements of R. Write ’None’ if there aren’t any.

(c) Find all maximal elements of R. Write ’None’ if there aren’t any.

(d) Find all least elements of R. Write ’None’ if there aren’t any.

(e) Find all greatest elements of R. Write ’None’ if there aren’t any.

12. (12 pts) Consider the “divides” relation on the set $A = \{1, 2, 3, 5, 10, 15\}$. That is, consider the relation R defined as follows for all $n, m \in A$:

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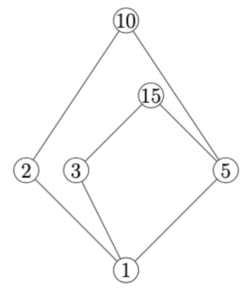
The relation R is a partial order relation.

(a) Draw the Hasse diagram for R.

Solution:



(b) Find all minimal elements of R. Write ’None’ if there aren’t any.

Solution: 1

(c) Find all maximal elements of R. Write ’None’ if there aren’t any.

Solution: 10, 15

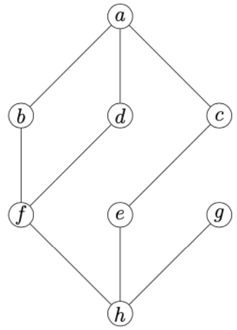
(d) Find all least elements of R. Write ’None’ if there aren’t any.

Solution: 1

(e) Find all greatest elements of R. Write ’None’ if there aren’t any.

Solution: None

6. (6 pts) Let $(\{a, b, c, d, e, f, g, h\}, \preceq)$ be the partially ordered set defined by the following Hasse diagram:



Which of the following statements is **true**?

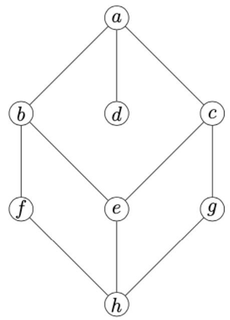
☐ a and f are incomparable.

☐ $g \preceq c$.

☐ a is the greatest element.

☐ h is the least element.

13. (8 pts) Let $(\{a, b, c, d, e, f, g, h\}, \preceq)$ be the partially ordered set defined by the following Hasse diagram:



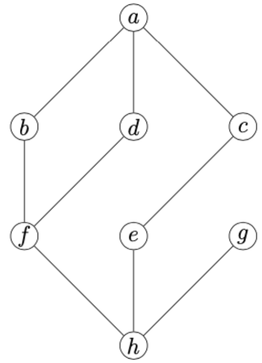
(a) Find all minimal elements. Write 'None' if there aren't any.

(b) Find all maximal elements. Write 'None' if there aren't any.

(c) Find all least elements. Write 'None' if there aren't any.

(d) Find all greatest elements. Write 'None' if there aren't any.

6. (6 pts) Let $(\{a, b, c, d, e, f, g, h\}, \preceq)$ be the partially ordered set defined by the following Hasse diagram:



Which of the following statements is **true**?

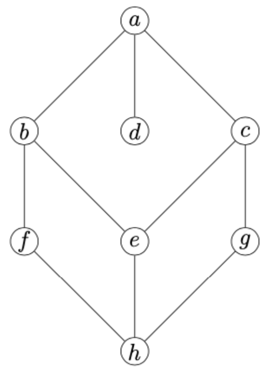
A a and f are incomparable.

B $g \preceq c$.

C a is the greatest element.

☑ h is the least element.

13. (8 pts) Let $(\{a, b, c, d, e, f, g, h\}, \preceq)$ be the partially ordered set defined by the following Hasse diagram:



(a) Find all minimal elements. Write 'None' if there aren't any.

Solution: d, h

(b) Find all maximal elements. Write 'None' if there aren't any.

Solution: a

(c) Find all least elements. Write 'None' if there aren't any.

Solution: None

(d) Find all greatest elements. Write 'None' if there aren't any.

Solution: a