Logic, Sequence & Induction (L)

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10. (12 pts) Consider the following compound proposition:

$$(q \wedge p) \vee ((\sim q) \rightarrow (p \vee q))$$

(a) Construct the truth table of the above compound proposition *including all subexpressions.*

(b) Is this compound proposition a tautology, a contradiction, or neither a tautology nor a contradiction?

11. (12 pts) Consider the following compound proposition:

$$(q \wedge p) \wedge (q \rightarrow \sim(p \vee q))$$

(a) Construct the truth table of the above compound proposition *including all subexpressions.*

(b) Is this compound proposition a tautology, a contradiction, or neither a tautology nor a contradiction?

10. (12 pts) Consider the following compound proposition:

$$(q \wedge p) \vee ((\sim q) \rightarrow (p \vee q))$$

(a) Construct the truth table of the above compound proposition *including all subexpressions.*

Solution:

p	q	$q \wedge p$	$p \vee q$	$\sim q$	$\sim q \rightarrow (p \vee q)$	$(q \wedge p) \vee (\sim q \rightarrow (p \vee q))$
T	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**	**T**	**T**
F	**F**	**F**	**F**	**T**	**F**	**F**

(b) Is this compound proposition a tautology, a contradiction, or neither a tautology nor a contradiction?

Solution: Because the compound proposition is false for some values of the propositional variables and true for others, it is neither a tautology nor contradiction.

11. (12 pts) Consider the following compound proposition:

$$(q \wedge p) \wedge (q \rightarrow \sim(p \vee q))$$

(a) Construct the truth table of the above compound proposition *including all subexpressions.*

Solution:

p	q	$q \wedge p$	$p \vee q$	$\sim(p \vee q)$	$q \rightarrow \sim(p \vee q)$	$(q \wedge p) \wedge (q \rightarrow \sim(p \vee q))$
T	**T**	**T**	**T**	**F**	**F**	**F**
T	**F**	**F**	**T**	**F**	**T**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**
F	**F**	**F**	**F**	**T**	**T**	**F**

(b) Is this compound proposition a tautology, a contradiction, or neither a tautology nor a contradiction?

Solution: Because the compound proposition is false for all values of the propositional variables, it is a contradiction.

1. (12 pts) Show that $\sim(q \to p) \leftrightarrow (\sim p \wedge q)$ is a tautology by constructing a truth table for all subexpressions.

1. (12 pts) Show that $\sim(q \to p) \leftrightarrow (\sim p \wedge q)$ is a tautology by constructing a truth table for all subexpressions.

Solution:

p	q	$\sim p$	$q \to p$	$\sim p \wedge q$	$\sim(q \to p)$	$\sim(q \to p) \leftrightarrow (\sim p \wedge q)$
T	**T**	**F**	**T**	**F**	**F**	**T**
T	**F**	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**F**	**F**	**T**

11. (12 pts) Show that $(q \wedge p) \wedge (p \to \sim(p \vee q))$ is a contradiction by constructing a truth table for all subexpressions.

11. (12 pts) Show that $(q \wedge p) \wedge (p \to \sim(p \vee q))$ is a contradiction by constructing a truth table for all subexpressions.

Solution:

p	q	$q \wedge p$	$p \vee q$	$\sim(p \vee q)$	$p \to \sim(p \vee q)$	$(q \wedge p) \wedge (p \to \sim(p \vee q))$
T	**T**	**T**	**T**	**F**	**F**	**F**
T	**F**	**F**	**T**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**T**	**F**
F	**F**	**F**	**F**	**T**	**T**	**F**

p	q	$q \wedge p$	$p \vee q$	$\sim(p \vee q)$	$p \to \sim(p \vee q)$	$(q \wedge p) \wedge (p \to \sim(p \vee q))$
T	**T**	**T**	**T**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**T**	**F**
T	**F**	**F**	**T**	**F**	**F**	**F**
F	**F**	**F**	**F**	**T**	**T**	**F**

11. (4 pts) Show that $\neg((q \to p) \to (p \vee \neg q))$ is a contradiction by constructing a truth table for all sub expressions.

11. Show that $\neg((q \to p) \to (p \vee \neg q))$ is a contradiction by constructing a truth table for all sub expressions.

Solution:

p	q	$q \to p$	$\neg q$	$p \vee \neg q$	$(q \to p) \to (p \vee \neg q)$	$\neg((q \to p) \to (p \vee \neg q))$
T	T	T	F	T	T	F
T	F	T	T	T	T	F
F	T	F	F	F	T	F
F	F	T	T	T	T	F

Since $\neg((q \to p) \to (p \vee \neg q))$ always is false for all true values of p og q the proposition is a contradiction.

11. (12 pts) Using the logical equivalences on page 11, prove the following logical equivalence:

$$(p \to q) \wedge p \equiv q \wedge p$$

In each step, indicate (by number) which equivalence from page 11 you used.

11. (12 pts) Using the logical equivalences on page 10, prove the following logical equivalence:

$$(p \to q) \wedge p \equiv q \wedge p$$

In each step, indicate (by number) which equivalence from page 10 you used.

Solution:

$$\begin{aligned}
& (p \to q) \wedge p & \\
\equiv\ & ({\sim}p \vee q) \wedge p & (12)\\
\equiv\ & p \wedge ({\sim}p \vee q) & (1)\\
\equiv\ & (p \wedge {\sim}p) \vee (p \wedge q) & (3)\\
\equiv\ & \mathbf{c} \vee (p \wedge q) & (5)\\
\equiv\ & (p \wedge q) \vee \mathbf{c} & (1)\\
\equiv\ & p \wedge q & (4)\\
\equiv\ & q \wedge p & (1)
\end{aligned}$$

10. (12 pts) Using the logical equivalences on page 9, prove the following logical equivalence:

$$(q \vee p) \to p \equiv p \vee {\sim}q$$

In each step, indicate (by number) which equivalence from page 9 you have used.

10. (12 pts) Using the logical equivalences on page 8, prove the following logical equivalence:

$$(q \vee p) \to p \equiv p \vee {\sim}q$$

In each step, indicate (by number) which equivalence from page 8 you have used.

Solution:

$$\begin{aligned}
& (q \vee p) \to p & \\
\equiv\ & {\sim}(q \vee p) \vee p & (12)\\
\equiv\ & ({\sim}q \wedge {\sim}p) \vee p & (9)\\
\equiv\ & p \vee ({\sim}q \wedge {\sim}p) & (1)\\
\equiv\ & (p \vee {\sim}q) \wedge (p \vee {\sim}p) & (3)\\
\equiv\ & (p \vee {\sim}q) \wedge \mathbf{t} & (5)\\
\equiv\ & p \vee {\sim}q & (4)
\end{aligned}$$

11. (12 pts) Using the logical equivalences on page 10, prove the following logical equivalence:

$$p \to (q \to r) \equiv (q \wedge p) \to r$$

In each step, indicate (by number) which equivalence from page 10 you used (only use one per step).

11. (12 pts) Using the logical equivalences on page 10, prove the following logical equivalence:

$$p \to (q \to r) \equiv (q \wedge p) \to r$$

In each step, indicate (by number) which equivalence from page 10 you used (only use one per step).

$$\begin{aligned}
& p \to (q \to r) & \\
\equiv\ & p \to ({\sim}q \vee r) & (12)\\
\equiv\ & {\sim}p \vee ({\sim}q \vee r) & (12)\\
\equiv\ & ({\sim}p \vee {\sim}q) \vee r & (2)\\
\equiv\ & {\sim}(p \wedge q) \vee r & (9)\\
\equiv\ & (p \wedge q) \to r & (12)\\
\equiv\ & (q \wedge p) \to r & (1)
\end{aligned}$$

11. (12 pts) Using the logical equivalences on page 10, prove the following logical equivalence:

$$(p \wedge {\sim}q) \rightarrow q \equiv p \rightarrow q$$

In each step, indicate (by number) which equivalence from page 10 you used.

6. (6 pts) One of the compound propositions below is logically equivalent to the compound proposition ${\sim}q \wedge p$. Which one?

☐ $(q \wedge p) \wedge {\sim}q$

☐ $q \vee {\sim}({\sim}p \vee q)$

☐ $(q \vee p) \wedge {\sim}q$

☐ ${\sim}({\sim}q \vee {\sim}p)$.

6. (6 pts) One of the compound propositions below is logically equivalent to the compound proposition ${\sim}p \vee q$. Which one?

☐ $(q \vee p) \rightarrow q$

☐ $(q \wedge p) \wedge {\sim}q$

☐ $({\sim}p \vee q) \rightarrow q$

☐ $(q \vee p) \wedge {\sim}q$

11. (12 pts) Using the logical equivalences on page 8, prove the following logical equivalence:

$$(p \wedge {\sim}q) \rightarrow q \equiv p \rightarrow q$$

In each step, indicate (by number) which equivalence from page 8 you used.

Solution:

$$
\begin{aligned}
& (p \wedge {\sim}q) \rightarrow q & \\
& \equiv {\sim}(p \wedge {\sim}q) \vee q & (12)\\
& \equiv ({\sim}p \vee {\sim}({\sim}q)) \vee q & (9)\\
& \equiv ({\sim}p \vee q) \vee q & (6)\\
& \equiv {\sim}p \vee (q \vee q) & (2)\\
& \equiv p \rightarrow (q \vee q) & (12)\\
& \equiv p \rightarrow q & (7)
\end{aligned}
$$

6. (6 pts) One of the compound propositions below is logically equivalent to the compound proposition ${\sim}q \wedge p$. Which one?

$\boxed{A}$ $(q \wedge p) \wedge {\sim}q$

$\boxed{B}$ $q \vee {\sim}({\sim}p \vee q)$

☑ $(q \vee p) \wedge {\sim}q$

$\boxed{D}$ ${\sim}({\sim}q \vee {\sim}p)$.

Solution:

$$
\begin{aligned}
& (q \vee p) \wedge {\sim}q \\
& \equiv (q \wedge {\sim}q) \vee (p \wedge {\sim}q) \\
& \equiv \mathbf{f} \vee (p \wedge {\sim}q) \\
& \equiv p \wedge {\sim}q \\
& \equiv {\sim}q \wedge p
\end{aligned}
$$

6. (6 pts) One of the compound propositions below is logically equivalent to the compound proposition ${\sim}p \vee q$. Which one?

☑ $(q \vee p) \rightarrow q$

$\boxed{B}$ $(q \wedge p) \wedge {\sim}q$

$\boxed{C}$ $({\sim}p \vee q) \rightarrow q$

$\boxed{D}$ $(q \vee p) \wedge {\sim}q$

Solution:

$$
\begin{aligned}
& (q \vee p) \rightarrow q \\
& \equiv {\sim}(q \vee p) \vee q \\
& \equiv ({\sim}q \wedge {\sim}p) \vee q \\
& \equiv ({\sim}q \vee q) \wedge ({\sim}p \vee q) \\
& \equiv \mathbf{t} \wedge ({\sim}p \vee q) \\
& \equiv {\sim}p \vee q
\end{aligned}
$$

4. (6 pts) Let the sequence $a_1, a_2, a_3, \ldots$ be recursively defined by

$$a_k = a_{k-1} + k \quad \text{for all } k > 1$$
$$a_1 = 1$$

Which of the following equations is **true** for all $n > 0$?

☐ $a_n = \frac{n(n-1)}{2}$

☐ $a_n = \frac{n(n+1)}{2}$

☐ $a_n = n!$

☐ $a_n = \sum_{i=1}^{n} n \cdot i$

4. (6 pts) Let the sequence $a_1, a_2, a_3, \ldots$ be recursively defined by

$$a_k = a_{k-1} + k \quad \text{for all } k > 1$$
$$a_1 = 1$$

Which of the following equations is **true** for all $n > 0$?

[A] $a_n = \frac{n(n-1)}{2}$

[B] ☑ $a_n = \frac{n(n+1)}{2}$

[C] $a_n = n!$

[D] $a_n = \sum_{i=1}^{n} n \cdot i$

Solution: A cannot be true since $a_1 = 1$ but $\frac{1(1-1)}{2} = 0$; C cannot be true since $a_2 = 1 + 2 = 3$ but $2! = 2 \cdot 1 = 2$; D cannot be true since $a_2 = 1 + 2 = 3$ but $\sum_{i=1}^{2} 2i = 2 \cdot 1 + 2 \cdot 2 = 6$.
Moreover, we can check that B is true by showing that it satisfies the initial condition and the recurrence. First, consider the initial condition:

$$a_1 = \frac{1(1+1)}{2} = 1$$

Secondly, we show that the recurrence is satisfied. That is we must prove the following equation:

$$\frac{k(k+1)}{2} = \frac{(k-1)((k-1)+1)}{2} + k$$

To this end, we simplify both sides and show that they are equal.
For the left-hand side, we have:

$$\frac{k(k+1)}{2} = \frac{k^2+k}{2}$$

For the right-hand side, we have:

$$\frac{(k-1)((k-1)+1)}{2} + k = \frac{k^2-k}{2} + \frac{2k}{2} = \frac{k^2+k}{2}$$

6. (6 pts) Let the sequence $a_1, a_2, a_3, \ldots$ be recursively defined by

$$\begin{aligned} a_k &= a_{k-1} + k \quad \text{for all } k > 1 \\ a_1 &= 1 \end{aligned}$$

Which of the following equations is **true** for all $n > 0$?

- [ ] $a_n = n(n+1)$
- [ ] $a_n = \sum_{i=1}^{n} i$
- [ ] $a_n = 2^{n-1}$
- [ ] $a_n = \prod_{i=1}^{n} i$

9. (6 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be recursively defined by

$$\begin{aligned} a_k &= a_{k-1} + 2^k \quad \text{for all } k > 0 \\ a_0 &= 1 \end{aligned}$$

Which of the following equations is **true** for all $n \geq 0$?

- [ ] $a_n = \sum_{i=0}^{n} 2^i$
- [ ] $a_n = n!$
- [ ] $a_n = 2^n$
- [ ] $a_n = n \cdot (n+1) + 1$

4. (6 pts) Let the sequence $a_1, a_2, a_3, \ldots$ be recursively defined by

$$\begin{aligned} a_k &= a_{k-1} + 2 \cdot k \quad \text{for all } k > 1 \\ a_1 &= 2 \end{aligned}$$

Which of the following equations is **true** for all $n > 0$?

- [ ] $a_n = n(n+1)$
- [ ] $a_n = \frac{n(n+1)}{2}$
- [ ] $a_n = 2 \cdot n!$
- [ ] $a_n = \prod_{i=1}^{n} i$

6. (6 pts) Let the sequence $a_1, a_2, a_3, \ldots$ be recursively defined by

$$\begin{aligned} a_k &= a_{k-1} + k \quad \text{for all } k > 1 \\ a_1 &= 1 \end{aligned}$$

Which of the following equations is **true** for all $n > 0$?

[A] $a_n = n(n+1)$

[x] $a_n = \sum_{i=1}^{n} i$

[C] $a_n = 2^{n-1}$

[D] $a_n = \prod_{i=1}^{n} i$

Solution: A cannot be true since $a_1 = 1$ but $1(1+1) = 2$; C cannot be true since $a_2 = 1+2 = 3$ but $2^{2-1} = 2$; D cannot be true since $a_2 = 3$ but but $\prod_{i=1}^{2} i = 1 \cdot 2 = 2$.

9. (6 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be recursively defined by

$$\begin{aligned} a_k &= a_{k-1} + 2^k \quad \text{for all } k > 0 \\ a_0 &= 1 \end{aligned}$$

Which of the following equations is **true** for all $n \geq 0$?

[x] $a_n = \sum_{i=0}^{n} 2^i$

[B] $a_n = n!$

[C] $a_n = 2^n$

[D] $a_n = n \cdot (n+1) + 1$

Solution: B cannot be true since $a_1 = 3$, but $2! = 2$; C cannot be true since $a_1 = 3$, but $2^1 = 2$; D cannot be true since $a_3 = 15$, but $3(3+1) + 1 = 3 \cdot 4 + 1 = 13$.

4. (6 pts) Let the sequence $a_1, a_2, a_3, \ldots$ be recursively defined by

$$\begin{aligned} a_k &= a_{k-1} + 2 \cdot k \quad \text{for all } k > 1 \\ a_1 &= 2 \end{aligned}$$

Which of the following equations is **true** for all $n > 0$?

[x] $a_n = n(n+1)$

[B] $a_n = \frac{n(n+1)}{2}$

[C] $a_n = 2 \cdot n!$

[D] $a_n = \prod_{i=1}^{n} i$

Solution: B cannot be true since $a_1 = 0$ but $\frac{1(1+1)}{2} = 1$; C cannot be true since $a_2 = 2+4 = 6$ but $2 \cdot 2! = 2 \cdot 2 \cdot 1 = 4$; D cannot be true since $a_1 = 2$ but $\prod_{i=1}^{1} i = 1$.

10. (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$\begin{aligned} a_k &= 2a_{k-1} + 1 \qquad \text{for all } k \geq 1 \\ a_0 &= 1 \end{aligned}$$

Prove by mathematical induction that $a_n = 2^{n+1} - 1$ for all $n \geq 0$.

10. (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$\begin{aligned} a_k &= 2a_{k-1} + 1 \qquad \text{for all } k \geq 1 \\ a_0 &= 1 \end{aligned}$$

Prove by mathematical induction that $a_n = 2^{n+1} - 1$ for all $n \geq 0$.

Solution: We want to prove the statement

$$a_n = 2^{n+1} - 1 \tag{$P(n)$}$$

for all $n \geq 0$.

Basis step: Let $n = 0$. We have

$$2^{0+1} - 1 = 2^1 - 1 = 2 - 1 = 1$$

and, by definition $a_0 = 1$. Hence, the basis step is verified.

Inductive step:
Suppose that $k \geq 0$ and that $P(k)$ holds, that is,

$$a_k = 2^{k+1} - 1 \tag{inductive hypothesis}$$

We must show that $P(k+1)$ holds, that is,

$$a_{k+1} = 2^{(k+1)+1} - 1 \tag{$P(k+1)$}$$

The following calculation shows that $P(k+1)$ holds:

$$\begin{aligned} a_{k+1} &= 2a_k + 1 && \text{(by definition of the sequence)} \\ &= 2(2^{k+1} - 1) + 1 && \text{(inductive hypothesis)} \\ &= 2 \cdot 2^{k+1} - 2 + 1 \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

12. (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$a_k = 3a_{k-1} - 1 \qquad \text{for all } k \geq 1$$
$$a_0 = 1$$

Prove by mathematical induction that $a_n = \frac{3^n+1}{2}$ for all $n \geq 0$.

12. (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$a_k = 3a_{k-1} - 1 \qquad \text{for all } k \geq 1$$
$$a_0 = 1$$

Prove by mathematical induction that $a_n = \frac{3^n+1}{2}$ for all $n \geq 0$.

Solution: We want to prove the statement

$$a_n = \frac{3^n + 1}{2} \tag{$P(n)$}$$

for all $n \geq 0$.

Basis step: Let $n = 0$. We have

$$\frac{3^0 + 1}{2} = \frac{1 + 1}{2} = 1$$

and, by definition $a_0 = 1$. Hence, the basis step is verified.

Inductive step:

Suppose that $k \geq 0$ and that $P(k)$ holds, that is,

$$a_k = \frac{3^k + 1}{2} \tag{inductive hypothesis}$$

We must show that $P(k+1)$ holds, that is,

$$a_{k+1} = \frac{3^{k+1} + 1}{2} \tag{$P(k+1)$}$$

The following calculation shows that $P(k+1)$ holds:

$$\begin{aligned} a_{k+1} &= 3a_k - 1 && \text{(by definition of the sequence)} \\ &= 3\frac{3^k + 1}{2} - 1 && \text{(inductive hypothesis)} \\ &= \frac{3^{k+1} + 3}{2} - \frac{2}{2} \\ &= \frac{3^{k+1} + 1}{2} \end{aligned}$$

2. (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$\begin{aligned} a_k &= a_{k-1} + 2k + 1 \qquad \text{for all } k \geq 1 \\ a_0 &= 0 \end{aligned}$$

Prove by mathematical induction that $a_n = n(n+2)$ for all $n \geq 0$.

2. (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$\begin{aligned} a_k &= a_{k-1} + 2k + 1 \qquad \text{for all } k \geq 1 \\ a_0 &= 0 \end{aligned}$$

Prove by mathematical induction that $a_n = n(n+2)$ for all $n \geq 0$.

Solution: We want to prove the statement

$$a_n = n(n+2) \tag{$P(n)$}$$

for all $n \geq 0$.

Basis step: Let $n = 0$. We have

$$0(0+2) = 0 \cdot 2 = 0$$

and, by definition $a_0 = 0$. Hence, the basis step is verified.

Inductive step:
Suppose that $k \geq 0$ and that $P(k)$ holds, that is,

$$a_k = k(k+2) \tag{inductive hypothesis}$$

We must show that $P(k+1)$ holds, that is,

$$a_{k+1} = (k+1)(k+3) \tag{$P(k+1)$}$$

We will show that the left-hand side of $P(k+1)$ equals the right-hand side. We start with the right-hand side:

$$(k+1)(k+3) = k^2 + 3k + 1k + 3 = k^2 + 4k + 3$$

For the left-hand side of $P(k+1)$ we have:

$$\begin{aligned} a_{k+1} &= a_k + 2(k+1) + 1 && \text{(by definition of the sequence)} \\ &= k(k+2) + 2(k+1) + 1 && \text{(inductive hypothesis)} \\ &= k^2 + 2k + 2k + 2 + 1 \\ &= k^2 + 4k + 3 \end{aligned}$$

That shows that both sides of $P(k+1)$ are equal.

12. (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$\begin{aligned} a_k &= 2a_{k-1} + 1 \qquad \text{for all } k \geq 1 \\ a_0 &= 1 \end{aligned}$$

Prove by mathematical induction that $a_n = 2^{n+1} - 1$ for all $n \geq 0$.

12. (12 pts) Let the sequence $a_0, a_1, a_2, \ldots$ be given by the following recursive definition

$$\begin{aligned} a_k &= 2a_{k-1} + 1 \qquad \text{for all } k \geq 1 \\ a_0 &= 1 \end{aligned}$$

Prove by mathematical induction that $a_n = 2^{n+1} - 1$ for all $n \geq 0$.

Solution: We want to prove the statement

$$a_n = 2^{n+1} - 1 \tag{$P(n)$}$$

for all $n \geq 0$.

Basis step: Let $n = 0$. We have

$$2^{0+1} - 1 = 2^1 - 1 = 2 - 1 = 1$$

and, by definition $a_0 = 1$. Hence, the basis step is verified.

Inductive step:

Suppose that $k \geq 0$ and that $P(k)$ holds, that is,

$$a_k = 2^{k+1} - 1 \tag{inductive hypothesis}$$

We must show that $P(k+1)$ holds, that is,

$$a_{k+1} = 2^{(k+1)+1} - 1 \tag{$P(k+1)$}$$

The following calculation shows that $P(k+1)$ holds:

$$\begin{aligned} a_{k+1} &= 2a_k + 1 && \text{(by definition of the sequence)} \\ &= 2(2^{k+1} - 1) + 1 && \text{(inductive hypothesis)} \\ &= 2 \cdot 2^{k+1} - 2 + 1 \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

10. (12 pts) Prove using mathematical induction that

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2 \qquad \text{for all } n \geq 0$$

10. (12 pts) Prove using mathematical induction that

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2 \qquad \text{for all } n \geq 0$$

Solution: We want to prove the statement

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2 \qquad (P(n))$$

for all $n \geq 0$.

Basis step: Let $n = 0$. The left-hand side of $P(0)$ simplifies to

$$\sum_{i=0}^{0}(2 \cdot i+1) = 2 \cdot 0 + 1 = 1$$

and the right-hand side of $P(0)$ simplifies to

$$(0+1)^2 = 1$$

Inductive step:
Suppose that $k \geq 0$ and that $P(k)$ holds, that is,

$$\sum_{i=0}^{k}(2i+1) = (k+1)^2$$

We must show that $P(k+1)$ holds, that is,

$$\sum_{i=0}^{k+1}(2i+1) = (k+2)^2$$

The following derivation proves this equation:

$$\begin{aligned}
&\sum_{i=0}^{k+1}(2i+1) \\
&= \sum_{i=0}^{k}(2i+1) + (2(k+1)+1) && \text{(write last summand separately)} \\
&= (k+1)^2 + (2(k+1)+1) && \text{(inductive hypothesis)} \\
&= (k^2+2k+1) + (2k+3) && \text{(binomial theorem \& distributivity)} \\
&= k^2+4k+4 && \text{(associativity \& factorisation)} \\
&= (k+2)^2 && \text{(binomial theorem)}
\end{aligned}$$

11. (12 pts) Prove the following statement by mathematical induction:

$$\sum_{i=1}^{n}(10i-8) = 5n^2 - 3n \qquad \text{for all } n \geq 1$$

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Solution: We want to prove the statement

$$\sum_{i=1}^{n}(10i-8) = 5n^2 - 3n$$

for all $n \geq 1$.

Basis step: Let $n = 1$. The left-hand side of $P(1)$ simplifies to

$$\sum_{i=1}^{1}(10i-8) = 10 \cdot 1 - 8 = 2$$

and the right-hand side of $P(1)$ simplifies to

$$5 \cdot 1^2 - 3 \cdot 1 = 5 - 3 = 2$$

Inductive step:

Suppose that $k \geq 1$ and that $P(k)$ holds, that is,

$$\sum_{i=1}^{k}(10i-8) = 5k^2 - 3k$$

We must show that $P(k+1)$ holds, that is,

$$\sum_{i=1}^{k+1}(10i-8) = 5(k+1)^2 - 3(k+1)$$

The following derivation proves this equation:

$$\begin{aligned}
&\sum_{i=1}^{k+1}(10i-8) \\
&= \sum_{i=1}^{k}(10i-8) + (10(k+1)-8) && \text{(write last summand separately)} \\
&= 5k^2 - 3k + 10(k+1) - 8 && \text{(inductive hypothesis)} \\
&= 5k^2 - 3k + 10k + 10 - 8 && \text{(distributivity)} \\
&= 5k^2 + 10k + 5 - 3k - 3 && \text{(10 - 8 = 2 = 5 - 3)} \\
&= 5(k^2 + 2k + 1) - 3k - 3 && \text{(factorisation)} \\
&= 5(k+1)^2 - 3(k+1) && \text{(binomial theorem)}
\end{aligned}$$

10. (12 pts) Prove by mathematical induction that $n^3 + 5n$ is divisible by 3 for all integers $n \geq 0$.

10. (12 pts) Prove by mathematical induction that $n^3 + 5n$ is divisible by 3 for all integers $n \geq 0$.

Solution: We want to prove the statement

$$3 \mid n^3 + 5n$$

for all $n \geq 0$.
Basis step: Let $n = 0$. We have

$$0(0^2 + 5) = 0$$

and 3 divides 0 since $0 \cdot 3 = 0$.

Inductive step:
Suppose that $k \geq 0$ and that $P(k)$ holds, that is,

$$3 \mid k^3 + 5k$$

We must show that $P(k+1)$ holds, that is,

$$3 \mid (k+1)^3 + 5(k+1)$$

We have that

$$\begin{aligned}(k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ &= k^3 + 5k + 3k^2 + 3k + 6 \\ &= k^3 + 5k + 3(k^2 + k + 3)\end{aligned}$$

By induction hypothesis we have that 3 divides $k^3 + 5k$, and by definition 3 divides $3(k^2+k+3)$. Hence, by Theorem 1, 3 also divides $k^3+5k+3(k^2+k+3)$. By the above calculation the latter is equal to $(k+1)^3 + 5(k+1)$, which is therefore also divisible by 3.

11. (12 pts) Prove, using mathematical induction, that 3 divides $n^3 + 2n$ for all integers $n \geq 0$.

11. (12 pts) Prove, using mathematical induction, that 3 divides $n^3 + 2n$ for all integers $n \geq 0$.

Solution: We want to prove the statement

$$3 \mid (n^3 + 2n) \qquad (P(n))$$

for all $n \geq 0$.

Basis step: Let $n = 0$. We have

$$0^3 + 2 \cdot 0 = 0 + 2 \cdot 0 = 0$$

and 3 divides 0. Hence, the basis step is verified.

Inductive step:
Suppose that $k \geq 0$ and that $P(k)$ holds, that is,

$$3 \mid k^3 + 2k \qquad \text{(inductive hypothesis)}$$

We must show that $P(k+1)$ holds, that is,

$$3 \mid (k+1)^3 + 2(k+1)$$

By the binomial theorem

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

Hence, we can rewrite the term $(k+1)^3 + 2(k+1)$ as follows:

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2(k+1) && \text{(binomial theorem)}\\
&= k^3 + 3k^2 + 3k + 1 + 2k + 2 && \text{(distributivity)}\\
&= k^3 + 2k + 3k^2 + 3k + 3 && \text{(commutativity of +)}\\
&= (k^3 + 2k) + 3(k^2 + k + 1)
\end{aligned}$$

According to the inductive hypothesis, 3 divides $k^3 + 2k$. Moreover, 3 also divides $3(k^2 + k + 1)$. Hence, 3 also divides the whole sum $(k^3 + 2k) + 3(k^2 + k + 1)$. Here we use the fact that given $a \mid b$ and $a \mid c$, we can conclude $a \mid (b + c)$.

12. (4 pts) Prove, using induction, that

$$\sum_{i=1}^{n}\left(i-\frac{1}{2}\right)=\frac{n^2}{2}$$

for all integers $n \geq 1$.

12. Prove, using induction, that

$$\sum_{i=1}^{n}\left(i-\frac{1}{2}\right)=\frac{n^2}{2}$$

for all integers $n \geq 1$.
Solution: We want to verify the statement

$$P(n) := \sum_{i=1}^{n}\left(i-\frac{1}{2}\right)=\frac{n^2}{2}$$

where n is an integer of value at least 1. So our base case is when $n = 1$.

Base case: Let $n = 1$. We have

$$\sum_{i=1}^{1}\left(i-\frac{1}{2}\right)=\left(1-\frac{1}{2}\right)=\frac{1}{2}=\frac{1^2}{2}.$$

Induction step: Suppose we know that $P(n)$ is true for some integer n that is equal or greater than 1. We now show that this implies $P(n+1)$ is true. We have that

$$\sum_{i=1}^{n+1}\left(i-\frac{1}{2}\right)=\left(\sum_{i=1}^{n}\left(i-\frac{1}{2}\right)\right)+\left((n+1)-\frac{1}{2}\right)=\frac{n^2}{2}+\frac{2n+1}{2}=\frac{(n+1)^2}{2}.$$