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Article *in* Physics of Fluids · January 2003

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Analyses of Kolmogorov's model of breakup and its application into Lagrangian computation of liquid sprays under air-blast atomization

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(Received 26 December 2001; accepted 11 October 2002; published 6 December 2002)

This paper considers the breakup of liquid drops at the large Weber number within the framework of Kolmogorov's scenario of breakup. The population balances equation for droplet radius distribution is written to be an invariant under the group of scaling transformations. It is shown that due to this symmetry, the long-time limit solution of this equation is a power function. When the standard deviation of droplet radius strongly increases and, consequently, the characteristic length scale disappears, the power asymptotic solution can be viewed as a further evolution of Kolmogorov's log-normal distribution. This new universality appears to be consistent with the experimental observation of fractal properties of droplets produced by air-blast breakup. The scaling properties of Kolmogorov's model at later times are also demonstrated in the case where the breakup frequency is a power function of instantaneous radius. The model completes the Liouville equation for distribution function of liquid particles in the phase space of droplet position, velocity, and radius. The numerical scheme is proposed for stochastic modeling of droplets production. Lagrangian simulation of the spray under air-blast atomization is performed using KIVA II code, which is a frequently used code for computation of turbulent flows with sprays. The qualitative agreement of simulation with measurements is demonstrated. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1527914]

I. INTRODUCTION

Modern computational approaches developed for flows with sprays are often based on the spray equation written by Williams[1] for the droplet distribution function. The Lagrangian Monte Carlo procedure was proposed by Dukowicz[2] to solve the spray equation coupling with computation of the gaseous phase. To complete the spray formulation, a model of droplets production is required. The wave breakup model of Reitz[3] is the most widely used model of droplets production. By assuming, in this model, that the breakup results from a hydrodynamic instability caused by surface tension, the newly formed droplets are characterized by a single Rayleigh mode[4] of atomization. At the same time, the Rayleigh's type of breakup takes place when the liquid jet is injected into quiescent environment at a relatively low velocity. When the liquid jet is injected into the flowing motion of gas at high relative velocity (large Weber number), the influence of interfacial forces is less pronounced and the mechanism of breakup becomes more complex. A wide range of turbulent eddies may impact on the liquid jet causing its breakup. The modeling of droplets formation under this type of breakup,[5–8] often referred to as the air-blast atomization, is the main subject of this paper.

Several experimental studies[5–9] have demonstrated the difficulty in clearly defining a dominant air-blast atomization mechanism. Each spray region produces droplets with a large spectrum of size, which is often independent of the breakup pre-existing properties. Kolmogorov[10] wrote a stochastic theory for the breakup of solid particles that describes the cascade of uncorrelated breakage events. This theory presents the breakup of solid particles as a random discrete process where the probability of breaking each parent particle into a given number of parts is independent of the size of the parent particle. It should be interesting to apply the Kolmogorov's scenario to the breakup of a liquid drop at the large Weber number, in the range from its initial size down to the size of stable droplets. Note that the cascade idea was involved earlier in the statistical description of breakup by Novikov *et al.*,[11] where Novikov's[12] multiplicative intermittency theory was implemented to obtain the stationary droplet size distribution.

The photographic examination of droplets produced by air-blast atomization (Liu *et al.*[6]) shows that there is no characteristic scale in the broad interval of droplet size variation between the initial radius and the maximum stable radius. In this situation, the ensemble of produced droplets may be presented within the framework of Mandelbrot's scaling principle[13] as a power distribution function of droplet size (Mandelbrot,[14] Procaccia[15]). The power function assumes size distribution similarity as droplet counts are realized at different resolution scales. The scaling properties of air-blast breakup was experimentally demonstrated by Zhou *et al.*[16]

The paper of Shavit and Chigier[17] also related the irregular and fragmentary physics of air-blast atomization to fractals and observed the self-similar liquid structures. To this end, there is another point of discussion concerning Kolmogorov's breakup model; from Lyapunov's theorem, Kolmogorov pointed out that, at large times, his physical breakup scenario corresponds to the log-normal distribution of particle size. At the same time, the log-normal distribution tends to the power distribution where the standard deviation goes to infinity and, consequently, the characteristic scale disappears. The next logical question is "Can the power asymptotic distribution be directly obtained by using Kolmogorov's hypotheses?" In addition to this question, another motivation arises. Kolomgorov stated, at the end of his paper[10] that "*it would be interesting to analyze such a mathematical scheme when the frequency of breakup is proportional to the particle size in some power. It seems to be that the log-normal law will be no more appropriated.*" This problem has not yet been treated in the literature.

The paper has two objectives. The first objective concerns the further theoretical analysis of Kolmogorov's model, reviewed for the liquid drop breaking up at the large Weber number. Formulating Kolmogorov's physical assumption, in terms of scaling breakup symmetry, Kolmogorov's discrete model was reproduced in the form of a balance equation for the distribution function of droplet size; analytical solutions for moments of this equiation were obtained. The asymptotic analyses performed in the paper, shows that the long-time limit solution of evolution equation is a power function through Kolmogorov's log-normal distribution. In the case of breakup with frequency taken as a power function of instantaneous radius, the asymptotic solution is verified to be a power function, as well as for breakup frequency independent of radius. Meanwhile, in this case, the critical exponent becomes different.

The second objective of the paper is to implement Kolmogorov's breakup model to evaluate the spray of liquid drops atomizing from an initial size (nozzle hole) down to the maximum stable size. The model supplied the right side of the spray equation, followed by the stochastic computation of droplet production. The resulting computations describe the round water jet atomized by a high-speed annular air jet.

II. KOLMOGOROV'S (1941) THEORY OF PARTICLE BREAKUP

Komogorov[10] considered an ensemble of breaking solid particles at the given moment t (scaled by the breakup frequency). The number of particles $N(r,t)$ of size $\rho \leqslant r$ was selected amongst all particles $N(t)$. The expected total number of particles was denoted as $\overline{N}(t)$ and of particles of size $\rho \leqslant r$ was denoted as $\overline{N}(r,t)$. Considering an outcome of breakup per unit time interval $[t,t+1]$ of a given parent particle of size r, the mean number $Q(\alpha)$ of secondary particles of size $\rho \leqslant \alpha r$ $(0 \leqslant \alpha \leqslant 1)$ was introduced. Kolmogorov assumed the probability of breaking each parent particle into a given number of parts to be independent of the parent particle size; in other words, $Q(\alpha)$ is assumed to not depend on conditions existing prior to breakup, and is not influenced by other parent particles. Following this assumption, Kolmogorov stated

$$\overline{N}(r,t+1) = \int_0^1 \overline{N}\left(\frac{r}{\alpha},t\right) dQ(\alpha). \tag{1}$$

Introducing the logarithm of particle-size $x = \ln r$, Kolmogorov pointed out that

$$T(x,t) = \frac{\overline{N}(e^x,t)}{\overline{N}(t)} = \frac{N(e^x,t)}{N(t)}, \tag{2}$$

where $T(x)$ is the cumulative probability distribution of logarithm of particle-size. Further, Eq. (1) was rewritten as follows:

$$T(x,t+1) = \int_{-\infty}^{0} T(x-\ln\alpha,t)\, dS(\ln\alpha). \tag{3}$$

By Lyapunov's theorem, Kolmogorov stated that, from discrete model (3), the long-time limit form of $T(x,t)$ appears to be a Gaussian distribution. Thus, the main result of Kolmogorov's work is that $N(r,t)$ is asymptotically governed by the log-normal law.

Note that Kolmogorov's discrete model considers the breakup process as an outcome at the finite time interval and, therefore, $Q(\alpha)$ characterizes the number of secondary particles averaged over many elementary break-up actions during a finite time interval. The next section describes the differential evolution equation that employs the parameters of the elementary hypothetical breakup act.

III. EVOLUTION EQUATION AND UNIVERSALITY AT LONG-TIME LIMIT

In the framework of Kolmogorov's scenario, consider the drop, which breaks up at the large Weber number: The breakup of the drop into secondary droplets is theorized to be independent of the instantaneous size of the individual parent drop. One can write the population balance equation (about the population balance equation see Ramakrishna,[18] for example). Hereafter, the full derivation of this equation is shown in order to clarify definitions that are used in the breakup modeling. A given drop is assumed to break with the frequency ν_0 (the number of breakups per unit time) to form, in mean, the number of new droplets (q_0) after each breakup action. Let r_1 be a characteristic length scale (or radius) of the parent drop. Suppose that the radius of each product droplet is within the interval $r_1[\alpha,(\alpha+d\alpha)]$ with the probability $q(\alpha)d\alpha$, and

$$\int_0^1 q(\alpha)d\alpha = 1, \tag{4}$$

where α is introduced in (1). Then, $q_0 q(\alpha)d\alpha$ is the probable number of new droplets in the interval $r_1[\alpha,(\alpha+d\alpha)]$. Here, according to Kolmogorov's hypotheses,[10] the probability density $q(\alpha)$ and the mean number of new born particles, q_0, are independent of radius r_1. Let the number of parent drops in the interval dr_1 be given by $F(r_1)dr_1$, where $F(r_1)$ is the number distribution function of parent

drops. Then, $\nu_0(r_1)F(r_1)dr_1$ is the probable number of parent drops undergoing breakup per unit time and $\nu_0(r_1)F(r_1)dr_1 q_0 q(\alpha)d\alpha$ is the probable number of new droplets formed per unit time in the interval $r_1[\alpha,(\alpha+d\alpha)]$. From all produced particles, we only select those that fall within the interval of length scales $[r,r+dr]$. This is achieved by the characteristic function of a small interval dr

$$dr\cdot\delta(r-\alpha r_1)=\begin{cases}1 & if\quad \alpha r_1\in[r,r+dr]\\ 0 & if\quad \alpha r_1\notin[r,r+dr]\end{cases}, \tag{5}$$

where $\delta(r-\alpha r_1)$ is Dirac delta function.

Recalling these definitions, the total number of new droplets that enter into $[r,r+dr]$ per unit time, writes

$$dr\int_0^\infty\int_0^1 q_0 q(\alpha)\delta(r-\alpha r_1)\nu_0(r_1)F(r_1)d\alpha dr_1. \tag{6}$$

After integration over r_1, the balance equation follows

$$\frac{\partial F(r)}{\partial t}=(q_0\hat{I}_+-1)\nu_0(r)F(r), \tag{7}$$

where the breakup operator is

$$\hat{I}_+F=\int_0^1 F\left(\frac{r}{\alpha}\right)q(\alpha)\frac{d\alpha}{\alpha}. \tag{8}$$

The breakup frequency, which is independent of instantaneous radius r, can be evaluated by writing an l-moment equation corresponding to (7). Multiplying (7) by r^l and integrating over the entire r-range gives

$$\frac{\partial}{\partial t}\langle r^l\rangle_F=\nu_0(q_0\langle\alpha^l\rangle-1)\langle r^l\rangle_F, \tag{9}$$

where

$$\langle r^l\rangle_F=\int_0^\infty r^l F(r)dr, \tag{10}$$

$$\langle\alpha^l\rangle=\int_0^1\alpha^l q(\alpha)d\alpha. \tag{11}$$

Putting $l=0$ in (9) leads to the equation for the droplet number density n

$$\frac{\partial n}{\partial t}=\nu_0(q_0-1)n. \tag{12}$$

The solution of this equation is

$$n=n_0\exp[\nu_0(q_0-1)t], \tag{13}$$

where n_0 is the initial drops density number. It is obvious that n holds a constant value when $q_0=1$. The total mass of droplets is conserved during breakup process; hence the third moment in (9) is constant, $\partial/\partial t\langle r^3\rangle_F=0$. This yields a relation between the third moment of distribution $q(\alpha)$ and the magnitude of q_0

$$\langle\alpha^3\rangle=\frac{1}{q_0}. \tag{14}$$

Combining (7) and (12), an equation for the normalized distribution function $f(r)=(1/n)F(r)$ is derived. Denoting the frequency of the production of new droplets as $\nu=\nu_0 q_0$ leads to

$$\frac{\partial f}{\partial t}=(\hat{I}_+-1)\nu f, \tag{15}$$

with

$$\hat{I}_+f=\int_0^1 f\left(\frac{r}{\alpha}\right)q(\alpha)\frac{d\alpha}{\alpha}, \tag{16}$$

$$\int_0^\infty f(r)dr=1. \tag{17}$$

Obtaining the solution of (15) requires knowledge of the distribution $q(\alpha)$; this information is not available due to the complexity of breakup process. At the same time, the operator $\hat{I}_+$ is invariant under the group of scaling transformations $(r\to\alpha r)$. Due to this symmetry, the evolution of the distribution function $f(r,t)$ to the ultimate steady-state delta function $\delta(r)$ goes through at least two intermediate asymptotics. Evaluating these intermediate asymptotics does not require knowledge of entire function $q(\alpha)$—only the knowledge of its first two logarithmic moments. In principle, this can be shown in several different ways, including the Lee groups method (Saveliev[19]), the Melling transform (Gorokhovski and Saveliev[20]), and the moment method. The most simple and illustrative (but may be not rigorous) is the moment method. One can calculate exactly the lth central logarithmic moment, $\langle z^l\rangle=\langle(\ln r-\langle\ln r\rangle)^l\rangle$, and further show that the asymptotic expansion verifies to be

$$\frac{\langle z^{2k}\rangle}{(\sqrt{\langle\ln^2\alpha\rangle\nu t})^{2k}}\underset{t\to\infty}{=}(2k-1)!!\left(1+O\left(\frac{1}{t}\right)\right), \tag{18}$$

$$\frac{\langle z^{2k+1}\rangle}{(\sqrt{\langle\ln^2\alpha\rangle\nu t})^{2k+1}}\underset{t\to\infty}{=}\frac{k(2k+1)!!}{\sqrt{\langle\ln^2\alpha\rangle\nu t}}\frac{\langle\ln^3\alpha\rangle}{3\langle\ln^2\alpha\rangle}\left(1+O\left(\frac{1}{t}\right)\right), \tag{19}$$

for even and odd powers, correspondingly. Here, $O(1/t)$ denotes smaller terms of order $1/t$.

Since the moments of Gaussian distribution $G(z)=(1/\sqrt{2\pi}\sigma)e^{-z^2/2\sigma^2}$ with variance σ^2, are $\langle z^{2k}\rangle/\sigma^{2k}=(2k-1)!!$ and $\langle z^{2k+1}\rangle/\sigma^{2k+1}=0$, the log-normal distribution function at large times can be written

$$f(r,t)\underset{t\to\infty}{=}\frac{1}{R}\frac{1}{\sqrt{2\pi\langle\ln^2\alpha\rangle\nu t}}\exp\left(-\frac{\langle\ln\alpha\rangle^2}{2\langle\ln^2\alpha\rangle}\nu t\right)\times\exp\left(-\frac{(\ln(r/R))^2}{2\langle\ln^2\alpha\rangle\nu t}\right)\left(\frac{R}{r}\right)^{1-\langle\ln\alpha\rangle/\langle\ln^2\alpha\rangle}, \tag{20}$$

where R denotes the initial length scale. This confirms Kolmogorov's main result concerning the log-normality of the distribution function where only two first moments of unknown decay spectrum $q(\alpha)$ are needed, regardless of the form of $q(\alpha)$.

At the same time, (20) leads to the observation that, at large times, *only one universal parameter* controls the last

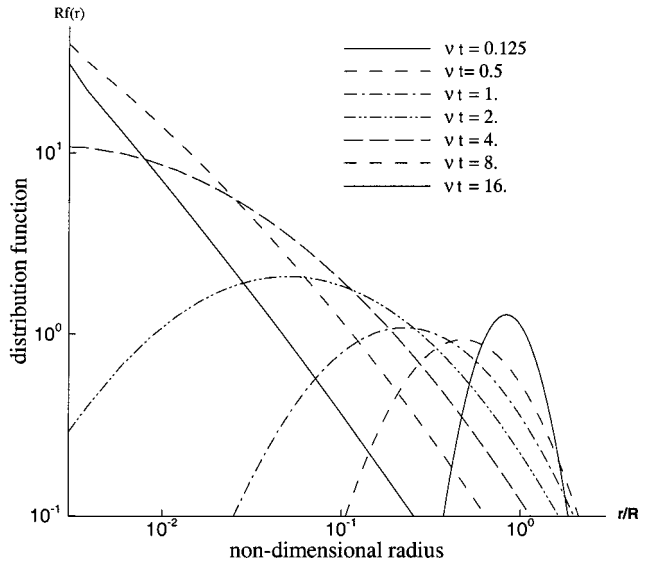

FIG. 1. The asymptotic size distribution at different nondimensional time moments νt. $\langle \ln \alpha \rangle / \langle \ln^2 \alpha \rangle = -1/2$ and $\langle \ln^2 \alpha \rangle = 1$. The log-normal distribution (36) goes to the power function (23).

stage of the breakup process. Indeed, the second multiplier in (20) tends to unity as time progresses further. This gives

$$f(r,t) \underset{t\to\infty}{=} \frac{1}{R} \frac{1}{\sqrt{2\pi \langle \ln^2 \alpha \rangle \nu t}} \times \exp\left(-\frac{\langle \ln \alpha \rangle^2}{2\langle \ln^2 \alpha \rangle} \nu t \right) \left(\frac{R}{r} \right)^{1-\langle \ln \alpha \rangle / \langle \ln^2 \alpha \rangle}.$$

Thus, Kolmogorov's log-normal distribution becomes a single-parameter $\langle \ln \alpha \rangle / \langle \ln^2 \alpha \rangle$ power function

$$f(r,t) \underset{t\to\infty}{=} \left(\frac{1}{r} \right)^{1-\langle \ln \alpha \rangle / \langle \ln^2 \alpha \rangle}. \tag{21}$$

This new universality of breakup process for the large Weber number is the primary result of this section.

The tendency to the power distribution on radius is demonstrated in Fig. 1, where (20) is given in log-scale plot at different nondimensional times νt. An interval of the linear shape appears to grow as time increases. The standard deviation increases and, consequently, the characteristic length scale disappears. This leads to the conclusion that the set of broken droplets is, asymptotically, the fractal object with the dimension defined by the ratio $\langle \ln \alpha \rangle / \langle \ln^2 \alpha \rangle$. Another conclusion is that the asymptotic power distribution in the breakup process plays the same role as the Boltzmann distribution in problems of statistical physics. Indeed, by setting (21) as $x = \ln r$, the power distribution (21) can be rewritten

$$T(x) = r \cdot f(r,t) \underset{t\to\infty}{\propto} e^{-x/h}, \tag{22}$$

where

$$h = -\frac{\langle \ln^2 \alpha \rangle}{\langle \ln \alpha \rangle}. \tag{23}$$

Equation (23) will be shown to represent the ratio between two coefficients in the Fokker–Planck equation. To some extent, the meaning of (23) is the same as the mobility–diffusion ratio given by Einstein in his theory of Brownian motion. Concerning the breakup process, finding h in (23) as a function of characteristic parameters, such as Weber and Reynolds numbers, presents an interesting problem. This paper, however, introduces the magnitude of h as a presumed value.

IV. MACROSCOPIC DESCRIPTION AT TIME SCALES LARGER THAN LIFE TIME OF DROPLET

The asymptotic distributions (20), obtained through an assumption of the scaling symmetry of the breakup process, depend only on two first moments $\langle \ln \alpha \rangle$ and $\langle \ln^2 \alpha \rangle$. This means that *changing of higher moments* $\langle \ln^k \alpha \rangle$, $k>2$ *in Eq. (15) does not affect its solution at large times* in comparison with the life time of the breaking drop. Due to this phenomena, the lattes can be simply equated to zero, rather than making (see, for example, Risken,[21] Iranpour[22]) the broadly used inconsistent assumption about the smallness of $\langle \ln^k \alpha \rangle$, $k>2$. More discussion on this important point can be found in the paper of Ryskin[23] on the Brownian motion.

By expanding $(1/\alpha) f(r/\alpha)$ on $\ln \alpha$ in (15)

$$\frac{1}{\alpha} f\left(\frac{r}{\alpha} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left(\frac{\partial}{\partial r} r \right)^n f(r) \ln^n \alpha,$$

and by setting to zero the third and all higher logarithmic moments, the integro-differential equation (15) becomes the Fokker–Planck-type equation

$$\frac{\partial f(r)}{\partial t} = \left[-\frac{\partial}{\partial r} r \langle \ln \alpha \rangle + \frac{1}{2!} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} r \langle \ln^2 \alpha \rangle \right] \nu f(r). \tag{24}$$

Here the differential operator is acting from the left on all r-dependent functions.

Additionally, the mass conservation condition (14) can be rewritten as

$$\frac{1}{q_0} = \langle \alpha^3 \rangle = \langle e^{3 \ln \alpha} \rangle = 1 + 3\langle \ln \alpha \rangle + \frac{9}{2!} \langle \ln^2 \alpha \rangle, \tag{25}$$

and, for the number distribution $F(r) = n f(r)$, one finally gives

$$\frac{\partial F(r)}{\partial t} = \left[-3\langle \ln \alpha \rangle - \frac{9}{2} \langle \ln^2 \alpha \rangle - \frac{\partial}{\partial r} r \langle \ln \alpha \rangle + \frac{1}{2} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} r \langle \ln^2 \alpha \rangle \right] \nu F(r). \tag{26}$$

Equation (26) describes the macro-scale dynamics of breakup resulting from many elementary breakup actions. The solution of (26) verifies to be

$$F(r,t)=\frac{\exp[3\left(-\langle\ln\alpha\rangle-(3/2)\langle\ln^2\alpha\rangle\right)\nu t]}{r\sqrt{2\pi\langle\ln^2\alpha\rangle\nu t}}\times\int_0^\infty\exp\left[-\frac{\left(\ln(r_0/r)+\langle\ln\alpha\rangle\nu t\right)^2}{2\langle\ln^2\alpha\rangle\nu t}\right]F_0(r_0)dr_0. \tag{27}$$

This solution will be used further in computations in Sec. VI.

V. POWER ASYMPTOTIC SOLUTION WITH BREAKUP FREQUENCY DEPENDENT OF INSTANTANEOUS RADIUS OF PARENT DROP

This section answers the following: ''Given that, in (15) the breakup frequency is a power function of radius

$$\nu(r)=cr^\beta,\quad \beta>0. \tag{28}$$

What is the long-time behavior of droplet-size distribution?''

In this case, it is possible to find an exact expression for mean values of integer powers l of $1/r^\beta$

$$\left\langle\left(\frac{1}{r^\beta}\right)^l\right\rangle=\left\langle\left(\frac{1}{r^\beta}\right)^l\right\rangle_0+\frac{ct}{1!}\left(\left\langle\left(\frac{1}{\alpha^\beta}\right)^{l-1}\right\rangle_q-1\right)\times\left\langle\left(\frac{1}{r^\beta}\right)^{l-1}\right\rangle_0+\cdots+\frac{c^lt^l}{l!}\left(\left\langle\left(\frac{1}{\alpha^\beta}\right)^l\right\rangle_q-1\right)\times\left(\left\langle\left(\frac{1}{\alpha^\beta}\right)^{l-1}\right\rangle_q-1\right)\cdots\left(\left\langle\left(\frac{1}{\alpha^\beta}\right)\right\rangle_q-1\right), \tag{29}$$

where $\langle\,\rangle_0$ and $\langle\,\rangle_q$ denote the averaging on initial distribution function $f_0(r)$ and decay spectrum $q(\alpha)$, correspondingly. It follows from (29) that moments $\langle(1/r^\beta)^l\rangle$ increase in long-time limit as t^l

$$\left\langle\left(\frac{1}{r^\beta}\right)^l\right\rangle\underset{t\to\infty}{\approx}\frac{c^lt^l}{l!}\left(\left\langle\left(\frac{1}{\alpha^\beta}\right)^l\right\rangle_q-1\right)\left(\left\langle\left(\frac{1}{\alpha^\beta}\right)^{l-1}\right\rangle_q-1\right)\cdots\times\left(\left\langle\left(\frac{1}{\alpha^\beta}\right)\right\rangle_q-1\right). \tag{30}$$

Equation (30) shows that the asymptotic solution of (15) can be sought in the following form:

$$f(r,t)\underset{t\to\infty}{\approx}(\lambda t+\tau_0)^{1/\beta}\varphi_\infty((\lambda t+\tau_0)^{1/\beta}r), \tag{31}$$

where $\lambda>0$ and τ_0 are constants. Indeed, the mean values $\langle(1/r^\beta)^l\rangle$ on distribution function (31) have the same asymptotic behavior as (30). Substituting (31) into (15) yields an equation for $\varphi_\infty(r)$

$$\left[-\frac{\lambda}{\beta}\frac{d}{dr}r+(\hat I_+-1)\nu\right]\varphi_\infty(r)=0. \tag{32}$$

In the case of Fokker–Planck approximation, the operator $\hat I_+$ has a form

$$\hat I_+=1-\langle\ln\alpha\rangle\frac{\partial}{\partial r}r+\frac{1}{2}\langle\ln^2\alpha\rangle\frac{\partial}{\partial r}r\frac{\partial}{\partial r}r. \tag{33}$$

Then the solution of (32) and (33) is verified as

$$\varphi_\infty(r)\propto\left(\frac{1}{r}\right)^{1+\beta-((2\langle\ln\alpha\rangle)/\langle\ln^2\alpha\rangle)}\exp\left(-\frac{2\lambda r^{-\beta}}{\beta^2\langle\ln^2\alpha\rangle c}\right). \tag{34}$$

Substituting (34) into (31), one yields the radius distribution function

$$f(r,t)\propto\left(\frac{1}{\lambda t+\tau_0}\right)^{1-((2\langle\ln\alpha\rangle)/\beta\langle\ln^2\alpha\rangle)}\times\exp\left(-\frac{2\lambda r^{-\beta}}{(\lambda t+\tau_0)\beta^2\langle\ln^2\alpha\rangle c}\right)\cdot\left(\frac{1}{r}\right)^{1+\beta-((2\langle\ln\alpha\rangle)/\langle\ln^2\alpha\rangle)},$$

which has the power form in the limit of $t\to\infty$

$$f(r,t)\underset{t\to\infty}{\propto}\left(\frac{1}{r}\right)^{1+\beta-((2\langle\ln\alpha\rangle)/\langle\ln^2\alpha\rangle)}. \tag{35}$$

Note that in the case of $\beta\to0$, expression (35) is not identical to (21). This remarkable fact can be explained by the different symmetry properties of Eq. (15) in two cases: $\beta=0$ and $\beta\neq0$.

VI. IMPLEMENTATION INTO COMPUTATIONAL CODE KIVA II

The KIVA II code (Amsden *et al.*[24]), is written for numerical computation of transient two and three-dimensional chemically reactive fluid turbulent flows with sprays. This numerical model incorporates the governing averaged equations for the gas phase and the stochastic particle method for computation of liquid sprays, including O'Rourke's[25] models of droplets dispersion and coalescence. The interaction between the spray droplets and the gas phase are also accounted for. Turbulence is modeled using standard $k-\varepsilon$ transport equations. The stochastic particle method[24] is based on the spray equation for distribution function $F(\mathbf{x},\mathbf{v},r;t)$. Adding in this paper the breakup source term in form of the right-hand side of (26), the spray equation looks as

$$\frac{\partial F}{\partial t}+\frac{\partial}{\partial\boldsymbol{x}}\boldsymbol{v}F+\frac{\partial}{\partial\mathrm{v}}\boldsymbol{a}F=\left[-3\langle\ln\alpha\rangle-\frac{9}{2}\langle\ln^2\alpha\rangle-\frac{\partial}{\partial r}r\langle\ln\alpha\rangle+\frac{1}{2}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}r\langle\ln^2\alpha\rangle\right]\nu F(r)+\Gamma_c, \tag{36}$$

where $\mathbf{v}$, $\mathbf{x}$ and $\mathbf{a}(\mathbf{x},\mathbf{v},r,t)$ are velocity, position, and droplet acceleration, respectively, and Γ_c denotes the model of coalescence between droplets. In this paper, the breakup is stochastically simulated along with Lagrangian tracking of parcels of drops; the initial size of drops is equal to the exit nozzle radius.

To simplify computations, the sampling procedure was based on the logarithm of radius, $x=\ln r$. Consider a given primary parcel that breaks into secondary droplets according to the following scenario. The starting distribution for the logarithm of droplet radius in this primary parcel is

$$T_0(x)=\delta(x_0-x). \tag{37}$$

After the passage of time ($\nu t = 1$), the new droplets arise due to the parcel breakup. The change of the initial distribution (37) is governed by Fokker–Planck equation (24) and, as time progresses to $\nu t = 1$, the distribution of $x = \ln r$ becomes

$$T(x) = \frac{1}{\sqrt{2\pi\langle \ln^2 \alpha\rangle}} \exp\left[- \frac{(x - x_0 - \langle \ln \alpha\rangle)^2}{2\langle \ln^2 \alpha\rangle}\right], \tag{38}$$

where $\langle \ln \alpha\rangle$ and $\langle \ln^2 \alpha\rangle$ are parameters of the model. Therefore, the sampling procedure for new droplets ($\int_{-\infty}^{x} T(x)dx = \eta$, $0 < \eta < 1$) is realized by the following function:

$$\int_{-\infty}^{x} T(x)dx = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x - x_0 - \langle \ln \alpha\rangle}{\sqrt{2\langle \ln^2 \alpha\rangle}}\right)\right], \tag{39}$$

where erf is the error function. The number of newborn droplets is computed from the mass conservation in the primary and secondary parcels. The breakup process is interrupted when the radius of secondary parcel becomes equal to or less than the maximum stable radius, r_{cr}. The breakup frequency[26–28] is

$$\nu = B \frac{\langle |\overline{\boldsymbol{v}}_g + \boldsymbol{v}_g' - \boldsymbol{v}_P| \rangle}{r} \sqrt{\frac{\rho_g}{\rho_l}}, \tag{40}$$

where $B = 1/\sqrt{3}$; $\boldsymbol{v}_g'$ is the local velocity fluctuation in gas;[24] and ρ_g, ρ_l are densities in the gaseous and liquid phases. The critical (or maximum stable) radius is computed by equating the critical Weber number to six.[29,30]

VII. EXAMPLES OF LAGRANGIAN COMPUTATION OF SPRAY UNDER AIR-BLAST ATOMIZATION

A schematic of the injector and an example of computed one-side spatial distributions of drops in the spray, are displayed in Figs. 2 and 3, respectively. The vertical lines in Fig. 3 are slightly moved outwards the symmetry axes. The inlet parameters are used from the experiment on air-blast atomization at atmospheric pressure (Stepowski *et al.*[31]). In this experiment, the round jet of water issues from the central tube ($D_l = 1.8$ mm) at low velocity and atomizes by a parallel flow of air issued at high velocity from an annular duct ($D_g = 3.4$ mm).

The spatial distributions of blobs in Fig. 3 and zooming displayed in Fig. 4 show that a broad spectrum of droplet size is presented at each spray location, with a co-existence of large and small droplets. In Fig. 5, the evolution over time of distributions of droplet-size is demonstrated at two cross sections in the near-nozzle region. The distributions at 3 mm from injector orifice shows mostly large unbroken drops of the size of injector orifice accompanied by small striped droplets. Simultaneously, it is seen that, from distributions at 5 mm from the injector, blobs that are the size of the injector orifice are essentially depleted and new droplets are produced with radius from 50 to 250 μm. These numerical distributions can be assessed only qualitatively since the existing experimental technique is not accessible to measurements of size distribution in the near-nozzle region. However, the diagnostics developed[31] that can determine the length of potential core of the liquid jet. In the case demonstrated in Fig. 2, this length was measured at 3.5 mm. Likewise, the computed distributions of size predict the jet disintegration between 3 and 5 mm.

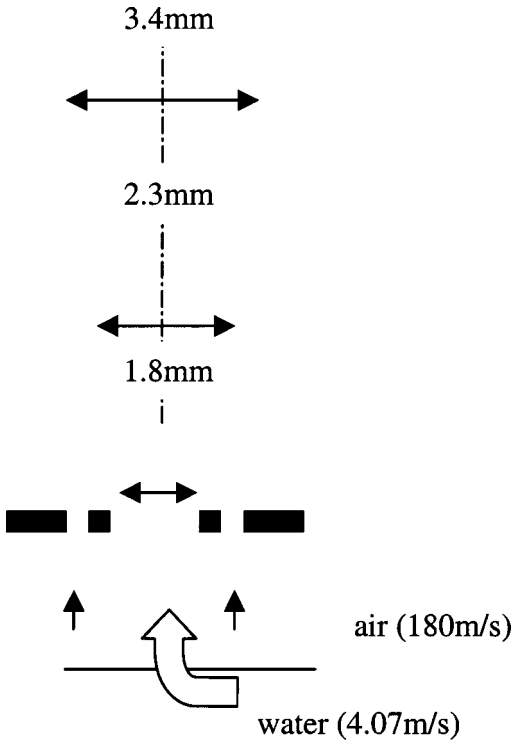

FIG. 2. Coaxial jet nozzle configuration from Ref. 31.

The ratio of gas-to-liquid momentum at the exit of injector, $M = \rho_g u_g^2/\rho_l u_l^2$, which correspond to inlet conditions in Fig. 3, equals 2.5. For the spray at $M = 26.4$ ($U_{\mathrm{liq}} = 0.8$ m/s and $U_g = 115$ m/s), the computed one-side spatial distributions of drops are shown in Fig. 6. It is seen that increasing parameter M intensifies the production of droplets, as it has been pointed out by Lasheras *et al.*[8] To evaluate the far field

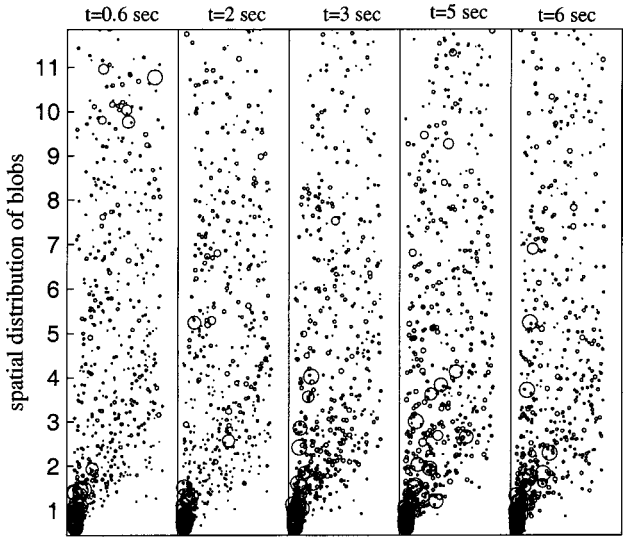

FIG. 3. Computed spatial distributions of drops at different time in the spray of water atomizing by the coaxial air jet (Fig. 2). The ratio of gas-to-liquid momentum at the exit of injector is equal to 2.5. The vertical lines (slightly moved outwards the symmetry axes) are given in cm.

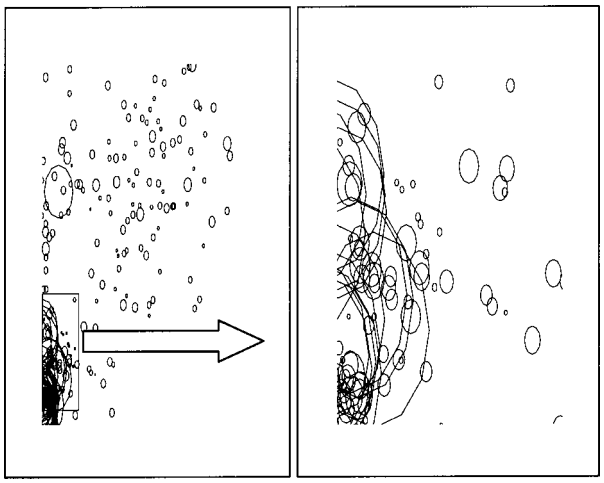

FIG. 4. An example of zooming of spatial distributions of drops in the spray presented in Fig. 3. A co-existence of droplets of different size is seen at every region of spray.

of a round water jet atomized by a high speed annular air jet, a series of measurements of the Sauter mean diameter was conducted by Lasheras *et al.*[8,32] In this experiment, the nozzle diameters of water and annular air jets are $D_l = 3.5$ mm and $D_g = 5.6$ mm, respectively. The measurements were taken at the constant gas velocity while varying the velocity of water. The computation of spray in the given geometry[32] was performed; the downstream variation of the

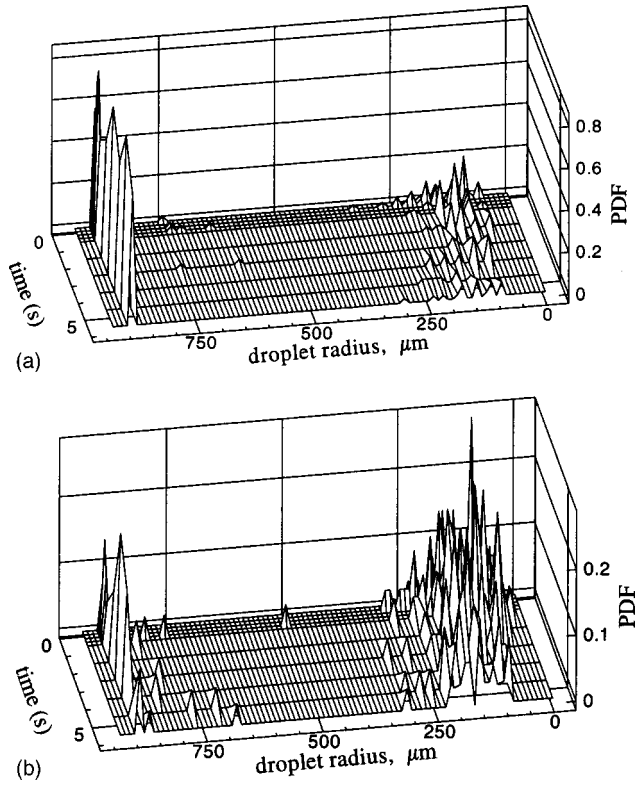

FIG. 5. The evolution in time of distributions of droplet size computed at two different distances from the injector orifice: 3 mm (upper part) and 5 mm (bottom part). Experiment (Ref. 31) shows the length of potential core equal to 3.5 mm.

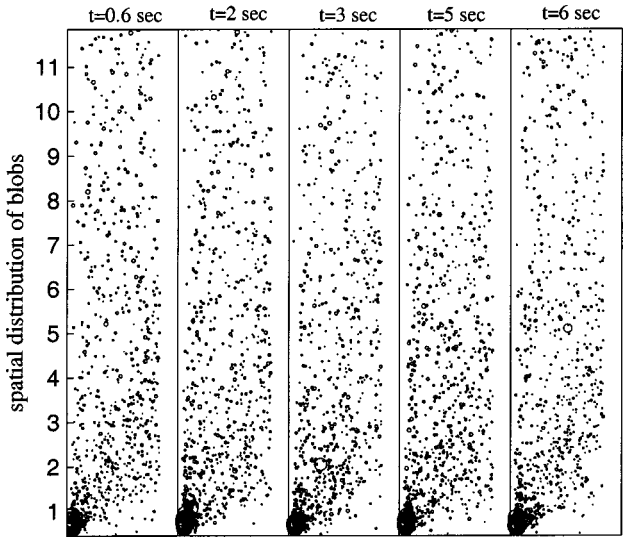

FIG. 6. Computed spatial distributions of drops at different time in the spray of water atomizing by the coaxial air jet ($U_{\text{liq}} = 0.8$ m/s, $U_g = 115$ m/s). The ratio of gas-to-liquid momentum at the exit of injector is equal to 26.4. The vertical lines (slightly moved outwards the symmetry axes) are given in cm.

Sauter mean diameter, D_{32}, at the given section was compared to the measurements.

The comparison is shown in Fig. 7 for three different liquid flow rates: $U_{\text{liq}} = 0.13$ m/s; 0.31 m/s; 0.55 m/s, and for the gaseous rate $U_g = 140$ m/s. It is seen that computed values of D_{32} are in general agreement with the experiment. The smaller Sauter mean diameters are computed as the velocity of annular air jet decreases. The nonmonotonic dependency of D_{32} on x/D_g, first decreasing (region dominant by breakup) and then increasing (region dominant by coalescence), was observed.[32] Such behavior can also be seen from our computations. However, instead of continuously increas-

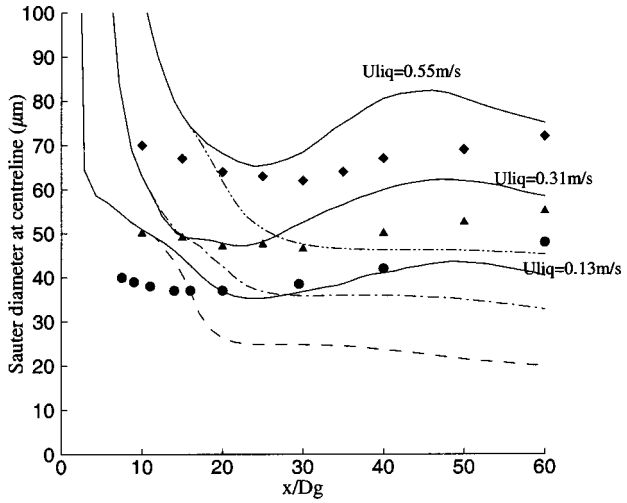

FIG. 7. Comparison of the Sauter mean diameter at the given section with measurements in Refs. 8 and 26 in the far-field of a round water jet atomized by a high speed annular air jet. The measurements and computations with and without coalescence model are presented by symbols, continuous and discontinuous lines, correspondingly. The gas velocity is held constant ($D_g = 5.6$ mm, $U_g = 140$ m/s), while varying the velocity of water, $U_{\text{liq}} = 0.13$; 0.31; 0.55 m/s.

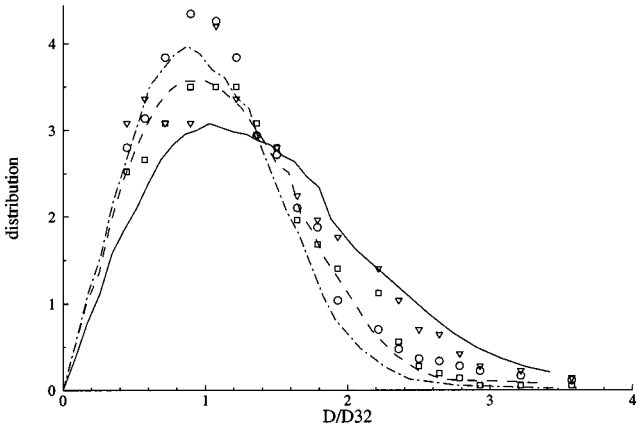

FIG. 8. Comparison of computed distributions of normalized diameter (D/D_{32}) with measured (Ref. 33) in the far field (15, 30, and 50 mm from injector orifice) of burning spray of methanol. Measurements: $\nabla-15$ mm; $\bigcirc-30$ mm; $\square-50$ mm; computations: Solid—15 mm; dashed—30 mm; dash–dot—50 mm, correspondingly.

ing, as was measured, the computed D_{32} tends to a constant value after 50 inlet diameters in the downstream direction. Alongside with computation accounting for the coalescence effects (O'Rourke's[25] model), the monotonic decreasing of D_{32} is shown when coalescence model is deactivated. This confirms the conclusion[32] about a significant role of coalescence effects.

Numerical simulations of burning spray in the rocket engine-like configuration is performed to compare with the model experiment.[33] In this experiment the round jet of liquid methanol issues from the central tube ($D_0=1.8$ mm) at low velocity (0.3 m/s) and atomizes by a parallel coflow of air issuing at high velocity (110 m/s) from an annular duct ($D=3.4$ mm). In Ref. 34, this flame-jet was computed excluding the atomization modeling; measured gas velocity, droplets size and droplets velocity distribution at 10 mm from injector were used as initial inlet conditions. One step global reaction was applied with kinetic constants given in Ref. 35. Methanol specific enthalpies were taken from JANAF tables[36] and latent heat, vapor pressure and laminar transport characteristics (diffusion and viscosity coefficients) were derived from Ref. 37. Spark ignition was provided by a local special energy deposition during a certain time: As soon as flame was developed, the special energy deposition was terminated. More details on computational procedure can be found in Refs. 34 and 38. In this paper, the numerical model of methanol spray combustion is complemented by the breakup modeling. Comparison of computed distributions of normalized diameter (D/D_{32}) with measured one in far field of spray (at 15, 30, and 50 mm from injector) are presented in Fig. 8. The computed distributions are obtained by averaging of 25 000 counts of droplets crossed the given distances from the injector orifice at different times. It is seen from the figure that the population of small droplets is increased with increasing of distance from injector and the shape of distribution is very similar to the measured one.

VIII. CONCLUSION

Kolmogorov described the breakup of solid particles in his early work (1941) as a discrete random process, where the probability that each parent particle breaks into a given number of parts is independent of particle size. It was pointed out by Kolmogorov that this general physical statement leads to the log-normal distribution of size of particle.

This paper discusses the further theoretical analyses of Kolmogorov's model. His model was reproduced in form of population balance equation for distribution of droplet size and the solution for moments was obtained; this solution was then used to derive the asymptotic analyses at latest times. We have shown that, in the long-time limit, the size distribution of droplet is described by a power function. The power asymptotic solution can be viewed as a further evolution of Kolmogorov's log-normal distribution when the standard deviation of droplet radius is strongly increasing and, consequently, the characteristic length scale disappears. This scaling property of Kolmogorov's model appears to be consistent with the fractal nature of air-blast breakup previously observed in experimental studies. The analyses of the case where the breakup frequency is a power function of radius, $\nu(r)=cr^{\beta}$, showed that the long-time solution is also verified as a power function. The discontinuity of the critical exponent of the asymptotic solution at $\beta=0$ was also demonstrated.

This paper also discusses the implementation of the considered model of breakup into the computation of spray under air-blast atomization. Considering the distribution function $F(\mathbf{x},\mathbf{v},r;t)$ of liquid particles in the phase space of droplet position, velocity and radius, the Liouville equation in the space $\mathbf{x}$, $\mathbf{v}$ was combined with Fokker–Planck equation in the space of droplet radius. The numerical scheme of stochastic computing of droplet breakup was implemented, along with Lagrangian tracking of parcels of liquid drops and integration of governing equations in the gas phase. The effects of interaction between droplets and gas motion, the coalescence and turbulent dispersion of droplets were also taken into account.

The computation of a round water jet atomized by a high-speed annular air jet, was also performed and the evolution in time of distributions of droplet-size was calculated. The spatial distributions of blobs showed a broad spectrum of droplet size at each spray location. The liquid core length was estimated and was in qualitative agreement with measurements. The significant influence of the ratio of inlet gas-to-liquid momentum on the breakup was shown. As to the far field of spray, the downstream variation of the Sauter mean diameter was compared with the experiment. The predicted values were relatively in agreement with experimental data. The nonmonotonic variation of Sauter mean diameter was shown in the downstream direction, first decreasing (region dominant by breakup) and then increasing (region dominant by coalescence). This behavior also follows experimental observation. The spray combustion of liquid methanol has been performed using developed breakup modeling and computed distributions of normalize diameter were compared with measurements. Increased population of small droplets with

increasing of distance from injector has been shown; computed distributions followed very closely the measured one.

The breakup model described in this paper can be easily implemented into any numerical code that computes dispersed reacting and nonreacting two-phase flows with Lagrangian formulation of spray.

ACKNOWLEDGMENTS

The authors are grateful for the support of this research by the European Society of Propulsion SEP/SNECMA and by the French National Center of Spatial Study CNES within the GdR ''Combustion in Rocket Engine.'' Acknowledgment is made to CNRS for partial support of this research.

[1]F. A. Williams, ''Spray combustion and atomization,'' Phys. Fluids **1**, 541 (1958).
[2]J. K. Dukowicz, ''A particle-fluid numerical model for liquid sprays,'' J. Comput. Phys. **35**, 229 (1980).
[3]R. D. Reitz, ''Modeling atomization processes in high-pressure vaporizing sprays,'' J. Atomization Spray Tech. **3**, 309 (1987).
[4]W. S. Rayleigh, ''On the instability of jets,'' Proc. London Math. Soc. **4**, 10 (1878).
[5]A. H. Lefebvre, *Atomization and Sprays* (Hemisphere, New York, 1989).
[6]A. B. Liu and R. D. Reitz, ''Mechanism of air-assisted liquid atomization,'' J. Atomization and Sprays **3**, 55 (1993).
[7]N. Chigier and R. D. Reitz, ''Regimes of jet breakup mechanisms (physical aspects),'' in *Recent Advances in Spray Combustion: Spray Atomization Drop Burning Phenomena*, edited by K. K. Kuo (AIAA, Reston, 1996), Vol. 1, pp. 109–135.
[8]J. C. Lasheras and E. J. Hopfinger, ''Liquid jet instability and atomization in a coaxial gas stream,'' Annu. Rev. Fluid Mech. **32**, 275 (2000).
[9]T. L. Georjion and R. D. Reitz, ''A drop-shattering collision model for multidimensional spray computations,'' J. Atomization and Sprays **9**, 231 (1999).
[10]A. N. Kolmogorov, ''On the log-normal distribution of particles sizes during break-up process,'' Dokl. Akad. Nauk SSSR **31**, 99 (1941).
[11]E. A. Novikov and D. G. Dommermuth, ''Distribution of droplets in a turbulent spray,'' Phys. Rev. E **56**, 5479 (1997).
[12]E. A. Novikov, ''The effects of intermittency on statistical characteristics of turbulence and scale similarity of breakdown coefficients,'' Phys. Fluids A **2**, 814 (1990).
[13]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[14]B. B. Mandelbrot, ''Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier,'' J. Fluid Mech. **62**, 331 (1974).
[15]I. Procaccia, ''Fractal structures in turbulence,'' J. Stat. Phys. **36**, 649 (1984).
[16]W. X. Zhou and Z. H. Yu, ''Multifractality of drop breakup in the air-blast nozzle atomization process,'' Phys. Rev. E **63**, 016302 (2001).
[17]U. Shavit and N. Chigier, ''Fractal dimensions of liquid jet interface under breakup,'' J. Atomization and Sprays **5**, 525 (1995).
[18]D. Ramkrishna and A. Fredrickson, *Population Balances. Theory and Applications to Particulate Systems Engineering* (Academic, San Diego, CA, 2000), p. 355.
[19]V. L. Saveliev, ''Temperature and mass dependence of the Boltzmann linear collision operator from the group theory point of view,'' J. Math. Phys. **37**, 6139 (1996).
[20]M. A. Gorokhovski and V. L. Saveliev, ''Similarities in stochastic modeling of breakup,'' in *Combustion in Rocket Engine* (CNES, Cepadues-Editions, Toulouse, France, 2001).
[21]H. Risken, *The Fokker–Planck Equation*, 2nd ed. (Springer, Berlin, 1989).
[22]R. Iranpour and P. Chacon, *Basic Stochastic Process: The Mark Kac Lectures* (Macmillan, New York, 1988).
[23]G. Ryskin, ''Simple procedure for correcting equations of evolution: Application to Markov processes,'' Phys. Rev. E **56**, 5123 (1997).
[24]A. A. Amsden, P. J. O'Rourke, and T. D. Butler, ''The KIVA-II: A computer program for chemically reactive flows with sprays,'' LA-11560-MS (1989).
[25]P. J. O'Rourke, ''Collective drop effects on vaporizing liquid sprays,'' Ph.D. thesis, Princeton University, 1981.
[26]G. M. Faeth, ''Structure and atomization properties of dense sprays,'' *Int. Symp. Combustion*, Pittsburgh, PA (Combustion Institute, Pittsburgh, 1990).
[27]N. A. Chigier, ''The physics of atomization,'' Plenary Lecture. ICLASS 91, Gaithersburg, MD, July 1991.
[28]P. J. O'Rourke and A. A. Amsden, ''The TAB method for *numerical* calculations of spray droplet breakup,'' SAE Tech. Pap. 872089, 1987.
[29]B. E. Gel'fand, S. A. Gubin, S. M. Kogarko, and S. P. Komar, ''Singularities of the breakup of viscous liquid droplets in shock waves,'' J. Eng. Phys. **25**, 1140 (1975).
[30]M. Pilch and C. A. Erdman, ''Use of breakup time data and velocity history data to predict the maximum size of stable fragments for acceleration-induced breakup of a liquid drop,'' Int. J. Multiphase Flow **13**, 741 (1987).
[31]D. Stepowski and O. Werquin, ''Diagnostics of scalars in two-phase dense jets by plane laser,'' in *Combustion in Rocket Engine* (CNES, Cepadues-Editions, Toulouse, France, 2001).
[32]J. C. Lasheras, E. Villermaux, and E. J. Hopfinger, ''Break-up and atomization of a round water jet by a high speed annular air jet,'' J. Fluid Mech. **357**, 351 (1998).
[33]P. J. Goix, C. F. Edwards, A. Cessou, C. M. Dunsky, and D. Stepowski, ''Structure of methanol–air coaxial reacting spray near stabilization region,'' Combust. Flame **98**, 205 (1994).
[34]P. Durand, M. Gorokhovski, and R. Borghi, ''Spray combustion computation using scalar PDF-equation approach,'' AIAA Pap. 99-0209 (1999).
[35]C. K. Westbrook and F. L. Dryer, ''Simplified reaction mechanism for the oxidation of hydrocarbon fuels in flames,'' Combust. Sci. Technol. **27**, 31 (1981).
[36]D. R. Stull and H. Prophet, in *JANAF Thermochemical Tables*, 2nd ed., National Standard Reference Data Series (U.S. National Bureau of Standards, Government Printing Office, Washington, DC, 1971).
[37]R. H. Perry and D. Green, *Perry's Chemical Engineers Handbook*, 6th ed. (McGraw-Hill, New York, 1984).
[38]P. Durand, M. Gorokhovski, and R. Borghi, ''An application of the pdf model to diesel engine combustion,'' Combust. Sci. Technol. **144**, 47 (1999).