DTU

31792 -- Advanced Optimization and Game Theory for Energy Systems

Lecture 1: Market clearing as an optimization problem

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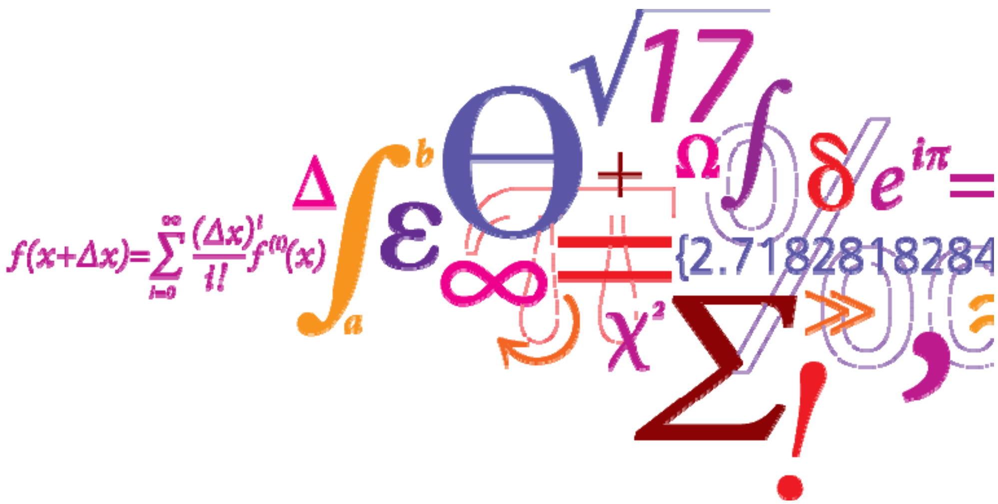

January 4, 2021

DTU Electrical Engineering
Department of Electrical Engineering

Market clearing: a simple example

Let us get started with a question. Assume an electricity market with a single generator (G1) and an elastic demand (D1). What are the market-clearing outcomes (production, consumption and market-clearing price)?

Capacity: 100 MW
Offer price: $12/MWh

Demand D1

Maximum load: 80 MW
Bid price: $40/MWh

Market outcomes:

- Production level of G1: **?**
- Consumption level of D1: **?**
- Market-clearing price: **?**

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Demand D1

Maximum load: 80 MW
Bid price: $40/MWh

Market outcomes:

- Production level of G1: **80 MW**
- Consumption level of D1: **80 MW**
- Market-clearing price: **$12/MWh**

Market clearing: a simple example

An extended example: two generators (G1 and G2) and two elastic demands (D1 and D2)

Capacity: 100 MW
Offer price: $12/MWh

Capacity: 80 MW
Offer price: $20/MWh

Demand D1

Maximum load: 100 MW
Bid price: $40/MWh

Demand D2

Maximum load: 50 MW
Bid price: $35/MWh

Market outcomes:

- Productions of G1 and G2: **?**
- Consumptions of D1 and D2: **?**
- Market-clearing price: **?**

Market clearing: a simple example

An extended example: two generators (G1 and G2) and two elastic demands (D1 and D2)

Capacity: 100 MW
Offer price: $12/MWh

Capacity: 80 MW
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Demand D1

Maximum load: 100 MW
Bid price: $40/MWh

Demand D2

Maximum load: 50 MW
Bid price: $35/MWh

Market outcomes:

- Productions of G1 and G2: **100 MW and 50 MW**
- Consumptions of D1 and D2: **100 MW and 50 MW**
- Market-clearing price: **$20/MWh**

Market clearing as an optimization problem

Question:
How to form the previous example as an optimization problem?

Market clearing as an optimization problem

Generic form:

Maximize social welfare (SW) of the market[1]

Subject to

- All technical constraints of generators and demands
- Power balance equality

[1] SW (also known as "*market surplus*") is equal to:
[total utility of demands based on their bid prices] – [total cost of generators based on their offer prices]

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \tag{1a}$$

subject to:

$$0 \leq p^{D1} \leq 100 \tag{1b}$$

$$0 \leq p^{D2} \leq 50 \tag{1c}$$

$$0 \leq p^{G1} \leq 100 \tag{1d}$$

$$0 \leq p^{G2} \leq 80 \tag{1e}$$

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \tag{1f}$$

Market clearing as an optimization problem

$$\underbrace{\underset{p^{\mathrm{G1}},p^{\mathrm{G2}},p^{\mathrm{D1}},p^{\mathrm{D2}}}{\text{Maximize}}}_{\text{Set of primal variables}} \quad SW = \underbrace{[40p^{\mathrm{D1}} + 35p^{\mathrm{D2}}]}_{\text{Utility of demands}} - \underbrace{[12p^{\mathrm{G1}} + 20p^{\mathrm{G2}}]}_{\text{Cost of generators}} \qquad \text{(1a)}$$

subject to:

$$\left.\begin{aligned} &0 \leq p^{\mathrm{D1}} \leq 100 && \text{(1b)} \\ &0 \leq p^{\mathrm{D2}} \leq 50 && \text{(1c)} \end{aligned}\right\} \text{Consumption limits}$$

$$\left.\begin{aligned} &0 \leq p^{\mathrm{G1}} \leq 100 && \text{(1d)} \\ &0 \leq p^{\mathrm{G2}} \leq 80 && \text{(1e)} \end{aligned}\right\} \text{Generation limits}$$

$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0 \qquad \text{(1f)}$$

Power balance

Market clearing as an optimization problem

$$\underbrace{\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}}}_{\text{Set of primal variables}} \quad SW = \underbrace{[40p^{D1} + 35p^{D2}]}_{\text{Utility of demands}} - \underbrace{[12p^{G1} + 20p^{G2}]}_{\text{Cost of generators}} \tag{1a}$$

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$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \tag{1f}$$

Power balance

Discussion:
Is this optimization problem convex? How to know it?

Market clearing as an optimization problem

Question:

How to obtain market-clearing price within the optimization problem?

Market clearing as an optimization problem

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Answer:
The dual variable (also known as "Lagrangian multiplier") of the power balance equality provides the market-clearing price!

Market clearing as an optimization problem

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How to obtain market-clearing price within the optimization problem?

Answer:
The dual variable (also known as "Lagrangian multiplier") of the power balance equality provides the market-clearing price!

Note: This is based on "uniform" pricing scheme, which is the most common practice in real-world electricity markets. There are other types of pricing schemes, such as "pay-as-bid" and "Vickrey–Clarke–Groves (VCG)", which derive market prices in a different way.

Market clearing as an optimization problem

$$\underset{p^{\mathrm{G1}},p^{\mathrm{G2}},p^{\mathrm{D1}},p^{\mathrm{D2}}}{\text{Maximize}} \quad SW = [40p^{\mathrm{D1}} + 35p^{\mathrm{D2}}] - [12p^{\mathrm{G1}} + 20p^{\mathrm{G2}}] \tag{1a}$$

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$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0 \qquad : \lambda \tag{1f}$$

Market clearing as an optimization problem

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Dual variable of power balance equality

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \tag{1a}$$

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$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda \tag{1f}$$

Discussion:

- What does a dual variable show in general (mathematical interpretation)?
- What is its sign (negative, or positive, or free)? Can the electricity market price be negative?

Market clearing as an optimization problem

Compact form:

Market clearing as an optimization problem

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}} \tag{1a}$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad \forall d \tag{1b}$$

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad \forall g \tag{1c}$$

$$\sum_d p_d^{\mathrm{D}} - \sum_g p_g^{\mathrm{G}} = 0 \qquad : \lambda \tag{1d}$$

U_d : bid price of demand d

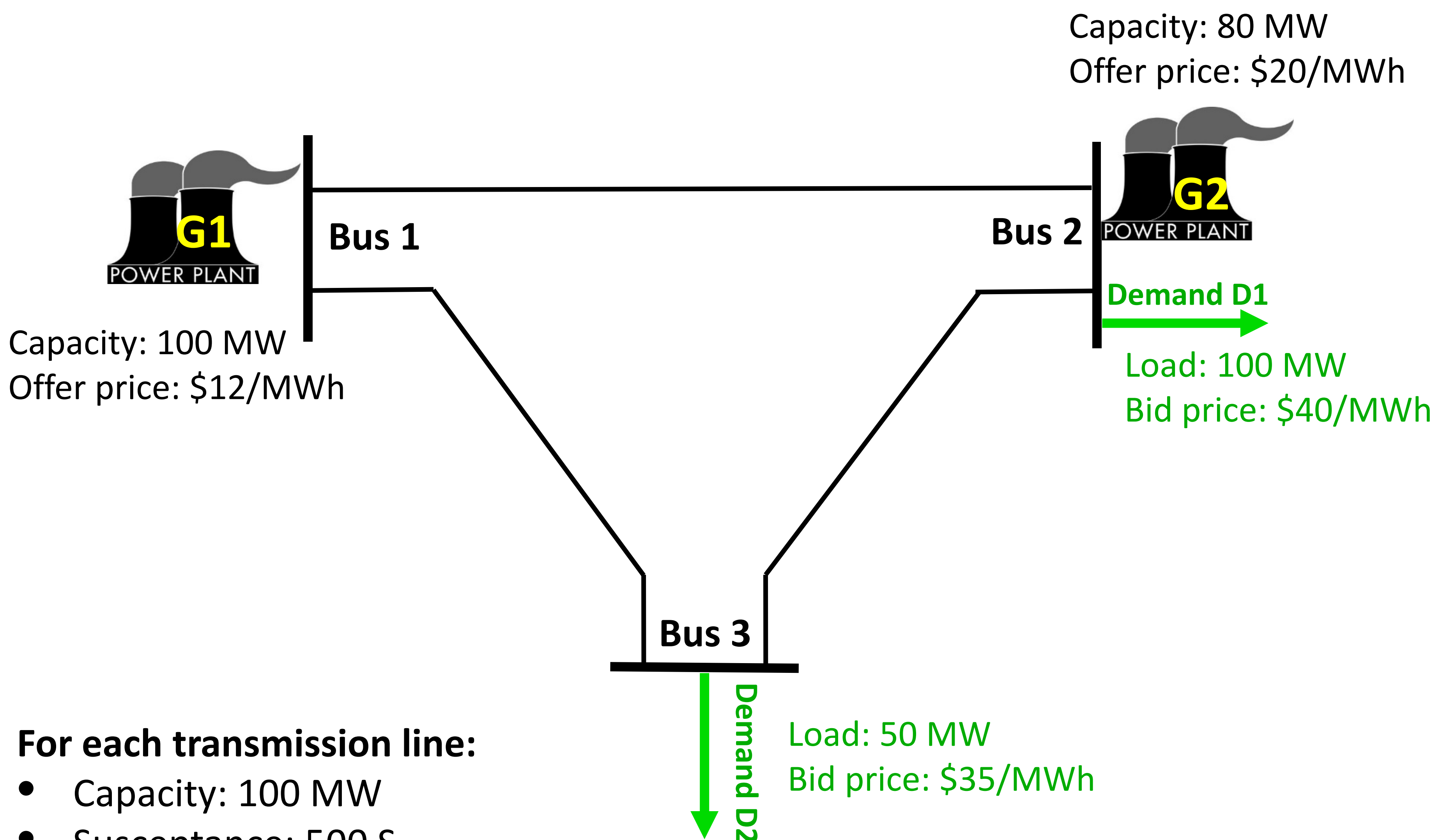
C_g : offer price of generator g

$\overline{P}_d^{\mathrm{D}}$: maximum load of demand d

$\overline{P}_g^{\mathrm{G}}$: capacity of generator g

Market clearing considering network

Market clearing considering network



For each transmission line:

- Capacity: 100 MW
- Susceptance: 500 S

Market clearing considering network

Let's use an approximate linearized representation of power flow equations (DC power flow). Accordingly, the following equation gives the power flow across the line connecting bus *n* to bus *m*:

$$f_{n,m} = \underbrace{B_{n,m}}_{\text{Susceptance (parameter)}} \underbrace{(\theta_n - \theta_m)}_{\text{Difference of voltage angles of buses } n \text{ and } m \text{ (variable)}}$$

Power flow from bus *n* to bus *m* (variable)

Market clearing considering network

Discussion:

- What are the power flow equations in reality?
- Why do we use the DC power flow equations?

$$f_{n,m} = \underbrace{B_{n,m}}_{\text{Susceptance (parameter)}}\underbrace{(\theta_n - \theta_m)}_{\text{Difference of voltage angles of buses } n \text{ and } m \text{ (variable)}}$$

Power flow from bus *n* to bus *m* (variable)

Market clearing considering network

Discussion:

- What are the power flow equations in reality?
- Why do we use the DC power flow equations?

Take course **31765** in case you are interested to learn more about power flow!

31765 Optimization in modern power systems — 2019/2020

Course information

Danish title	Optimering i moderne power systemer
Language of instruction	English
Point(ECTS)	5
Course type	MSc Technological specialization course, see more
Schedule	Autumn E3A (Tues 8-12)
Location	Campus Lyngby
Scope and form	Lectures, exercises, computer exercises, project work
Duration of Course	13 weeks
Date of examination	E3A
Type of assessment	Oral examination and reports
Aid	All Aid
Evaluation	7 step scale , internal examiner
Recommended prerequisites	31730 , 31730 – Fundamentals of Electric Power Engineering or equivalent Linear Algebra or equivalent Complex Analysis or equivalent Programming in Matlab; Python or another programming language also ok.
Responsible	Spyros Chatzivasileiadis , Lyngby Campus, Building 325, Ph. (+45) 4525 5621 , spchatz@elektro.dtu.dk Pierre Pinson , Lyngby Campus, Building 325, Ph. (+45) 4525 3541 , ppin@elektro.dtu.dk
Department	31 Department of Electrical Engineering
Registration Sign up	At the Studyplanner
Green challenge participation	This course gives the student an opportunity to prepare a project that may participate in DTU's Study Conference on sustainability, climate technology, and the environment (GRØN DYST). More information http://www.groendyst.dtu.dk/english

General course objectives

Optimization is a powerful tool that has several applications in power system operation. Optimization tools are used by electricity market operators, power system operators, and other players. Such tools define the market clearing, identify optimal bidding strategies for generators, determine optimal control actions for operators to e.g. minimize losses, and help devise optimal investment strategies for the future electricity grid. This course introduces the students to general optimization algorithms, explains their principles, and shows them how to formulate and solve the relevant problems in power systems. The knowledge acquired through this course could be applied to any decision making process, e.g. devise the optimal stock portfolio for a bank, find the fastest transportation route, and others.

Learning objectives

A student who has met the objectives of the course will be able to:

- Recognize and formulate problems for operation and investments in power systems
- Describe the basic principles of Linear programming, Quadratic programming, Nonlinear programming, and Semidefinite programming
- Formulate the dual of an optimization problem and the optimality conditions (KKT)
- Explain what locational marginal price is in electricity markets
- Design and solve optimal power flow problems (DC-OPF, AC-OPF)
- Understand and apply convex relaxations (e.g. semidefinite programming)
- Describe three advantages and disadvantages of each formulation
- Organize, plan, and carry out work in a group project
- Analyze and present the results in front of an audience

Content

This course focuses on how to take optimal decisions that deal with both the economic and the technical operation of power systems. We learn how to analyze and formulate optimization problems, for different objectives and accuracy. From an economic point of view, we cover electricity market operation, optimal bidding strategies for power producers, and optimal investment strategies for transmission owners. From a technical point of view, we cover the minimization of losses, minimization of reactive power needs, and optimal location of grid reinforcements. The course also focuses on the basic principles of how an optimization solver works, their strengths and weaknesses. This will lead to a better understanding of how to formulate a general optimization problem, which can be applied to any decision making process in the real world.

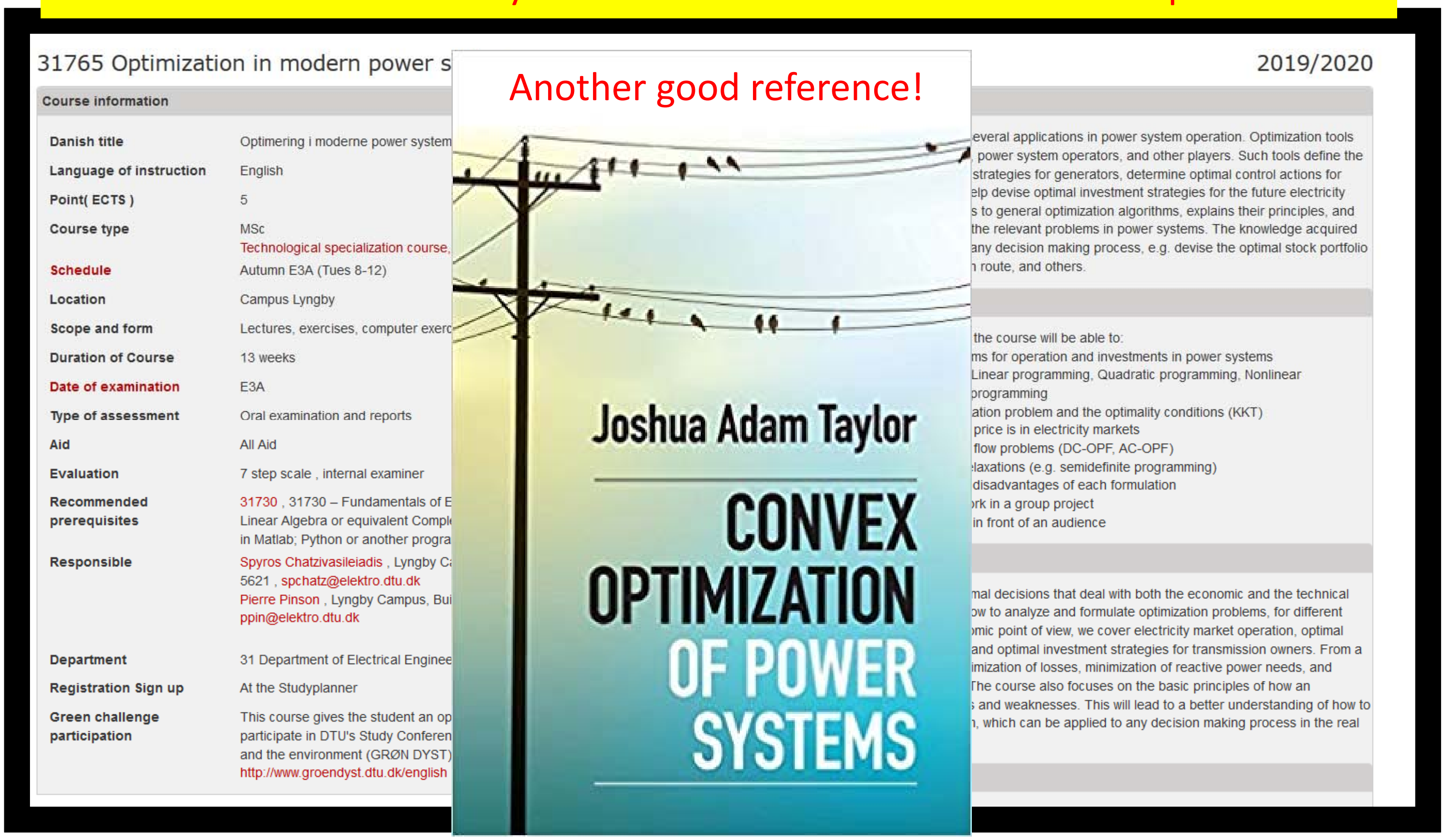
Last updated

Market clearing considering network

Discussion:

- What are the power flow equations in reality?
- Why do we use the DC power flow equations?

Take course **31765** in case you are interested to learn more about power flow!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the relevant problems in power systems. The knowledge acquired
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Market clearing considering network

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subject to:

$$0 \leq p^{D1} \leq 100$$

$$0 \leq p^{D2} \leq 50$$

$$0 \leq p^{G1} \leq 100$$

$$0 \leq p^{G2} \leq 80$$

$$p^{G1} - 500(\theta^{N1} - \theta^{N2}) - 500(\theta^{N1} - \theta^{N3}) = 0 \quad : \lambda^{N1}$$

$$p^{G2} - p^{D1} - 500(\theta^{N2} - \theta^{N1}) - 500(\theta^{N2} - \theta^{N3}) = 0 \quad : \lambda^{N2}$$

$$-p^{D2} - 500(\theta^{N3} - \theta^{N1}) - 500(\theta^{N3} - \theta^{N2}) = 0 \quad : \lambda^{N3}$$

$$-100 \leq 500(\theta^{N1} - \theta^{N2}) \leq 100$$

$$-100 \leq 500(\theta^{N1} - \theta^{N3}) \leq 100$$

$$-100 \leq 500(\theta^{N2} - \theta^{N3}) \leq 100$$

$$\theta^{N1} = 0$$

Market clearing considering network

$$\underset{p^{\mathrm{G1}},p^{\mathrm{G2}},p^{\mathrm{D1}},p^{\mathrm{D2}},\theta^{\mathrm{N1}},\theta^{\mathrm{N2}},\theta^{\mathrm{N3}}}{\text{Maximize}} \quad SW = [40p^{\mathrm{D1}} + 35p^{\mathrm{D1}}] - [12p^{\mathrm{G1}} + 20p^{\mathrm{G2}}]$$

subject to:

$$0 \le p^{\mathrm{D1}} \le 100$$

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$$0 \le p^{\mathrm{G1}} \le 100$$

$$0 \le p^{\mathrm{G2}} \le 80$$

$$\left.\begin{aligned}
&p^{\mathrm{G1}} - 500(\theta^{\mathrm{N1}} - \theta^{\mathrm{N2}}) - 500(\theta^{\mathrm{N1}} - \theta^{\mathrm{N3}}) = 0 \quad : \lambda^{\mathrm{N1}}\\
&p^{\mathrm{G2}} - p^{\mathrm{D1}} - 500(\theta^{\mathrm{N2}} - \theta^{\mathrm{N1}}) - 500(\theta^{\mathrm{N2}} - \theta^{\mathrm{N3}}) = 0 \quad : \lambda^{\mathrm{N2}}\\
&- p^{\mathrm{D2}} - 500(\theta^{\mathrm{N3}} - \theta^{\mathrm{N1}}) - 500(\theta^{\mathrm{N3}} - \theta^{\mathrm{N2}}) = 0 \quad : \lambda^{\mathrm{N3}}
\end{aligned}\right\} \text{Power balance at each bus}$$

$$\left.\begin{aligned}
&-100 \le 500(\theta^{\mathrm{N1}} - \theta^{\mathrm{N2}}) \le 100\\
&-100 \le 500(\theta^{\mathrm{N1}} - \theta^{\mathrm{N3}}) \le 100\\
&-100 \le 500(\theta^{\mathrm{N2}} - \theta^{\mathrm{N3}}) \le 100
\end{aligned}\right\} \text{Capacity of each transmission line}$$

$$\theta^{\mathrm{N1}} = 0 \longrightarrow \text{Reference bus}$$

Market clearing considering network

Compact form:

Market clearing considering network

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}, \theta_n}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad \forall d$$

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{d \in \Psi_n} p_d^{\mathrm{D}} + \sum_{m \in \Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g \in \Psi_n} p_g^{\mathrm{G}} = 0 \qquad : \lambda_n \quad \forall n$$

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad \forall n, \forall m$$

$$\theta_{ref} = 0$$

Market clearing considering network

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}, \theta_n}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

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All demands located at bus *n*

Market clearing considering network

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$$- F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad \forall n, \forall m$$

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All buses *m* connected to bus *n* through transmission lines

Market clearing considering network

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All generators located at bus *n*

Market clearing considering network

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$$\theta_{ref} = 0$$

Nodal price at bus *n*
(locational marginal price, LMP)

Market clearing considering network

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$$\theta_{ref} = 0$$

Capacity of line connecting bus *n* to bus *m*

Market clearing considering network

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$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad \forall n, \forall m$$

$$\theta_{ref} = 0$$

Voltage angle at the reference bus

GAMS code

```
Sets
g  generators /G1*G2/
n  buses      /N1*N3/
d  demands    /D1*D2/
alias (n,m)

sets
MapN(n,n) Network topology /
N1.N2
N1.N3
N2.N3
N2.N1
N3.N1
N3.N2/

MapG(g,n) Location of generators /
G1.N1
G2.N2
/

MapD(d,n) Location of demands /
D1.N2
D2.N3
/
;
```

GAMS code

```
Parameter PGmax(g) Capacity of generators [MW] /
G1  100
G2  80
/;

Parameter C(g) Offer price of generators [$ per MWh] /
G1  12
G2  20
/;

Parameter L(d) Maximum load of demands [MW] /
D1  100
D2  50
/;

Parameter U(d) Utility of demands [$ per MWh] /
D1  40
D2  35
/
;
```

GAMS code

```
Table Fmax(n,n)  Capacity of transmission lines [MW]
    N1          N2            N3
N1  0           100           100
N2  100         0             100
N3  100         100           0
;

Table B(n,n) Susceptance of transmission lines [Ohm^{-1}]
    N1          N2            N3
N1  0           500           500
N2  500         0             500
N3  500         500           0
;
```

GAMS code

```
Free variable
SW          Social welfare of the market [$]
f(n,m)      Power flow from bus n to m [MW]
theta(n)    Voltage angle of bus n [rad.];

Positive variable
p_D(d)      Consumption level of demand d [MW]
p_G(g)      Production level of generator g [MW];

Equations
objective,cons1,cons2,cons3,cons4,cons5,cons6;

objective..   SW          =e= sum(d, U(d)*p_D(d)) - sum(g, C(g)*p_G(g));
cons1(g)..    p_G(g)      =l= PGmax(g);
cons2(d)..    p_D(d)      =l= L(d);
cons3(n,m)..  f(n,m)      =e= B(n,m)*(theta(n)-theta(m));
cons4(n,m)..  f(n,m)      =l= Fmax(n,m);
cons5..       theta('N1') =e=0;
cons6(n)..    -sum(g$MapG(g,n),p_G(g))+sum(d$MapD(d,n),p_D(d))
              +sum(m$MapN(n,m),f(n,m)) =e=0;

Model Market_clearing /all/;
Solve Market_clearing using lp maximizing SW;
Display SW.l,p_G.l,p_D.l,f.l,cons6.m;
```

Exercise 1

In GAMS code (or your own code in Python or in Julia or in MATLAB), change the transmission capacity of line connecting buses 1 and 3 to 40 MW. Please run the code, and compare the market-clearing outcomes to the original ones. Interpret the new outcomes.

Market-clearing problem: primal optimization

$$\underset{p_g^{\mathrm{G}}\geq 0,\ p_d^{\mathrm{D}}\geq 0,\ \theta_n}{\text{Maximize}} \quad \sum_d U_d\, p_d^{\mathrm{D}} - \sum_g C_g\, p_g^{\mathrm{G}}$$

subject to:

$$p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad : \mu_d^{\mathrm{D}} \quad \forall d$$

$$p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad : \mu_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{d\in\Psi_n} p_d^{\mathrm{D}} + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^{\mathrm{G}} = 0 \quad : \lambda_n \quad \forall n$$

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

$$\theta_{(n=ref)} = 0 \quad : \gamma$$

Market-clearing problem: dual optimization

$$\underset{\mu^{\mathrm{D}}_d \geq 0,\ \mu^{\mathrm{G}}_g \geq 0,\ \underline{\eta}_{n,m} \geq 0,\ \overline{\eta}_{n,m} \geq 0,\ \lambda_n,\ \gamma}{\text{Minimize}} \quad \sum_d \mu^{\mathrm{D}}_d\ \overline{P}^{\mathrm{D}}_d + \sum_g \mu^{\mathrm{G}}_g\ \overline{P}^{\mathrm{G}}_g + \sum_{n,(m \in \Omega_n)} F_{n,m}(\underline{\eta}_{n,m} + \overline{\eta}_{n,m})$$

subject to:

$$-U_d + \mu^{\mathrm{D}}_d + \lambda_{n \in \Psi_d} \geq 0 \quad : p^{\mathrm{D}}_d \quad \forall d$$

$$C_g + \mu^{\mathrm{G}}_g - \lambda_{n \in \Psi_g} \geq 0 \quad : p^{\mathrm{G}}_g \quad \forall g$$

$$\sum_{m \in \Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) + \gamma = 0 \quad : \theta_n \quad n = ref$$

$$\sum_{m \in \Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) = 0 \quad : \theta_n \quad \forall n \ / \ ref$$

Market-clearing problem: dual optimization

$$\underset{\mu_d^{\mathrm{D}}\geq 0,\ \mu_g^{\mathrm{G}}\geq 0,\ \underline{\eta}_{n,m}\geq 0,\ \overline{\eta}_{n,m}\geq 0,\ \lambda_n,\ \gamma}{\text{Minimize}} \quad \sum_d \mu_d^{\mathrm{D}}\ \overline{P}_d^{\mathrm{D}} + \sum_g \mu_g^{\mathrm{G}}\ \overline{P}_g^{\mathrm{G}} + \sum_{n,(m\in\Omega_n)} F_{n,m}(\underline{\eta}_{n,m} + \overline{\eta}_{n,m})$$

subject to:

$$-U_d + \mu_d^{\mathrm{D}} + \lambda_{n\in\Psi_d} \geq 0 \quad : p_d^{\mathrm{D}} \quad \forall d$$

$$C_g + \mu_g^{\mathrm{G}} - \lambda_{n\in\Psi_g} \geq 0 \quad : p_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) + \gamma = 0 \quad : \theta_n \quad n = ref$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) = 0 \quad : \theta_n \quad \forall n\ /\ ref$$

Exercise 2: Derive this formulation yourself!

Market-clearing problem: dual optimization

$$\underset{\mu_d^{\mathrm{D}}\geq 0,\ \mu_g^{\mathrm{G}}\geq 0,\ \underline{\eta}_{n,m}\geq 0,\ \overline{\eta}_{n,m}\geq 0,\ \lambda_n,\ \gamma}{\text{Minimize}} \quad \sum_d \mu_d^{\mathrm{D}}\ \overline{P}_d^{\mathrm{D}} + \sum_g \mu_g^{\mathrm{G}}\ \overline{P}_g^{\mathrm{G}} + \sum_{n,(m\in\Omega_n)} F_{n,m}(\underline{\eta}_{n,m} + \overline{\eta}_{n,m})$$

subject to:

$$-U_d + \mu_d^{\mathrm{D}} + \lambda_{n\in\Psi_d} \geq 0 \quad : p_d^{\mathrm{D}} \quad \forall d$$

$$C_g + \mu_g^{\mathrm{G}} - \lambda_{n\in\Psi_g} \geq 0 \quad : p_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) + \gamma = 0 \quad : \theta_n \quad n = ref$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) = 0 \quad : \theta_n \quad \forall n \,/\, ref$$

Exercise 2: Derive this formulation yourself!

How to derive a dual optimization? Next session!

Exercise 3

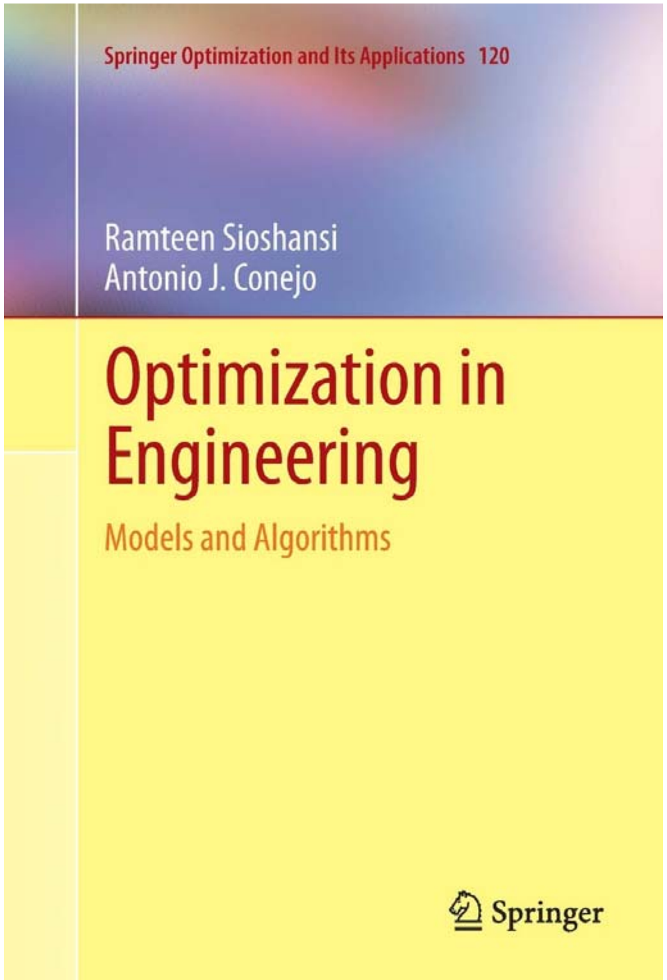
Please check Step 1 of the course project. Think of your problem!

Thanks for your attention!

Email: seykaz@elektro.dtu.dk

How to derive <u>optimality conditions</u> and <u>dual problem</u> of a linear optimization problem?

References


Springer Optimization and Its Applications 120
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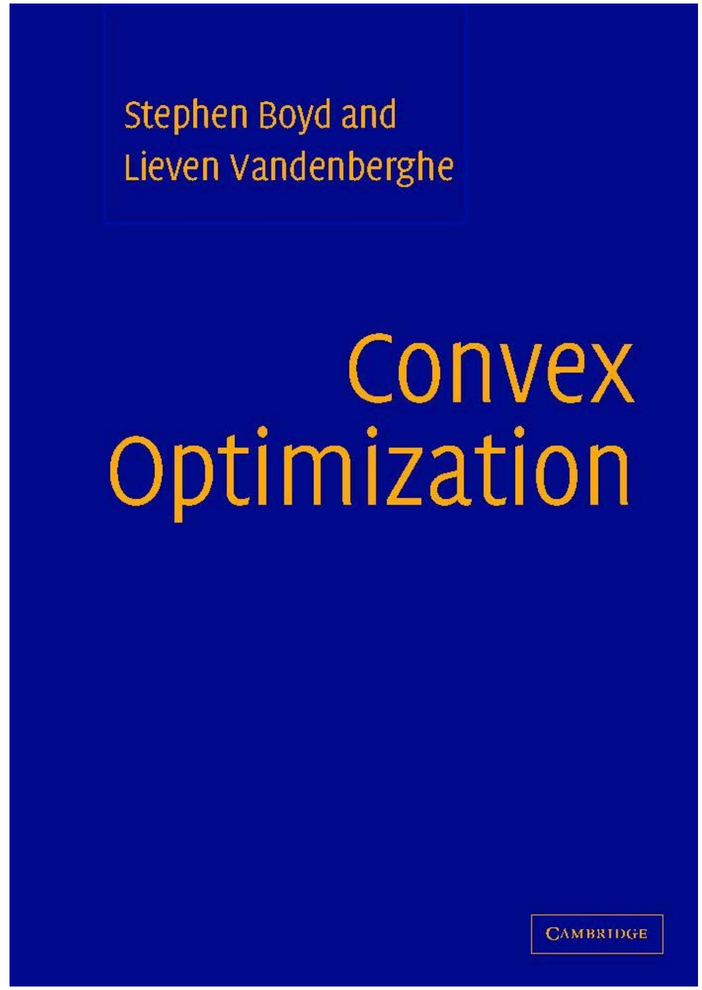
Stephen Boyd and
Lieven Vandenberghe
Convex Optimization
CAMBRIDGE

Stephen Boyd at DTU

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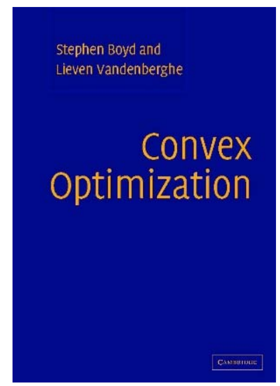

A mathematician on a mission

Mathematical analysis | Operations analysis | Mathematics

TUESDAY 27 NOV 18 | By Morten Andersen

Stanford University Professor Stephen Boyd applies convex optimization to a wide range of engineering problems. With astounding results.

"DTU should teach a course on convex optimization. And all students should be obliged to take it!"

HC Ørsted lectures

Twice a year, DTU invites prominent foreign researchers to lecture on their work, research findings, and the prospects within their field of research at the so-called Ørsted Lectures. The lectures are open to all.

Link:

https://www.dtu.dk/english/news/2018/11/dynamo-a-mathematician-on-a-mission?id=4969767c-da34-416b-8720-ea3f1fb2009e

Convex optimization course at DTU

02953 Convex optimization

2019/2020

Course information

Danish title	Konveks optimering
Language of instruction	English
Point(ECTS)	5
Course type	Ph.D., Professionally focussed course
Schedule	June
Location	Campus Lyngby
Scope and form	Lectures and exercises, followed by a final project.
Duration of Course	3 weeks
Type of assessment	Evaluation of experiments and reports
Aid	All Aid
Evaluation	pass / not pass , internal examiner
Recommended prerequisites	01005/02601/02610 , Coursework in linear algebra (e.g. 01005) and numerical algorithms (e.g., 02601), introductory-level coursework in optimization (e.g., 02610), a certain degree of mathematical maturity, and proficiency in a high-level programming language such as MATLAB, Python, or Julia.
Participants restrictions	Minimum 10 Maximum: 30
Responsible	Martin Skovgaard Andersen , Lyngby Campus, Building 303B, Ph. (+45) 4525 3036 , mskan@dtu.dk
Department	01 Department of Applied Mathematics and Computer Science
Home page	http://people.compute.dtu.dk/mskan/convexopt.html
Registration Sign up	At the Studyplanner
Green challenge participation	Please contact the teacher for information on whether this course gives the student the opportunity to prepare a project that may participate in DTU´s Study Conference on sustainability, climate technology, and the environment (GRØN DYST). More infor http://www.groendyst.dtu.dk/english

General course objectives

The aim of the course is to provide students with a general overview of convex optimization theory, its applications, and computational methods for large-scale optimization. The students will learn how to recognize convex optimization problems and how to solve these numerically using either an existing software library or by deriving/implementing a suitable method that exploits problem structure. As part of the course, the students will work on a project which aims to provide students with the opportunity to put theory to work in a practical and application-oriented context.

Learning objectives

A student who has met the objectives of the course will be able to:

- recognize and characterize convex functions and sets
- explain/characterize the subdifferential of a convex function
- describe basic concepts of convex analysis
- derive the Lagrange dual of a convex optimization problem
- recognize and formulate conic constraints
- derive a convex relaxation of nonconvex quadratic problems
- implement a first-order method for a large-scale optimization problem with structure
- construct and implement a splitting method for a convex–concave saddle-point problem
- evaluate the computational performance of an optimization algorithm

Content

Convex analysis (convex sets and functions, convex conjugate, duality, dual norms, composition rules, subgradient calculus), conic optimization (linear optimization, second-order cone optimization, semidefinite optimization), first-order methods for smooth and nonsmooth optimization (proximal gradient methods, acceleration), splitting methods (Douglas–Rachford splitting, ADMM, Chambolle–Pock algorithm), stochastic methods, incremental methods and coordinate descent methods.

CourseLiterature

S. Boyd and L. Vandenberghe: "Convex Optimization", Cambridge University Press, 2003.
A. Ben-Tal and A. Nemirovski: "Lectures on Modern Convex Optimization", lecture notes, 2013.

Last updated

20. juni, 2019

How to derive Lagrangian function?

$$\underset{x}{\text{Minimize}} \quad f(x)$$

subject to:

$$h(x) = 0 \quad : \quad \lambda$$

$$g(x) \leq 0 \quad : \quad \mu$$

This is a standard form of an optimization problem!

How to derive Lagrangian function?

$$\begin{aligned}
&\underset{x}{\text{Minimize}} \quad f(x) \\
&\text{subject to:} \\
&h(x) = 0 \quad : \quad \lambda \\
&g(x) \leq 0 \quad : \quad \mu
\end{aligned}$$

This is a standard form of an optimization problem!

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

How to derive optimality conditions?

Original (primal) problem

$$
\begin{aligned}
&\underset{x}{\text{Minimize}} \quad f(x) \\
&\text{subject to:} \\
&h(x) = 0 \quad : \quad \lambda \\
&g(x) \leq 0 \quad : \quad \mu
\end{aligned}
$$

Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^{\top} h(x) + \mu^{\top} g(x)$$

How to derive optimality conditions?

Original (primal) problem

$$\underset{x}{\text{Minimize}} \quad f(x)$$

subject to:

$$h(x) = 0 \quad : \quad \lambda$$

$$g(x) \leq 0 \quad : \quad \mu$$

Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

$$\frac{\partial \mathcal{L}(x, \lambda, \mu)}{\partial x} = 0$$

$$h(x) = 0$$

$$0 \leq -g(x) \perp \mu \geq 0$$

$$\lambda \in \text{free}$$

Optimality
Karush–Kuhn–Tucker (KKT)
conditions

How to derive optimality conditions?

Original (primal) problem

$$\begin{aligned} &\underset{x}{\text{Minimize}} \quad f(x) \\ &\text{subject to:} \\ &h(x) = 0 \quad : \quad \lambda \\ &g(x) \leq 0 \quad : \quad \mu \end{aligned}$$

Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

$$\boxed{\begin{aligned} &\frac{\partial \mathcal{L}(x, \lambda, \mu)}{\partial x} = 0 \\ &h(x) = 0 \\ &0 \leq -g(x) \perp \mu \geq 0 \\ &\lambda \in \text{free} \end{aligned}}$$

Optimality
Karush–Kuhn–Tucker (KKT)
conditions

Complementarity condition

Example

Let us consider the following linear optimization problem:

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$x_1 \geq 0 \quad : \quad \mu_3$$

$$-x_2 \geq 0 \quad : \quad \mu_4$$

$$x_3 \geq 0 \quad : \quad \mu_5$$

$$x_4 \geq 0 \quad : \quad \mu_6$$

Example

Let us consider the following linear optimization problem:

$$\underset{\boxed{x_1, x_2, x_3, x_4}}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

Four primal variables

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$x_1 \geq 0 \quad : \quad \mu_3$$

$$-x_2 \geq 0 \quad : \quad \mu_4$$

$$x_3 \geq 0 \quad : \quad \mu_5$$

$$x_4 \geq 0 \quad : \quad \mu_6$$

Example

Let us consider the following linear optimization problem:

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \mu_2$$

$$x_1 \geq 0 \quad : \mu_3$$

$$-x_2 \geq 0 \quad : \mu_4$$

$$x_3 \geq 0 \quad : \mu_5$$

$$x_4 \geq 0 \quad : \mu_6$$

Six dual variables, one per constraint

Example

Let us consider the following linear optimization problem:

$$\underset{x_1, x_2, x_3, x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \mu_2$$

$$x_1 \geq 0 \quad : \mu_3$$

$$-x_2 \geq 0 \quad : \mu_4$$

$$x_3 \geq 0 \quad : \mu_5$$

$$x_4 \geq 0 \quad : \mu_6$$

Six dual variables, one per constraint

Recall:
When we derive Lagrangian function, the inequality constraints should be in form of

$$g(x) \leq 0$$

Example

Original (primal) problem

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$x_1 \geq 0 \quad : \quad \mu_3$$

$$-x_2 \geq 0 \quad : \quad \mu_4$$

$$x_3 \geq 0 \quad : \quad \mu_5$$

$$x_4 \geq 0 \quad : \quad \mu_6$$

Lagrangian function

$$\begin{aligned}\mathcal{L}(\mathbf{x},\mu) = {} & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1\left(\frac{2}{3}x_1 + 2x_2 + x_3 - 1\right) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4\end{aligned}$$

Example

Lagrangian function

$$\begin{aligned}\mathcal{L}(\mathbf{x},\mu) = {} & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4\end{aligned}$$

Optimality KKT conditions

$$\begin{aligned}&\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_1} = 18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0 \\ &\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_2} = 8 - 2\mu_1 - \mu_2 + \mu_4 = 0 \\ &\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_3} = 12 - \mu_1 - \mu_5 = 0 \\ &\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_4} = 16 - \mu_2 - \mu_6 = 0 \\ &0 \le (\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \perp \mu_1 \ge 0 \\ &0 \le (x_1 + x_2 + x_4 - 1) \perp \mu_2 \ge 0 \\ &0 \le x_1 \perp \mu_3 \ge 0 \\ &0 \le -x_2 \perp \mu_4 \ge 0 \\ &0 \le x_3 \perp \mu_5 \ge 0 \\ &0 \le x_4 \perp \mu_6 \ge 0\end{aligned}$$

Example

Lagrangian function

$$
\begin{aligned}
\mathcal{L}(\mathbf{x},\mu) = {} & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\
& - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\
& - \mu_2(x_1 + x_2 + x_4 - 1) \\
& - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4
\end{aligned}
$$

Optimality KKT conditions

$$
\begin{aligned}
&\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_1} = 18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0 \\
&\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_2} = 8 - 2\mu_1 - \mu_2 + \mu_4 = 0 \\
&\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_3} = 12 - \mu_1 - \mu_5 = 0 \\
&\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_4} = 16 - \mu_2 - \mu_6 = 0 \\
&0 \leq (\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \perp \mu_1 \geq 0 \\
&0 \leq (x_1 + x_2 + x_4 - 1) \perp \mu_2 \geq 0 \\
&0 \leq x_1 \perp \mu_3 \geq 0 \\
&0 \leq -x_2 \perp \mu_4 \geq 0 \\
&0 \leq x_3 \perp \mu_5 \geq 0 \\
&0 \leq x_4 \perp \mu_6 \geq 0
\end{aligned}
$$

Can we write KKT conditions in a more compact way?

Example

Lagrangian function

$$
\begin{aligned}
\mathcal{L}(\mathbf{x}, \mu) = &\ 18x_1 + 8x_2 + 12x_3 + 16x_4 \\
& - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\
& - \mu_2(x_1 + x_2 + x_4 - 1) \\
& - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4
\end{aligned}
$$

Optimality KKT conditions

$$
\begin{aligned}
&\frac{\partial \mathcal{L}(\mathbf{x}, \mu)}{\partial x_1} = 18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0 \\
&\frac{\partial \mathcal{L}(\mathbf{x}, \mu)}{\partial x_2} = 8 - 2\mu_1 - \mu_2 + \mu_4 = 0 \\
&\frac{\partial \mathcal{L}(\mathbf{x}, \mu)}{\partial x_3} = 12 - \mu_1 - \mu_5 = 0 \\
&\frac{\partial \mathcal{L}(\mathbf{x}, \mu)}{\partial x_4} = 16 - \mu_2 - \mu_6 = 0 \\
&0 \leq (\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \perp \mu_1 \geq 0 \\
&0 \leq (x_1 + x_2 + x_4 - 1) \perp \mu_2 \geq 0 \\
&0 \leq x_1 \perp \mu_3 \geq 0 \\
&0 \leq -x_2 \perp \mu_4 \geq 0 \\
&0 \leq x_3 \perp \mu_5 \geq 0 \\
&0 \leq x_4 \perp \mu_6 \geq 0
\end{aligned}
$$

Can we write KKT conditions in a more compact way? Yes!

Example

Lagrangian function

$$\begin{aligned}\mathcal{L}(\mathbf{x},\mu) = &\ 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ &- \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ &- \mu_2(x_1 + x_2 + x_4 - 1) \\ &- \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4\end{aligned}$$

Optimality KKT conditions

$$\boxed{\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_1} = 18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0}$$

$$\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_2} = 8 - 2\mu_1 - \mu_2 + \mu_4 = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_3} = 12 - \mu_1 - \mu_5 = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_4} = 16 - \mu_2 - \mu_6 = 0$$

$$0 \leq (\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \perp \mu_1 \geq 0$$

$$0 \leq (x_1 + x_2 + x_4 - 1) \perp \mu_2 \geq 0$$

$$\boxed{0 \leq x_1 \perp \mu_3 \geq 0}$$

$$0 \leq -x_2 \perp \mu_4 \geq 0$$

$$0 \leq x_3 \perp \mu_5 \geq 0$$

$$0 \leq x_4 \perp \mu_6 \geq 0$$

For example, we can merge these two conditions to get rid of dual variable \mu_{3} corresponding to the non-negativity condition of x_{1}, i.e.,

$$0 \leq x_1 \perp (18 - \frac{2}{3}\mu_1 - \mu_2) \geq 0$$

Example

Eventually, the optimality KKT conditions are

Original (primal) problem

$$\begin{aligned}
&\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4\\
&\text{subject to:}\\
&\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1\\
&x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2\\
&x_1 \geq 0\\
&-x_2 \geq 0\\
&x_3 \geq 0\\
&x_4 \geq 0
\end{aligned}$$

$\Rightarrow$

Optimality KKT conditions

$$\begin{aligned}
&0 \leq (\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \perp \mu_1 \geq 0\\
&0 \leq (x_1 + x_2 + x_4 - 1) \perp \mu_2 \geq 0\\
&0 \leq x_1 \perp (18 - \frac{2}{3}\mu_1 - \mu_2) \geq 0\\
&0 \leq -x_2 \perp (-8 + 2\mu_1 + \mu_2) \geq 0\\
&0 \leq x_3 \perp (12 - \mu_1) \geq 0\\
&0 \leq x_4 \perp (16 - \mu_2) \geq 0
\end{aligned}$$

Example

How to write a code to directly solve KKT conditions (as a system of equations)?

Option 1: Solve using PATH solver (http://pages.cs.wisc.edu/~ferris/path.html)

Option 2: Define an auxiliary objective function (e.g., minimize 1), consider KKT conditions as the constraints, and then solve the resulting optimization problem using a non-linear solver (nonlinearity comes from complementarity conditions) ---- we will discuss later in this course how to linearize the complementarity conditions using auxiliary binary (0/1) variables!

How to derive dual problem?

How to derive dual problem?

Discussion:

Why is it appealing to derive dual problem?

How to derive dual problem?

Recall that

Original (primal) problem

$$\begin{aligned}&\underset{x}{\text{Minimize}} \quad f(x)\\ &\text{subject to:}\\ &h(x) = 0 \quad : \quad \lambda\\ &g(x) \leq 0 \quad : \quad \mu\end{aligned}$$

Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

Step 1: derive "dual function" as $\underset{x}{\text{Minimize}} \quad \mathcal{L}(x, \lambda, \mu)$

How to derive dual problem?

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Step 1: derive "dual function" as $\underset{x}{\text{Minimize}} \quad \mathcal{L}(x, \lambda, \mu)$

- Dual function is an unconstrained optimization problem. For arbitrarily given values of dual variables (\mu should be non-negative), the dual function minimizes the (relaxed) Lagrangian function. Primal variables are the only variables to be optimized.
- Why "relaxed"? Because constraints in the original primal problem are relaxed, and the fixed dual variables in the dual function "penalize" the violation of relaxed constraints.
- The optimal value of the dual function provides a "**lower bound**" for the optimal value of objective function of the original primal problem.
- More info? Watch this short video: https://www.youtube.com/watch?v=4OifjG2kIJQ

How to derive dual problem?

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Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

Step 2: derive "dual problem" which provides the <u>best possible lower bound</u>, i.e.,

$$\underset{\lambda \in \text{free} \;,\; \mu \geq 0}{\text{Maximize}} \quad \underbrace{\underset{x}{\text{Minimize}} \quad \mathcal{L}(x, \lambda, \mu)}_{\substack{\text{dual function} \\ \text{(i.e., lower bound)}}}$$

Example

Recall our previous example

Original (primal) problem

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$
$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$
$$x_1 \geq 0 \quad : \quad \mu_3$$
$$-x_2 \geq 0 \quad : \quad \mu_4$$
$$x_3 \geq 0 \quad : \quad \mu_5$$
$$x_4 \geq 0 \quad : \quad \mu_6$$

$\Rightarrow$

Lagrangian function

$$\begin{aligned}\mathcal{L}(\mathbf{x},\mu) = &\, 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ &- \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ &- \mu_2(x_1 + x_2 + x_4 - 1) \\ &- \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4\end{aligned}$$

Example

Recall our previous example

Original (primal) problem

$$\begin{aligned}
&\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4\\
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&\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1\\
&x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2\\
&x_1 \geq 0 \quad : \quad \mu_3\\
&-x_2 \geq 0 \quad : \quad \mu_4\\
&x_3 \geq 0 \quad : \quad \mu_5\\
&x_4 \geq 0 \quad : \quad \mu_6
\end{aligned}$$

Lagrangian function

$$\begin{aligned}
\mathcal{L}(\mathbf{x}, \mu) = {} & 18x_1 + 8x_2 + 12x_3 + 16x_4\\
& - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1)\\
& - \mu_2(x_1 + x_2 + x_4 - 1)\\
& - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4
\end{aligned}$$

Dual problem

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \left\{ \begin{aligned}
\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4\\
& - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1)\\
& - \mu_2(x_1 + x_2 + x_4 - 1)\\
& - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4
\end{aligned} \right\}$$

Example

Recall our previous example

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$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$
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Lagrangian function

$$\begin{aligned}\mathcal{L}(\mathbf{x},\mu) = {} & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4\end{aligned}$$

These two terms (-1 times -\mu) are constants in the inner (i.e., minimization) problem, but variables in the outer (i.e., maximization) problem!

Dual problem

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \left\{ \begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{aligned} \right\}$$

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6\geq 0}{\text{Maximize}} \left\{ \begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{aligned} \right\}$$

Example

Dual problem:

$$\underset{\mu_1,\dots,\mu_6 \geq 0}{\text{Maximize}} \left\{ \begin{array}{l} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ \qquad - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ \qquad - \mu_2(x_1 + x_2 + x_4 - 1) \\ \qquad - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{array} \right\}$$

$$\Downarrow$$

$$\underset{\mu_1,\dots,\mu_6 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0$$

$$8 - 2\mu_1 - \mu_2 + \mu_4 = 0$$

$$12 - \mu_1 - \mu_5 = 0$$

$$16 - \mu_2 - \mu_6 = 0$$

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6\geq 0}{\text{Maximize}} \left\{ \begin{array}{l} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ \qquad - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ \qquad - \mu_2(x_1 + x_2 + x_4 - 1) \\ \qquad - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{array} \right\}$$

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Can we write the dual problem in a more compact way?

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \left\{ \begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{aligned} \right\}$$

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

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$$16 - \mu_2 - \mu_6 = 0$$

Can we write the dual problem in a more compact way? Yes!

Note:

- Dual variables \mu_{3} to \mu_{6} are isolated, since they do not appear in the objective function, and do not link constraints!
- We also know that they are non-negative.
- So, we can get rid of them by converting equalities to inequalities.

Example

Dual problem:

$$\begin{aligned}
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&\text{subject to:}\\
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$$\Rightarrow$$

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\end{aligned}$$

Example

Dual problem:

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$$\Rightarrow$$

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Example

Primal problem

Two options, both are equivalent:

Option 1

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$x_1 \geq 0 \quad : \quad \mu_3$$

$$-x_2 \geq 0 \quad : \quad \mu_4$$

$$x_3 \geq 0 \quad : \quad \mu_5$$

$$x_4 \geq 0 \quad : \quad \mu_6$$

Dual problem

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Option 2 (preferred, due to less number of variables/constraints)

$$\underset{x_1,x_3,x_4 \geq 0,\ x_2 \leq 0}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$\underset{\mu_1,\mu_2 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 \geq 0$$

$$8 - 2\mu_1 - \mu_2 \leq 0$$

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Important points

- ✓ Number of variables in the primal problem = Number of constraints in the dual problem

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- ✓ Dual problem of the dual problem is the primal problem!

- ✓ Dual variables of the dual problem are the primal variables!

Important points

✓ **Weak duality theorem**:

The value of objective function of the dual problem at any point of its feasible region is *lower than or equal to* that of the primal problem at any point of its feasible region.

In our example:

$$18x_1 + 8x_2 + 12x_3 + 16x_4 \geq \mu_1 + \mu_2$$

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In our example:

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Important points

✓ **Strong duality theorem**:

In the optimal point, if Slater's condition holds, the value of objective function of the dual problem is *equal to* that of the primal problem.

In our example [note that superscript * denotes the optimal value]:

$$\underbrace{18x_1^* + 8x_2^* + 12x_3^* + 16x_4^*}_{\text{The value of the objective function of the primal problem}} = \underbrace{\mu_1^* + \mu_2^*}_{\text{The value of the objective function of the dual problem}}$$

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