

A note on basic mathematics

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1 Naive set theory

A set is a collection of objects. An object a belonging to A , is called an *element* of A . We write $a \in A$ to denote that a belongs to A .

We are especially interested in certain sets of numbers:

- The natural numbers:

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

- The integers (whole numbers):

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

- The rational numbers:

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N} \right\}$$

Two rational numbers $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are equal, if $a_1 \cdot b_2 = b_1 \cdot a_2$.

- The real numbers denoted \mathbb{R} . It is difficult to give a short and easy to understand description of real numbers. Naively we could say that real numbers are all the numbers missing from the rational numbers. A more correct definition would be “all the numbers that can be approximated arbitrarily precise by a sequence of rational numbers”, or “the algebraic completion of the rational numbers”. Neither of these are easy to get to, so we shall stick with the first, naive, description.¹

2 Rules for addition and multiplication

Addition and multiplication satisfy a number of rules that allow us to simplify arithmetic expressions.

¹In reality, these definition are not *that* hard to get to. An interested student would be able to get to them in a couple of weeks

1. Associativity: Both addition and multiplication are *associative*, i.e. for all $a, b, c \in \mathbb{R}$, we have that

$$\begin{aligned}a + (b + c) &= (a + b) + c \\a \cdot (b \cdot c) &= (a \cdot b) \cdot c\end{aligned}$$

For example, $2 + (3 + 4) = (2 + 3) + 4 = 9$

2. Commutativity: Both addition and multiplication are *commutative*, i.e. for all $a, b \in \mathbb{R}$, we have that

$$\begin{aligned}a + b &= b + a \\a \cdot b &= b \cdot a\end{aligned}$$

For example, $4 + 2 = 2 + 4 = 6$.

3. Distributivity: Multiplication *distributes* over addition, i.e. for all $a, b, c \in \mathbb{R}$, we have that

$$\begin{aligned}a \cdot (b + c) &= a \cdot b + a \cdot c \\(b + c) \cdot a &= b \cdot a + c \cdot a\end{aligned}$$

4. Additive identity 0, i.e. for all $a \in \mathbb{R}$

$$a + 0 = a \quad \text{and} \quad 0 + a = a$$

5. Multiplicative identity 1, i.e. for all $a \in \mathbb{R}$

$$a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a$$

6. Additive inverse: Every $a \in \mathbb{R}$ has an *additive inverse* $-a$ such that

$$a + (-a) = 0$$

We also write $a - b$ instead of $a + (-b)$.

7. Multiplicative inverse: Every non-zero $a \in \mathbb{R}$ has a *multiplicative inverse* a^{-1} (also written $\frac{1}{a}$) such that

$$a \cdot a^{-1} = 1$$

We also write $\frac{a}{b}$ or a/b (“ a divided by b ”) instead of $a \cdot b^{-1}$.

3 Fractions

A *fraction* is a powerful way to describe division of two numbers. Let $a, b \in \mathbb{R}$ with $b \neq 0$. We then write

$$\frac{a}{b}$$

We call a the *numerator* and b the *denominator*.

3.1 Rules for fractions

1. Addition and subtraction with common denominator:

$$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$$

where $a, b, c \in \mathbb{R}$ with $b \neq 0$.

2. The general formula for addition and subtraction is:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

where $a, b, c, d \in \mathbb{R}$ with $b, d \neq 0$.

3. General formula for multiplication:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

where $a, b, c, d \in \mathbb{R}$ with $b, d \neq 0$.

4. General formula for division:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

where $a, b, c, d \in \mathbb{R}$ with $b, c, d \neq 0$.

3.2 Simplifying fractions

A fraction is in *simplest form* when the numerator and denominator have no common factors (or divisors) other than 1.

Example: The rational number $\frac{2}{4}$ is not in its simplest form. But $\frac{1}{2}$ is.

Every rational number is equal to one in the simplest form. The following rules can be used to simplify fractions

1. If the numerator and the denominator are equal, $\frac{a}{a} = 1$

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \dots$$

2. A fraction $\frac{a}{1}$ with denominator 1 is written simply as a .

$$\frac{5}{1} = 5 \quad \frac{24}{1} = 24 \quad \frac{-7}{1} = -7$$

3. A fraction $\frac{0}{a}$ is equal to 0.

$$\frac{0}{8} = 0 \quad \frac{0}{56} = 0 \quad \frac{0}{-11} = 0$$

4. **Note:** A fraction $\frac{a}{0}$ is undefined

$$\frac{7}{0} \text{ is undefined} \quad \frac{-18}{0} \text{ is undefined}$$

5. Let a, m, n be real numbers with $a, n \neq 0$. Then

$$\frac{a \cdot m}{a \cdot n} = \frac{m}{n}$$

4 Exponentiation

Let $n \in \mathbb{N}$ be a natural numbers, and $a \in \mathbb{R}$ a real. Define

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}, \quad a^0 = 1, \quad a^{-n} = \frac{1}{\underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}}$$

We call a the *base number* and n the *exponent*.

NOTE: This is the *definition* of what exponents mean. Hence any time we see a^b we can substitute the above if we want.

4.1 Rules for exponentiation:

In the following, let $a, b \in \mathbb{R}$, while $m, n \in \mathbb{N}$.

1. Multiplication with common base number:

$$a^n \cdot a^m = a^{n+m}$$

2. Division with common base number:

$$\frac{a^n}{a^m} = a^{n-m}$$

3. Multiplication with different base, same exponent:

$$a^n \cdot b^n = (a \cdot b)^n$$

4. Division with different base, same exponent:

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

5. Exponentiation of exponentiation :

$$(a^n)^m = a^{n \cdot m}$$

4.2 Rational exponents

In the following, let $a \in \mathbb{R}_+$, and $m, n \in \mathbb{N}$.

Definition: We define exponentiation with rational exponents as:

$$a^{m/n} = \sqrt[n]{a^m}$$

The above rules for natural exponents are also valid for rational exponents

5 Binomial theorems for squares

Let $a, b \in \mathbb{R}$ be given. Then we have the following:

First formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Second formula:

$$(a - b)^2 = a^2 - 2ab + b^2$$

Third formula:

$$(a + b)(a - b) = a^2 - b^2$$

6 Equations

An equation is a *predicate* with one or more free variables (typically denoted by x, y, z, \dots)

The goal in solving an equation is to get concrete values for the free variables, such that the equations holds for these concrete values.

We talk about the *variable set* as the set each variable comes from. For instance, which values can x take in the following?

$$\frac{1}{x} = \sqrt{x} \quad \text{and} \quad \frac{x+5}{x+2} = \frac{6}{2x+4}$$

The basic technique is applying **identical** algebraic transformations on both sides of the equality sign.

We denote the set of *all solutions* as the solution set. There may be one, several, none or infinitely many solutions to any given equation.

6.1 An example

We look at the equation:

$$\frac{x}{\sqrt{4x-1}} = \frac{1}{2}$$

What values can x take?

$$4x - 1 > 0 \Leftrightarrow x > \frac{1}{4}$$

We can now do the following transformations:

$$\begin{aligned} \frac{x}{\sqrt{4x-1}} = \frac{1}{2} &\Rightarrow \frac{x^2}{4x-1} = \frac{1}{4} \\ &\Leftrightarrow x^2 = \frac{1}{4}(4x-1) \\ &\Leftrightarrow 4x^2 = 4x-1 \\ &\Leftrightarrow 4x^2 - 4x + 1 = 0 \\ &\Leftrightarrow (2x-1)^2 = 0 \\ &\Leftrightarrow 2x-1 = 0 \\ &\Leftrightarrow 2x = 1 \\ &\Leftrightarrow x = \frac{1}{2} \end{aligned}$$

We note that $x = \frac{1}{2} > \frac{1}{4}$, and hence we can accept the solution.

6.2 Absolute value

The *absolute value* of a real number a , written as $|a|$, is defined as follows: If a is negative, then $|a| = -a$, otherwise $|a| = a$. That is, $|a|$ is always positive.

Examples:

1. $|3| = 3$ and $|-3| = 3$.

2. Consider the equation $|x + 5| = 2$. What values can x take?

By the definition of absolute value, we know that

$$x + 5 = 2 \text{ or } x + 5 = -2$$

Therefore,

$$x = -3 \text{ or } x = -7$$

7 Systems of linear equations:

Equations can be viewed as *restrictions* on the free variables x, y, z, \dots . Given two equations:

$$y = -2x + 3 \quad \text{and} \quad y = -\frac{3}{2}x + 2$$

We want to find values for x and y such that both equations are solved **simultaneously**.

The basic technique for systems of equations, is the do *substitution* from one equation to the other. We isolate one variable, and the substitute it into the other. In the above example the variable y is isolated in both equations and hence we can substitute in from one to the other:

$$\begin{aligned} y = -2x + 3 &\Leftrightarrow -\frac{3}{2}x + 2 = -2x + 3 \\ &\Leftrightarrow -\frac{3}{2}x = -2x + 1 \\ &\Leftrightarrow -3x = -4x + 2 \\ &\Leftrightarrow x = 2 \end{aligned}$$

We can now substitute the (concrete) value for x into the second equation, to get a (concrete) value for y :

$$\begin{aligned} y = -\frac{3}{2}x + 2 &\Leftrightarrow y = -\frac{3}{2} \cdot 2 + 2 \\ &\Leftrightarrow y = -3 + 2 \\ &\Leftrightarrow y = -1 \end{aligned}$$

To check our solution we insert the concrete values into our two equations, and see if they reduce to identities:

$$y = -2x + 3 \Leftrightarrow -1 = -2 \cdot 2 + 3 \Leftrightarrow -1 = -4 + 3$$

and

$$y = -\frac{3}{2}x + 2 \Leftrightarrow -1 = -\frac{3}{2} \cdot 2 + 2 \Leftrightarrow -1 = -3 + 2$$

Since this holds we accept the solution $x = 2, y = -1$.

8 Quadratic equations

A quadratic equation is a equation on the form:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Theorem: The equation:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

has the solutions:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 - 4ac \geq 0$.

8.1 Substitution in quadratic equations

Sometimes we can rearrange higher order equations to look like quadratic equations. We call this principle **substitution**.

In general, we substitute one variable for another to reduce *the degree*² of the equation. If we look at

$$x^4 - x^2 - 12 = 0, \quad \text{and} \quad 2x^4 - 5x^2 + \frac{9}{8} = 0$$

we see that if we choose $z = x^2$ we can change these to:

$$z^2 - z - 12 = 0, \quad \text{and} \quad 2z^2 - 5z + \frac{9}{8} = 0$$

Both equations are now quadratic, and hence we can use the theorem for quadratic equations. We can then obtain a solution for z and use the solution to get a solution for x .

Note that we will often get more than two solutions when doing equations of higher degrees. In general, for a n degree equation we can get up to n solutions.

²The degree of an equation is the highest exponentiation of a free variable. In the examples the degree is 4