The Consistency of Peano Arithmetic

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Abstract

Mathematics has always had an undoubtedly special place in the universe of things. However, among the countless ideas that regard it, one stands out for its incessant recurrence: the myth, partly attributable to everyday life, according to which everything which is mathematical must inevitably also be true. The very essence of this discipline seems to be what, more or less implicitly, provides a measure of certainty for all other sciences. And it is no coincidence that one of the most common synonyms of "it's certain" is "it's mathematical", to emphasize its objectivity as opposed to the subjective, which does not seem to earn the same ontological dignity.

But reality is far more complex than that: for example, there are a variety of theorems whose provability is based on the very subjective acceptance of the axiom of choice. Yet no one would dare to deny the status of a mathematical theorem to these. And it is precisely this subjectivism that constitutes, at once, both a strength and a possible weakness for mathematics itself: while it opens the way to the innumerable possibilities of human (not only?) creativity, it let us face a frightening question. If even the axioms, the premises of every deduction, are subjective and therefore hand-picked by one or more individuals whose fallibility we cannot rule out, how do we guarantee the meaningfulness of what comes after them? What if, given premises that seem reasonable to us and on which we have based all of our efforts, we could prove *correctly* both a certain statement and its negation?

Could we consider "mathematical" all this? By the logical principle of *ex false*, every thinkable statement would be derived correctly from the above premises and, in fact, given the axioms to be true, every single expressible statement would be considered true. What mathematics could be done if 2 were both even and odd at the same time? What if 0 and 1 were simultaneously the same number and two different numbers?

These reflections are the roots of a story frequently told: that of the foundations of mathematics, which accompanied much of the last century. Its fame is due to the unsettling results to which the simple questions from which it starts led. We'll briefly set the scene.

The 20 and 30 of the $'900$ are driven by the well-known *Hilbert's program*, partly justified by the *Russell's Paradox*, discovered not long before, which pointed out the contradictory nature of elementary set theory. The idea was to assure mathematics a solid foundation from which to start: it was intended to justify the meaningfulness of infinite mathematics using only finitary tools

whose safety was not in question, a concept never fully clarified by Hilbert. Another well-known part of this story is the somewhat destructive contribution of Gödel: Hilbert's goal has been proved to be impossible to achieve. This is because if infinite mathematics is a proper extension of finitary mathematics and the latter contains sufficient, in fact minimal, arithmetic capabilities, the *incompleteness' theorems* guarantee the impossibility of exhibiting a finitary proof of consistency of infinite mathematics that preserves the consistency of finitary mathematics.

Obviously, these theorems not only thwart Hilbert's Program, but place great limitations to any research that seeks, in some other way, to ensure the soundness of the foundations of mathematics. A solution therefore had to be sought elsewhere, going down a road that the theorems of Gödel had not already interrupted.

In 1936 Gerhard Gentzen published a revolutionary demonstration, that of the Consistency of Peano's Arithmetic, paving the way for a new discipline called Proof Theory. The first version of his paper, that of 1936, was in some ways still immature, although it already contained all the fundamental ideas. A clearer and more refined version was published in 1938 and can be found, along with the first, in his "textitCollected Papers": it is this second one on which we will rely. The purpose of this thesis is, in fact, to retrace the steps of that demonstration from its logical and mathematical foundations.

During the writing of this paper we'll take some deviations from the path followed by Gentzen, incorporating new keys to the original concepts and some personal reworking. One example is the main theorem of section 2.1: the 1938 paper adopts an unusual method for representing ordinal numbers, and this choice seems reasonable since, in doing so, he introduces a structure similar to that with which we formalize the proofs; however, for the proof of a fundamental property of theirs, this paper refers back to the 1936 version, in which the usual set representation was adopted. I have taken the opportunity to present my own version of the demonstration hoping that the exposition would somehow gain additional value from it.

The first chapter develops around the formalization of Peano's Arithmetic, both in its mathematical aspects and in the logical rules governing deductions. These constitute both the indispensable tools for dealing with the actual demonstration and a subject of considerable expository interest. The second is a more technical chapter which contains the details of the demonstration. It opens by outlining a scheme of the path that is to be followed. Unlike the original article, which postponed the section abount ordinal numbers, here we preferred to introduce them at the beginning hoping that they would provide an additional guideline for grasping the meaning and beauty of the ideas in Gentzen's proof.

What should remain with the reader, in my opinion, is the beauty of a mathematical achievement that proves something *on* mathematics and lays the foundation for a variety of branches from foundations of mathematics to computer language theory, proof mining and even automated deduction.