Characterizing PSSs through graph theory

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1. Structural equivalence

2. Characterizing PSSs

3. Characterizing positive bases

Why should we care ?

Question

How to minimize a smooth function f ? (∇f is not available).

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Through direct search:

- **Inputs**: $x \in \mathbb{R}^n$, $M = \{c_1, \dots, c_s\} \subset \mathbb{R}^n$.
- Compare $f(x)$ to each $f(x + \alpha c_i)$.
- If $f(x)$ is the smallest, decrease α.
- Otherwise, $x \leftarrow x_0 + \alpha c_i$.

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- Otherwise, $x \leftarrow x_0 + \alpha c_i$.

If the elements of M are *'well spread'* in $\mathbb{R}^n$, the algorithm converges.

Positive spanning sets and positive bases

Definition

A **PSS** is a set of vectors well spread in $\mathbb{R}^n$.
A **positive basis** is minimal for this property.

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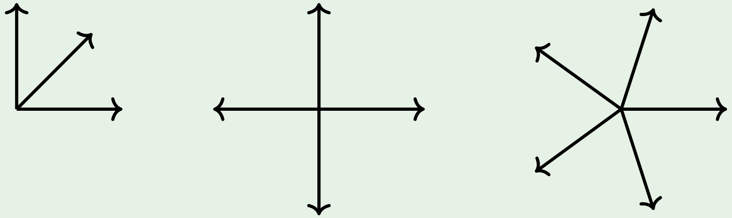
Examples in $\mathbb{R}^2$



One of these is not a PSS...

Positive spanning sets and positive bases

Definition

A **PSS** is a matrix M such that $y^\top M \leq 0^\top$ implies $y = 0$.
A **positive basis** is minimal for this property.

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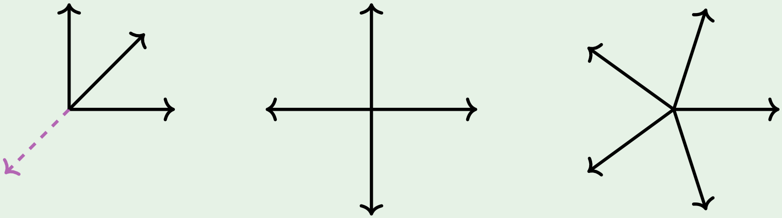

No close neighbor for the dashed vector.

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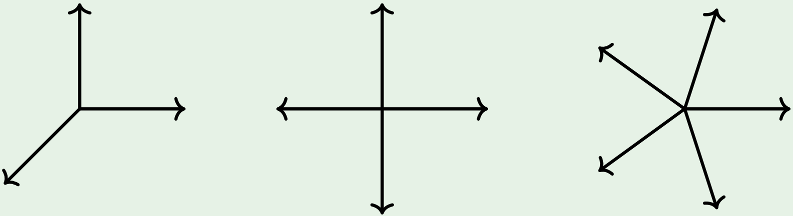

Which is not a positive basis ?

Positive spanning sets and positive bases

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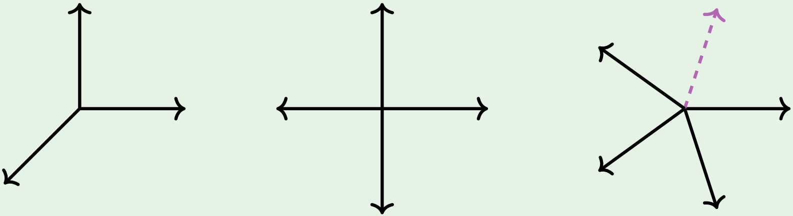
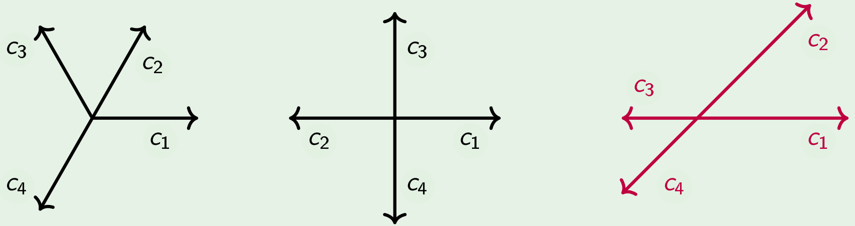


The dashed vector can be removed.

Structural equivalence

Quiz

Examples in $\mathbb{R}^2$



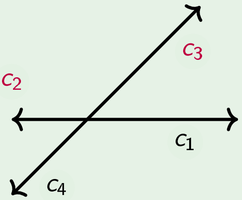
Which set is 'equivalent' to the red ?

We're looking for transformations that leave the 'positively spanning' property invariant.

Structural equivalence

Quiz

Examples in $\mathbb{R}^2$



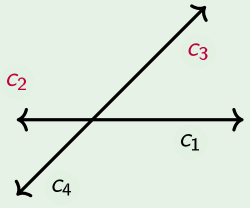
Reorder the elements.

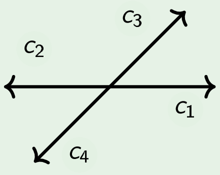
Definition (Structural equivalence $\equiv$)

M is **structurally equivalent** to N if $M = \quad NP \quad$ where P is a permutation matrix,

Quiz

Examples in $\mathbb{R}^2$





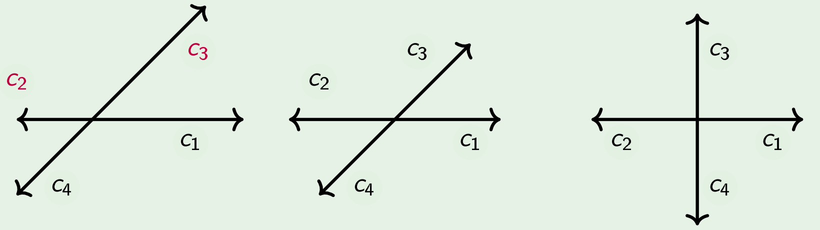
Rescale them.

Definition (Structural equivalence $\equiv$)

M is **structurally equivalent** to N if $M = \quad NPD$ where P is a permutation matrix, D is diagonal with positive entries

Quiz

Examples in $\mathbb{R}^2$



Change bases.

Definition (Structural equivalence $\equiv$)

M is **structurally equivalent** to N if $M = BNPD$ where P is a permutation matrix, D is diagonal with positive entries and B is invertible.

An algebraic approach

Remark

- *If* $M \equiv N$*:*
$$\begin{cases} M \text{ PSS} & \iff & N \text{ PSS.} \\ M \text{ positive basis} & \iff & N \text{ positive basis.} \end{cases}.$$

An algebraic approach

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- *If* $M \equiv N$: $$\begin{cases} M \text{ PSS} & \iff & N \text{ PSS.} \\ M \text{ positive basis} & \iff & N \text{ positive basis.} \end{cases}.$$
- *M positive basis* $\implies$ $n+1 \leq |M| \leq 2n.$

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- M *positive basis* $\implies n+1 \leq |M| \leq 2n$.
- $\begin{bmatrix} I_n & -1_n \end{bmatrix}$ *and* $\begin{bmatrix} I_n & -I_n \end{bmatrix}$ *are positive bases, where* $-1_n = \begin{bmatrix} -1 \\ \dots \\ -1 \end{bmatrix}$.

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- $\begin{bmatrix} I_n & -1_n \end{bmatrix}$ *and* $\begin{bmatrix} I_n & -I_n \end{bmatrix}$ *are positive bases, where* $-1_n = \begin{bmatrix} -1 \\ \dots \\ -1 \end{bmatrix}$.
- *PSSs span* $\mathbb{R}^n$ *with positive combinations.* *obviously...*

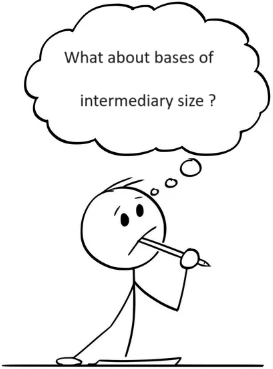
Characterizing positive bases

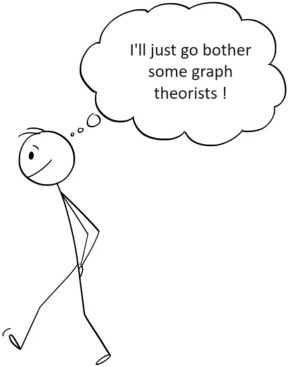
Proposition

M positive basis and $|M| = n+1 \quad \Longrightarrow \quad M \equiv \begin{bmatrix} I_n & -1_n \end{bmatrix}$.
M positive basis and $|M| = 2n \quad \Longrightarrow \quad M \equiv \begin{bmatrix} I_n & -I_n \end{bmatrix}$.

Application

In $\mathbb{R}^3$, any non-minimal PSS of size 5 satisfies $M \equiv \begin{bmatrix} 1 & 0 & 0 & -1 & \times \\ 0 & 1 & 0 & -1 & \times \\ 0 & 0 & 1 & -1 & \times \end{bmatrix}$.


What about bases of
intermediary size ?


I'll just go bother
some graph
theorists !

1 Structural equivalence

2 Characterizing PSSs

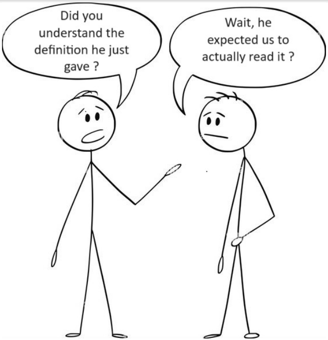
3 Characterizing positive bases

nec matrices

Definition (Negative echelon column matrix)

$\mathrm{N} \in \mathbb{R}^{n \times s}$ is a **nec matrix** if there exists a sequence $z_0 = 1 < z_1 < z_2 < \cdots < z_{s-1} \leq n$ of integers satisfying

1. For all $j \in [\![1, s-1]\!]$, for all $i \in [\![z_j, n]\!]$, $\mathrm{N}_{i,j} = 0$.
2. For all $j \in [\![1, s-1]\!]$, for all $i \in [\![z_{j-1}, z_j - 1]\!]$, $\mathrm{N}_{i,j} < 0$.
3. $\mathrm{N}_{i,s} < 0$, for all $i \geq z_{s-1}$.



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1. Each column ends with zeros. except the last.
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Example

$$\begin{bmatrix} -1 & \times & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & \times & \times & \times \\ -1 & \times & \times & \times \\ 0 & -1 & \times & \times \\ 0 & -1 & \times & \times \\ 0 & 0 & -1 & \times \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & \times & \times \\ -1 & \times & \times \\ 0 & -1 & \times \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Nec and PSS

Theorem

$M \equiv \begin{bmatrix} I_n & N & X \end{bmatrix}$ (with N nec) $\implies$ M PSS

Proof.

Simply note that $-1_n \in cone(M)$! $\square$

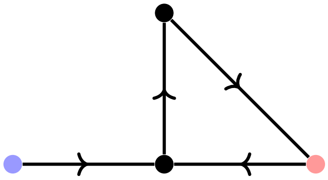
Example

$$\begin{bmatrix} -1 & \times & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & \times & \times & \times \\ -1 & \times & \times & \times \\ 0 & -1 & \times & \times \\ 0 & -1 & \times & \times \\ 0 & 0 & -1 & \times \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & \times & \times \\ -1 & \times & \times \\ 0 & -1 & \times \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Strong edge-connection

Definition (Strongly connected)

A digraph G is **strongly connected** if for each two vertices u and v, an oriented path joins u to v.



Not strongly connected: no path from red to blue.

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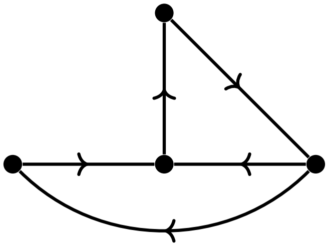
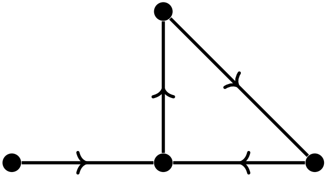


A strongly connected digraph G.

Ears

Definition (Ear)

An **ear** is a directed path with no vertices in G, except the extremities.



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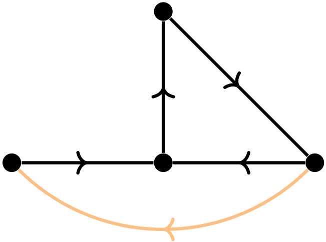

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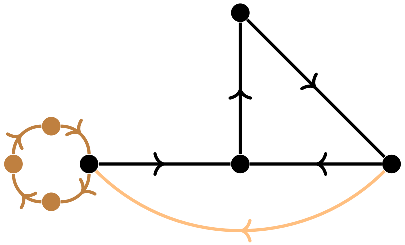

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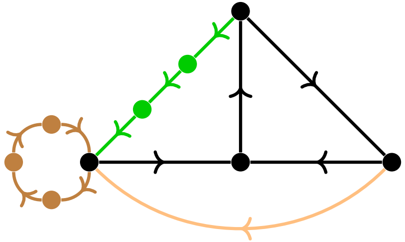

Three ears of the black digraph G.

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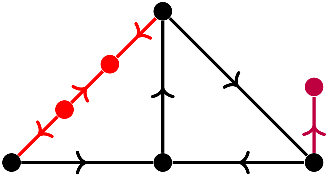

Not ears !

Ear-decomposition

Characterization

G strongly connected $\iff$ *G can be built from a sequence of ears.*

An ear-decomposition of the boat G.

Ear-decomposition

Characterization

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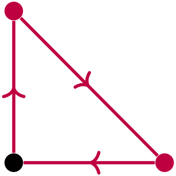

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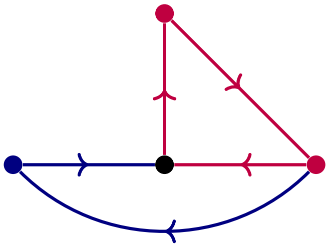

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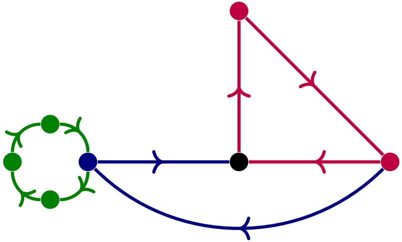
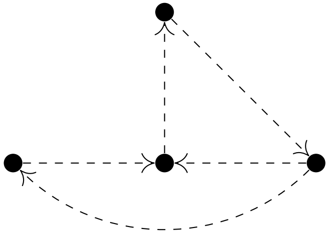


An ear-decomposition of the boat G.

Network matrices

PSSs arise from strongly connected digraphs through **network matrices**.

G = 

Draw your favourite connected digraph G.

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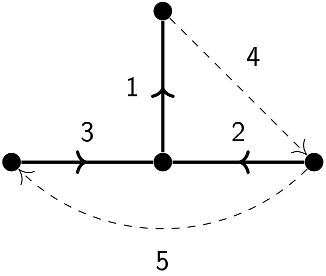
G = 

Choose a spanning tree. That's your basis.

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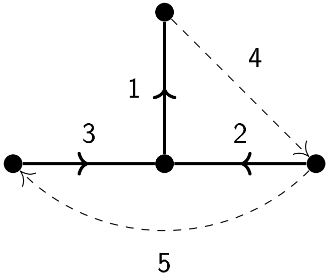
G = 

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

The n^{th} column is the n^{th} arc expressed in the basis.

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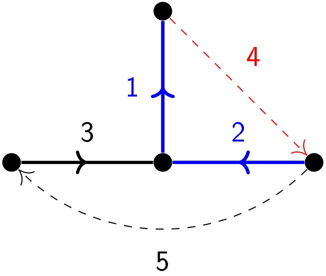
G = 

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

arcs: $-1-2=4$ **columns:** $-c_1-c_2=c_4$

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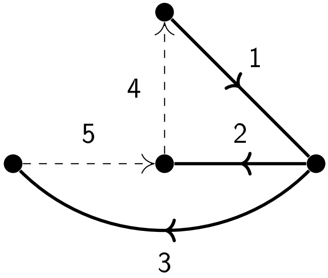
G = 

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

All network matrices are equivalent.

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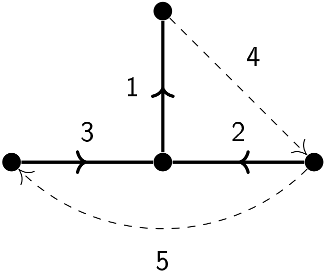
G = 

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

G is strongly connected $\implies$ M is a PSS.

Network matrices

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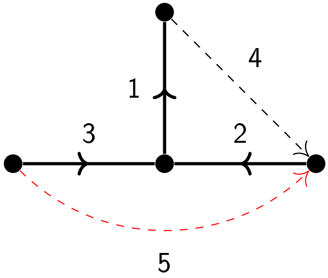
G = 

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

G is **not** strongly connected $\implies$ M is **not** a PSS.

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Example

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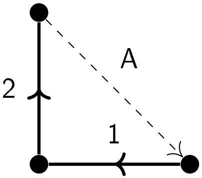
Strongly connected $\iff$ PSS ?

G strongly connected $\iff$ G associated to $\begin{bmatrix} \mathtt{I}_n & N & X \end{bmatrix}$.

Let's prove it.

Strongly connected $\iff$ PSS ?

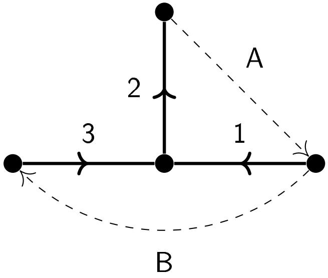
G cycle $\implies$ network matrix $\begin{bmatrix} \mathrm{I}_n & -\mathbb{1}_n \end{bmatrix}$.



$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

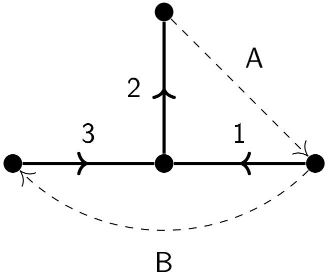
Strongly connected $\iff$ PSS ?

Add an ear. What happens ?



Strongly connected $\iff$ PSS ?

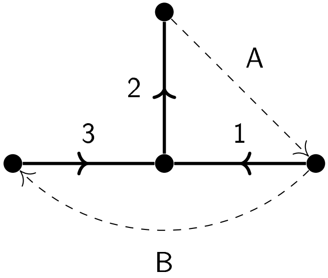
The previous matrix is contained in the new one...



$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Strongly connected $\iff$ PSS ?

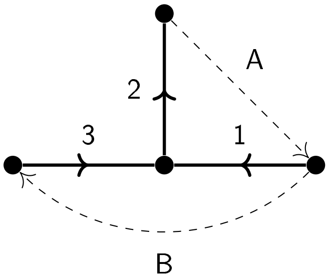
...with zeros ending its columns.



$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Strongly connected $\iff$ PSS ?

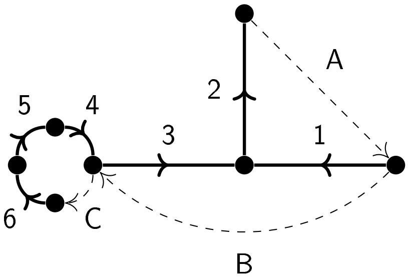
The new column ends with -1.



$$\begin{bmatrix} & & -1 & 1 \\ & I_3 & -1 & 0 \\ & & 0 & -1 \end{bmatrix}$$

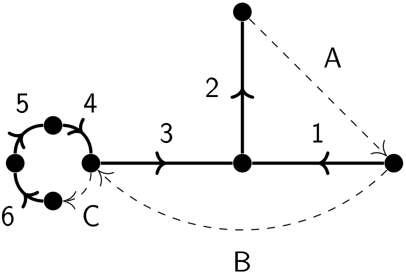
Strongly connected $\iff$ PSS ?

Add another ear.



Strongly connected $\iff$ PSS ?

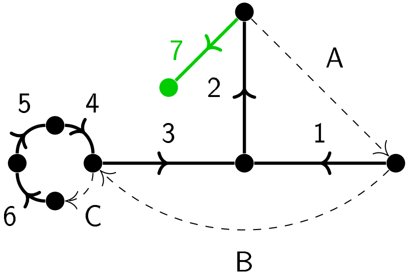
Same phenomenon ! **nec** matrix !



$$\begin{bmatrix} I_6 & \begin{matrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{matrix} \end{bmatrix}$$

Strongly connected $\iff$ PSS ?

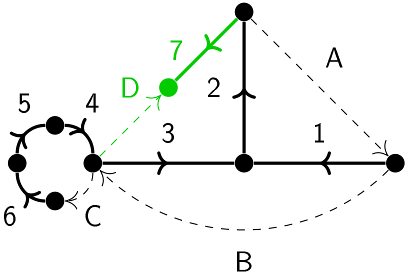
Non-ears break the pattern.



$$\left[\begin{array}{cccc} & -1 & 1 & 0 \\ & -1 & 0 & 0 \\ & 0 & -1 & 0 \\ I_7 & 0 & 0 & -1 \\ & 0 & 0 & -1 \\ & 0 & 0 & -1 \\ & 0 & 0 & 0 \end{array}\right]$$

Strongly connected $\iff$ PSS ?

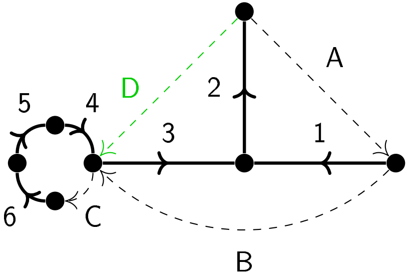
Non-ears break the pattern.



$$\begin{bmatrix} & -1 & 1 & 0 & 0 \\ & -1 & 0 & 0 & 1 \\ & 0 & -1 & 0 & 1 \\ I_7 & 0 & 0 & -1 & 0 \\ & 0 & 0 & -1 & 0 \\ & 0 & 0 & -1 & 0 \\ & 0 & 0 & 0 & 1 \end{bmatrix}$$

Strongly connected $\iff$ PSS ?

Trivial ears add useless columns.



$$\left[\begin{array}{c|cccc} & -1 & 1 & 0 & 0 \\ & -1 & 0 & 0 & -1 \\ & 0 & -1 & 0 & -1 \\ I_6 & 0 & 0 & -1 & 0 \\ & 0 & 0 & -1 & 0 \\ & 0 & 0 & -1 & 0 \end{array}\right]$$

Graphs and PSSs

Theorem

Graph strongly connected $\iff$ *its network matrices are PSSs !*

Remark

Characterizations of strongly connected digraphs can be restated in the language of linear algebra !

Restating properties

Characterization

G strongly connected $\iff$ *a network matrix is* $\begin{bmatrix} I & N & X \end{bmatrix}$ *(N nec).*

G not strongly connected $\iff$ *it is* $\begin{bmatrix} I & N & X \\ 0 & 0 & A \end{bmatrix}$ *(A non-negative).*

Restating properties

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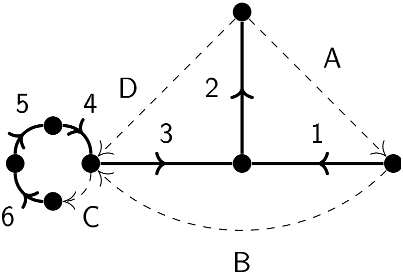

G is strongly connected.

$$\begin{array}{c|ccc|c} \text{I} & & \text{N} & & \text{X} \\ \end{array}$$
$$\left[\begin{array}{c|ccc|c} & -1 & 1 & 0 & 0 \\ & -1 & 0 & 0 & -1 \\ I_6 & 0 & -1 & 0 & -1 \\ & 0 & 0 & -1 & 0 \\ & 0 & 0 & -1 & 0 \\ & 0 & 0 & -1 & 0 \end{array}\right]$$

Restating properties

Characterization

G strongly connected $\iff$ *a network matrix is* $\begin{bmatrix} I & N & X \end{bmatrix}$ *(N nec).*

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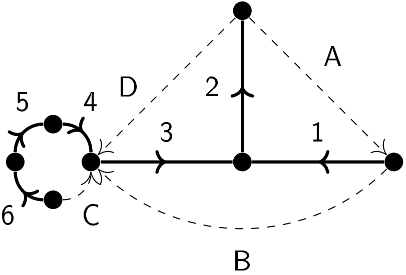


G is not strongly connected.

$$\begin{array}{cccc} \mathrm{I} & \mathrm{N} & \mathrm{X} & \\ \multicolumn{4}{c}{\left[\begin{array}{c|c|cc} I_3 & \begin{matrix}-1 & 1\\ -1 & 0\\ 0 & -1\end{matrix} & \begin{matrix}0 & 0\\ 0 & -1\\ 0 & -1\end{matrix} & O_3 \\ \hline O_3 & \begin{matrix}0 & 0\\ 0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}1 & 0\\ 1 & 0\\ 1 & 0\end{matrix} & I_3 \end{array}\right]} \\ 0 & 0 & \mathrm{A} & \end{array}$$

Restating properties

Characterization

G strongly connected $\iff$ a network matrix is $\begin{bmatrix} I & N & X \end{bmatrix}$ (N nec).

G not strongly connected $\iff$ it is $\begin{bmatrix} I & N & X \\ 0 & 0 & A \end{bmatrix}$ (A non-negative).

The property can be generalized !

Characterization

M PSS $\iff$ $M \equiv \begin{bmatrix} I & N & X \end{bmatrix}$ (N nec).

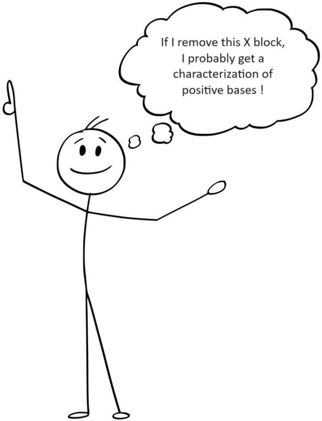
M not PSS $\iff$ $M \equiv \begin{bmatrix} I & N & X \\ 0 & 0 & A \end{bmatrix}$ (A non-negative).

A very nice conjecture

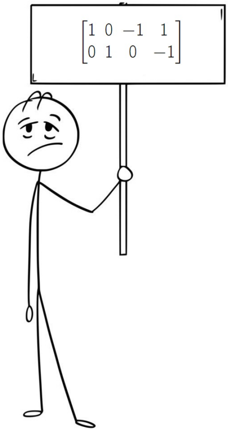
Characterization

M PSS $\iff$ *it is equivalent to* $\begin{bmatrix} I & N & X \end{bmatrix}$ (N *nec*).

M not PSS $\iff$ *it is equivalent to* $\begin{bmatrix} I & N & X \\ 0 & 0 & A \end{bmatrix}$ (A *non-negative*).



...well, maybe not



Nec matrices

Definition (Negative echelon column matrix)

$N \in \mathbb{R}^{n\times s}$ is a **nec matrix** if

1. Each column ends with zeros. except the last.
2. Each has less zeros than its predecessor.
3. Each has a block of -1 above the zeros. the first column starts with it.
4. The other values are arbitrary.

Example

$$\begin{bmatrix} -1 & \times & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & -1 & \times \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & \times & \times & \times \\ -1 & \times & \times & \times \\ 0 & -1 & \times & \times \\ 0 & -1 & \times & \times \\ 0 & 0 & -1 & \times \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & \times & \times \\ -1 & \times & \times \\ 0 & -1 & \times \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Creating a basis

$$
\begin{array}{cl}
\left[\begin{array}{ccccc}
 & -1 & \times & \times & \times \\
 & -1 & \times & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
\mathrm{I}_{10} & 0 & 0 & -1 & \times \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1
\end{array}\right]
&
\begin{array}{l}
X_2 \\
 \\
 \\
X_3 \\
 \\
X_1 \\
 \\
 \\
 \\
 \\
\end{array}
\end{array}
$$

We see three 'arbitrary' blocks.

Creating a basis

$$
\left[\begin{array}{ccccc}
 & -1 & \times & \times & \times \\
 & -1 & \times & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
\mathrm{I}_{10} & 0 & 0 & -1 & \times \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1
\end{array}\right]
\begin{array}{l}
X_2 \\ \\ \\ X_3 \\ \\ X_1 \\ \\ \\ \\ \\
\end{array}
$$

Let's list the restrictions on these blocks.

Creating a basis

$$
\begin{array}{cc}
\left[\begin{array}{ccccc}
 & -1 & \times & \times & \times \\
 & -1 & \times & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
\mathrm{I}_{10} & 0 & 0 & -1 & \times \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1
\end{array}\right]
&
\begin{array}{c}
X_2 \\
 \\
 \\
X_3 \\
 \\
X_1 \\
 \\
 \\
 \\

\end{array}
\end{array}
$$

- Vectors in $cone(X_i)$ do not have <u>one</u> positive coordinate.

Creating a basis

$$
\left[\begin{array}{ccccc}
 & -1 & \times & \times & \times \\
 & -1 & \times & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
\mathrm{I}_{10} & 0 & 0 & -1 & \times \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1
\end{array}\right]
\begin{array}{l}
X_2 \\ \\ \\ X_3 \\ \\ X_1 \\ \\ \\ \\ \\
\end{array}
$$

- Vectors in $cone(X_i)$ do not have <u>one</u> positive coordinate.
- Vectors in $cone(X_i)$ are not negative.

Creating a basis

$$
\left[\begin{array}{ccccc}
 & -1 & \times & \times & \times \\
 & -1 & \times & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
\mathrm{I}_{10} & 0 & 0 & -1 & 0 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1
\end{array}\right]
\begin{array}{l}
X_2 \\
\\
\\
X_3 \\
\\
\\
\\
\\
\\
\\
\end{array}
$$

- Vectors in $cone(X_i)$ do not have <u>one</u> positive coordinate.
- Vectors in $cone(X_i)$ are not negative.

$\Longrightarrow$ In X_1, each cross is a 0.

Creating a basis

$$
\left[\begin{array}{ccccc}
 & -1 & \times & \times & \times \\
 & -1 & \times & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
\mathrm{I}_{10} & 0 & 0 & -1 & 0 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1
\end{array}\right]
\begin{array}{l}
X_2 \\
\\
\\
X_3 \\
\\
\\
\\
\\
\\
\\
\end{array}
$$

- The max entry of $x \in cone(X_i)$, when positive, is <u>not</u> unique.
- Vectors in $cone(X_i)$ are not negative.

Creating a basis

$$
\begin{array}{cc}
\left[\begin{array}{ccccc}
 & -1 & 3 & 9 & 2 \\
 & -1 & 3 & 9 & 2 \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
 & 0 & -1 & \times & \times \\
\mathrm{I}_{10} & 0 & 0 & -1 & 0 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1 \\
 & 0 & 0 & 0 & -1
\end{array}\right] & X_3
\end{array}
$$

- The max entry of $x \in cone(X_i)$, when positive, is <u>not</u> unique.
- Vectors in $cone(X_i)$ are not negative.

$\Longrightarrow$ Columns of X_2 are positive multiples of a same $y \in \{1_2, -e_1, -e_2\}$.

Creating a basis

$$\left[\begin{array}{c|cccc} & -1 & 3 & 9 & 2 \\ & -1 & 3 & 9 & 2 \\ & 0 & -1 & 0 & 0 \\ & 0 & -1 & -3 & 5 \\ & 0 & -1 & -3 & 5 \\ \mathrm{I}_{10} & 0 & 0 & -1 & 0 \\ & 0 & 0 & 0 & -1 \\ & 0 & 0 & 0 & -1 \\ & 0 & 0 & 0 & -1 \\ & 0 & 0 & 0 & -1 \end{array}\right]$$

- The max entry of $x \in cone(X_i)$, when positive, is not unique.
- Vectors in $cone(X_i)$ are not negative.

There are no other restrictions !

Characterizing bases

Definition (Critical set)

$K(\mathbb{R}^n)$ is the set of vectors v satisfying condition 1 or 2.

1. $\mathrm{v} \leq 0_n \quad \& \quad \exists i, v_i = 0.$
2. $\exists i, j, v_i = v_j = \max_{k \leq n} v_k > 0.$

Characterizing bases

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2. $\exists i, j, v_i = v_j = \max_{k \leq n} v_k > 0.$

Theorem

$$\left[\begin{array}{ccc} I_n & \boxed{\begin{array}{cc} -1 & \\ \ldots & X_{n_1} \\ -1 & \end{array}} & \\ & 0 & \boxed{\begin{array}{cc} -1 & \\ \ldots & X_{n_2} \\ -1 & \end{array}} \\ & & 0 \quad \boxed{\ldots} \end{array}\right] \textit{is a basis} \iff \forall i, cone(X_{n_i}) \subset K(\mathbb{R}^{n_i}).$$

...recursive

Near-maximal bases

Proposition

$$M \text{ has size } 2n-1 \iff M \equiv \begin{bmatrix} & -1 & 0 \\ I_n & -1 & x^\top \\ & 0 & -I_{n-2} \end{bmatrix},\ x \in \mathbb{R}^{n-2},\ x \leq 0_{n-2}.$$

Example

Basis of size 7 in $\mathbb{R}^4$ $\iff$ equivalent to $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -x & -y \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$,

where $x \leq 0$ and $y \leq 0$.

Near-minimal bases

Proposition

$$|M| = n+2 \iff M \equiv \left[\begin{array}{ccc} I_n & -1_k & x \\ & 0_{n-k} & -1_{n-k} \end{array}\right], \quad \begin{cases} 1 \le k < n, x \in \mathbb{R}^k \\ x \le 0, x_1 = 0 \end{cases}.$$

Example

Basis of size 5 in $\mathbb{R}^3 \iff$ equivalent to $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & x \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}, x \le 0.$

Conclusion

In a nutshell

- Digraphs are pretty cool.
- The notion of PSS generalizes that of strongly connected digraph.
- Knowing so allows to find new properties of PSSs.

Perspectives

- Characterizing blocks with even more rows.
- Characterizing blocks with few columns.
- Characterizing PkSSs in a similar fashion.

Thanks

for your attention !