# Functions (F)

| | Chapter | Lecture | Assignment |
|---|---|---|---|
| **Functions** | 7 | 5 | 3 |

**3.** (6 pts) Let the two functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$f(n) = (-1)^n \cdot n \qquad \text{for all } n \in \mathbb{Z}$$
$$g(n) = 2n \qquad \text{for all } n \in \mathbb{Z}$$

Which of the following statements is **false**?

☐ The range of $g$ is $\{n \in \mathbb{Z} \mid n \text{ is even}\}$.

☐ $f$ is onto.

☐ $f \circ g$ is one-to-one.

☐ The range of $g \circ f$ is $\{n \in \mathbb{Z} \mid n \text{ is even and } n \geq 0\}$.

**2.** (6 pts) Let the function $f : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$f(n) = 3n + 2 \qquad \text{for all } n \in \mathbb{Z}$$

Which of the following statements is **true**?

☐ $f$ is onto.

☐ $f$ is one-to-one.

☐ $(f \circ f)(n) = 9n + 4$.

☐ The range of $f$ is a finite set.

**2.** (6 pts) Let the two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$f(n) = n + 1 \qquad \text{for all } n \in \mathbb{N}$$
$$g(n) = 2 \cdot n \qquad \text{for all } n \in \mathbb{N}$$

where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$. Which of the following statements is **true**?

☐ The range of $f \circ g$ is $\{n \in \mathbb{N} \mid n \text{ is odd}\}$.

☐ The range of $f \circ g$ is $\{n \in \mathbb{N} \mid n \text{ is even}\}$.

☐ The range of $f \circ g$ is $\{n \in \mathbb{N} \mid n > 0\}$.

☐ $f \circ g$ is onto.

**3.** (6 pts) Let the two functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$f(n) = (-1)^n \cdot n \qquad \text{for all } n \in \mathbb{Z}$$
$$g(n) = 2n \qquad \text{for all } n \in \mathbb{Z}$$

Which of the following statements is **false**?

[A] The range of $g$ is $\{n \in \mathbb{Z} \mid n \text{ is even}\}$.

[B] $f$ is onto.

[C] $f \circ g$ is one-to-one.

☑ The range of $g \circ f$ is $\{n \in \mathbb{Z} \mid n \text{ is even and } n \geq 0\}$.

*Solution*: The range of $g \circ f$ is $\{n \in \mathbb{Z} \mid n \text{ is even}\}$.

**2.** (6 pts) Let the function $f : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$f(n) = 3n + 2 \qquad \text{for all } n \in \mathbb{Z}$$

Which of the following statements is **true**?

[A] $f$ is onto.

☑ $f$ is one-to-one.

[C] $(f \circ f)(n) = 9n + 4$.

[D] The range of $f$ is a finite set.

*Solution*: Let $n, m$ such that $f(m) = f(n)$, that is, $3m + 2 = 3n + 2$. By subtracting 2 on both sides of the equation we obtain $3m = 3n$, and by dividing by 3 on both sides of this new equation we obtain $m = n$.

**2.** (6 pts) Let the two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$f(n) = n + 1 \qquad \text{for all } n \in \mathbb{N}$$
$$g(n) = 2 \cdot n \qquad \text{for all } n \in \mathbb{N}$$

where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$. Which of the following statements is **true**?

☑ The range of $f \circ g$ is $\{n \in \mathbb{N} \mid n \text{ is odd}\}$.

[B] The range of $f \circ g$ is $\{n \in \mathbb{N} \mid n \text{ is even}\}$.

[C] The range of $f \circ g$ is $\{n \in \mathbb{N} \mid n > 0\}$.

[D] $f \circ g$ is onto.

**2.** (6 pts) Let the two functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$\begin{aligned} f(n) &= n^2 && \text{for all } n \in \mathbb{Z} \\ g(n) &= n + 1 && \text{for all } n \in \mathbb{Z} \end{aligned}$$

where $\mathbb{Z}$ is the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Which of the following statements is **true**?

☐ $f$ is onto.

☐ $g$ is a one-to-one correspondence.

☐ $f$ is one-to-one.

☐ $(f \circ g)(n) = n^2 + 1$ for all $n \in \mathbb{Z}$.

**2.** (6 pts) Let the two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be defined as follows:

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☐ $f$ is one-to-one.

☐ $(g \circ f)(n) = 2 \cdot n + 1$ for all $n \in \mathbb{N}$.

**5.** (2 pts) Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be recursively defined by

$$\begin{aligned} f(1) &= 1, \\ f(n) &= nf(n-1). \end{aligned}$$

What expression does the function calculate?

☐ $f(n) = \frac{n(n+1)}{2}$

☐ $f(n) = n!$

☐ $f(n) = 2^n$

☐ $f(n) = n^n$

**2.** (6 pts) Let the two functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$\begin{aligned} f(n) &= n^2 && \text{for all } n \in \mathbb{Z} \\ g(n) &= n + 1 && \text{for all } n \in \mathbb{Z} \end{aligned}$$

where $\mathbb{Z}$ is the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Which of the following statements is **true**?

[A] $f$ is onto.

☑ $g$ is a one-to-one correspondence.

[C] $f$ is one-to-one.

[D] $(f \circ g)(n) = n^2 + 1$ for all $n \in \mathbb{Z}$.

*Solution*: To show that $g$ is a one-to-one correspondence, we show that it is both one-to-one and onto: If $g(n) = g(m)$, then $n+1 = m+1$. By subtracting 1 from both sides of the latter equation we obtain that $n = m$. Hence, $g$ is one-to-one. Moreover, given any $n \in \mathbb{Z}$, we have that $g(n-1) = (n-1)+1 = n$. Hence, $g$ is onto.

**2.** (6 pts) Let the two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$\begin{aligned} f(n) &= n + 1 && \text{for all } n \in \mathbb{N} \\ g(n) &= 2 \cdot n && \text{for all } n \in \mathbb{N} \end{aligned}$$

where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$. Which of the following statements is **true**?

[A] $f$ is onto.

[B] $g$ is a one-to-one correspondence.

☑ $f$ is one-to-one.

[D] $(g \circ f)(n) = 2 \cdot n + 1$ for all $n \in \mathbb{N}$.

*Solution*: If $f(n) = f(m)$, then $n + 1 = m + 1$. By subtracting 1 from both sides of the latter equation we obtain that $n = m$. Hence, $f$ is one-to-one.

**5.** Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be recursively defined by

$$\begin{aligned} f(1) &= 1, \\ f(n) &= nf(n-1). \end{aligned}$$

What expression does the function calculate?

[A] $f(n) = \frac{n(n+1)}{2}$

☑ $f(n) = n!$

[C] $f(n) = 2^n$

[D] $f(n) = n^n$

**2.** (6 pts) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$f(n) = 2 \cdot (n \bmod 12) \qquad\qquad \text{for all } n \in \mathbb{Z}$$

where $\mathbb{Z}$ is the set of integers. Which of the following statements is **true**?

☐ $f$ is not a function because 12 and 24 have the same image.

☐ $f$ is one-to-one.

☐ The range of $f$ is has cardinality 12.

☐ $f$ is onto.

**3.** (6 pts) Let $f : \mathbb{N} \to \mathbb{N}$ be the function from the set of natural numbers to the set of natural numbers defined as follows:

$$f(n) = n \bmod 3$$

How many elements are there in the range of $f$?

☐ 6

☐ Infinite number of elements

☐ 3

☐ None; The range of $f$ is the empty set

**4.** (6 pts) How many one-to-one functions are there from the set $\{1, 2, 3\}$ to the set $\{a, b, c, d\}$?

☐ 12

☐ 24

☐ 36

☐ 48

**2.** (6 pts) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$f(n) = 2 \cdot (n \bmod 12) \qquad\qquad \text{for all } n \in \mathbb{Z}$$

where $\mathbb{Z}$ is the set of integers. Which of the following statements is **true**?

[A] $f$ is not a function because 12 and 24 have the same image.

[B] $f$ is one-to-one.

☑ The range of $f$ is has cardinality 12.

[D] $f$ is onto.

**3.** (6 pts) Let $f : \mathbb{N} \to \mathbb{N}$ be the function from the set of natural numbers to the set of natural numbers defined as follows:

$$f(n) = n \bmod 3$$

How many elements are there in the range of $f$?

[A] 6

[B] Infinite number of elements

☑ 3

[D] None; The range of $f$ is the empty set

*Solution*: The range of $f$ is the set $\{0, 1, 2\}$.

**4.** (6 pts) How many one-to-one functions are there from the set $\{1, 2, 3\}$ to the set $\{a, b, c, d\}$?

[A] 12

☑ 24

[C] 36

[D] 48

*Solution*: A one-to-one function from $\{1, 2, 3\}$ to $\{a, b, c, d\}$ is a 3-permutation of the set $\{a, b, c, d\}$. Hence, there are $P(n, r)$ such functions, with $n = 4$ and $r = 3$:

$$P(4, 3) = \frac{4!}{(4-3)!} = \frac{24}{1} = 24$$