Discrete Mathematics - Assignment 3

1

Description

Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined as $f(n) = 2n - 5$ for all integers n.

Note: To justify your answers in (b) and (c), if the answer is positive you need to provide a proof, and if it is negative, give a counterexample.

a) List three elements that are in the range of f

Solution

$$
\begin{aligned}
f(2) &= 2 * 2 - 5 \\
&= 4 - 5 \\
&= -1 \\
f(3) &= 2 * 3 - 5 \\
&= 6 - 5 \\
&= 1 \\
f(4) &= 2 * 4 - 5 \\
&= 8 - 5 \\
&= 3
\end{aligned}
$$

The elements -1, 1 and 3 are in the range of f following that f is a function from $\mathbb{Z}$ (the set of all integers) to $\mathbb{Z}$.

b) Is f one-to-one? Justify your answer

Solution

If the function $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(n) = 2n - 5$ for each integer n, then f is one-to-one.

Proof:

Suppose n_1 and n_2 are integers such that $f(n_1) = f(n_2)$.

By definition of f, we have $f(n_1) = 2n_1 - 5$ and $f(n_2) = 2n_2 - 5$.

Since $f(n_1) = f(n_2)$, we have $2n_1 - 5 = 2n_2 - 5$.

Simplifying (adding 5 to both sides), we have $2n_1 = 2n_2$.

Dividing both sides by 2, we have $n_1 = n_2$.

Therefore, n_1 and n_2 are equal, and f is one-to-one.

□

Definition of one-to-one:

A function f from a set X to a set Y is one-to-one (or injective) if, and only if, for all elements $x_1, x_2 \in X$

$$F(x_1) = F(x_2) \implies x_1 = x_2$$

Symbolically:
$F : X \to Y$ is one-to-one $\iff \forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$ then $x_1 = x_2$

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c) Is f onto? Justify your answer

Solution

If the function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(n) = 2n - 5$ for each integer n, then f is **not** onto.

Counterexample:

The co-domain of n is $\mathbb{Z}$, and $0 \in \mathbb{Z}$.
But $f(n) \neq 0$ for any integer n.

For if $f(n) = 0$, then $2n - 5 = 0$, which implies $2n = 5$, which implies $n = 5/2$

But $5/2$ is not an integer. Hence there is no integer n for which $f(n) = 0$, and thus f is not onto.

$\square$

Definition of onto:

Let F be a function from a set X to a set Y. F is onto (or surjective) if, and only if, given any element $y \in Y$, there exists an element $x \in X$ with the property $y = F(x)$.

Symbolically:
$F : X \to Y$ is onto $\iff \forall y \in Y, \exists x \in X \text{ such that } F(x) = y$

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d) Write an explicit formula for the composition $f \circ f$

Solution

(Composing function f with itself)

$$\begin{aligned}(f \circ f)(n) &= f(f(n)) \\ &= f(2n - 5) \\ &= 2(2n - 5) - 5 \\ &= 4n - 10 - 5 \\ &= 4n - 15\end{aligned}$$

Thus, the explicit formula for the composition is the following:

$$f \circ f = 4n - 15$$

2

Description

Let $a_k = 2k - 5$ and $b_k = 2 - k$.

Simplify each expression to only use a single summation ($\sum$) or product ($\prod$) using the properties of summations and products listed below. List intermediate steps.

$$\sum_{k=m}^{n} a_k - 3 \cdot \sum_{k=m}^{n} b_k$$

$$\prod_{k=m}^{n} a_k \cdot \prod_{k=m}^{n} b_k$$

Theorem 5.1.1
If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
2. $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ (generalized distributive law)
3. $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$

Example:
Given sequences $a_k = k$ and $b_k = 2k + 1$, simplifying an expression $\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$ could be done in the following way:

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} k + 2 \cdot \sum_{k=m}^{n} (2k + 1)$$

by (substitution)

$$= \sum_{k=m}^{n} k + \sum_{k=m}^{n} 2 \cdot (2k + 1)$$

by (2)

$$= \sum_{k=m}^{n} k + \sum_{k=m}^{n} (4k + 2)$$

by (algebraic simplification)

$$= \sum_{k=m}^{n} (k + (4k + 2))$$

by (1)

$$= \sum_{k=m}^{n} (5k + 2)$$

by (algebraic simplification)

Solution

$$\sum_{k=m}^{n} a_k - 3 \cdot \sum_{k=m}^{n} b_k$$

by (substitution)

$$= \sum_{k=m}^{n} 2k - 5 - 3 \cdot \sum_{k=m}^{n} 2 - k$$

by (2)

$$= \sum_{k=m}^{n} 2k - 5 \cdot \sum_{k=m}^{n} -3 \cdot (2 - k)$$

by (algebraic simplification)

$$= \sum_{k=m}^{n} 2k - 5 \cdot \sum_{k=m}^{n} -6 + 3k$$

by (1)

$$= \sum_{k=m}^{n} (2k - 5 + 3k - 6)$$

by (algebraic simplification)

$$= \sum_{k=m}^{n} 5k - 11$$

$$\prod_{k=m}^{n} a_k \cdot \prod_{k=m}^{n} b_k$$

by (substitution)

$$= \prod_{k=m}^{n} 2k - 5 \cdot \prod_{k=m}^{n} 2 - k$$

by (3)

$$= \prod_{k=m}^{n} (2k - 5) \cdot (2 - k)$$

by (algebraic simplification)

$$= \prod_{k=m}^{n} (9k - 2k^2 - 10)$$

3

Description

Prove, using mathematical induction, that 3 divides $n^3 + 5n - 6$ for all integers $n \geq 0$.

Hint: You can use the fact that given $a|b$ and $a|c$, we can conclude $a|(b+c)$ for all integers a,b,c. Moreover, you can use the binomial theorem for exponent 3, which states that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ for all real numbers a,b.

Solution

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