MATH 480W D100 Assignment 5

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1. Consider an infinite chessboard, and on each square a positive integer is written. If each entry is the average of its four neighbours (to the north, east, west, and south), show that all entries are the same.

Solution: Since any non-empty set of positive integers has a least element, there is a minimum value m. Let's consider a square with value m, and let its four neighbors have values a, b, c, d. The question states that:

$$\frac{a+b+c+d}{4} = m. \tag{1}$$

Rewriting (1) gives us:

$$(a-m)+(b-m)+(c-m)+(d-m) = 0. \tag{2}$$

Because we already know that m is minimal, each of a, b, c, d is at least m, so every term in (2) is nonnegative.
Since their sum is 0 and each term is nonnegative, each term has to equal 0. Therefore:

$$a = b = c = d = m.$$

$\square$

2. Place the integers $1, 2, \ldots, n^2$ onto an $n \times n$ chessboard (without repeating a number). Show that there exist two adjacent squares whose entries differ by at least $n+1$ (adjacent means horizontally, vertically, or diagonally).

Solution:

Example for $n = 2$ ($n + 1 = 3$).

1	2
3	**4**

The diagonal neighbors 1 and 4 differ by $4 - 1 = 3 \geq n + 1$.

Example for $n = 3$ ($n + 1 = 4$).

1	2	3
4	**5**	6
7	8	9

The diagonal neighbors 1 and 5 differ by $5 - 1 = 4 \geq n + 1$.

For $n = 1$, a 1×1 chessboard doesn't make sense, so let's assume $n \geq 2$. Let's label the $n \times n$ grid cells by coordinates $(i, j) \in \{1, \ldots, n\}^2$. For cells u, v, let $\mathrm{dist}(u, v)$ be the minimal number of steps in a path $u = x_0, x_1, \ldots, x_t = v$ where each consecutive pair x_k, x_{k+1} is adjacent. On an $n \times n$ grid:

$$\mathrm{dist}(u, v) \leq n - 1 \text{ for all } u, v,$$

since in each step we can reduce both the row and column differences by at most 1.

(proof by contradiction) Assume that every adjacent pair of cells has values that differ by at most n. Fix any u, v and take a shortest path $u = x_0, \ldots, x_t = v$ with $t = \mathrm{dist}(u, v)$. By the triangle inequality:

$$\left|\mathrm{val}(u) - \mathrm{val}(v)\right| \;\leq\; \sum_{k=0}^{t-1} \left|\mathrm{val}(x_{k+1}) - \mathrm{val}(x_k)\right| \;\leq\; tn \;\leq\; n(n-1).$$

Choose u to be the cell containing 1 and v the cell containing n^2. Then we have:

$$n^2 - 1 = \left|\mathrm{val}(v) - \mathrm{val}(u)\right| \leq n(n-1) = n^2 - n,$$

which is a contradiction for $n \geq 2$. Therefore, some adjacent pair must differ by at least $n + 1$.

$\square$

3. There are n people on a field such that for each person the distances to all the other people are different. Each person has a water pistol and, on mark, shoots the person closest to her. Show that when n is odd, at least one person is dry. What happens when n is even?

Solution:

Show that when n is odd, at least one person is dry
We can do proof by induction on n:
Base case $n = 3$: Let A, B be the closest pair. The question states that A and B will shoot each other because they are the closest, so the third person is dry.
Induction step: Assume the claim holds for some odd n. Consider a configuration with $n + 2$ people (also odd). Let A, B be a closest pair, the question states that they will shoot each other, again. Then there are two possibilities:

- If some other person also shoots A or B, then at least one of A, B is hit twice. Since exactly $n + 2$ shots are fired, the number of distinct people hit is at most $n + 1$. So at least one of the $n + 2$ people is not hit.

- If no one else shoots A or B, remove A, B from the configuration. Then the remaining n people still have unique nearest neighbors among themselves, so by the induction hypothesis at least one of them is not hit.

In either case there is an unhit person for $n + 2$. By induction, for all odd n at least one person is not hit. $\square$

What happens when n is even?
Example with everyone getting wet: Partition the n people into $n/2$ disjoint pairs, place the two members of each pair very close together, and place different pairs far apart from each other, adjusting distances to keep nearest neighbors unique. Within each pair, the two nearest neighbors are each other, so they shoot one another. Everyone is hit exactly once, and no one stays dry.
Example with a lucky dude not getting wet. Place $n - 1$ people very close together in a small group, while keeping nearest neighbors unique, and place one person far away from the group. Each of the grouped-up people shoots each other within the group, while no one targets the dude far away from the group.

Therefore, for an even n it is possible that someone escapes and also possible that no one escapes. There is no concrete answer.

$\square$

4. Let A be a set of n elements. Let E and O be the sets of even and odd subsets, respectively. Show that $|E| = |O|$ in two ways: one "easy" way, and one combinatorial.

Solution: Let A be a set with $|A| = n$. Let

$$E = \{S \subseteq A : \ |S| \text{ is even}\}, \qquad O = \{S \subseteq A : \ |S| \text{ is odd}\}.$$

Easy way: By the binomial theorem:

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad (1-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

therefore:

$$\sum_{k \text{ even}} \binom{n}{k} - \sum_{k \text{ odd}} \binom{n}{k} = 0 \quad \Longrightarrow \quad \sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k}.$$

But $\sum_{k \text{ even}} \binom{n}{k} = |E|$ and $\sum_{k \text{ odd}} \binom{n}{k} = |O|$. Therefore $|E| = |O|$.

Combinatorial way: Take any element $a \in A$. Let:

$$\varphi : \mathcal{P}(A) \to \mathcal{P}(A), \qquad \varphi(S) = \begin{cases} S \cup \{a\}, & a \notin S, \\ S \setminus \{a\}, & a \in S. \end{cases}$$

Equivalently, $\varphi(S) = S \triangle \{a\}$. Then $\varphi(\varphi(S)) = S$ and flips the parity of $|S|$. Therefore, φ is a bijection $E \leftrightarrow O$, and $|E| = |O|$.

$\square$

5. A gardener plants three maple trees, four oak trees, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Find the probability that no two birch trees are next to one another.

Solution: Plant the 7 maple and oak trees (MMMOOOO). The number of arrangements is:

$$\frac{7!}{3!\,4!}$$

These 7 trees create 8 places where the 5 birch trees can be planted, like that one time in class with the binary number example.
To avoid planting adjacent B's, plant at most one B in each gap. So we can choose 5 of the 8 gaps:

$$\binom{8}{5}$$

Then the number of good arrangements is:

$$N_{\text{good}} = \frac{7!}{3!\,4!}\binom{8}{5}$$

and the total number of arrangements is:

$$N_{\text{total}} = \frac{12!}{3!\,4!\,5!}$$

therefore:

$$\Pr(\text{no two B adjacent}) = \frac{N_{\text{good}}}{N_{\text{total}}} = \frac{\frac{7!}{3!\,4!}\binom{8}{5}}{\frac{12!}{3!\,4!\,5!}} = \frac{7!\,\binom{8}{5}\,5!}{12!} = \frac{7}{99}$$

$\square$

6. A group of 20 math students are offered 6 courses in a particular semester. Each student chooses between zero and six courses to take. Prove or disprove: there exist five students and two courses such that all five students chose both courses or all five chose neither of the two courses.

Solution: (proof by counterexample) Assign each student a distinct 3-course schedule:

$$\binom{6}{3} = 20.$$

Fix any ordered pair of courses (C, C'). A 3-course choice contains both C and C' iff it also contains exactly one of the remaining four courses. So there are:

$$\binom{4}{1} = 4$$

students who take both C and C'. Similarly, a 3-course choice contains neither C nor C' iff it is chosen entirely from the other four courses, so there are:

$$\binom{4}{3} = 4$$

students who take neither C nor C'.

Therefore, for every pair (C, C') there are exactly 4 students taking both and exactly 4 students taking neither, so no pair has 5 students in either category. Therefore the assumption is false. $\square$