31761 – Renewables in Electricity Markets

Lecture 10: Sequential Games and Bilevel Models*

*A significant portion of the content is inspired by Jalal's slides in Course #31792

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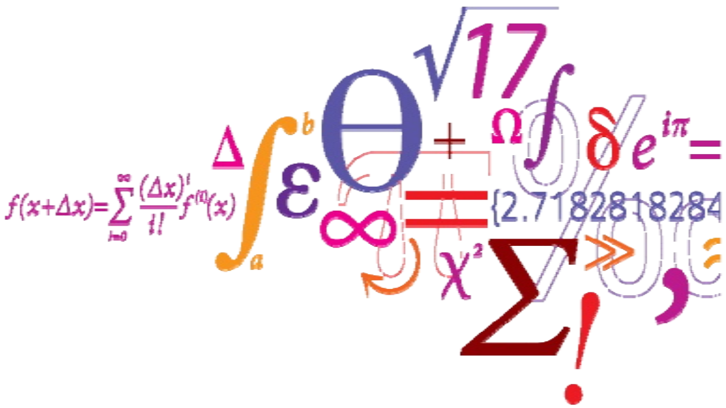

April, 2022

Technical University of Denmark
Department of Wind and Energy Systems

Announcement

Note about Assignment 2 Step 3:

You DO need to provide the code for this step, and the payoff matrix should be directly generated in the results of your program. Please be reminded that programming account for 25% of your whole grade, and the readability of your code is an important part of it.

For this step specifically, given the assignment of the generator bundles as an input, the model should automatically run all scenarios needed for the payoff matrix and generates the payoff matrix as an output. Using your code, we should be able to manually change your assignment of bundles and rerun your code to generate updated payoff matrices

Clarification

Recal exercise from Lecture 8:

Practice **iterated elimination of dominated strategies** to find the Nash equilibrium!

Any equilibrium in a reduced game after iterated elimination of dominated strategies **is a Nash equilibrium** in the original game*.

Player B

Player A

Payoff Matrix	Left	Center	Right
Top	13 4	0 11	4 12
Middle	-1 5	-3 9	17 -1
Bottom	8 6	6 0	4 1

Clarification

Recal exercise from Lecture 8:

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Player A

Payoff Matrix	Left	Center	Right
Top	13 4	0 11	4 12
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Bottom	8 6	6 10	4 13

Any equilibrium in a reduced game after iterated elimination of dominated strategies **is a Nash equilibrium** in the original game*.

Learning Objectives

Today's lecture will (re-)introduce **sequential (Stackelberg) games**, and their relevance in the energy market.

After this lecture, you should be able to

1. Provide the definition of a **Stackelberg equilibrium** in a sequential game, and identify it in a payoff matrix;
2. Formulate a sequential game of an energy market application using a **bilevel model**, identify the **leader** and the **follower**, and match the leader with the **upper level problem**, and the follower with the **lower level problem**;
3. Derive the **Karush-Kuhn-Tucker conditions (KKTs)** of the lower lever problem of the bilevel model, given which, update the upper level problem as a **Mathematical Program with Complementarity Constraint (MPCC).**

Recap

a) **“Stag Hunt”** game:

		Hunter B	
		Stag	Hare
Hunter A	Stag	3 3	2 0
	Hare	0 2	1 1

b) **“Matching pennies”** game:

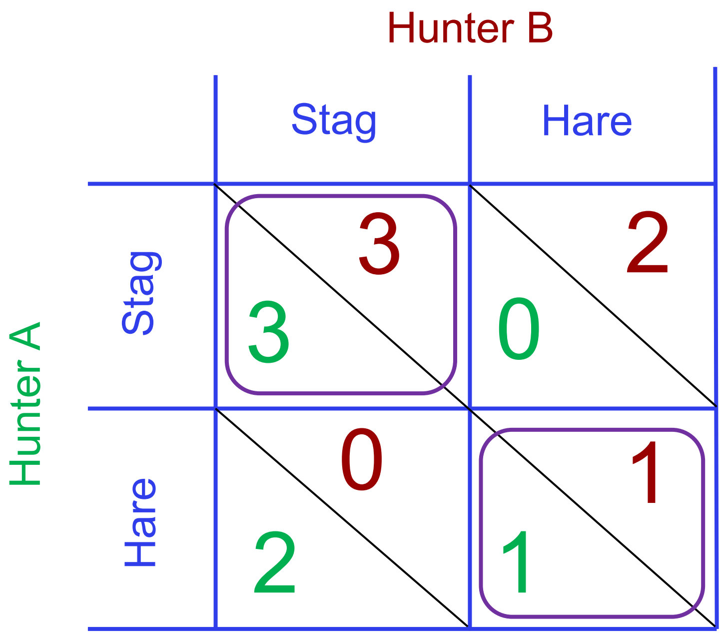
		ODD	
		Heads	Tails
EVEN	Heads	-1 1	1 -1
	Tails	1 -1	-1 1

Discussion (3 min)

1. What are the stories behind these two simultaneous games? How do they translate to their payoff matrices?
2. What are the NE’s in these two games in pure strategies if any?
3. In both games, if one of the players gets to move first knowing that the other player will give their best response, will it change the equilibrium / create new equilibria?

Recap

a) **“Stag Hunt”** game:



		Hunter B	
		Stag	Hare
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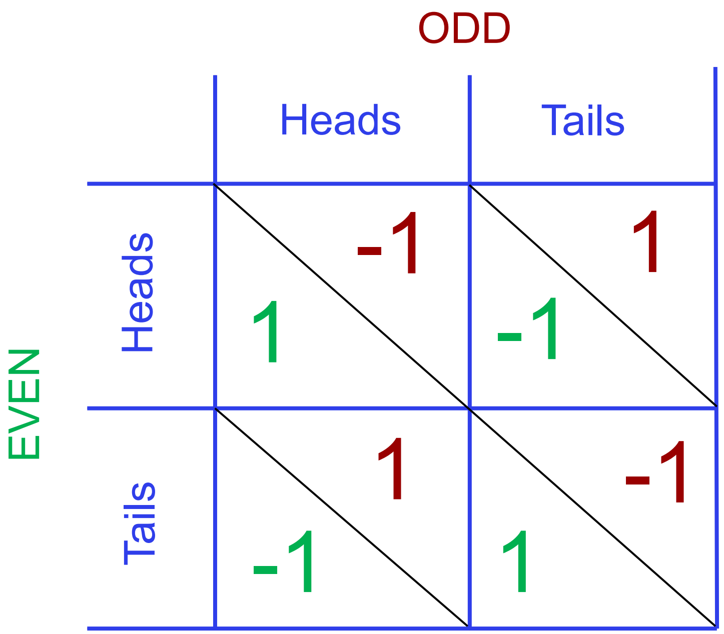
b) **“Matching pennies”** game:



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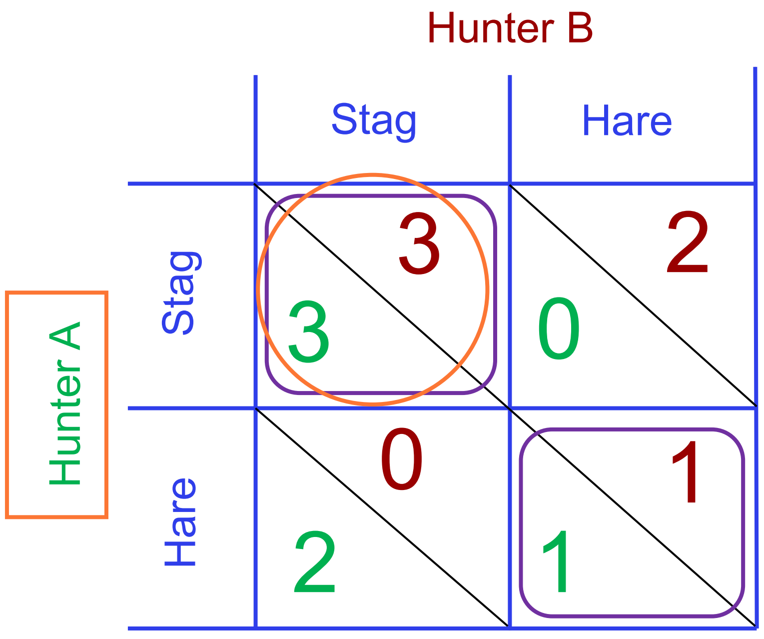

		Hunter B	
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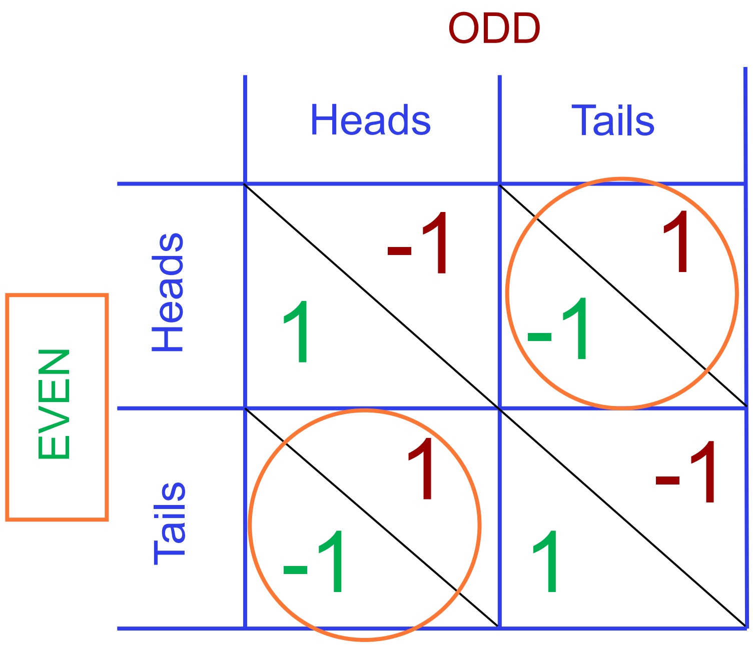
b) **“Matching pennies”** game:



		ODD	
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Discussion (3 min)

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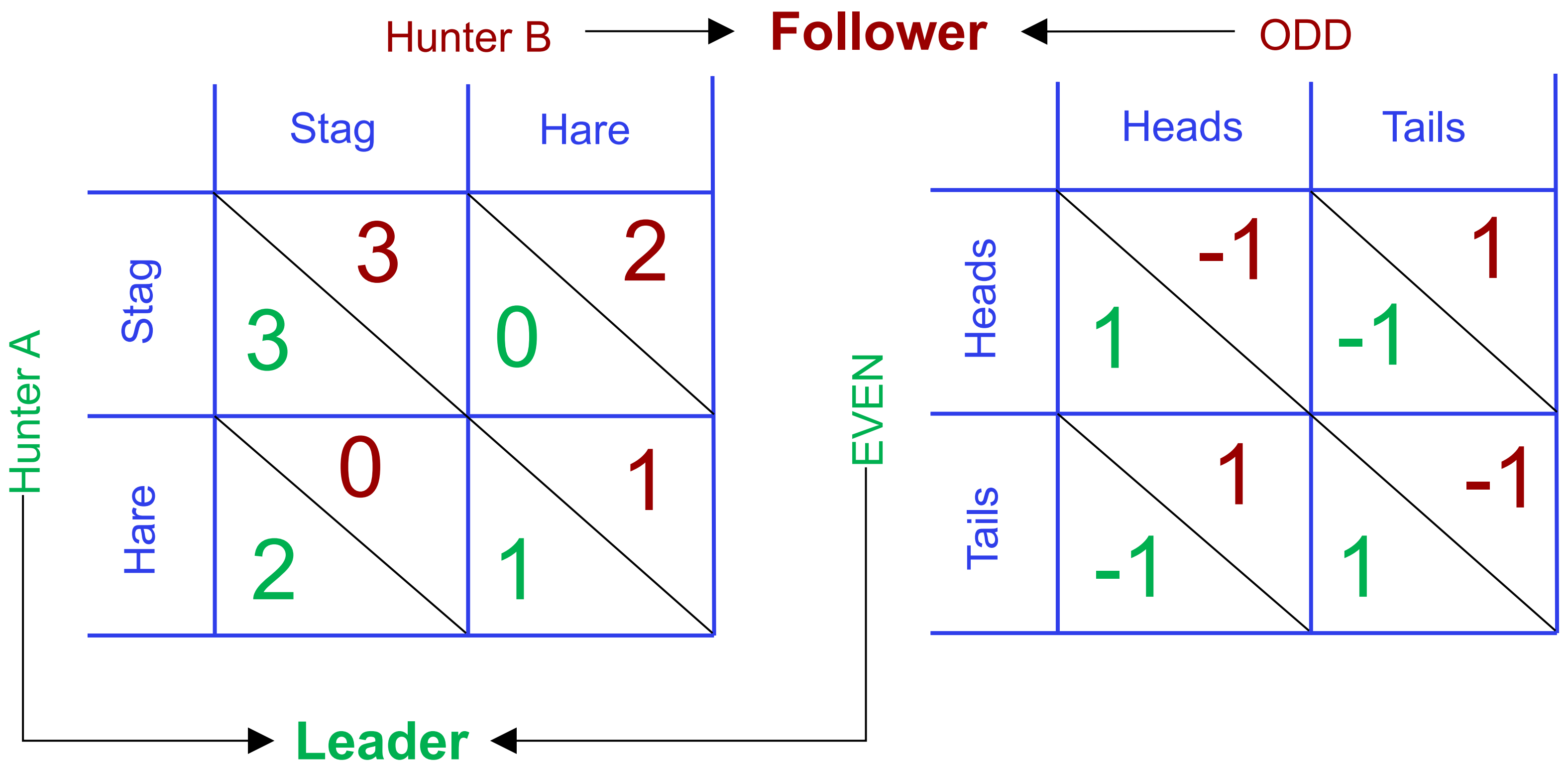
Game 4

a) **“Stag Hunt”** game:

b) **“Matching pennies”** game:



Hunter B → **Follower** ← ODD

Hunter A	Stag	Hare
Stag	3 3	2 0
Hare	0 2	1 1

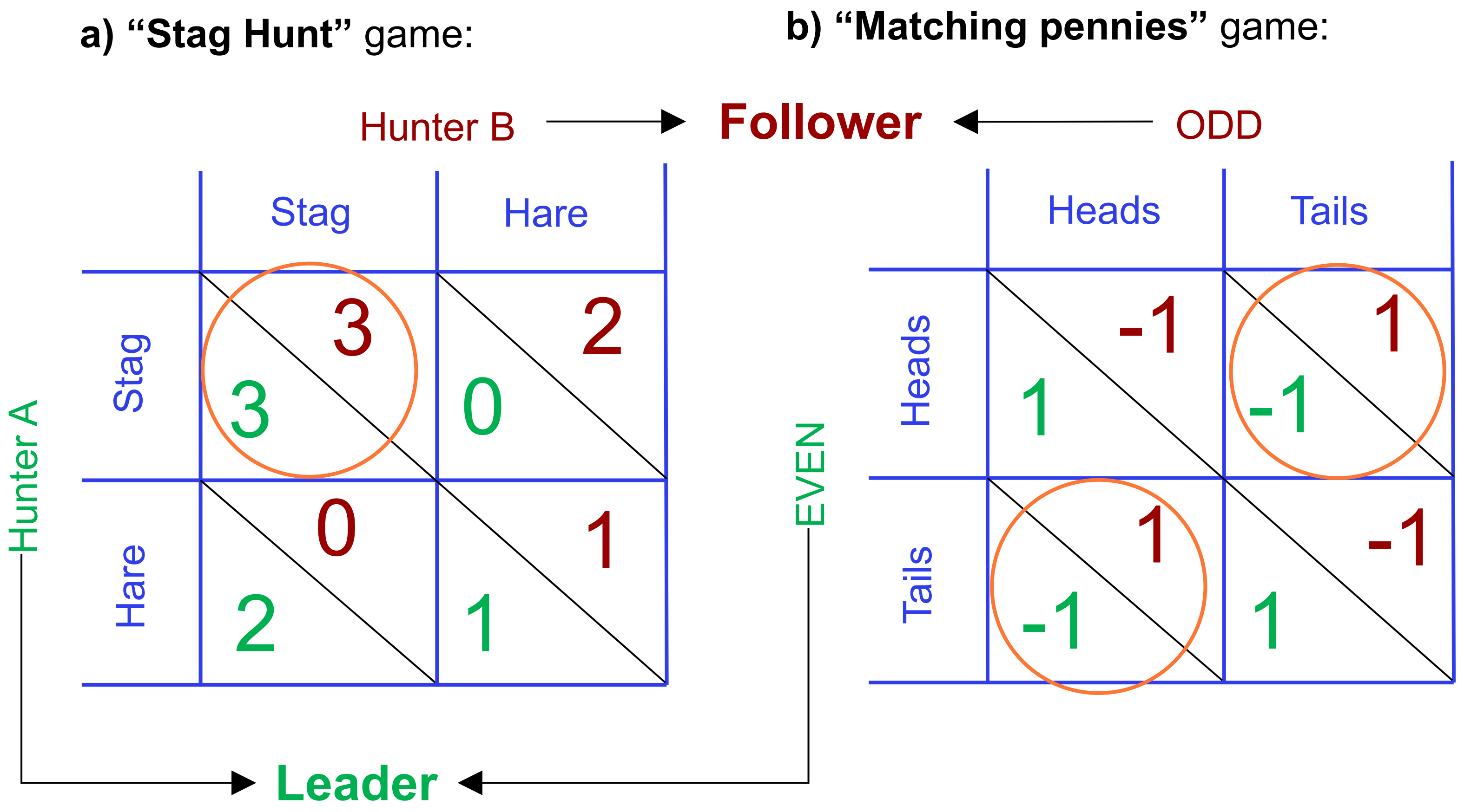
EVEN	Heads	Tails
Heads	-1 1	1 -1
Tails	1 -1	-1 1

Hunter A → **Leader** ← EVEN

Sequential Game:

a game where one player chooses their action before the others choose theirs.

Game 4



a) **“Stag Hunt”** game:

b) **“Matching pennies”** game:

Hunter B → **Follower** ← ODD

Hunter A → **Leader** ← EVEN

		Stag	Hare
Hunter A	Stag	3 3	2 0
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		Heads	Tails
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Sequential Game:

a game where one player chooses their action before the others choose theirs.

Stackelberg Equilibrium:

the strategy profile that serves best each player, given the best action of the leading player.

Game 4

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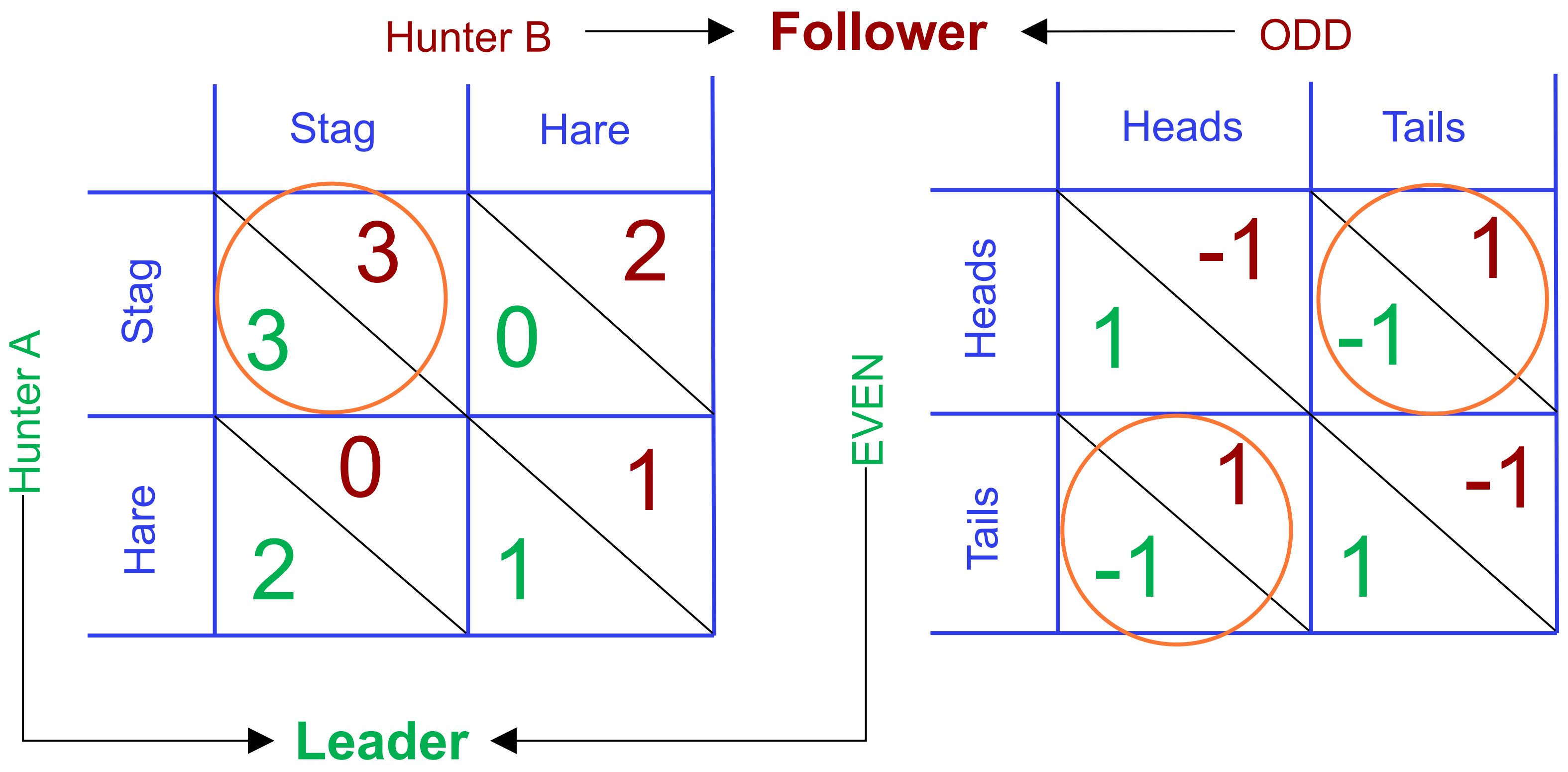
b) **“Matching pennies”** game:



Sequential Game:

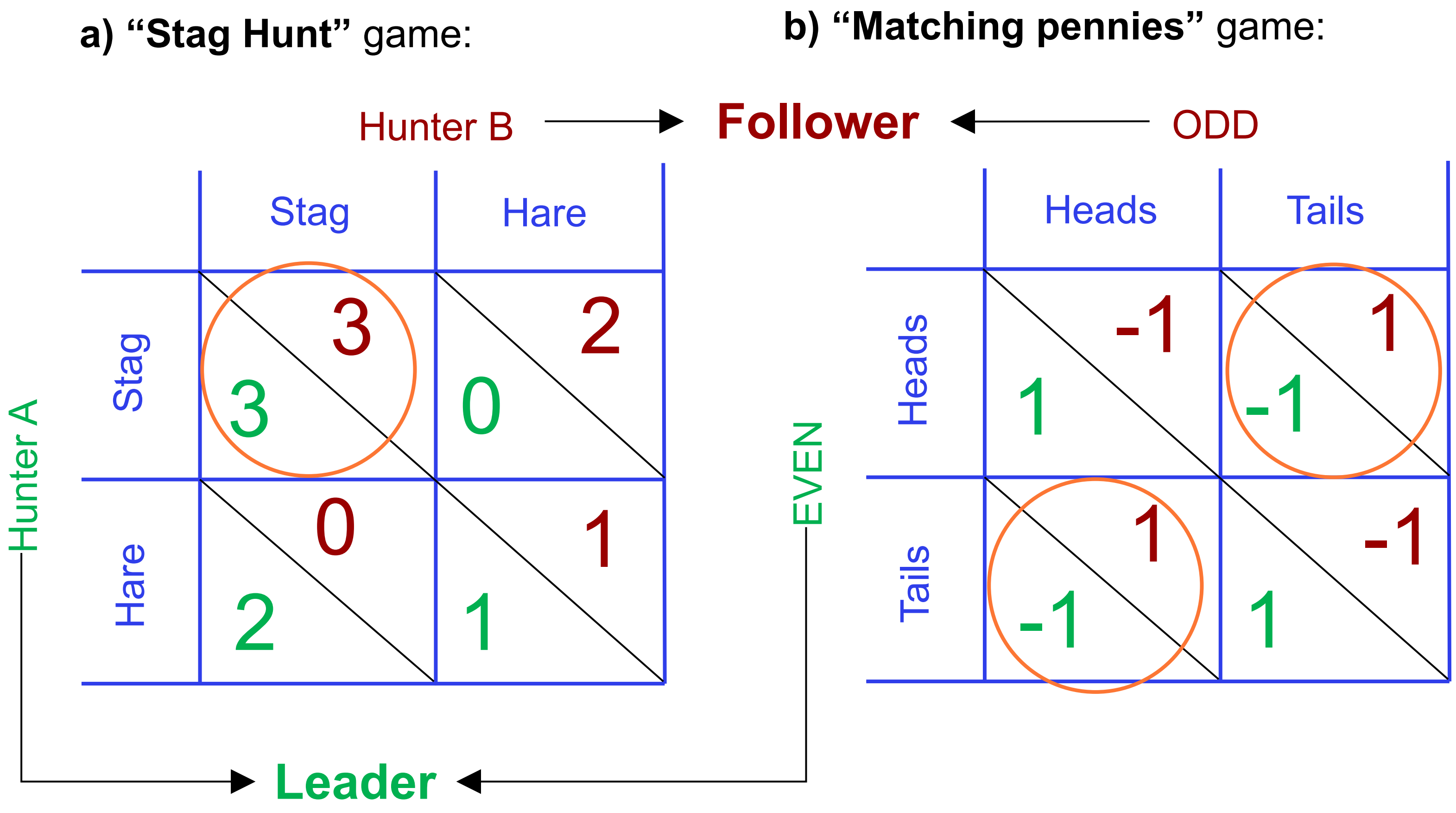
a game where one player chooses their action before the others choose theirs.

Stackelberg Equilibrium:

the strategy profile that serves best each player, given the best action of the leading player.

Can the leader exercise a mixed strategy to achieve better outcomes?

Game 4



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b) **“Matching pennies”** game:

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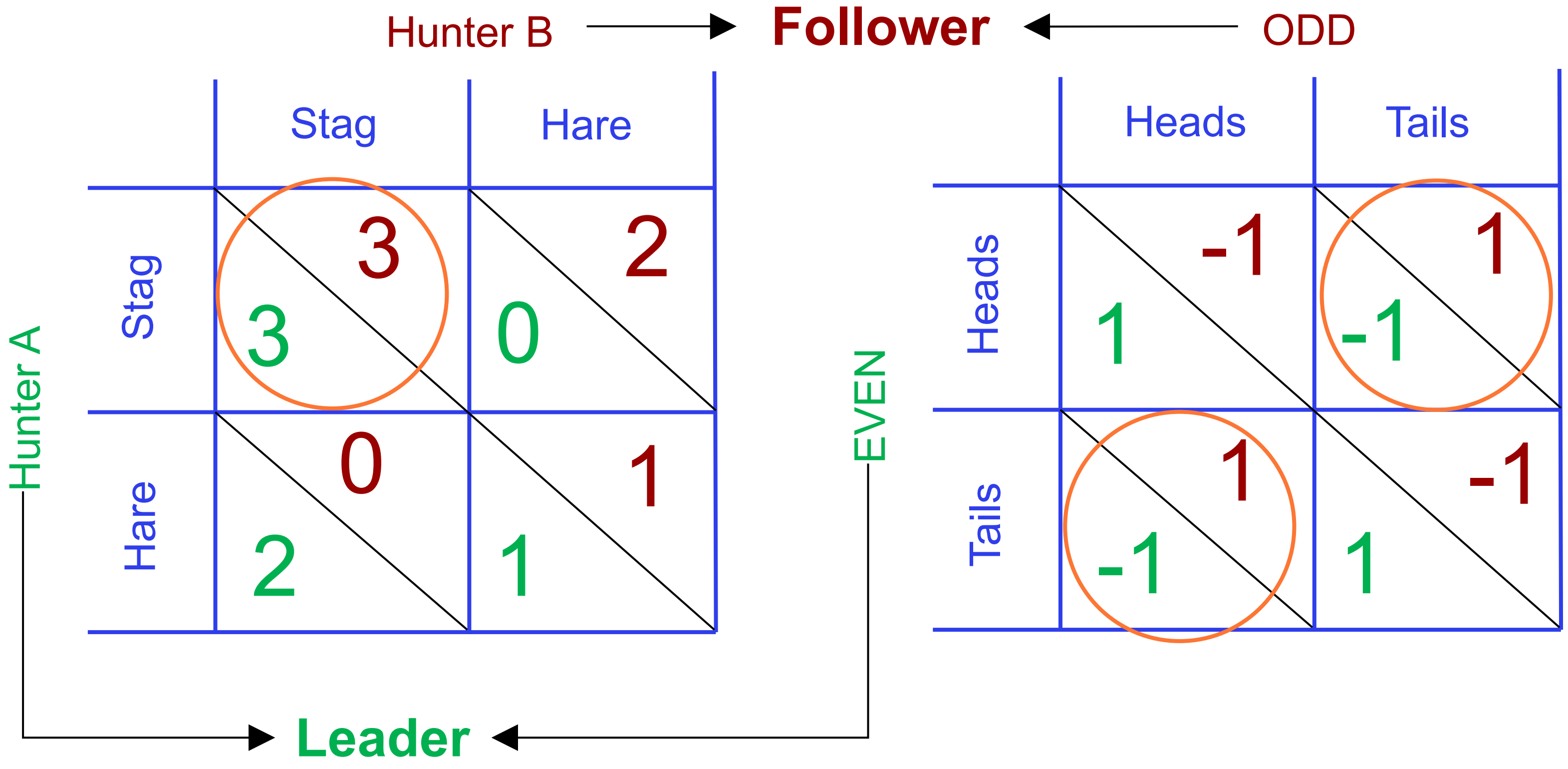
Who benefits from this sequential set-up, the leader or the follower?

Game 4

a) **"Stag Hunt"** game:

b) **"Matching pennies"** game:



	Stag	Hare
Stag	3 3	2 0
Hare	0 2	1 1

	Heads	Tails
Heads	-1 1	1 -1
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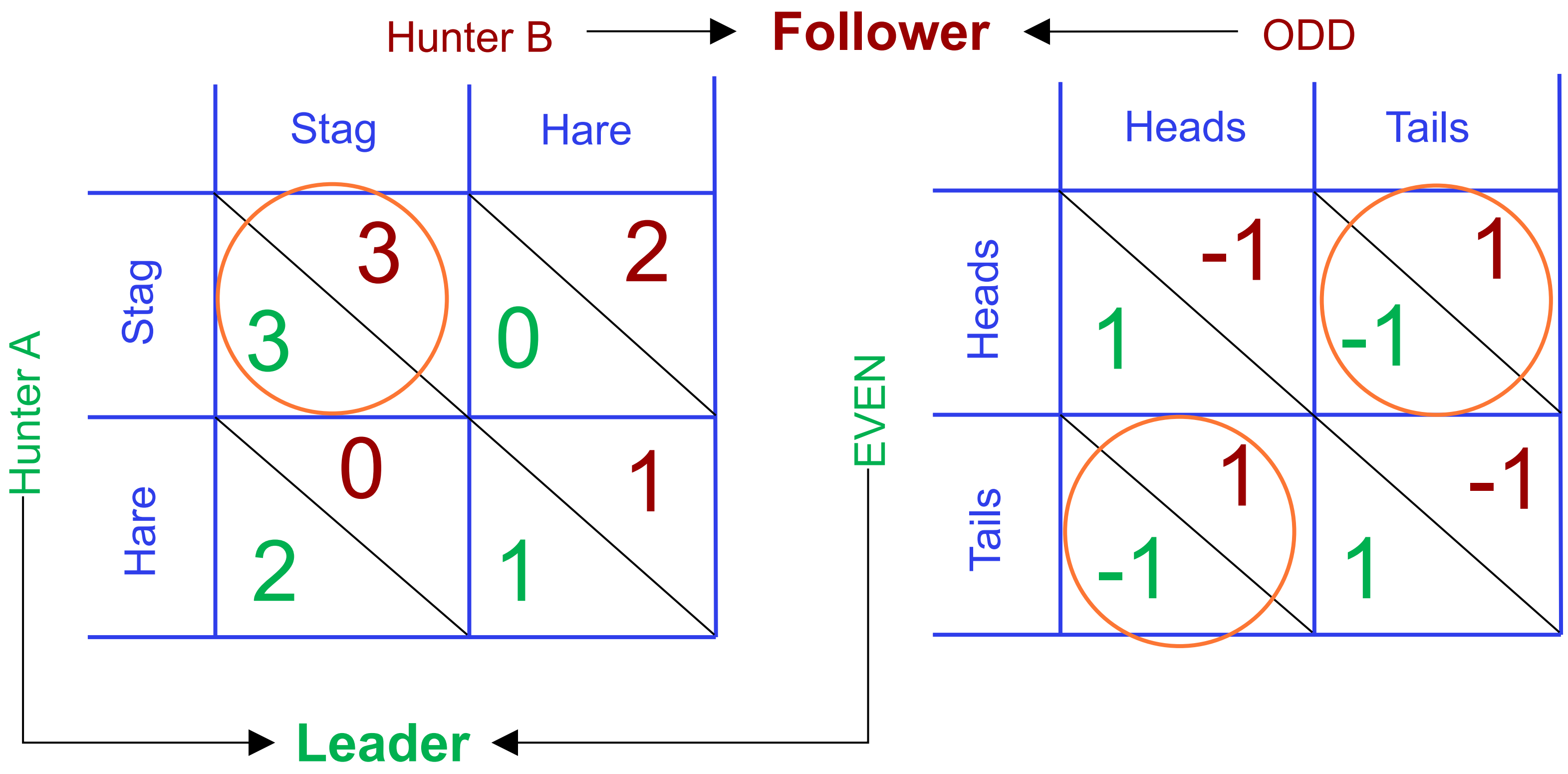
Who benefits from this sequential set-up, the leader or the follower?

a) Both.
b) Follower.

Game 4

a) **“Stag Hunt”** game:

b) **“Matching pennies”** game:



Hunter B → **Follower** ← ODD

Hunter A → **Leader** ← EVEN

Hunter A		Stag	Hare
	Stag	3 3	2 0
	Hare	0 2	1 1

EVEN		Heads	Tails
	Heads	-1 1	1 -1
	Tails	1 -1	-1 1

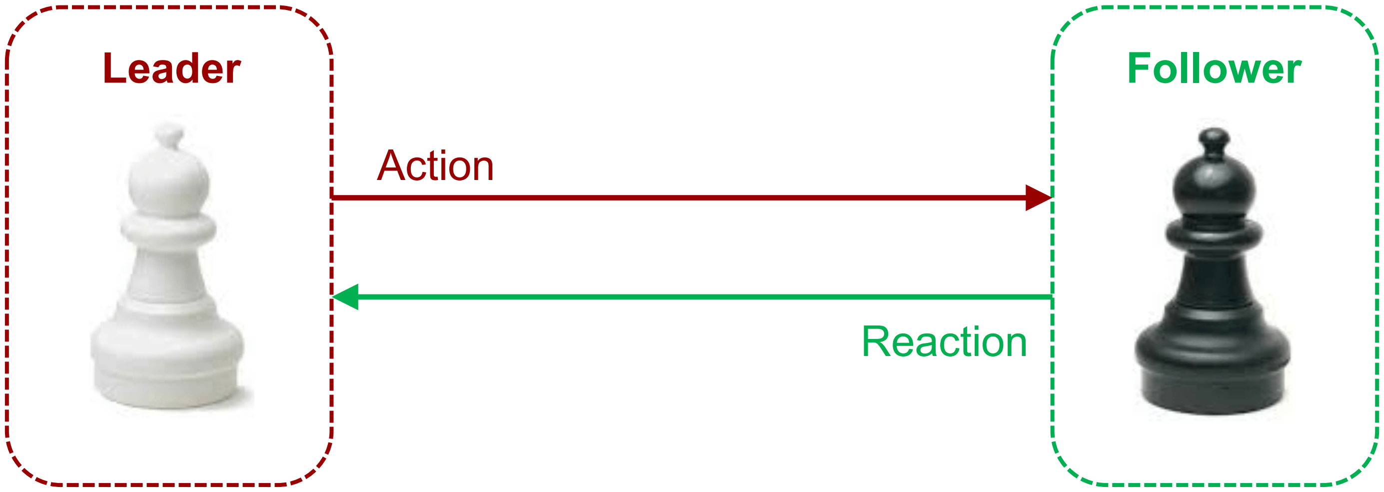
Who benefits from this sequential set-up, the leader or the follower?

a) Both.
b) Follower.

Are there situations where the leader benefits more and the follower benefits less compared to the NE?
→ Assignment 2 Step 5

Sequential Games

Two-stage sequential game



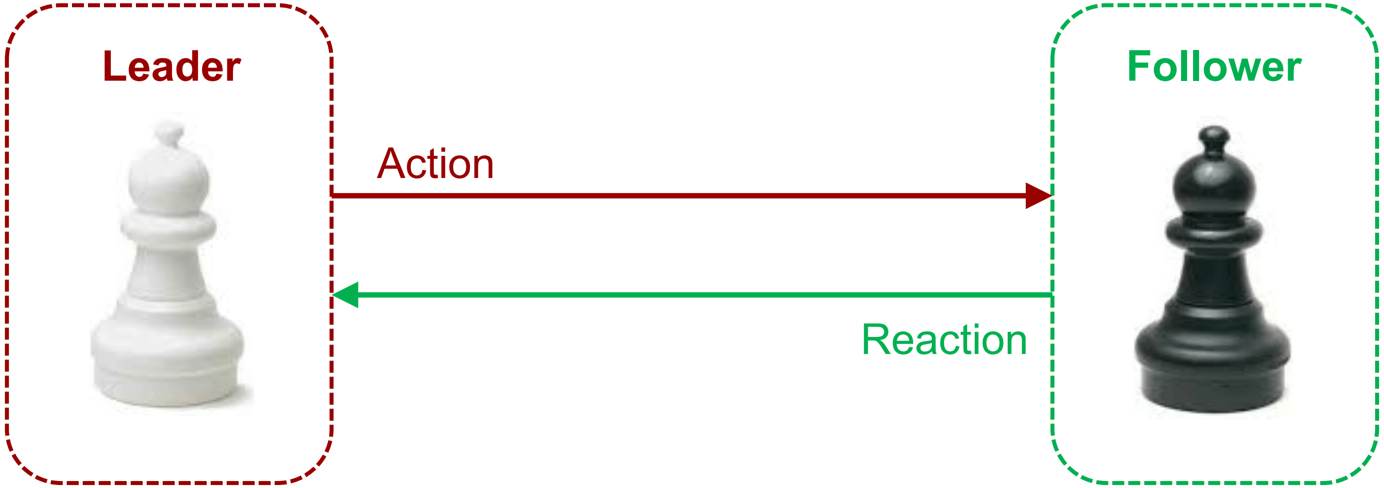
- Plays first

- Plays second

*They are also called **Dynamic Games** and **Stackelberg games**

Sequential Games

Two-stage sequential game



Leader

Action

Reaction

Follower

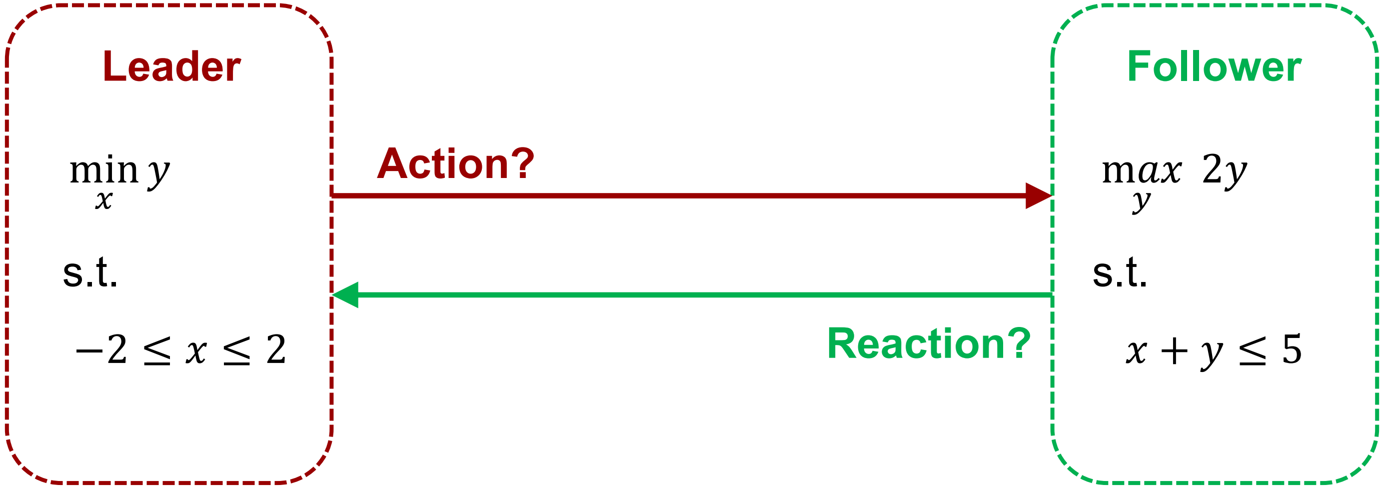
- Plays first
- Action influences optimal reaction of the follower.
- Tries to anticipate the follower’s reaction

- Plays second
- Reaction influences the payoffs of both the follower and the leader.

How should the leader play?

Sequential Games

Two-stage sequential game



Leader

$$\min_{x} y$$

s.t.

$$-2 \leq x \leq 2$$

Action?

Reaction?

Follower

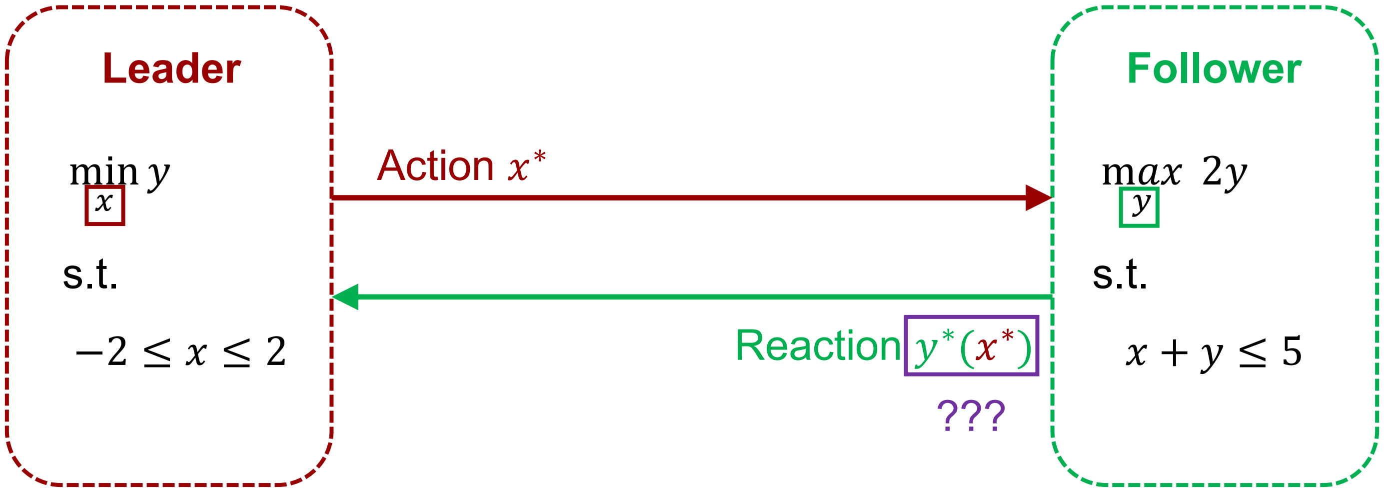
$$\max_{y} \; 2y$$

s.t.

$$x + y \leq 5$$

Sequential Games

Two-stage sequential game



Leader

$$\min_{\boxed{x}} y$$

s.t.

$$-2 \leq x \leq 2$$

Action x^*

Reaction $\boxed{y^*(x^*)}$

???

Follower

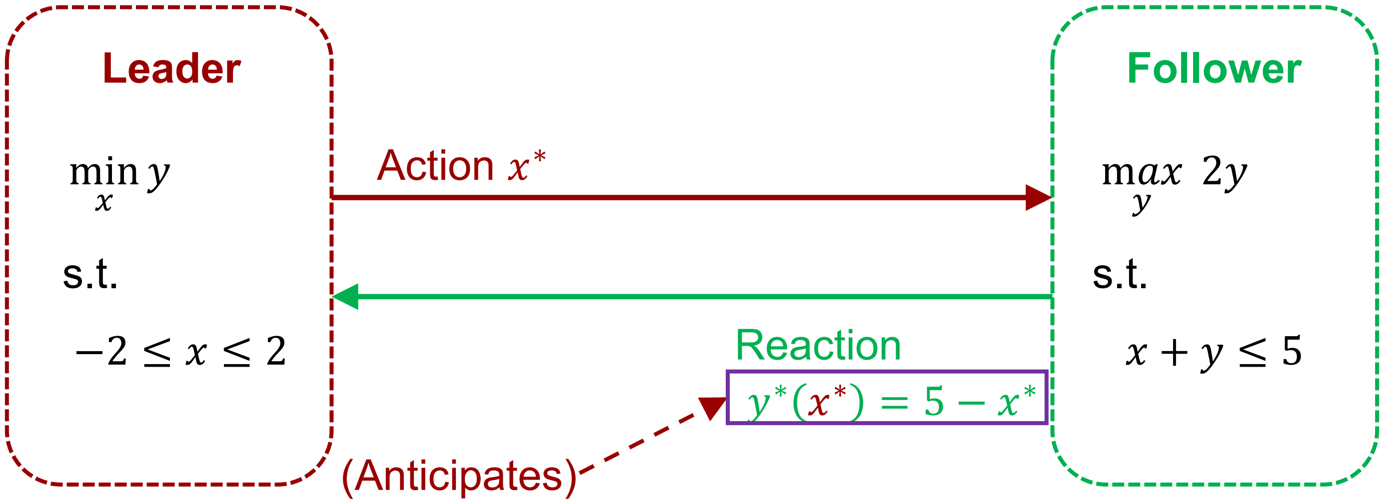
$$\max_{\boxed{y}} \; 2y$$

s.t.

$$x + y \leq 5$$

Sequential Games

Two-stage sequential game



Leader

$$\min_{x} y$$

s.t.

$$-2 \le x \le 2$$

Action x^*

Reaction

$$\boxed{y^*(x^*) = 5 - x^*}$$

(Anticipates)

Follower

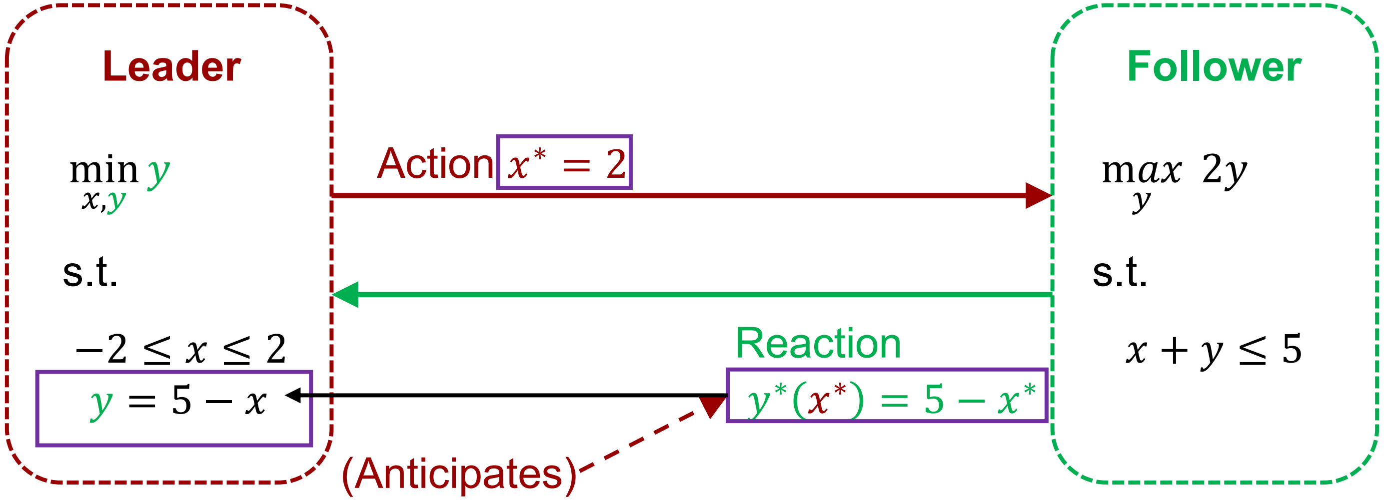
$$\max_{y}\ 2y$$

s.t.

$$x + y \le 5$$

Sequential Games

Two-stage sequential game



Leader

$\min_{x,y} y$

s.t.

$-2 \leq x \leq 2$

$\boxed{y = 5 - x}$

Action $\boxed{x^* = 2}$

Reaction

$\boxed{y^*(x^*) = 5 - x^*}$

(Anticipates)

Follower

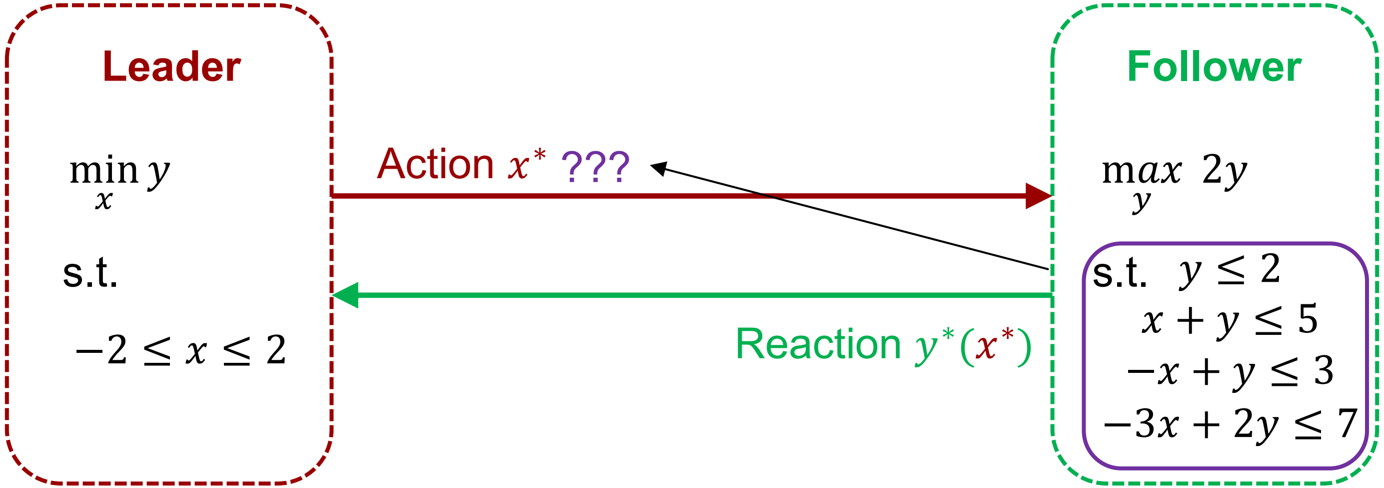
$\max_{y} \ 2y$

s.t.

$x + y \leq 5$

Sequential Games

Two-stage sequential game



Leader

$$\min_{x} y$$

s.t.

$$-2 \le x \le 2$$

Action x^* ???

Reaction $y^*(x^*)$

Follower

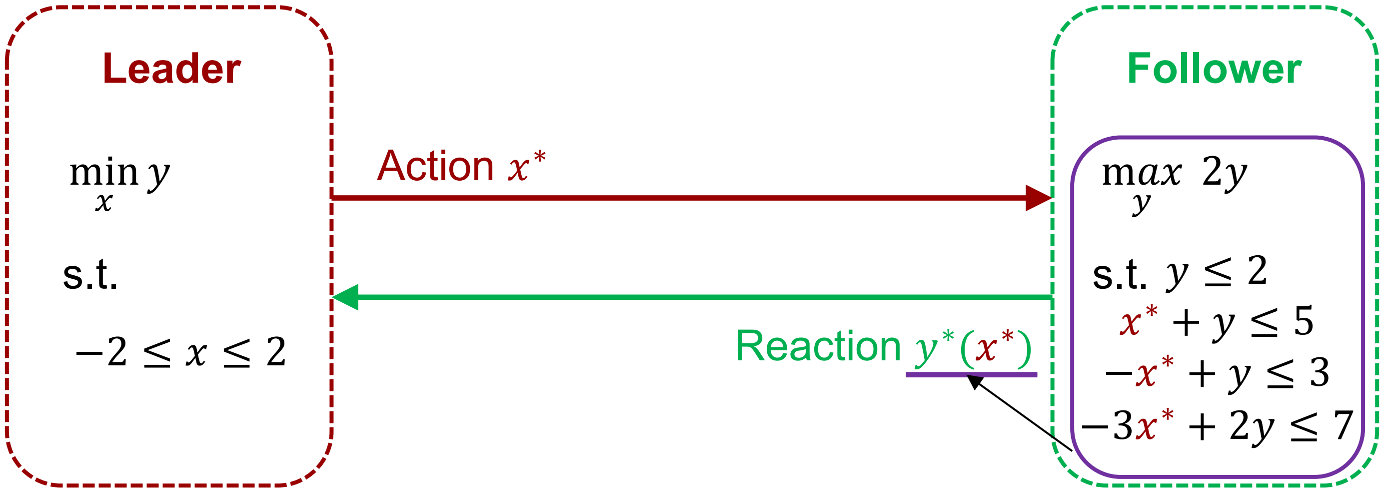
$$\max_{y} \; 2y$$

$$\begin{aligned} \text{s.t.} \quad y &\le 2 \\ x + y &\le 5 \\ -x + y &\le 3 \\ -3x + 2y &\le 7 \end{aligned}$$

How can the leader integrate the follower's optimal reaction in its own strategy **in general**?

Sequential Games

Two-stage sequential game



Leader

$$\min_{x} y$$

s.t.

$$-2 \le x \le 2$$

Action x^*

Reaction $y^*(x^*)$

Follower

$$\max_{y} \ 2y$$

s.t. $y \le 2$

$x^* + y \le 5$

$-x^* + y \le 3$

$-3x^* + 2y \le 7$

How can the leader integrate the follower's optimal reaction in its own strategy **in general**?

We can **constrain** the leader's optimization problem by the follower's optimization problem!

Sequential Games

Two-stage sequential game

Leader

$$\min_{x} \quad y$$

$$\text{s.t.} \quad -2 \le x \le 2$$

$$\max_{y} \; 2y$$

$$\begin{aligned} \text{s.t.} \quad y &\le 2 \\ x + y &\le 5 \\ -x + y &\le 3 \\ -3x + 2y &\le 7 \end{aligned}$$

Follower problem as a constraint

We can **constrain** the leader's optimization problem by the follower's optimization problem!

Sequential Games

Two-stage sequential game

Missing anything?

Leader

$$\min_{x} \quad y$$

$$\text{s.t.} \quad -2 \leq x \leq 2$$

$$\max_{y} \; 2y$$

$$\begin{aligned} \text{s.t.} \quad y &\leq 2 \\ x + y &\leq 5 \\ -x + y &\leq 3 \\ -3x + 2y &\leq 7 \end{aligned}$$

Follower
problem as a
constraint

We can **constrain** the leader's optimization problem by the follower's optimization problem!

Sequential Games

Two-stage sequential game

Missing anything?

Follower's primal variable

Leader

$$\min_{x,\boxed{y}} \quad y$$

$$\text{s.t.} \quad -2 \leq x \leq 2$$

$$\begin{aligned} \max_{y} \quad & 2y \\ \text{s.t.} \quad & y \leq 2 \\ & x + y \leq 5 \\ & -x + y \leq 3 \\ & -3x + 2y \leq 7 \end{aligned}$$

Follower problem as a constraint

We can **constrain** the leader's optimization problem by the follower's optimization problem!

Sequential Games

Two-stage sequential game

$$\min_{x,y} \quad y$$

$$\text{s.t.} \quad -2 \le x \le 2$$

$$\max_{y} \; 2y$$

$$\text{s.t.} \quad y \le 2$$

$$x + y \le 5$$

$$-x + y \le 3$$

$$-3x + 2y \le 7$$

Leader

Follower problem as a constraint

Bilevel model

In the bilevel model, an optimization problem is constrained by another optimization problem.

We can **constrain** the leader's optimization problem by the follower's optimization problem!

Bilevel Model

Two-stage sequential game

Upper-Level Problem
(Leader)

Leader

$$\min_{x,y} \quad y$$
$$\text{s.t.} \quad -2 \le x \le 2$$

Lower-Level Problem
(Follower)

$$\max_{y} \ 2y$$
$$\begin{aligned} \text{s.t.} \quad y &\le 2 \\ x + y &\le 5 \\ -x + y &\le 3 \\ -3x + 2y &\le 7 \end{aligned}$$

Follower problem as a constraint

Bilevel model

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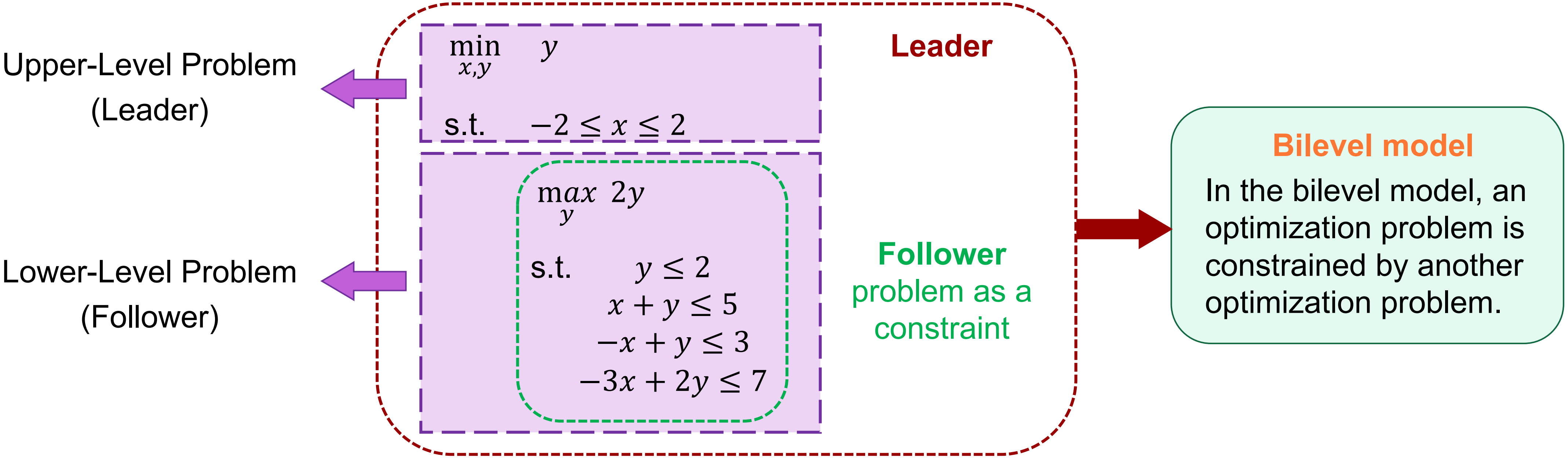
Bilevel Model

Two-stage sequential game



Upper-Level Problem
(Leader)

$$\min_{x,y} \quad y$$

$$\text{s.t.} \quad -2 \le x \le 2$$

Leader

Lower-Level Problem
(Follower)

$$\max_{y} \; 2y$$

$$\begin{aligned} \text{s.t.} \quad y &\le 2 \\ x + y &\le 5 \\ -x + y &\le 3 \\ -3x + 2y &\le 7 \end{aligned}$$

Follower problem as a constraint

Bilevel model

In the bilevel model, an optimization problem is constrained by another optimization problem.

How do we solve this problem?

We can **constrain** the leader's optimization problem by the follower's optimization problem!

Bilevel Model

Two-stage sequential game

Upper-Level Problem
(Leader)

Leader

$$\min_{x,y} \quad y$$

$$\text{s.t.} \quad -2 \le x \le 2$$

Lower-Level Problem
(Follower)

Follower problem as a constraint

$$\max_{y} \; 2y$$

$$\begin{aligned} \text{s.t.} \quad & y \le 2 \\ & x + y \le 5 \\ & -x + y \le 3 \\ & -3x + 2y \le 7 \end{aligned}$$

Bilevel model

In the bilevel model, an optimization problem is constrained by another optimization problem.

Replace the lower problem with the Optimality (KKT) conditions!

(Lecture 2)

How do we solve this problem?

We can **constrain** the leader's optimization problem by the follower's optimization problem!

Bilevel Model

Standard format of optimization problem

$$\begin{aligned} &\underset{x}{\text{Minimize}} \quad f(x) \\ &\text{subject to:} \\ &h(x) = 0 \quad : \quad \lambda \\ &g(x) \leq 0 \quad : \quad \mu \end{aligned}$$

Two-stage sequential game

Upper-Level Problem
(Leader)

$$\begin{aligned} &\min_{x,y} \quad y \\ &\text{s.t.} \quad -2 \leq x \leq 2 \end{aligned}$$

Lower-Level Problem
(Follower)

$$\begin{aligned} &\max_{y} \ 2y \\ &\text{s.t.} \quad y \leq 2 \\ &\quad x + y \leq 5 \\ &\quad -x + y \leq 3 \\ &\quad -3x + 2y \leq 7 \end{aligned}$$

Replace the lower problem with the Optimality (KKT) conditions!

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Two-stage sequential game

Upper-Level Problem
(Leader)

$$\begin{aligned} &\min_{x,y} \quad y \\ &\text{s.t.} \quad -2 \leq x \leq 2 \end{aligned}$$

Lower-Level Problem
(Follower)

$$\begin{aligned} &\min_{y} \quad -2y \\ \text{s.t.} \quad & y - 2 \leq 0: && \mu_1 \\ & x + y - 5 \leq 0: && \mu_2 \\ & -x + y - 3 \leq 0: && \mu_3 \\ & -3x + 2y - 7 \leq 0: && \mu_4 \end{aligned}$$

Replace the lower problem with the Optimality (KKT) conditions!

(Lecture 2)

Bilevel Model

Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

Two-stage sequential game

Upper-Level Problem
(Leader)

$$\min_{x,y} \quad y$$

$$\text{s.t.} \quad -2 \leq x \leq 2$$

Lower-Level Problem
(Follower)

$$\min_{y} \ -2y$$

$$\begin{aligned} \text{s.t.} \quad & y - 2 \leq 0: && \mu_1 \\ & x + y - 5 \leq 0: && \mu_2 \\ & -x + y - 3 \leq 0: && \mu_3 \\ & -3x + 2y - 7 \leq 0: && \mu_4 \end{aligned}$$

Replace the lower problem with the Optimality (KKT) conditions!

(Lecture 2)

Bilevel Model

Lagrangian function

$$\boxed{\mathcal{L}(x,\lambda,\mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)}$$

Two-stage sequential game

Upper-Level Problem
(Leader)

$$\min_{x,y} \quad y$$

$$\text{s.t.} \quad -2 \le x \le 2$$

Lower-Level Problem
(Follower)

$$\min_{y} \; -2y$$

$$\begin{aligned} \text{s.t.} \quad & y-2 \le 0: && \mu_1 \\ & x+y-5 \le 0: && \mu_2 \\ & -x+y-3 \le 0: && \mu_3 \\ & -3x+2y-7 \le 0: && \mu_4 \end{aligned}$$

Replace the lower problem with the Optimality (KKT) conditions!

(Lecture 2)

Lagrangian function of lower-level problem:

$$\begin{aligned} &\mathcal{L}(y,\mu_1,\mu_2,\mu_3,\mu_4) \\ &= -2y + \mu_1(y-2) + \mu_2(x+y-5) \\ &+ \mu_3(-x+y-3) + \mu_4(-3x+2y-7) \end{aligned}$$

Bilevel Model

KKT conditions

$$\frac{\partial \mathcal{L}(x,\lambda,\mu)}{\partial x} = 0$$
$$h(x) = 0$$
$$0 \leq -g(x) \perp \mu \geq 0$$
$$\lambda \in \text{free}$$

Two-stage sequential game

Upper-Level Problem
(Leader)

$$\min_{x,y} \quad y$$
$$\text{s.t.} \quad -2 \leq x \leq 2$$

Lower-Level Problem
(Follower)

$$\min_{y} \ -2y$$
$$\begin{aligned}
\text{s.t.} \quad & y - 2 \leq 0: && \mu_1 \\
& x + y - 5 \leq 0: && \mu_2 \\
& -x + y - 3 \leq 0: && \mu_3 \\
& -3x + 2y - 7 \leq 0: && \mu_4
\end{aligned}$$

Replace the lower problem with the Optimality (KKT) conditions!

(Lecture 2)

Lagrangian function of lower-level problem:

$$\begin{aligned}
& \mathcal{L}(y,\mu_1,\mu_2,\mu_3,\mu_4) \\
& = -2y + \mu_1(y-2) + \mu_2(x+y-5) \\
& + \mu_3(-x+y-3) + \mu_4(-3x+2y-7)
\end{aligned}$$

Bilevel Model

KKT conditions

$$\frac{\partial \mathcal{L}(x,\lambda,\mu)}{\partial x} = 0$$
$$h(x) = 0$$
$$0 \leq -g(x) \perp \mu \geq 0$$
$$\lambda \in \text{free}$$

Two-stage sequential game

Upper-Level Problem
(Leader)

$$\min_{x,y} \quad y$$
$$\text{s.t.} \quad -2 \leq x \leq 2$$

Lower-Level Problem
(Follower)

$$\min_{y} \; -2y$$

$$\begin{aligned} \text{s.t.} \quad & y-2 \leq 0: && \mu_1 \\ & x+y-5 \leq 0: && \mu_2 \\ & -x+y-3 \leq 0: && \mu_3 \\ & -3x+2y-7 \leq 0: && \mu_4 \end{aligned}$$

Replace the lower problem with the Optimality (KKT) conditions!

(Lecture 2)

Lagrangian function of lower-level problem:

$$\begin{aligned} &\mathcal{L}(y,\mu_1,\mu_2,\mu_3,\mu_4) \\ &= -2y + \mu_1(y-2) + \mu_2(x+y-5) \\ &+ \mu_3(-x+y-3) + \mu_4(-3x+2y-7) \end{aligned}$$

KKT conditions of the lower-level problem:

$$\frac{\partial \mathcal{L}}{\partial y} = -2 + \mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 0$$

$$0 \leq 2 - y \perp \mu_1 \geq 0$$
$$0 \leq 5 - x - y \perp \mu_2 \geq 0$$
$$0 \leq x - y + 3 \perp \mu_3 \geq 0$$
$$0 \leq 3x - 2y + 7 \perp \mu_4 \geq 0$$

Bilevel Model

KKT conditions

$$\frac{\partial \mathcal{L}(x,\lambda,\mu)}{\partial x} = 0$$
$$h(x) = 0$$
$$0 \leq -g(x) \perp \mu \geq 0$$
$$\lambda \in \text{free}$$

Two-stage sequential game

Upper-Level Problem
(Leader)

$$\min_{x,y} \quad y$$
$$\text{s.t.} \quad -2 \leq x \leq 2$$

Lower-Level Problem
(Follower)

$$\min_{y} \; -2y$$
$$\begin{aligned} \text{s.t.} \quad & y - 2 \leq 0: & \mu_1 \\ & x + y - 5 \leq 0: & \mu_2 \\ & -x + y - 3 \leq 0: & \mu_3 \\ & -3x + 2y - 7 \leq 0: & \mu_4 \end{aligned}$$

Replace the lower problem with the Optimality (KKT) conditions!

(Lecture 2)

Lagrangian function of lower-level problem:

$$\begin{aligned} & \mathcal{L}(y,\mu_1,\mu_2,\mu_3,\mu_4) \\ & = -2y + \mu_1(y-2) + \mu_2(x+y-5) \\ & + \mu_3(-x+y-3) + \mu_4(-3x+2y-7) \end{aligned}$$

KKT conditions of the lower-level problem:

$$\frac{\partial \mathcal{L}}{\partial y} = -2 + \mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 0$$
$$0 \leq 2 - y \perp \mu_1 \geq 0$$
$$0 \leq 5 - x - y \perp \mu_2 \geq 0$$
$$0 \leq x - y + 3 \perp \mu_3 \geq 0$$
$$0 \leq 3x - 2y + 7 \perp \mu_4 \geq 0$$

Bilevel Model

Two-stage sequential game

Leader
(variables: x, y, $\mu_1, \mu_2, \mu_3, \mu_4$)

$$\min_{x,y,\mu_1,\mu_2,\mu_3,\mu_4} \quad y$$

s.t. $-2 \le x \le 2$

$$\boxed{\begin{array}{l} -2+\mu_1+\mu_2+\mu_3+2\mu_4=0 \\ 0 \le 2-y \perp \mu_1 \ge 0 \\ 0 \le 5-x-y \perp \mu_2 \ge 0 \\ 0 \le x-y+3 \perp \mu_3 \ge 0 \\ 0 \le 3x-2y+7 \perp \mu_4 \ge 0 \end{array}}$$

Lagrangian function of lower-level problem:

$$\begin{aligned} &\mathcal{L}(y,\mu_1,\mu_2,\mu_3,\mu_4) \\ &= -2y+\mu_1(y-2)+\mu_2(x+y-5) \\ &+\mu_3(-x+y-3)+\mu_4(-3x+2y-7) \end{aligned}$$

KKT conditions of the lower-level problem:

$$\boxed{\begin{array}{c} \dfrac{\partial \mathcal{L}}{\partial y} = -2+\mu_1+\mu_2+\mu_3+2\mu_4=0 \\ 0 \le 2-y \perp \mu_1 \ge 0 \\ 0 \le 5-x-y \perp \mu_2 \ge 0 \\ 0 \le x-y+3 \perp \mu_3 \ge 0 \\ 0 \le 3x-2y+7 \perp \mu_4 \ge 0 \end{array}}$$

Bilevel Model

Two-stage sequential game

Leader

(variables: x, y, $\mu_1, \mu_2, \mu_3, \mu_4$)

$$\min_{x,y,\mu_1,\mu_2,\mu_3,\mu_4} \quad y$$

s.t. $-2 \leq x \leq 2$

$$\boxed{\begin{array}{l} -2 + \mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 0 \\ 0 \leq 2 - y \perp \mu_1 \geq 0 \\ 0 \leq 5 - x - y \perp \mu_2 \geq 0 \\ 0 \leq x - y + 3 \perp \mu_3 \geq 0 \\ 0 \leq 3x - 2y + 7 \perp \mu_4 \geq 0 \end{array}}$$

Follower's primal variables

Follower's dual variables

Lagrangian function of lower-level problem:

$$\mathcal{L}(y, \mu_1, \mu_2, \mu_3, \mu_4)$$
$$= -2y + \mu_1(y - 2) + \mu_2(x + y - 5)$$
$$+ \mu_3(-x + y - 3) + \mu_4(-3x + 2y - 7)$$

KKT conditions of the lower-level problem:

$$\boxed{\begin{array}{c} \dfrac{\partial \mathcal{L}}{\partial y} = -2 + \mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 0 \\ 0 \leq 2 - y \perp \mu_1 \geq 0 \\ 0 \leq 5 - x - y \perp \mu_2 \geq 0 \\ 0 \leq x - y + 3 \perp \mu_3 \geq 0 \\ 0 \leq 3x - 2y + 7 \perp \mu_4 \geq 0 \end{array}}$$

Bilevel Model

Two-stage sequential game

$$\min_{x,y,\mu_1,\mu_2,\mu_3,\mu_4} \quad y$$

Leader
(variables: $x, y,$
$\mu_1, \mu_2, \mu_3, \mu_4$)

s.t. $-2 \leq x \leq 2$

$$-2 + \mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 0$$
$$0 \leq 2 - y \perp \mu_1 \geq 0$$
$$0 \leq 5 - x - y \perp \mu_2 \geq 0$$
$$0 \leq x - y + 3 \perp \mu_3 \geq 0$$
$$0 \leq 3x - 2y + 7 \perp \mu_4 \geq 0$$

Updated leader's problem with
complementarity constraints

Mathematical Program with
Complementarity Constraint (MPCC)

Bilevel Model

Two-stage sequential game

$$\min_{x,y,\mu_1,\mu_2,\mu_3,\mu_4} \quad y$$

Leader
(variables: $x, y, \mu_1, \mu_2, \mu_3, \mu_4$)

$$\begin{aligned}
\text{s.t.} \quad & -2 \leq x \leq 2 \\
& -2 + \mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 0 \\
& 0 \leq 2 - y \perp \mu_1 \geq 0 \\
& 0 \leq 5 - x - y \perp \mu_2 \geq 0 \\
& 0 \leq x - y + 3 \perp \mu_3 \geq 0 \\
& 0 \leq 3x - 2y + 7 \perp \mu_4 \geq 0
\end{aligned}$$

Updated leader's problem with **complementarity constraints**

Mathematical Program with Complementarity Constraint (MPCC)

Difficult to solve due to the term that contains the product of the primal and the dual variables.

("Big M" approach: Course #31792 https://www.youtube.com/watch?v=STQRFr4praA)

Bilevel Model

Two-stage sequential game

Leader (variables: $x, y, \mu_1, \mu_2, \mu_3, \mu_4$)

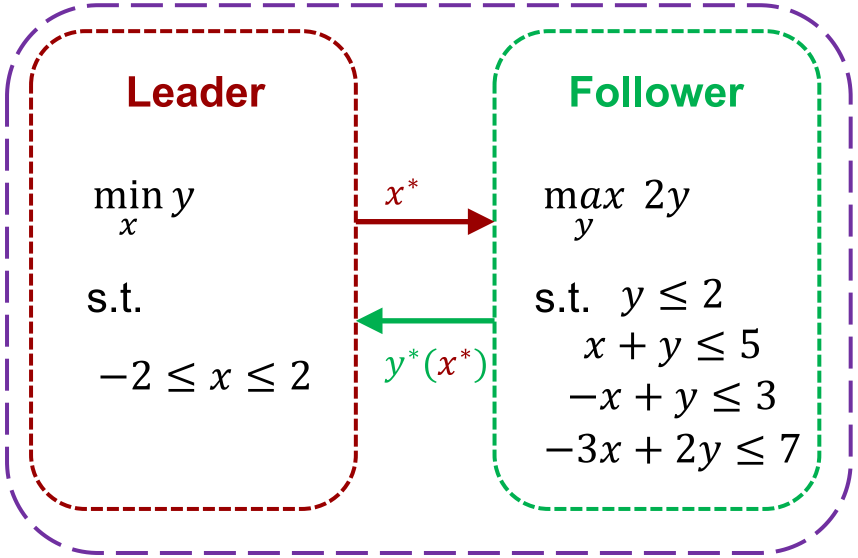
$$\min_{x,y,\mu_1,\mu_2,\mu_3,\mu_4} \quad y$$

$$\begin{aligned}
\text{s.t.} \quad & -2 \leq x \leq 2 \\
& -2 + \mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 0 \\
& 0 \leq 2 - y \perp \mu_1 \geq 0 \\
& 0 \leq 5 - x - y \perp \mu_2 \geq 0 \\
& 0 \leq x - y + 3 \perp \mu_3 \geq 0 \\
& 0 \leq 3x - 2y + 7 \perp \mu_4 \geq 0
\end{aligned}$$

Updated leader's problem with **complementarity constraints**

Mathematical Program with Complementarity Constraint (MPCC)

Original problem



Leader

$$\min_x y$$

s.t.

$$-2 \leq x \leq 2$$

x^*

$y^*(x^*)$

Follower

$$\max_y \ 2y$$

$$\begin{aligned}
\text{s.t.} \quad & y \leq 2 \\
& x + y \leq 5 \\
& -x + y \leq 3 \\
& -3x + 2y \leq 7
\end{aligned}$$

Bilevel Model

Two-stage sequential game

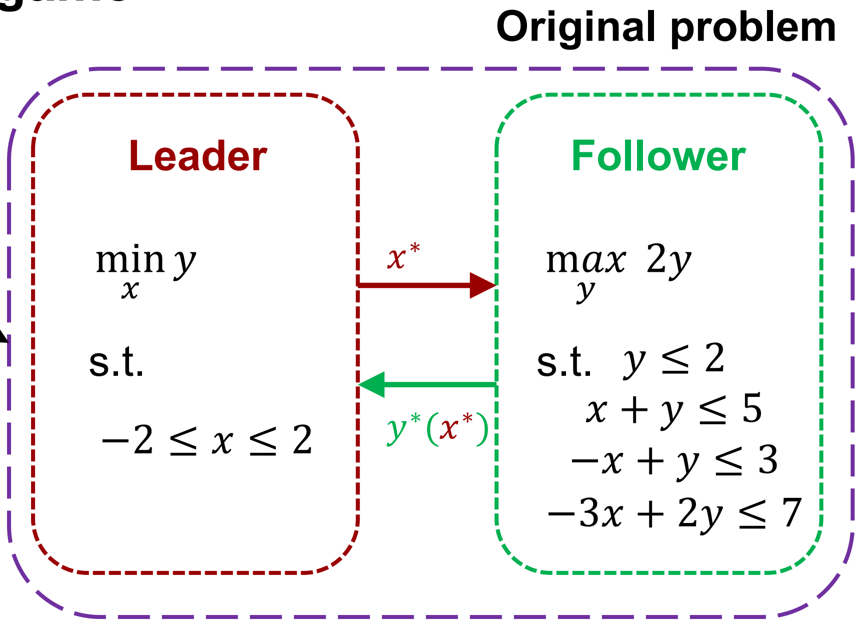
Leader
(variables: $x, y, \mu_1, \mu_2, \mu_3, \mu_4$)

$$\min_{x,y,\mu_1,\mu_2,\mu_3,\mu_4} \quad y$$

$$\begin{aligned}
\text{s.t.} \quad & -2 \le x \le 2 \\
& -2 + \mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 0 \\
& 0 \le 2 - y \perp \mu_1 \ge 0 \\
& 0 \le 5 - x - y \perp \mu_2 \ge 0 \\
& 0 \le x - y + 3 \perp \mu_3 \ge 0 \\
& 0 \le 3x - 2y + 7 \perp \mu_4 \ge 0
\end{aligned}$$

Optimal solution
x^*, y^*

leader's best strategy for the first move.


Original problem

Leader

$$\min_x y$$

s.t.

$$-2 \le x \le 2$$

x^*

$y^*(x^*)$

Follower

$$\max_y \ 2y$$

$$\begin{aligned}
\text{s.t.} \quad & y \le 2 \\
& x + y \le 5 \\
& -x + y \le 3 \\
& -3x + 2y \le 7
\end{aligned}$$

Updated leader's problem with **complementarity constraints**

Mathematical Program with Complementarity Constraint (MPCC)

Bilevel Model

Upper-Level Problem
(Leader)

minimize *Upper-Level Objective Function (leader's primal variables)*
subject to *Uppler-Level Constraints*

Lower-Level Problem
(Follower)

minimize *Lower-Level Objective Function (follower's primal variables)*
subject to *Lower-Level Constraints*

Bilevel Model

Upper-Level Problem
(Leader)

minimize *Upper-Level Objective Function (leader's primal variables)*
subject to *Uppler-Level Constraints*

Lower-Level Problem
(Follower)

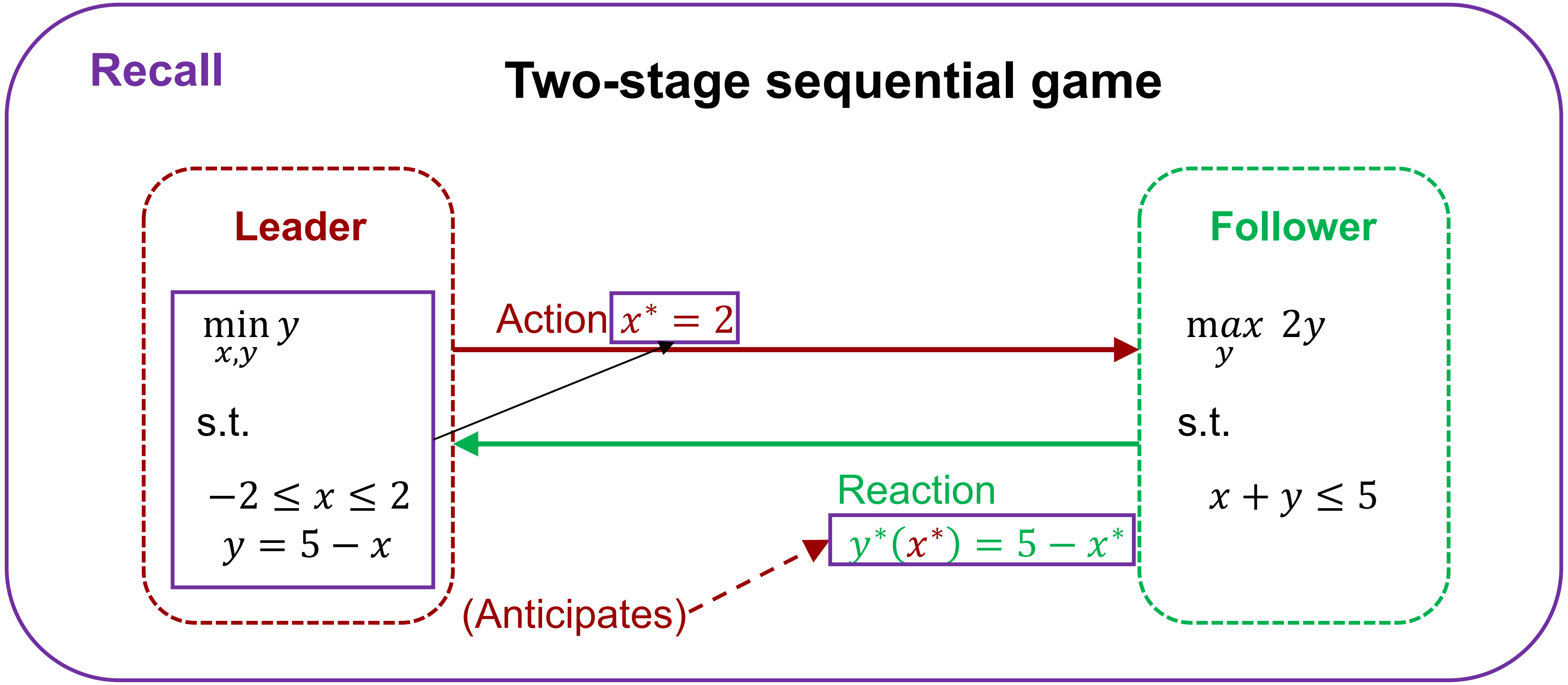
minimize *Lower-Level Objective Function (follower's primal variables)*
subject to *Lower-Level Constraints*

⇕

Mathematical Program
with Complementarity
Constraints (MPCC)

minimize *Upper-Level Objective Function (leader's primal variables,*
follower's primal and *dual variables)*
subject to *Uppler-Level Constraints*
Optimality (KKT) Conditions of the Lower-Level Problem(s)

Bilevel Model



Recall

Two-stage sequential game

Leader

$$\min_{x,y} y$$

s.t.

$$-2 \le x \le 2$$
$$y = 5 - x$$

Action $x^* = 2$

Follower

$$\max_{y} \ 2y$$

s.t.

$$x + y \le 5$$

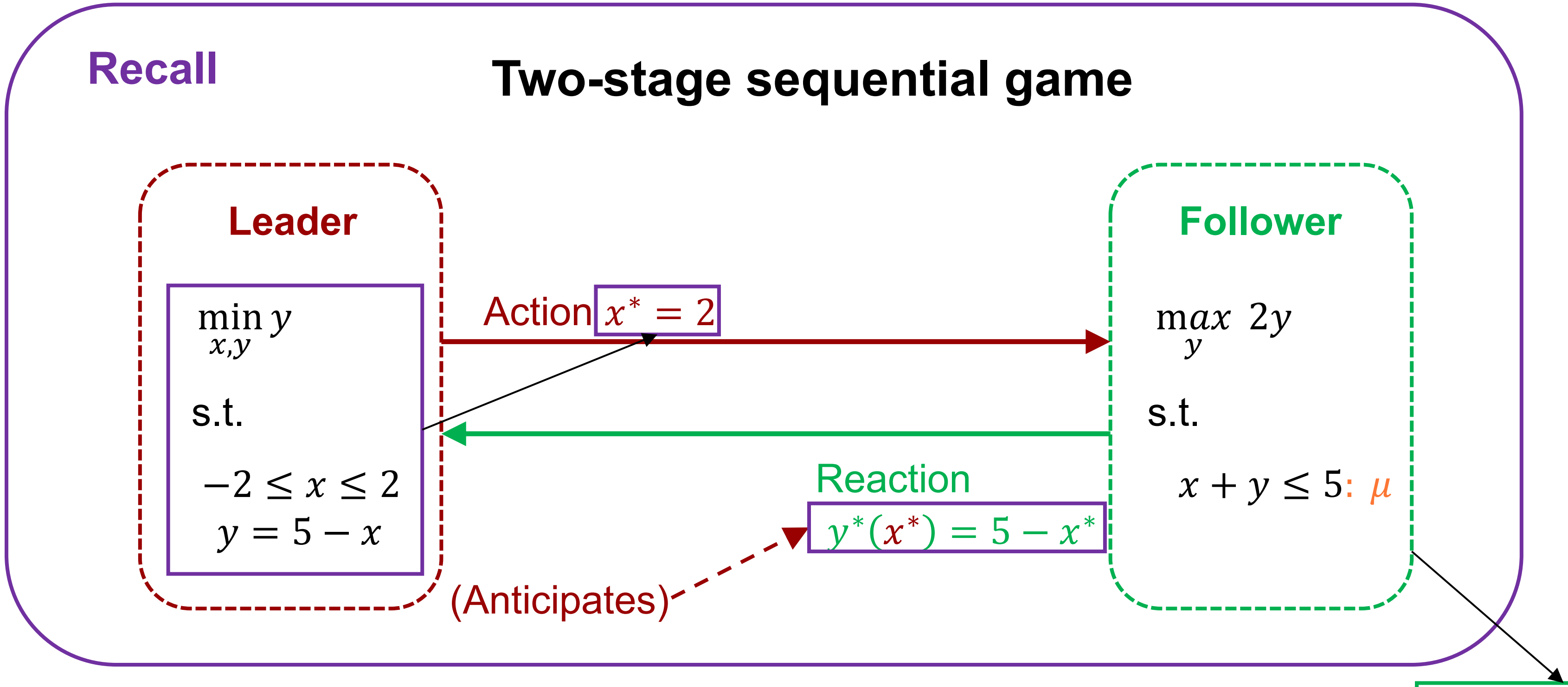
Reaction $y^*(x^*) = 5 - x^*$

(Anticipates)

Question:

Why is the updated leader’s problem missing the follower’s dual variables?

Bilevel Model



Recall

Two-stage sequential game

Leader

$$\min_{x,y} y$$

s.t.

$$-2 \leq x \leq 2$$
$$y = 5 - x$$

Action $\boxed{x^* = 2}$

Follower

$$\max_{y} \; 2y$$

s.t.

$$x + y \leq 5 : \mu$$

Reaction

$\boxed{y^*(x^*) = 5 - x^*}$

(Anticipates)

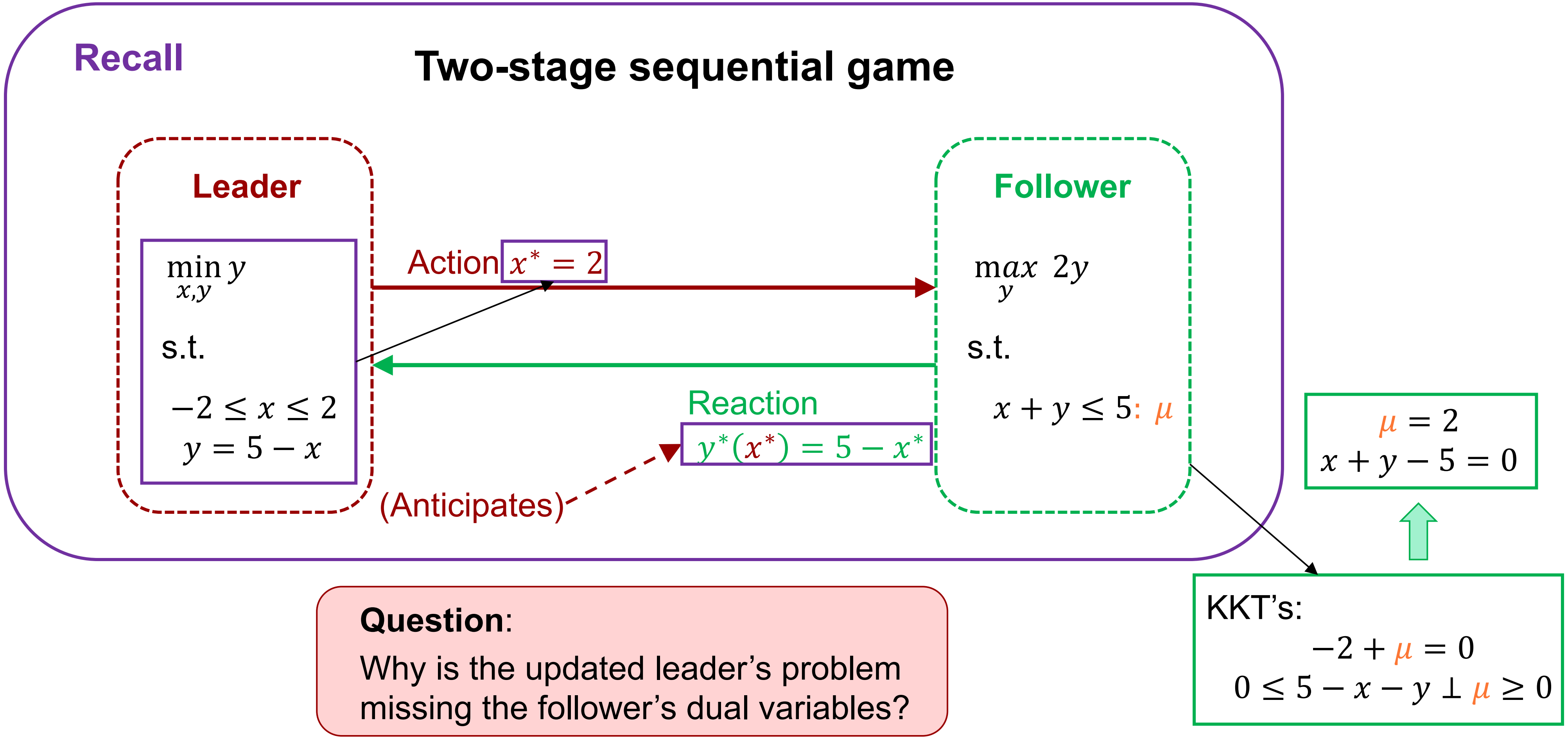
KKT's:

$$-2 + \mu = 0$$
$$0 \leq 5 - x - y \perp \mu \geq 0$$

Question:

Why is the updated leader's problem missing the follower's dual variables?

Bilevel Model



Recall

Two-stage sequential game

Leader

$$\min_{x,y} y$$

s.t.

$$-2 \leq x \leq 2$$
$$y = 5 - x$$

Action $x^* = 2$

Follower

$$\max_{y} \; 2y$$

s.t.

$$x + y \leq 5: \mu$$

Reaction

$$y^*(x^*) = 5 - x^*$$

(Anticipates)

KKT's:

$$-2 + \mu = 0$$
$$0 \leq 5 - x - y \perp \mu \geq 0$$

$$\mu = 2$$
$$x + y - 5 = 0$$

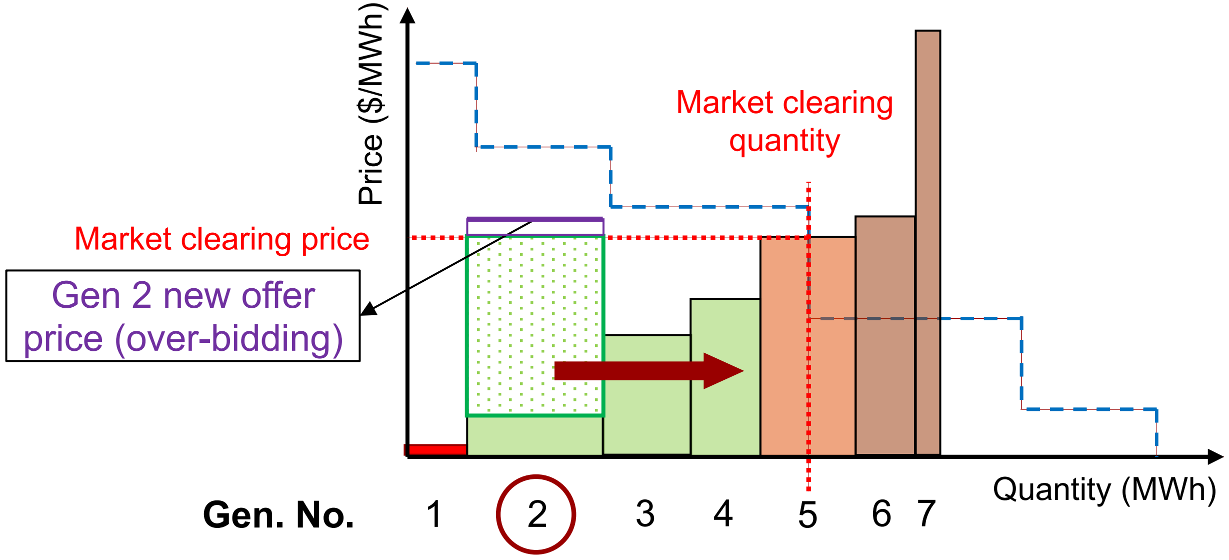
Question:

Why is the updated leader's problem missing the follower's dual variables?

Strategic Offering Problem

What happens to the profit of Gen 2?

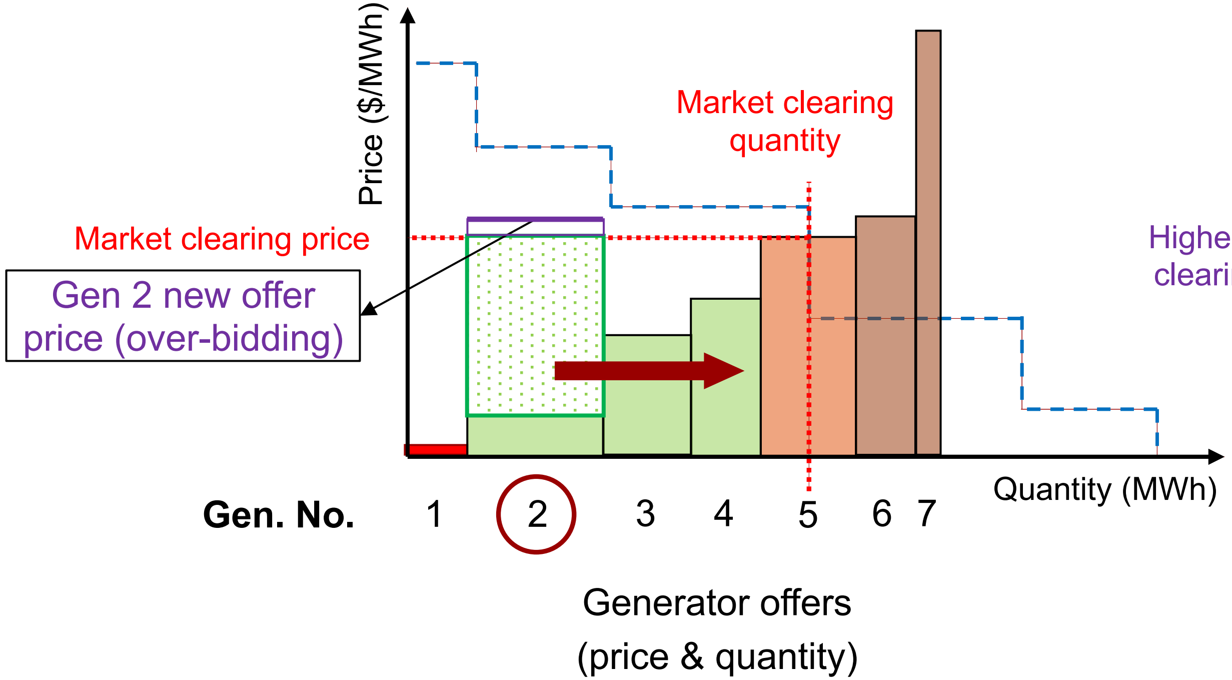
Recall from Lecture 9:



Generator offers
(price & quantity)

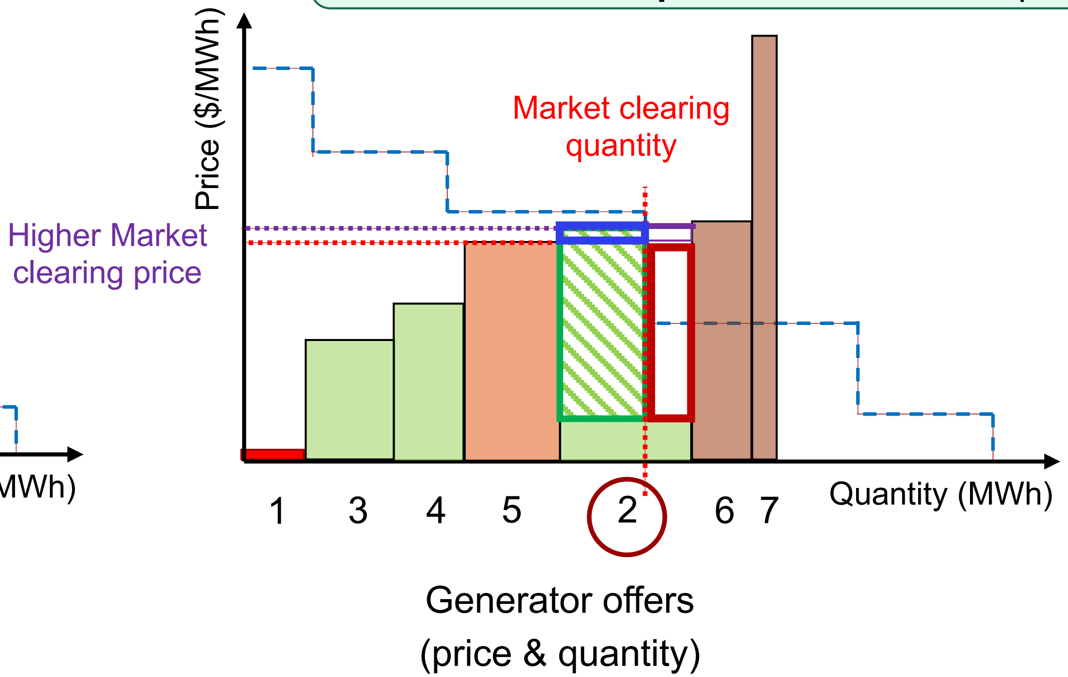
Strategic Offering Problem

Recall from Lecture 9:



What happens to the profit of Gen 2?

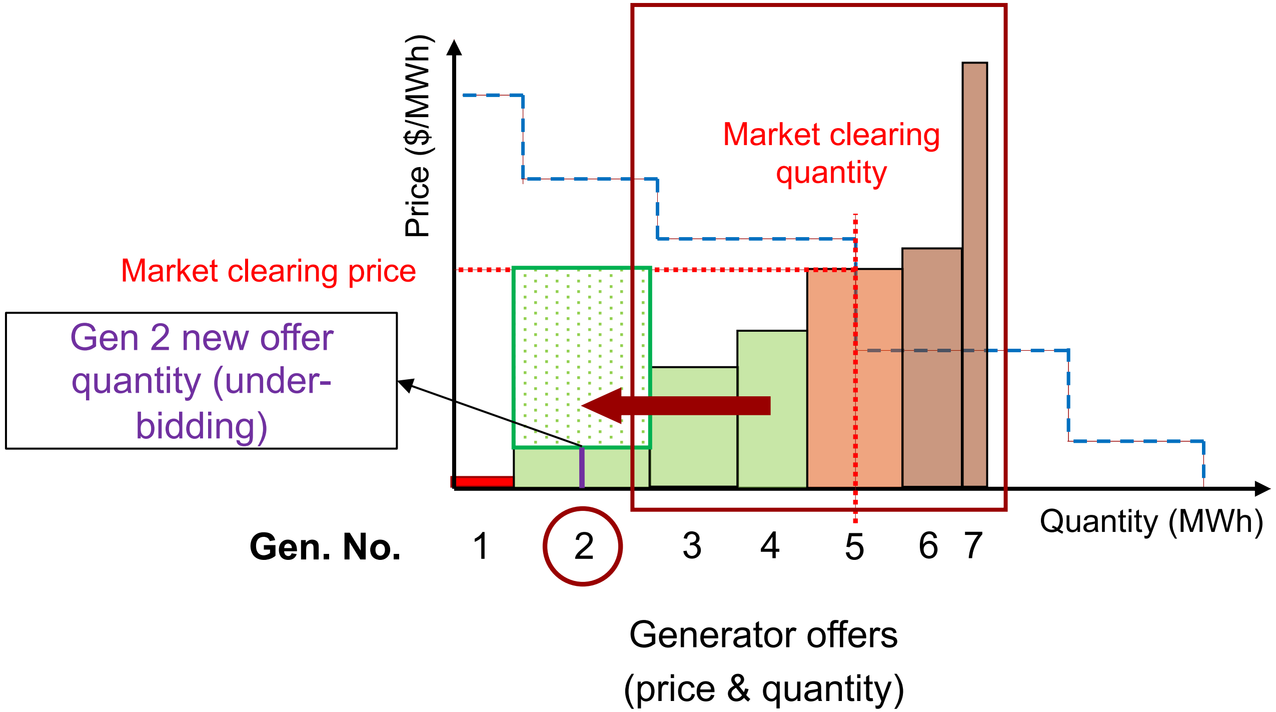
Higher market clearing price but lower quantity --> **possible increase** in profit



Strategic Offering Problem

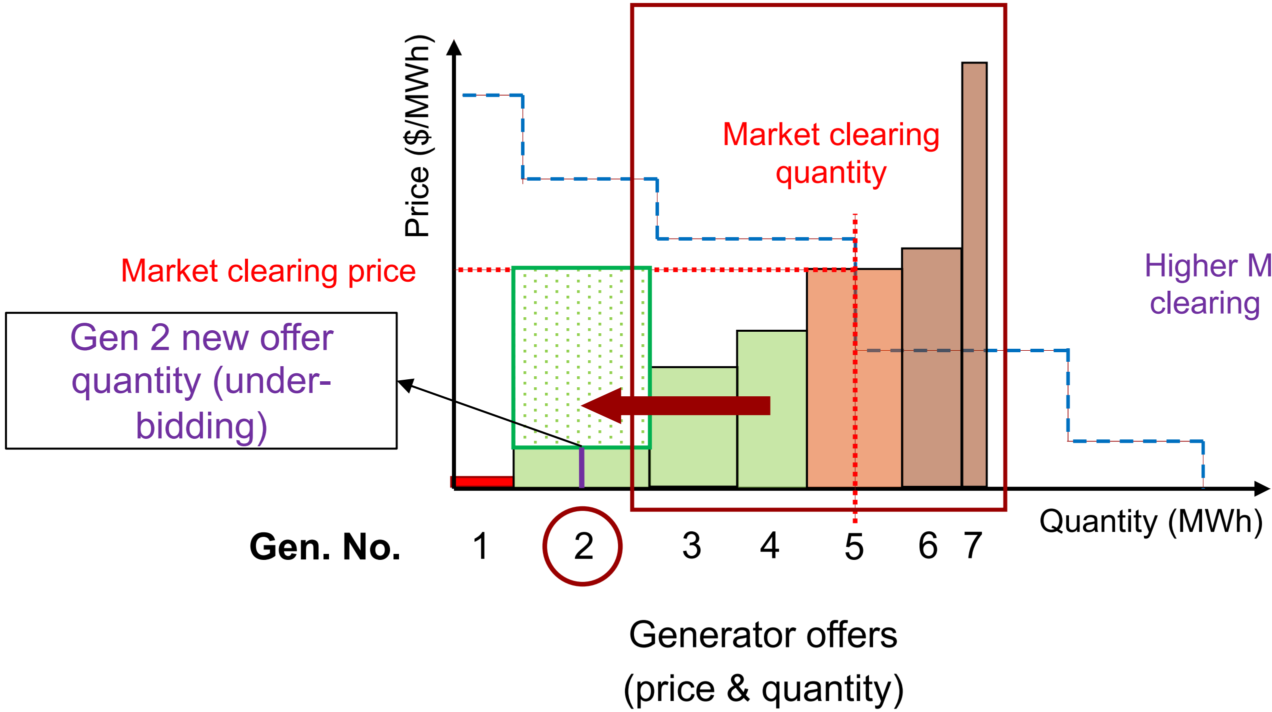
Recall from Lecture 9:

What happens to the profit of Gen 2?



Strategic Offering Problem

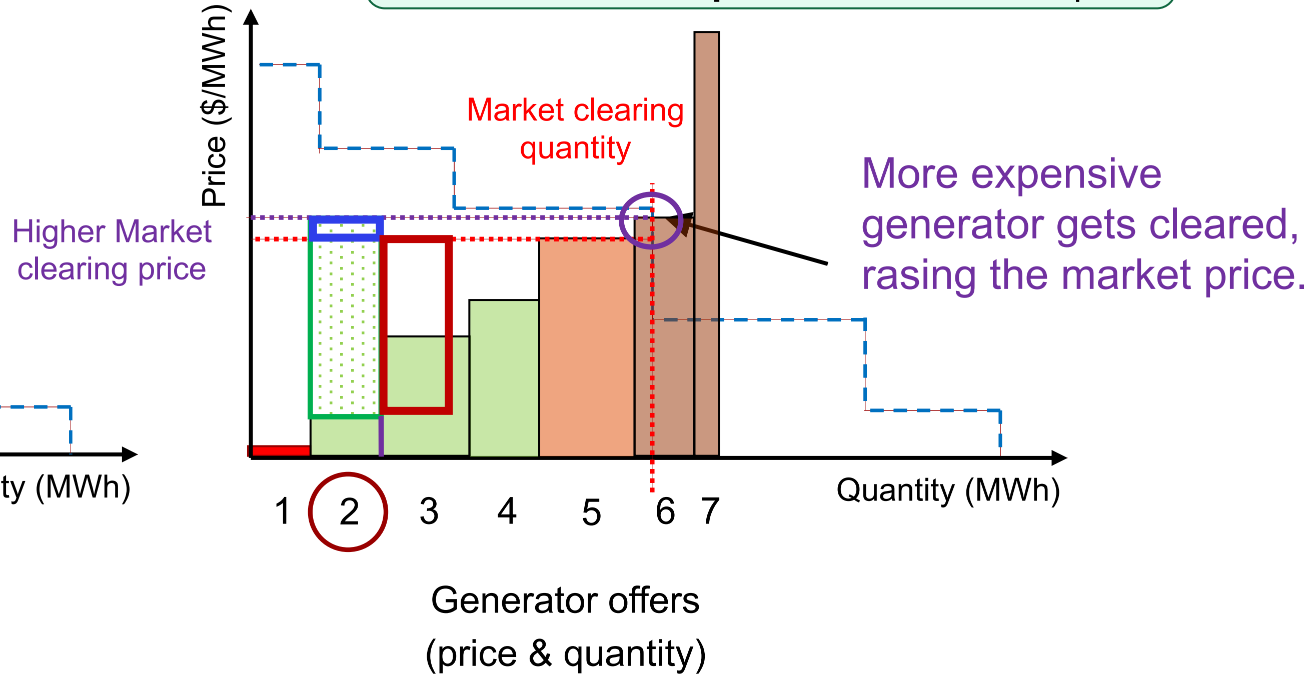
Recall from Lecture 9:



Generator offers (price & quantity)

What happens to the profit of Gen 2?

Higher market clearing price but lower quantity --> **possible increase** in profit



Generator offers (price & quantity)

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader?

Follower?

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Follower: market clearing

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Follower: market clearing

Optimization problem:

$$\max_{p_d, p_g} \ \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

subject to:

$$0 \le p_d \le \overline{p}_d, \forall d \in D : \ \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \overline{p}_g, \forall g \in G : \ \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \ : \ \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Follower: market clearing

Optimization problem:

bid price of demand d   offer price of generator g

$$\max_{p_d, p_g} \ \mathrm{SW} = \sum_{d \in D} \boxed{U_d} p_d - \sum_{g \in G} \boxed{C_g} p_g$$

subject to:

$$0 \le p_d \le \overline{p}_d, \forall d \in D : \ \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \overline{p}_g, \forall g \in G : \ \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \ : \ \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Follower: market clearing

Optimization problem:

$$\max_{p_d, p_g} \ \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

subject to:

maximum load of demand d

$$0 \leq p_d \leq \overline{p}_d, \ \forall d \in D \ : \ \underline{\mu}_d, \overline{\mu}_d$$

$$0 \leq p_g \leq \overline{p}_g, \ \forall g \in G \ : \ \underline{\mu}_g, \overline{\mu}_g$$

offered capacity of generator g

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \ : \ \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Maximize profit: $\max_{?} \; p_i(\lambda - C_i')$

Follower: market clearing

$$\max_{p_d, p_g} \; \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

subject to:

$$0 \le p_d \le \overline{p}_d, \forall d \in D : \; \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \overline{p}_g, \forall g \in G : \; \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \; : \; \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Maximize profit: $\max_{?} \boxed{p_i}(\boxed{\lambda} - \boxed{C_i'})$

The **dispatch** and the **market price** are determined by the follower's decision.

True production cost

(parameter)

Follower: market clearing

$$\max_{p_d,p_g} \ \text{SW} = \sum_{d\in D} U_d p_d - \sum_{g\in G} C_g \, p_g$$

subject to:

$$0 \le p_d \le \overline{p}_d, \forall d \in D : \ \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \overline{p}_g, \forall g \in G : \ \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d\in D} p_d - \sum_{g\in G} p_g = 0 \ : \ \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Maximize profit: $\max_{?} \; p_i(\lambda - C_i')$

How do you reflect this in the leader's problem?

Follower: market clearing

$$\max_{p_d, p_g} \; \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

subject to:

$$0 \le p_d \le \overline{p}_d, \forall d \in D \; : \; \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \overline{p}_g, \forall g \in G \; : \; \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \quad : \quad \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Maximize profit: $\max_{C_i, \overline{p}_i} p_i(\lambda - C_i')$

How do you reflect this in the leader's problem?

Offer price and **offer quantity**!

Follower: market clearing

$$\max_{p_d, p_g} \ \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

subject to:

$$0 \le p_d \le \overline{p}_d, \forall d \in D : \ \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \overline{p}_g, \forall g \in G : \ \underline{\mu}_g, \overline{\mu}_g$$

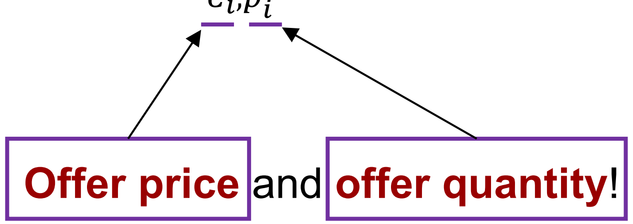
$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \ : \ \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Maximize profit: $\max_{C_i, \overline{p}_i} \; p_i(\lambda - C_i')$



Offer price and **offer quantity**!

Follower: market clearing

$$\max_{p_d, p_g} \; \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

subject to:

$$0 \leq p_d \leq \overline{p}_d, \forall d \in D \; : \; \underline{\mu}_d, \overline{\mu}_d$$

$$0 \leq p_g \leq \overline{p}_g, \forall g \in G \; : \; \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \quad : \quad \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Maximize profit: $\max_{C_i,\overline{p}_i} p_i(\lambda - C_i')$

subject to:

$0 \leq \overline{p}_i \leq \boxed{\overline{p}_i'}$

True capacity (parameter)

Follower: market clearing

$$\max_{p_d,p_g} \ \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

subject to:

$$0 \leq p_d \leq \overline{p}_d, \forall d \in D : \ \underline{\mu}_d, \overline{\mu}_d$$

$$0 \leq p_g \leq \overline{p}_g, \forall g \in G : \ \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \ : \ \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?

Leader: strategic generator i

Maximize profit:

$$\max_{\boxed{C_i,\overline{p}_i}} p_i(\lambda - C_i')$$

subject to:

$$0 \le \overline{p}_i \le \overline{p}_i'$$

C_i is a **parameter** in the follower's problem because $i \in G$

$\overline{p}_i$ is a **parameter** in the follower's problem because $i \in G$

Follower: market clearing

$$\max_{p_d,p_g} \ \mathrm{SW} = \sum_{d\in D} U_d p_d - \boxed{\sum_{g\in G} C_g p_g}$$

subject to:

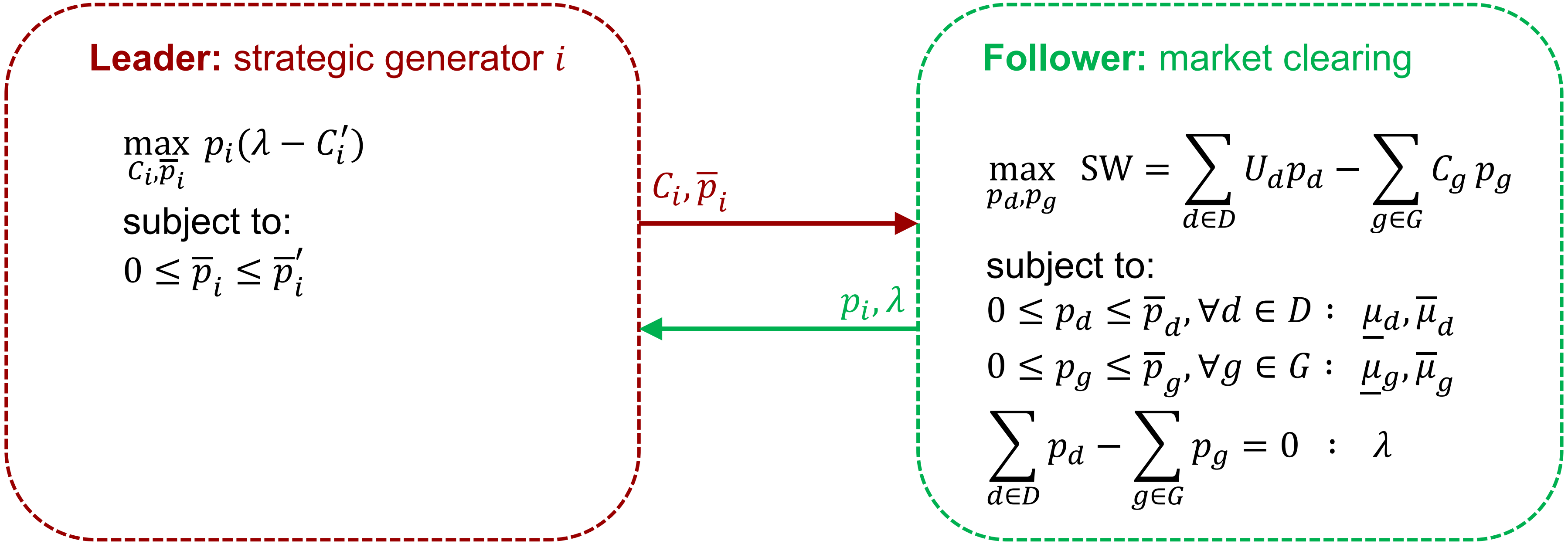
$$0 \le p_d \le \overline{p}_d, \forall d \in D : \ \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \boxed{\overline{p}_g}, \forall g \in G : \ \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d\in D} p_d - \sum_{g\in G} p_g = 0 \ : \ \lambda$$

Strategic Offering Problem

Can we use a bilevel model to find the best offering strategy of a generator?



Leader: strategic generator i

$$\max_{C_i, \overline{p}_i} \; p_i(\lambda - C_i')$$

subject to:

$$0 \leq \overline{p}_i \leq \overline{p}_i'$$

Follower: market clearing

$$\max_{p_d, p_g} \;\; \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

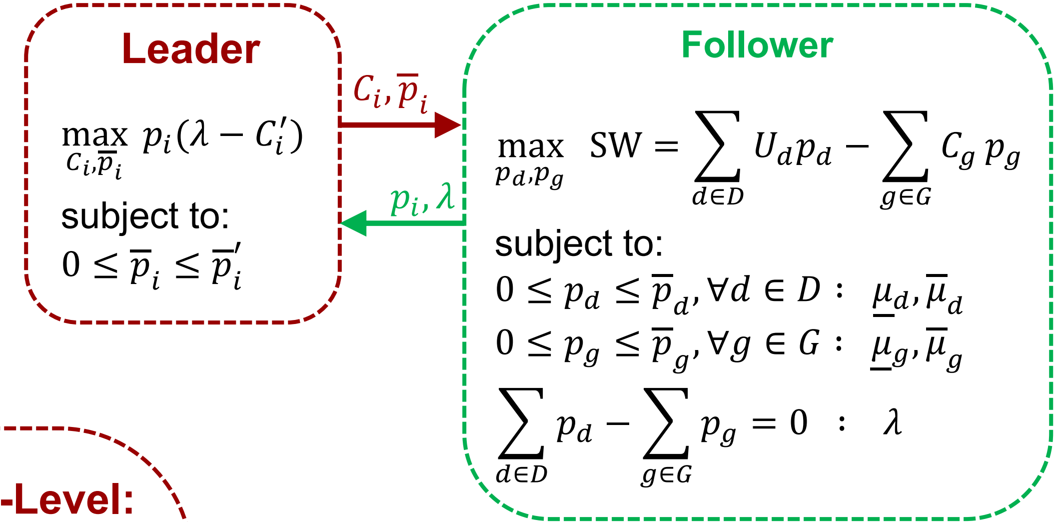
subject to:

$$0 \leq p_d \leq \overline{p}_d, \forall d \in D \; : \;\; \underline{\mu}_d, \overline{\mu}_d$$

$$0 \leq p_g \leq \overline{p}_g, \forall g \in G \; : \;\; \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \quad : \quad \lambda$$

Strategic Offering Problem



Leader

$$\max_{C_i,\overline{p}_i} \; p_i(\lambda - C_i')$$

subject to:

$$0 \le \overline{p}_i \le \overline{p}_i'$$

$C_i, \overline{p}_i$

p_i, λ

Follower

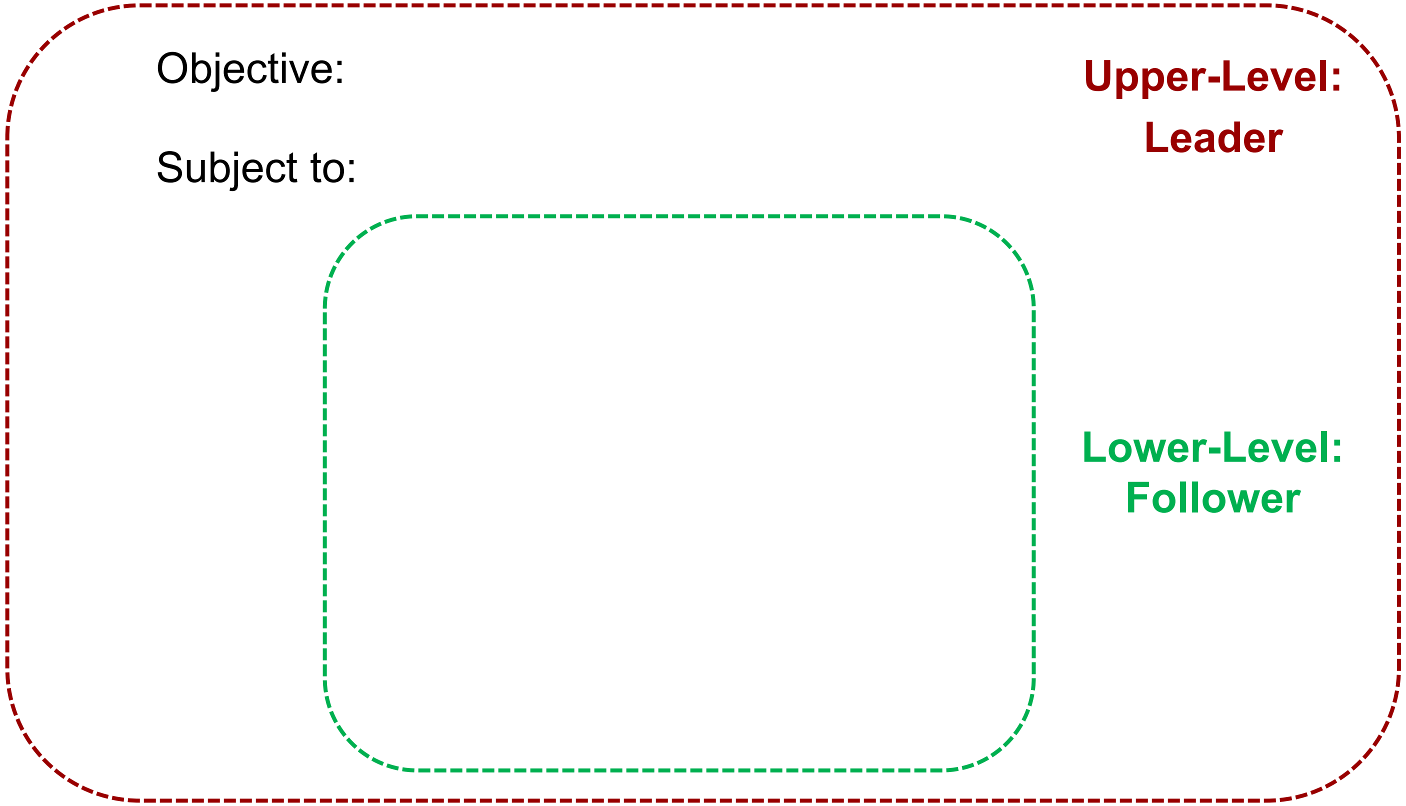
$$\max_{p_d,p_g} \; \mathrm{SW} = \sum_{d\in D} U_d p_d - \sum_{g\in G} C_g \, p_g$$

subject to:

$$0 \le p_d \le \overline{p}_d, \forall d \in D : \quad \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \overline{p}_g, \forall g \in G : \quad \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d\in D} p_d - \sum_{g\in G} p_g = 0 \quad : \quad \lambda$$



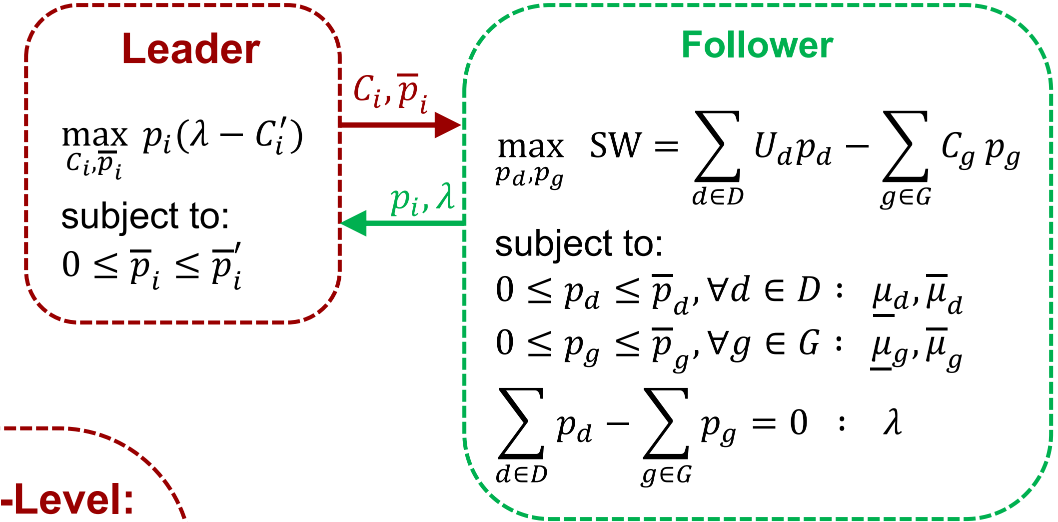
Objective:

Subject to:

Upper-Level: Leader

Lower-Level: Follower

Strategic Offering Problem



Leader

$$\max_{C_i,\overline{p}_i} \; p_i(\lambda - C_i')$$

subject to:

$$0 \le \overline{p}_i \le \overline{p}_i'$$

$C_i, \overline{p}_i$

p_i, λ

Follower

$$\max_{p_d,p_g} \; \mathrm{SW} = \sum_{d \in D} U_d p_d - \sum_{g \in G} C_g \, p_g$$

subject to:

$$0 \le p_d \le \overline{p}_d, \forall d \in D : \; \underline{\mu}_d, \overline{\mu}_d$$

$$0 \le p_g \le \overline{p}_g, \forall g \in G : \; \underline{\mu}_g, \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \; : \; \lambda$$

Upper-Level: Leader

Objective: $$\min_{C_i,\overline{p}_i,p_d,p_g} \; -p_i(\lambda - C_i')$$

Subject to: $$0 \le \overline{p}_i \le \overline{p}_i'$$

Lower-Level: Follower

$$\min_{p_d,p_g} \sum_{g \in G} C_g \, p_g - \sum_{d \in D} U_d p_d$$

s.t.

$$-p_d \le 0, \forall d \in D \; : \; \underline{\mu}_d$$

$$p_d - \overline{p}_d \le 0, \forall d \in D : \; \overline{\mu}_d$$

$$-p_g \le 0, \forall g \in G \; : \; \underline{\mu}_g$$

$$p_g - \overline{p}_g \le 0, \forall g \in G : \; \overline{\mu}_g$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \; : \; \lambda$$

Strategic Offering Problem

Objective: $\min_{C_i, \overline{p}_i, p_d, p_g} \; -p_i(\lambda - C_i')$

Subject to: $0 \leq \overline{p}_i \leq \overline{p}_i'$

Upper-Level: Leader

$$\min_{p_d, p_g} \sum_{g \in G} C_g \, p_g - \sum_{d \in D} U_d p_d$$

s.t.

$$\begin{aligned}
-p_d &\leq 0, \forall d \in D &&: \underline{\mu}_d \\
p_d - \overline{p}_d &\leq 0, \forall d \in D &&: \overline{\mu}_d \\
-p_g &\leq 0, \forall g \in G &&: \underline{\mu}_g \\
p_g - \overline{p}_g &\leq 0, \forall g \in G &&: \overline{\mu}_g \\
\sum_{d \in D} p_d - \sum_{g \in G} p_g &= 0 &&: \lambda
\end{aligned}$$

Lower-Level: Follower

Lagrangian function of lower-level problem:

KKT conditions of the lower-level problem:

Strategic Offering Problem

Upper-Level: Leader

Objective: $\min_{C_i, \overline{p}_i, p_d, p_g} \quad -p_i(\lambda - C_i')$

Subject to: $0 \leq \overline{p}_i \leq \overline{p}_i'$

Lower-Level: Follower

$$\min_{p_d, p_g} \sum_{g \in G} C_g \, p_g - \sum_{d \in D} U_d p_d$$

$$\text{s.t.} \quad \begin{aligned} -p_d &\leq 0, \forall d \in D &&: \underline{\mu}_d \\ p_d - \overline{p}_d &\leq 0, \forall d \in D &&: \overline{\mu}_d \\ -p_g &\leq 0, \forall g \in G &&: \underline{\mu}_g \\ p_g - \overline{p}_g &\leq 0, \forall g \in G &&: \overline{\mu}_g \\ \sum_{d \in D} p_d - \sum_{g \in G} p_g &= 0 &&: \lambda \end{aligned}$$

Lagrangian function of lower-level problem:

$$\mathcal{L}\left(p_d, p_g, \underline{\mu}_d, \overline{\mu}_d, \underline{\mu}_g, \overline{\mu}_g, \lambda\right)$$
$$= \sum_{g \in G} C_g \, p_g - \sum_{d \in D} U_d p_d - p_d \underline{\mu}_d + \overline{\mu}_d (p_d - \overline{p}_d)$$
$$- p_g \underline{\mu}_g + \overline{\mu}_g \left(p_g - \overline{p}_g\right) + \lambda (\sum_{d \in D} p_d - \sum_{g \in G} p_g)$$

KKT conditions of the lower-level problem:

Strategic Offering Problem

Upper-Level: Leader

Objective: $\min\limits_{C_i, \overline{p}_i, p_d, p_g} \; -p_i(\lambda - C_i')$

Subject to: $0 \leq \overline{p}_i \leq \overline{p}_i'$

Lower-Level: Follower

$$\min_{p_d, p_g} \sum_{g \in G} C_g \, p_g - \sum_{d \in D} U_d p_d$$

s.t.

$$\begin{aligned}
-p_d &\leq 0, \forall d \in D &&: \underline{\mu}_d \\
p_d - \overline{p}_d &\leq 0, \forall d \in D &&: \overline{\mu}_d \\
-p_g &\leq 0, \forall g \in G &&: \underline{\mu}_g \\
p_g - \overline{p}_g &\leq 0, \forall g \in G &&: \overline{\mu}_g \\
\sum_{d \in D} p_d - \sum_{g \in G} p_g &= 0 &&: \lambda
\end{aligned}$$

Lagrangian function of lower-level problem:

$$\mathcal{L}\left(p_d, p_g, \underline{\mu}_d, \overline{\mu}_d, \underline{\mu}_g, \overline{\mu}_g, \lambda\right)$$
$$= \sum_{g \in G} C_g \, p_g - \sum_{d \in D} U_d p_d - p_d \underline{\mu}_d + \overline{\mu}_d (p_d - \overline{p}_d)$$
$$- p_g \underline{\mu}_g + \overline{\mu}_g \left(p_g - \overline{p}_g\right) + \lambda (\sum_{d \in D} p_d - \sum_{g \in G} p_g)$$

KKT conditions of the lower-level problem:

$$\frac{\partial \mathcal{L}}{\partial p_d} = -U_d - \underline{\mu}_d + \overline{\mu}_d + \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial p_g} = C_g - \underline{\mu}_g + \overline{\mu}_g - \lambda = 0$$
$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0$$
$$0 \leq p_d \perp \underline{\mu}_d \geq 0$$
$$0 \leq \overline{p}_d - p_d \perp \overline{\mu}_d \geq 0$$
$$0 \leq p_g \perp \underline{\mu}_g \geq 0$$
$$0 \leq \overline{p}_g - p_g \perp \overline{\mu}_g \geq 0$$

Strategic Offering Problem

Upper-Level: Leader

Objective: $\min_{C_i, \overline{p}_i, p_d, p_g} \; -p_i(\lambda - C_i')$

Subject to: $0 \le \overline{p}_i \le \overline{p}_i'$

Lower-Level: Follower

$$\min_{p_d, p_g} \sum_{g \in G} C_g \, p_g - \sum_{d \in D} U_d p_d$$

s.t.

$$
\begin{aligned}
-p_d \le 0, \forall d \in D \quad &: \quad \underline{\mu}_d \\
p_d - \overline{p}_d \le 0, \forall d \in D \quad &: \quad \overline{\mu}_d \\
-p_g \le 0, \forall g \in G \quad &: \quad \underline{\mu}_g \\
p_g - \overline{p}_g \le 0, \forall g \in G \quad &: \quad \overline{\mu}_g \\
\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \quad &: \quad \lambda
\end{aligned}
$$

Lagrangian function of lower-level problem:

$$
\begin{aligned}
&\mathcal{L}\left(p_d, p_g, \underline{\mu}_d, \overline{\mu}_d, \underline{\mu}_g, \overline{\mu}_g, \lambda\right) \\
&= \sum_{g \in G} C_g \, p_g - \sum_{d \in D} U_d p_d - p_d \underline{\mu}_d + \overline{\mu}_d (p_d - \overline{p}_d) \\
&- p_g \underline{\mu}_g + \overline{\mu}_g \left(p_g - \overline{p}_g\right) + \lambda (\sum_{d \in D} p_d - \sum_{g \in G} p_g)
\end{aligned}
$$

KKT conditions of the lower-level problem:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial p_d} = -U_d - \underline{\mu}_d + \overline{\mu}_d + \lambda = 0 \quad & \times|D| \\
\frac{\partial \mathcal{L}}{\partial p_g} = C_g - \underline{\mu}_g + \overline{\mu}_g - \lambda = 0 \quad & \times|G| \\
\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \quad & \times 1 \\
0 \le p_d \perp \underline{\mu}_d \ge 0 \quad & \times|D| \\
0 \le \overline{p}_d - p_d \perp \overline{\mu}_d \ge 0 \quad & \times|D| \\
0 \le p_g \perp \underline{\mu}_g \ge 0 \quad & \times|G| \\
0 \le \overline{p}_g - p_g \perp \overline{\mu}_g \ge 0 \quad & \times|G|
\end{aligned}
$$

Strategic Offering Problem

Which variables should be added?

Objective: $\min_{C_i,\overline{p}_i,\boxed{???}} -p_i(\lambda - C_i')$

Subject to: $0 \le \overline{p}_i \le \overline{p}_i'$

$$\frac{\partial \mathcal{L}}{\partial p_d} = -U_d - \underline{\mu}_d + \overline{\mu}_d + \lambda = 0 \qquad \times|D|$$

$$\frac{\partial \mathcal{L}}{\partial p_g} = C_g - \underline{\mu}_g + \overline{\mu}_g - \lambda = 0 \qquad \times|G|$$

$$\sum_{d\in D} p_d - \sum_{g\in G} p_g = 0 \qquad \times 1$$

$$0 \le p_d \perp \underline{\mu}_d \ge 0 \qquad \times|D|$$

$$0 \le \overline{p}_d - p_d \perp \overline{\mu}_d \ge 0 \qquad \times|D|$$

$$0 \le p_g \perp \underline{\mu}_g \ge 0 \qquad \times|G|$$

$$0 \le \overline{p}_g - p_g \perp \overline{\mu}_g \ge 0 \qquad \times|G|$$

Updated Leader's Problem under MPCC

Lagrangian function of lower-level problem:

$$\mathcal{L}\left(p_d, p_g, \underline{\mu}_d, \overline{\mu}_d, \underline{\mu}_g, \overline{\mu}_g, \lambda\right)$$
$$= \sum_{g\in G} C_g\, p_g - \sum_{d\in D} U_d p_d - p_d \underline{\mu}_d + \overline{\mu}_d\left(p_d - \overline{p}_d\right)$$
$$- p_g \underline{\mu}_g + \overline{\mu}_g\left(p_g - \overline{p}_g\right) + \lambda\left(\sum_{d\in D} p_d - \sum_{g\in G} p_g\right)$$

KKT conditions of the lower-level problem:

$$\frac{\partial \mathcal{L}}{\partial p_d} = -U_d - \underline{\mu}_d + \overline{\mu}_d + \lambda = 0 \qquad \times|D|$$

$$\frac{\partial \mathcal{L}}{\partial p_g} = C_g - \underline{\mu}_g + \overline{\mu}_g - \lambda = 0 \qquad \times|G|$$

$$\sum_{d\in D} p_d - \sum_{g\in G} p_g = 0 \qquad \times 1$$

$$0 \le p_d \perp \underline{\mu}_d \ge 0 \qquad \times|D|$$

$$0 \le \overline{p}_d - p_d \perp \overline{\mu}_d \ge 0 \qquad \times|D|$$

$$0 \le p_g \perp \underline{\mu}_g \ge 0 \qquad \times|G|$$

$$0 \le \overline{p}_g - p_g \perp \overline{\mu}_g \ge 0 \qquad \times|G|$$

Strategic Offering Problem

$C_i^*, \overline{p}_i^*$: Optimal offering strategy

Updated Leader's Problem under MPCC

Objective:

$$\min_{\boxed{C_i,\overline{p}_i}, p_d, p_g, \underline{\mu}_d, \overline{\mu}_d, \underline{\mu}_g, \overline{\mu}_g, \lambda} \; -p_i(\lambda - C_i')$$

Subject to:

$$0 \leq \overline{p}_i \leq \overline{p}_i'$$

$$\frac{\partial \mathcal{L}}{\partial p_d} = -U_d - \underline{\mu}_d + \overline{\mu}_d + \lambda = 0 \qquad \times|D|$$

$$\frac{\partial \mathcal{L}}{\partial p_g} = C_g - \underline{\mu}_g + \overline{\mu}_g - \lambda = 0 \qquad \times|G|$$

$$\sum_{d \in D} p_d - \sum_{g \in G} p_g = 0 \qquad \times 1$$

$$0 \leq p_d \perp \underline{\mu}_d \geq 0 \qquad \times|D|$$

$$0 \leq \overline{p}_d - p_d \perp \overline{\mu}_d \geq 0 \qquad \times|D|$$

$$0 \leq p_g \perp \underline{\mu}_g \geq 0 \qquad \times|G|$$

$$0 \leq \overline{p}_g - p_g \perp \overline{\mu}_g \geq 0 \qquad \times|G|$$

Learning Objectives

Today's lecture will (re-)introduce **sequential (Stackelberg) games**, and their relevance in the energy market.

After this lecture, you should be able to

1. Provide the definition of a **Stackelberg equilibrium** in a sequential game, and identify it in a payoff matrix;
2. Formulate a sequential game of an energy market application using a **bilevel model**, identify the **leader** and the **follower**, and match the leader with the upper level problem, and the follower with the lower level problem;
3. Derive the **Karush-Kuhn-Tucker conditions (KKTs)** of the lower lever problem of the bilevel model, given which, update the upper level problem as a **Mathematical Program with Complementarity Constraint (MPCC).**

Now you can do Assignment 2 Step 3 (c), Step 4, and Step 5…

Well, basically the whole assignment… Hooray!!

Thanks for your attention!

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April, 2022

Technical University of Denmark
Department of Wind and Energy Systems