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1. Shifting to the continuous realm, let $P(x_i)$ represent $\prod_{a\in A} a$ and $S(x_i)$ represent $\sum_{a\in A} a = 2025$, and model the question as a constrained optimization problem maximizing

$$P(x_i) = \prod_{i=1}^{n} x_i \text{ with constraint } S: 2025 = \sum_{i=1}^{n} x_i \implies S(x_i) = (\sum_{i=1}^{n} x_i) - 2025.$$

This can be solved with the method of Lagrange multipliers:

$$\frac{\partial P}{\partial x_i} = \lambda \frac{\partial S}{\partial x_i} \implies \frac{\prod_{k=1}^{n} x_k}{x_i} = \lambda(1-0) \implies \frac{P}{x_i} = \lambda \quad \forall i.$$

Then, comparing any two indices i, j:

$$\frac{P}{x_i} = \frac{P}{x_j} \implies x_i = x_j, P \neq 0.$$

Therefore the product P is maximized with respect to constrain S when all terms x_i are the same. Letting $x_i = c$ with $c \in \mathbb{N}$, integer solutions can be represented with the constraint $2025 = \sum_{i=1}^{n} c = cn$ and the product to maximize is given by $f(c) = \prod_{i=1}^{n} c = c^n$. Rearranging gives $n = \frac{2025}{c}$ so $f(c) = c^{\frac{2025}{c}}$. The maximum of $f(c)$ is the maximum of $\prod_{a\in A} a$. Solving first for argmax$(f(x))$ in the continuous realm:

$$f(x) = x^{\frac{2025}{x}} \implies \ln(f(x)) = \frac{2025\ln(x)}{x}, \text{ let } h(x) = \frac{\ln(x)}{x}$$
$$\text{then } \operatorname{argmax}(f(x)) = \operatorname{argmax}(\ln(f(x))) = \operatorname{argmax}(h(x)).$$

argmax$(h(x))$ can be found by solving $h'(x) = 0$:

$$h'(x) = \frac{xx^{-1} - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2}$$

$\ln(x) = 1$ only when $x = e$

$$h(e) = \frac{1-\ln(e)}{e^2} = \frac{1-1}{e^2} = \frac{0}{e^2} = 0$$

so argmax$(f(x)) = e$. The nearest integer to e is 3 so in the discrete realm argmax$(f(c)) = 3$ and $n = \frac{2025}{3} = 675 \in \mathbb{Z}^+$. Since 2025 is divided perfectly by c there are no special cases to check. Therefore the maximum product of the elements of A, under the constraint, is

$$f(3) = 3^{675}. \quad \square$$

2. Observe the first few terms of the sequence:

$$\begin{aligned} a_1 &\leq a_2 + a_3 \\ a_2 &\leq a_4 + a_5 \\ a_3 &\leq a_6 + a_7 \\ a_4 &\leq a_8 + a_9 \end{aligned}$$

and so on infinitely. This can be rearranged to

$$a_1 \leq a_2 + a_3 \leq a_4 + a_5 + a_6 + a_7 \leq \dots$$

Hence it can be conjectured that each sum of terms grouped together between inequalities (the $2^n - n$th term to the 2^nth term) are greater than or equal to a_1. Leading to the following crux move.

Crux move: prove that, for all n,

$$\sum_{k=2^n-n}^{2^n} a_k \geq a_1, \quad n = 0, 1, 2, \dots$$

Proving that this is true would show that

$$\sum_i a_i \geq \sum_{i=1}^{\infty} a_1 = \infty$$

This is because the sum of any constant $c \in \mathbb{R}^+ > 0$ can grow arbitrarily large, so summing $c = a_1$ an infinite number of times is divergent. Therefore $\sum_i ai$ diverges. □

3. **Conjecture 1:** $f_0 = 0, f_1 = 1, f_2 = 1, f_{12} = 144$ are the only perfect squares in f_n. By inspecting the first hundred or so terms I noticed that these were the only perfect squares present. Since they all appear early in the sequence, and given perfect squares become less frequent as you move "right" along the number line, it seems plausible that there aren't any more. I then wrote a python script to check the first 500000 terms for perfect squares and it failed to find any additional ones.

   Conjecture 2: Ignoring $f_0 = 0$ (because I'm not sure how to count its parity) the terms of f_n follow the pattern "odd-odd-even". This is again noticeable by inspecting early terms of the sequence, but also feels intuitive given the form of $f_n = f_{n-1} + f_{n-2}$ where if either f_{n-1} or f_{n-2} are odd then f_n will be odd, but if both are odd then f_n will be even. Since the initial condition $f_1 = 1$ is odd the sequence has a starting point and looks like it would be stable (conversely if there were only even starting conditions the sequence would generate only even terms).

4. Every $n \geq 1$ lies between consecutive squares such that $k^2 \leq n < (k+1)^2$ for some unique $k \geq 1$. For this problem it is useful to use this equivalent bound:

$$k^2 \leq n \leq (k+1)^2 - 1.$$

Given this inequality, the denominator of $q(n)$ has the following behaviors:

$$\lfloor \sqrt{n} \rfloor = k, \qquad \lfloor \sqrt{n+1} \rfloor = \begin{cases} k, & n \leq (k+1)^2 - 2, \\ k+1, & n = (k+1)^2 - 1. \end{cases}$$

Since if $n \leq (k+1)^2 - 2$, then $n + 1 \leq (k+1)^2 - 1 < (k+1)^2$, so $\lfloor \sqrt{n+1} \rfloor = k$.
And if $n = (k+1)^2 - 1$, then $n + 1 = (k+1)^2$ and $\lfloor \sqrt{n+1} \rfloor = k + 1$.
In the case where $\lfloor \sqrt{n} \rfloor = \lfloor \sqrt{n+1} \rfloor = k$, we have

$$q(n) = \left\lfloor \frac{n}{k} \right\rfloor \leq \left\lfloor \frac{n+1}{k} \right\rfloor = q(n+1),$$

so $f(n)$ is not decreasing.

But when $n = k^2 - 1$ (i.e. any square minus one) with $k \geq 2$, we get

$$q(k^2-1) = \left\lfloor \frac{k^2-1}{k-1} \right\rfloor = \left\lfloor \frac{(k-1)(k+1)}{k-1} \right\rfloor = k+1 > k = \left\lfloor \frac{k^2}{k} \right\rfloor = q(k^2) = q\big((k^2-1)+1\big).$$

Thus $q(n) > q(n+1)$ holds exactly at $n = k^2 - 1$. $\square$

5. $C_n = A_n + B_n$ when $A_n = 2^n$ and $B_n = n!$. Proof:

$$\begin{aligned}
&\text{Verify } C_n = 2^n : \\
&\qquad 2^n = (n+4)2^{n-1} - 4n\,2^{n-2} + (4n-8)2^{n-3} \\
&\qquad\quad = 2^{\,n-3}(4(n+4) - 8n + (4n-8)) = 2^{\,n-3} \cdot 8 = 2^n. \\
&\text{Verify } C_n = n! : \\
&\qquad n! = (n+4)(n-1)! - 4n(n-2)! + (4n-8)(n-3)! \\
&\qquad\quad = n! + 4(n-1)! - 4n(n-2)! + 4n(n-3)! - 8(n-3)! \\
&\qquad\quad = n! + (n-3)![4(n-1)(n-2) - 4n(n-2) + 4n - 8] \\
&\qquad\quad = n! + (n-3)!\big(4n^2 - 12n + 8 - 4n^2 + 8n + 4n - 8\big) \\
&\qquad\quad = n! + (n-3)! \cdot 0 = n!.
\end{aligned}$$

Because the recurrence is linear and homogeneous, the sum of two solutions is a solution. Therefore $C_n = A_n + B_n = 2^n + n!$ $\square$

6. The full set of positive integer solutions can be found by parameterizing the curve and restricting its domain to valid integers. Begin by rearranging:

$$\begin{aligned}
\frac{1}{x} + \frac{1}{y} = \frac{1}{n} &\implies \frac{x+y}{xy} = \frac{1}{n} \\
&\implies \frac{xy}{x+y} = n \\
&\implies xy = xn + yn \\
&\implies xy - xn - yn = 0 \\
&\implies (x-n)(y-n) - n^2 = 0 \\
&\implies (x-n)(y-n) = n^2.
\end{aligned}$$

Then to parameterize:

$$\text{Let } t = x - n \text{ with } t \in \mathbb{N},$$

$$\text{so that } t(y-n) = n^2 \implies y - n = \frac{n^2}{t} \implies y = \frac{n^2}{t} + n, \quad x = t + n.$$

Therefore positive integer solution pairs are given by $(x,y) = (t+n, \frac{n^2}{t} + n)$ with $t, n, \frac{n^2}{t} \in \mathbb{Z}^+$. $\square$