

SIMPLE INTEGRAL FUSION RINGS OF RANK 4

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A non-trivial perfect integral fusion ring has at least four distinct basic FPdims. In particular, its rank is at least 4. Moreover, every perfect integral fusion ring of rank 4 is simple and has type $[1, d_1, d_2, d_3]$ with $3 \leq d_1 < d_2 < d_3$. The generic form of the fusion data is as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & x_1 & x_2 & x_3 \\ 0 & x_2 & x_4 & x_5 \\ 0 & x_3 & x_5 & x_6 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & x_2 & x_4 & x_5 \\ 1 & x_4 & x_7 & x_8 \\ 0 & x_5 & x_8 & x_9 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & x_3 & x_5 & x_6 \\ 0 & x_5 & x_8 & x_9 \\ 1 & x_6 & x_9 & x_{10} \end{pmatrix}.$$

Below is the exact count $\#$ of simple integral fusion rings of rank 4 with $\text{FPdim} < 10^i$. These data suggest that, in the absence of any bound on FPdim , the total number is likely infinite. However, recall that there is **no** 1-Frobenius example with $\text{FPdim} < 10^{10}$, and no Drinfeld example at all.

FPdim <	10^3	10^4	10^5	10^6	10^7
#	1	2	6	12	66

Below is the complete list of the 66 such rings with $\text{FPdim} < 10^7$. This collection may help in identifying the structure of a potential infinite family.

FPdim	Factors	d_1	d_2	d_3	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
574	$2 \times 7 \times 41$	11	14	16	0	4	4	1	6	3	12	1	9	6
7315	$5 \times 7 \times 11 \times 19$	35	40	67	30	1	2	9	15	25	12	12	25	39
63436	$2^2 \times 15859$	103	149	175	57	13	16	67	23	74	51	44	98	48
65971	37×1783	65	89	232	51	5	2	4	22	56	60	10	79	186
68587	107×641	103	116	211	48	27	12	19	33	79	30	38	79	129
90590	$2 \times 5 \times 9059$	142	180	195	39	0	75	77	60	32	25	87	56	120
113310	$2 \times 3^2 \times 5 \times 1259$	90	172	275	77	2	3	25	40	64	39	75	112	184
310730	$2 \times 5 \times 7 \times 23 \times 193$	312	317	336	62	139	101	48	120	105	168	96	115	130
311343	$3 \times 59 \times 1759$	286	315	361	95	76	85	115	89	141	206	4	241	39
494102	$2 \times 7 \times 29 \times 1217$	396	399	422	40	219	127	20	150	135	248	124	141	162
532159	532159	211	409	566	84	24	30	63	98	129	299	56	332	278
585123	$3 \times 7 \times 11 \times 17 \times 149$	288	397	587	159	30	43	179	59	227	208	40	341	245
1012810	$2 \times 5 \times 101281$	305	622	730	144	32	40	125	140	169	414	125	457	270
1043710	$2 \times 5 \times 29 \times 59 \times 61$	574	587	608	186	165	207	220	186	199	144	220	199	228
1107139	11×100649	256	671	769	12	61	28	23	183	87	582	70	549	261
1152907	7×164701	443	679	704	237	95	38	296	82	340	264	214	421	84
1185558	$2 \times 3 \times 11^2 \times 23 \times 71$	348	722	737	55	32	107	206	124	176	244	371	300	360
1343202	$2 \times 3 \times 7 \times 31981$	203	626	954	72	12	20	65	88	141	451	101	541	569
1411338	$2 \times 3 \times 71 \times 3313$	415	714	854	40	84	112	115	210	185	85	470	219	581
1419779	197×7207	573	720	757	317	66	131	388	126	354	59	335	306	198
1425114	$2 \times 3^4 \times 19 \times 463$	578	727	750	382	31	121	444	106	382	48	316	339	127
1483682	$2 \times 53 \times 13997$	610	645	834	417	39	111	224	270	320	128	236	265	395
1745610	$2 \times 3 \times 5 \times 31 \times 1877$	577	774	902	224	100	140	309	166	345	74	403	322	405
1898038	2×949019	268	842	1057	171	2	23	182	68	208	474	247	628	504
1916226	$2 \times 3^2 \times 13 \times 19 \times 431$	234	715	1162	200	3	5	64	104	169	200	304	507	816
2112179	$71^2 \times 419$	603	763	1080	378	158	14	45	306	379	651	54	554	477
2161715	5×432343	660	925	933	413	32	143	501	135	425	156	408	425	211
2175315	$3 \times 5 \times 145021$	647	879	992	1	199	245	136	323	201	376	357	352	549
2630642	$2 \times 7 \times 41 \times 4583$	384	583	1464	179	17	47	192	72	343	336	48	545	1157
2753395	5×550679	832	919	1103	511	233	48	303	265	575	185	383	400	336
2898555	$3 \times 5 \times 11^2 \times 1597$	709	1092	1097	104	216	176	183	384	213	933	40	804	159
2903702	$2 \times 17 \times 41 \times 2083$	777	984	1154	224	52	328	363	318	285	179	442	393	627
3033052	$2^2 \times 11 \times 29 \times 2377$	943	1009	1061	426	51	411	460	414	184	108	448	215	693
3108087	$3^2 \times 41 \times 8423$	769	793	1374	291	117	200	160	286	492	572	38	611	746
3139986	$2 \times 3 \times 419 \times 1249$	762	1090	1171	219	342	35	76	416	352	793	227	608	376
3185630	$2 \times 5 \times 7 \times 17 \times 2677$	572	1194	1197	39	120	135	310	204	304	52	991	108	944
3325370	$2 \times 5 \times 17 \times 31 \times 631$	983	1074	1098	240	312	360	403	288	379	58	633	197	566
3397890	$2 \times 3 \times 5 \times 191 \times 593$	360	1208	1345	167	23	31	175	160	208	961	175	1008	384
3416428	$2^2 \times 19 \times 44953$	403	1177	1367	191	18	47	139	222	198	148	845	384	978
3483967	109×31963	965	1105	1154	396	240	246	369	370	405	35	716	110	710
3784371	$3 \times 71 \times 109 \times 163$	455	769	1728	133	13	79	108	151	367	204	223	630	1351
3802502	$2 \times 11 \times 307 \times 563$	1106	1119	1152	207	555	324	88	456	352	752	272	417	409
3805498	$2 \times 31 \times 61379$	620	1184	1421	531	19	23	250	300	360	435	515	624	744
3852015	$3 \times 5 \times 256801$	858	1193	1301	198	45	394	565	239	379	56	670	421	665
3939260	$2^2 \times 5 \times 13 \times 109 \times 139$	603	1163	1491	146	169	53	0	402	268	1113	39	970	626
4135971	$3 \times 7 \times 37 \times 5323$	615	1223	1504	373	27	77	460	115	490	228	621	671	758
4245382	2×2122691	919	1252	1354	208	72	416	373	456	215	325	604	384	853
4430367	$3^2 \times 211 \times 2333$	821	1249	1482	588	114	33	593	129	694	101	639	639	559
5146446	$2 \times 3 \times 857741$	926	1412	1515	40	95	453	471	366	308	366	687	548	816
5457862	2×2728931	542	1484	1721	210	25	83	178	306	252	279	983	540	1176
5656406	$2 \times 7 \times 404029$	951	1352	1710	480	220	88	561	186	755	411	432	907	573
6191519	$17 \times 59 \times 6173$	1050	1573	1617	535	242	99	651	231	761	534	588	851	295
6478290	$2 \times 3^2 \times 5 \times 7^2 \times 13 \times 113$	1415	1492	1500	400	472	488	525	440	517	39	950	132	881
6510658	$2 \times 7 \times 11 \times 67 \times 631$	1426	1450	1541	274	643	461	207	552	480	782	437	528	600
6559662	$2 \times 3 \times 31 \times 35267$	1005	1106	2080	496	124	180	535	190	817	460	85	969	1170
6641015	5×1328203	1105	1458	1815	588	108	228	575	360	677	711	250	1038	569
7024422	$2 \times 3 \times 59 \times 19843$	1294	1353	1876	707	273	208	384	468	813	176	584	609	876
7557886	$2 \times 7 \times 539849$	934	1540	2077	444	18	207	594	244	660	237	699	912	1104
7622140	$2^2 \times 5 \times 223 \times 1709$	1433	1555	1775	461	325	500	363	675	438	79	1000	134	1304
7738923	$3 \times 101 \times 25541$	1369	1540	1869	471	399	329	649	301	880	815	122	1219	220
8240068	$2^2 \times 563 \times 3659$	595	1679	2251	71	120	49	405	110	500	1266	201	1500	1000
8712844	$2^2 \times 7 \times 311173$	1313	1615	2093	57	355	514	182	650	489	254	936	485	1412
8847555	$3 \times 5 \times 31 \times 53 \times 359$	768	1591	2393	26	110	165	199	343	487	597	597	1084	1516
8888746	$2 \times 977 \times 4549$	767	1310	2566	96	244	76	13	312	585	513	403	1011	1875
9279060	$2^2 \times 3 \times 5 \times 7 \times 22093$	1403	1825	1995	711	450	75	188	795	623	1628	48	1222	439
9863787	$3 \times 349 \times 9421$	1079	2064	2107	337	344	43	548	344	720	919	841	1064	696