

# CLASSIFICATION OF MODULAR DATA OF INTEGRAL MODULAR FUSION CATEGORIES UP TO RANK 12

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**ABSTRACT.** We classify all the modular data of integral modular fusion categories up to rank 12. The types come from the Egyptian fractions with squared denominators, listed using SageMath. The fusion rings come from solving the dimension and associativity equations, using Normaliz. The  $S$ -matrices come from making the character table self-transpose. The  $T$ -matrices come from solving the Anderson-Moore-Vafa equations. Finally, we keep what satisfies the definition of a modular data. We get that every perfect integral modular fusion category up to rank 12 is trivial. We also prove that every simple integral modular fusion category of rank 13 is pointed, up to 12 types (in progress). Finally, we prove that every odd-dimensional perfect modular fusion category up to rank 25 is trivial, up to 21 types (of rank 25).

## 1. INTRODUCTION

In this paper, all the fusion categories are assumed to be over the complex field. The notion of integral modular fusion category was highly studied, see the list of references at the beginning of [6]. In [2] they are classified up to rank 6 (all pointed) where the notion of Egyptian fraction is the key. The idea making possible the classification up to rank 12 in the present paper is the fact that (as we will explain in the next paragraph) we only need to consider Egyptian fractions *with squared denominators*, which is combinatorially much more restrictive. To get an idea, just for a sum equal to 1, the number of Egyptian fractions of length  $n = 1, 2, \dots, 8$  is 1, 1, 3, 14, 147, 3462, 294314, 159330691, respectively (see [21]), whereas with squared denominators, the number is just 1, 0, 0, 1, 0, 1, 1, 4, respectively (see [1]).

The notion of fusion ring and basics results are recalled in §2.1, where we refer to [10, Chapter 3]. As introduced in [12], a fusion ring  $\mathcal{F}$  is called *s-Frobenius* if for every basis element  $b$  then  $\text{FPdim}(\mathcal{F})^s / \text{FPdim}(b)$  is an algebraic integer. By [10, Proposition 8.14.6], the Grothendieck ring of a modular fusion category is  $\frac{1}{2}$ -Frobenius. Let  $\mathcal{F}$  be a  $\frac{1}{2}$ -Frobenius integral fusion ring of basis  $\{b_1, \dots, b_r\}$ ,  $\text{FPdim } D$  and type  $[d_1, \dots, d_r]$ , with  $1 = d_1 \leq d_2 \leq \dots \leq d_r$  and  $d_i = \text{FPdim}(b_i)$ . By assumption,  $d_i^2$  divides  $D$ , for all  $i$ . Now there is a unique square-free positive integer  $q$  such that  $D = qs^2$ . Thus  $d_i$  divides  $s$ , for all  $i$ . Let  $s_i$  be the positive integer  $s/d_i$ , recall that  $D = \sum_{i=1}^r d_i^2$ , so we get the following Egyptian fraction with squared denominators:

$$q = \sum_{i=1}^r \frac{1}{s_i^2}.$$

All such Egyptian fractions were classified up to  $r = 13$  using SageMath [20], see §6 where we will see how to reduce to  $q \leq r/4$ . Note that  $s_1 = s$ , so that  $d_i = s_1/s_i$ , and we can assume that  $s_i$  divides  $s_1$ , for all  $i$ . Then as detailed in §6, up to rank 12, we are reduced to consider 1028 types (and 9025 ones up to rank 13).

The next step is to apply some criteria for a type to come from a fusion ring, as explained in §3. These criteria are proved using elementary number theory, mainly modular arithmetic. They exclude about 35% of the types up to rank 13. For the remaining types, we apply our fusion ring solver, see §4; the idea is to solve the dimension equations

$$d_i d_j = \sum_k N_{i,j}^k d_k,$$

which are positive linear Diophantine equations, using Normaliz [4], after reducing the number of variables using the unit axiom of a fusion data  $(N_{i,j}^k)$  and the Frobenius reciprocity, see 2.1; together with exploiting as efficiently as possible the associativity equations (non-linear) during the solving of above linear equations, *which is very challenging*. This step provides a classification of all the  $\frac{1}{2}$ -Frobenius integral fusion rings up to rank 12, there are exactly 10628 ones (from 71 types), and prove that there is no (non-trivial) perfect one up to rank 12, see §5, but also no (non-pointed) simple one at rank 13, up to 12 types (in progress).

Now, we want to classify all the possible modular data associated to these fusion rings. The definition of modular data we use (see §2.2) is contrained by the main properties of a modular fusion category, in fact a pseudounitary one, because this paper focuses on the integral case, see [10, Proposition 9.6.5]. We only need to consider the commutative fusion rings because a modular fusion category is braided, so has a commutative Grothendieck ring (note that our classification also contains 213 noncommutative fusion rings, see §5). Let us first consider the  $S$ -matrices: from a

given commutative fusion ring, consider its eigentable (see Definition 2.6) as a matrix, and keep only the ones with cyclotomic entries (such fusion ring can be called *cyclotomic*). If an appropriate renormalization and permutation make a self-transpose matrix (see §2.3.1), then the fusion ring can be called *self-transposable*, otherwise it can be excluded. We deduce that there are exactly 68 cyclotomic  $\frac{1}{2}$ -Frobenius integral self-transposable fusion rings up to rank 12 (from 27 types), i.e. less than 0.7% of the 10628 ones in the previous step. Now about the  $T$ -matrices: for the remaining fusion rings  $\mathcal{R}$ , we solve the Anderson-Moore-Vafa equations (see §2.2) on the finite ring  $\mathbb{Z}/m\mathbb{Z}$ , where  $m$  is the biggest integer dividing  $\text{FPdim}(\mathcal{R})^{5/2}$ , and we keep only the  $S$ - and  $T$ -matrices satisfying all the assumptions of Definition 2.10. We end up with 19 + 62 modular data, from 5 + 17 fusion rings and 3 + 12 types (non-pointed + pointed).

**Remark 1.1.** *The pointed modular fusion categories are well-known to be given by a metric group  $(G, q)$ , i.e. a finite abelian group  $G$  together with a non-degenerate quadratic form  $q : G \rightarrow \mathbb{C}^*$ , see [10, Section 8.4], given by the  $T$ -matrix.*

The following theorem is just a summary of §8 where we will mention (in the non-pointed case) all the modular data (MD):  $S$ - and  $T$ -matrices, but also central charge, fusion data, 2nd Frobenius-Schur indicators (but just the  $T$ -matrices in the pointed case).

**Theorem 1.2.** *There are 19 MD of non-pointed integral modular fusion categories up to rank 12, given by:*

- Rank 8,  $\text{FPdim}$  36, type  $[1, 1, 2, 2, 2, 2, 3, 3]$ :
  - 6 MD with central charge  $c = 0$  from  $\mathcal{Z}(\text{Vec}_{S_3}^\omega)$ , see [14],
  - 2 MD with  $c = 4$  from  $(C_3^2 + 0)^{C_2}$ , see [13, point (b) on page 983].
- Rank 10,  $\text{FPdim}$  36, type  $[1, 1, 1, 2, 2, 2, 2, 2, 2, 3]$ :
  - 3 MD with  $c = 4$  from  $SU(3)_3$ , its complex conjugate and a zesting, see [8, §6.3.1].
- Rank 11,  $\text{FPdim}$  32, type  $[1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2]$ :
  - 8 MD with  $c = \pm 1$  from  $SO(8)_2$ , conjugates and zestings, see §2.4.

There are 62 MD of pointed modular fusion categories up to rank 12; here are their number per group  $G$ :

$G$	$C_1$	$C_2$	$C_3$	$C_2^2$	$C_4$	$C_5$	$C_6$	$C_7$	$C_3^3$	$C_2 \times C_4$	$C_8$	$C_3^2$	$C_9$	$C_{10}$	$C_{11}$	$C_2 \times C_6$	$C_{12}$
#MD	1	2	2	5	4	2	4	2	4	4	4	2	2	4	2	10	8

There is no other integral modular data up to rank 12 (i.e. all categorifiable as above).

**Question 1.3.** *Is there a modular data without categorification?*

Note that the Drinfeld center of the representation category of any non-abelian simple finite group  $G$  (more generally, centerless perfect) is a perfect (non-simple) integral modular fusion category of  $\text{FPdim } |G|^2$  denoted  $\mathcal{Z}(\text{Rep}(G))$ , more generally, see [5, §11.1]. Note that for  $G = A_5$ , it is of rank 22.

At the fusion ring level, motivated by [17, Corollary 6.16], we proved the following:

**Theorem 1.4.** *Every perfect integral  $\frac{1}{2}$ -Frobenius fusion ring up to rank 12, without basis element of prime-power  $\text{FPdim}$ , is trivial.*

Note that the Grothendieck ring of  $\mathcal{Z}(\text{Rep}(A_7))$  is a perfect integral  $\frac{1}{2}$ -Frobenius fusion ring of rank 74 without basis element of prime-power  $\text{FPdim}$ . So, let us ask:

**Question 1.5.** *Is there a non-pointed simple integral  $\frac{1}{2}$ -Frobenius fusion ring without basis element of prime-power  $\text{FPdim}$ ?*

The proof of Theorem 1.4 requires the use of type criteria (see §3) and fusion ring solver (see §4). In fact, we can almost extend it up to rank 13 (there remain only 12 types to check, see §5.2). We deduce directly that:

**Corollary 1.6.** *Every simple integral modular fusion category up to rank 12 is pointed.*

Motivated by [11, Question 2], it is proven in [15] that every simple integral fusion category is weakly group-theoretical if and only if every simple integral modular fusion category is pointed. Thus, we ask the following:

**Question 1.7.** *Is there a non-pointed simple integral modular fusion category?*

At the fusion ring level, we proved the following:

**Theorem 1.8.** *Every perfect integral  $\frac{1}{2}$ -Frobenius fusion ring up to rank 12 is trivial.*

The proof of Theorem 1.8 up to rank 9 is immediate from the list in §6.1 and the Nichols-Richmond theorem extended to the fusion ring, see the proof of [9, Theorem 3.4], because there is always a non-trivial basis element of  $\text{FPdim} \leq 2$ . The proof up to rank 12 is as for Theorem 1.4, but a bit harder. We deduce directly that:

**Corollary 1.9.** *Every perfect integral modular fusion category up to rank 12 is trivial.*

The application of techniques developed in this paper (together with results in [6]) permits also to reduce the research of odd-dimensional perfect modular fusion categories up to rank 25, to 21 types (of rank 25) mentioned in §7.

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**Availability of data and materials.** Data for the computations in this paper are available on reasonable request from the authors. The softwares used for the computations can be downloaded from the URLs listed in the references.

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## 2. FUSION DATA AND MODULAR DATA

In this section, we recall the notions of fusion data and modular data, and the basic results. We refer to [10].

**2.1. Fusion data.** The notion of fusion data slightly augments the notion of finite group.

**Definition 2.1.** *A fusion data is a finite set  $\{1, 2, \dots, r\}$  with an involution  $i \rightarrow i^*$ , and nonnegative integers  $N_{i,j}^k$  such that for all  $i, j, k, t$ :*

- (Associativity)  $\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t$ ,
- (Unit)  $N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$ ,
- (Dual)  $N_{i^*,j}^1 = N_{j,i^*}^1 = \delta_{i,j}$

The fusion data can be denoted just as  $(N_{i,j}^k)$ .

Observe that  $1^* = 1$ . The following result can be deduced from Definition 2.1, see [10, Proposition 3.1.6].

**Proposition 2.2** (Frobenius reciprocity). *For all  $i, j, k$ , then  $N_{i,j}^k = N_{i^*,k}^j = N_{k,j^*}^i$ .*

A *fusion ring*  $\mathcal{R}$  is a free  $\mathbb{Z}$ -module together with a finite basis  $\mathcal{B} = \{b_1, \dots, b_r\}$  and a fusion product defined by

$$b_i b_j = \sum_k N_{i,j}^k b_k,$$

where  $(N_{i,j}^k)$  is a fusion data, and a  $*$ -structure defined by  $b_i^* := b_{i^*}$ , such that  $(b_i b_j)^* = b_j^* b_i^*$ . Then the three axioms for a fusion data reformulate as follows. For all  $i, j, k$ ,

- $(b_i b_j) b_k = b_i (b_j b_k)$ ,
- $b_1 b_i = b_i b_1 = b_i$ ,
- $\tau(b_i b_j^*) = \delta_{i,j}$ ,

with  $\tau(x)$  the coefficient of  $b_1$  in the decomposition of  $x \in \mathcal{R}$ . It follows that  $\mathcal{R}_{\mathbb{C}} := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{C}$  has a structure of finite dimensional unital  $*$ -algebra, where  $\tau$  extends linearly as a trace (i.e.  $\tau(xy) = \tau(yx)$ ), and define an inner product as  $\langle x, y \rangle := \tau(xy^*)$ . Note that  $\langle x, b_i \rangle$  is the coefficient of  $b_i$  in the decomposition of  $x$ .

**Theorem 2.3** (Frobenius-Perron theorem). *Let  $\mathcal{R}$  be a fusion ring with basis  $\mathcal{B}$ , and let  $\mathcal{R}_{\mathbb{C}}$  be the finite dimensional unital  $*$ -algebra defined as above. There is a unique  $*$ -homomorphism  $d : \mathcal{R}_{\mathbb{C}} \rightarrow \mathbb{C}$  such that  $d(\mathcal{B}) \subset \mathbb{R}_{>0}$ .*

The number  $d(b_i)$  is called the *Frobenius-Perron dimension* of  $b_i$ , noted  $\text{FPdim}(b_i)$  or just  $d_i$ , whereas  $\sum_i d_i^2$  is called the Frobenius-Perron dimension of  $\mathcal{R}$ , noted  $\text{FPdim}(\mathcal{R})$ . The list  $[d_1, d_2, \dots, d_r]$  is called the *type* of  $\mathcal{R}$ .

The fusion ring  $\mathcal{R}$  is called:

- *Frobenius* (or 1-Frobenius, or of Frobenius type) if  $\frac{\text{FPdim}(\mathcal{R})}{\text{FPdim}(b_i)}$  is an algebraic integer, for all  $i$ ,
- *integral* if the number  $\text{FPdim}(b_i)$  is an integer, for all  $i$ ,
- *pointed* if  $\text{FPdim}(b_i) = 1$ , for all  $i$ ,
- *commutative* if  $b_i b_j = b_j b_i$ , for all  $i, j$ , i.e.  $N_{i,j}^k = N_{j,i}^k$ ,

The *multiplicity* of  $\mathcal{R}$  is  $\max_{i,j,k}(N_{i,j}^k)$ , and its *rank* is  $r$ , the cardinal of the basis.

**Remark 2.4.** *The fusion data provides a representation of its fusion ring as follows. Consider the matrices  $M_i = (N_{i,j}^k)_{k,j}$ , then by the Associativity axiom in Definition 2.1, we can prove that  $M_i M_j = \sum_k N_{i,j}^k M_k$ . Moreover,  $M_1$  is the identity matrix, and by Frobenius reciprocity, the matrix adjoint  $M_i^*$  is  $M_{i^*}$ . Finally, by Frobenius-Perron theorem, the operator norm  $\|M_i\|$  is equal to  $\text{FPdim}(b_i)$ .*

**Remark 2.5.** *The notion of fusion data is just a combinatorial reformulation of the notion of fusion ring, so all the attributes qualifying a fusion ring can be used to qualify a fusion data.*

**Definition 2.6** (Eigentable). *Let  $(N_{i,j}^k)$  be a commutative fusion data. Consider the fusion matrices  $M_i = (N_{i,j}^k)_{k,j}$ . Because  $M_i^* = M_{i^*}$ , the commutativity makes these matrices normal, so simultaneously diagonalizable. Let  $(D_i)$  be their simultaneous diagonalization, then  $D_i = \text{diag}(\lambda_{i,j})$ . In fact, we can always choose  $\lambda_{i,1} = \|M_i\| = d_i$ . The matrix  $(\lambda_{i,j})$  is called the eigentable (or character table) of the fusion data. Let  $c_j := \sum_i |\lambda_{i,j}|^2$  be the formal codegrees.*

**Lemma 2.7.** *Let  $M \in M_n(\mathbb{Z}_{\geq 0})$ . Then  $M$  is a permutation matrix if and only if  $\|M\| = 1$ .*

*Proof.* Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis for which the matrix  $M$  has non-negative integer entries. If  $M$  is not a permutation matrix then one of the following holds:

- (1)  $\exists i, j, k$  with  $j \neq k$  such that  $\langle M e_i, e_j \rangle, \langle M e_i, e_k \rangle \neq 0$ ,
- (2)  $\exists i, j, k$  with  $i \neq j$  such that  $M e_i = M e_j = e_k$ ,

but (1) implies that  $\|M e_i\| / \|e_i\| \geq \sqrt{2}$ , whereas (2) implies that  $\|M(e_i + e_j)\| / \|e_i + e_j\| = \sqrt{2}$ . Both cases implies that  $\|M\| > 1$ . Next, if  $M$  is a permutation matrix then trivially  $\|M\| = 1$ .  $\square$

**Corollary 2.8.** *Let  $x, y$  be two basis elements of a fusion ring such that  $\text{FPdim}(x) = 1$ . Then  $xy$  and  $yx$  are basis elements and  $\text{FPdim}(xy) = \text{FPdim}(yx) = \text{FPdim}(y)$ .*

*Proof.* Immediate from Remark 2.4, Lemma 2.7 and the fact that  $\text{FPdim}$  is a ring homomorphism.  $\square$

**Corollary 2.9.** *A fusion ring is pointed if and only if its basis has a structure of finite group for the fusion product.*

**2.2. Modular data.** Roughly speaking, a modular data is a fusion data with two matrices  $S$  and  $T$  generating a representation of the modular group  $\text{SL}(2, \mathbb{Z})$ . More precisely, following [16, Theorem 1.2] and [10, §8.18]:

**Definition 2.10.** *Let  $\mathcal{R}$  be a fusion ring of rank  $r$ , type  $[d_1, \dots, d_r]$  and fusion data  $(N_{i,j}^k)$ . Let  $D := \sqrt{\text{FPdim}(\mathcal{R})}$  and  $\zeta_n := \exp(2i\pi/n)$ . A (pseudounitary) modular data on  $\mathcal{R}$  is given by two matrices  $S, T \in M_r(\mathbb{C})$  such that:*

- $S, T$  are symmetric matrices,  $T$  is unitary, diagonal,  $T_{1,1} = 1$ ,  $S_{1,i} = d_i$  and  $SS^* = D^2 \mathbf{1}$ .
- Verlinde formula:  $N_{i,j}^k = \frac{1}{D^2} \sum_l \frac{S_{li} S_{lj} \overline{S_{lk}}}{d_l}$ .

- *Central charge*: let  $\theta_i := T_{i,i}$  and  $p_{\pm} := \sum_{i=1}^r d_i^2(\theta_i)^{\pm 1}$ . Then  $p_+/p_-$  is a root of unity, and  $p_+ = D\zeta_8^c$  for some rational number  $c$ , called the **central charge**, only defined modulo 8.
- *Projective representation of  $\mathrm{SL}(2, \mathbb{Z})$* :  $(ST)^3 = p_+ S^2$ ,  $\frac{S^2}{D^2} = C$ ,  $C^2 = \mathbf{1}$ , where  $C$  is the permutation matrix for the involution  $i \rightarrow i^*$ , satisfying  $\mathrm{Tr}(C) > 0$ .
- *Cauchy theorem*: the set of distinct prime factors of  $\mathrm{ord}(T)$  coincides with the distinct prime factors of  $\mathrm{norm}(D^2)$ , where  $\mathrm{norm}(x)$  is the product of the distinct Galois conjugates of the algebraic number  $x$ .
- *Cyclotomic integers*: for all  $i, j$  the numbers  $S_{i,j}$ ,  $S_{i,j}/d_j$  and  $T_{i,i}$  are cyclotomic integers. Moreover, the conductor of  $S_{i,j}$  divides  $\mathrm{ord}(T)$ , and there is  $j$  such that  $S_{i,j}/d_j \in \mathbb{R}_{\geq 1}$ , for all  $i$ .
- *Frobenius-Schur indicators*: for all  $i$  and for all  $n \geq 1$ , the number  $\nu_n(i) := \sum_{j,k} N_{j,k}^i (d_j \theta_j^n) \overline{(d_k \theta_k^n)}$  is a cyclotomic integer whose conductor divides  $n$  and  $\mathrm{ord}(T)$ . Moreover,  $\nu_1(i) = \delta_{i,1}$  and  $\nu_2(i) = \pm \delta_{i,i^*}$ .
- *Anderson-Moore-Vafa equations*:  $T_{i,i} = \zeta_m^{v_i}$ , with  $0 \leq v_i < m$  and  $m$  the biggest integer dividing  $D^5$ . Let  $t_i$  be  $v_i \bmod m$ . For all  $i, j, k, l$ , the following equation holds on  $\mathbb{Z}/m\mathbb{Z}$ , involving its structure of  $\mathbb{Z}$ -module:

$$\left( \sum_{p=1}^r N_{i,j}^p N_{p,k}^l \right) (t_i + t_j + t_k + t_l) = \sum_{p=1}^r \left( N_{i,j}^p N_{p,k}^l + N_{i,k}^p N_{j,p}^l + N_{j,k}^p N_{i,p}^l \right) t_p.$$

Let  $s_i := f(v_i/m)$  be the **topological spin** of the  $i$ -th basis element, with  $f(x) = x\delta_{x \leq \frac{1}{2}} + (x-1)\delta_{x > \frac{1}{2}}$ .

We wonder about the redundancy in Definition 2.2, in particular, whether the Anderson-Moore-Vafa equations can be deduced from the other assumptions.

**Remark 2.11.** *Verlinde formula and the results in [15, Section 2] implies that the fusion ring  $\mathcal{R}$  is commutative, and together with  $S$  symmetric and  $SS^* = \mathrm{FPdim}(\mathcal{R})\mathbf{1}$ , we can deduce that  $\mathcal{R}$  is self-transposable (in the sense of §2.3.1). Finally, by the proof of [10, Proposition 8.14.6],  $\mathcal{R}$  is also  $\frac{1}{2}$ -Frobenius.*

Note that this paper focuses on integral fusion categories, so  $D^2$  is an integer and  $\mathrm{norm}(D^2) = D^2$ . Moreover such a category is pseudounitary, so spherical, so pivotal (see [10]). In a non-pseudounitary setting, Definition 2.10 should be modified (as in [16, Theorem 1.2]) because the equality  $S_{1,i} = \mathrm{FPdim}(b_i)$  may not hold.

Note that such a definition of modular data is so strong that we do not even know one without categorification (yet), which leads us to Question 1.3.

**2.3. From fusion data to modular data.** This subsection is dedicated to explain how to classify all the possible modular data (if any) associated to a fusion data. First of all, the fusion data can be assumed commutative and  $\frac{1}{2}$ -Frobenius (see Remark 2.11). Up to rank 12, the number of  $\frac{1}{2}$ -Frobenius integral fusion rings is exactly 10628, with exactly 213 noncommutative ones, see the details per rank in §5.

**2.3.1.  $S$ -matrix.** Let  $(N_{i,j}^k)$  be a commutative fusion data of rank  $r$ . Let  $(\lambda_{i,j})$  be its eigentable and  $(c_j)$  its formal codegrees (see Definition 2.6). The goal here is to find all the possible permutations  $q$  of  $\{1, \dots, r\}$  such that:

- $q(1) = 1$ ,
- $d_{q(i)} = d_i$ ,
- the matrix  $S = (\sqrt{c_1/c_j} \lambda_{i,q(j)})$  is symmetric (i.e. self-transpose).

**Remark 2.12.** *Above symmetric assumption implies that*

$$\sqrt{c_1/c_j} = \sqrt{c_1/c_j} \lambda_{1,q(j)} = \sqrt{c_1/c_1} \lambda_{j,q(1)} = d_j,$$

so we can assume that  $\sqrt{c_1/c_j} = d_j$ , i.e.,  $c_1/d_j^2 = c_j$ , for all  $j$ , see [18, Example 2.9].

If such a permutation  $q$  exists, then the fusion data will be called *self-transposable*, but it is very rare, so this step is a very strong filter. We can recover the fusion data from  $S$  by the Verlinde formula. Note that we only need to consider the fusion data whose eigentable has all its entries cyclotomic (such fusion data will be called *cyclotomic*). The addition of the assumptions self-transposable and cyclotomic permits to exclude more than 99.3% of the  $\frac{1}{2}$ -Frobenius integral fusion rings (up to rank 12) found in §5. There remain 68 ones, the number of types and fusion rings per rank is written below:

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Types	1	1	1	1	1	1	2	2	2	4	5	6
#FR	1	1	1	2	1	1	3	7	4	11	12	24

Here is the list of types counted above:

- rank 1: [1],
- rank 2: [1,1],
- rank 3: [1,1,1],

- rank 4:  $[1,1,1,1]$ ,
- rank 5:  $[1,1,1,1,1]$ ,
- rank 6:  $[1,1,1,1,1,1]$ ,
- rank 7:  $[1,1,1,1,1,1,1], [1,1,1,1,2,2,2]$ ,
- rank 8:  $[1,1,1,1,1,1,1,1], [1,1,2,2,2,2,3,3]$ ,
- rank 9:  $[1,1,1,1,1,1,1,1,1], [1,1,1,1,4,4,6,6,6]$ ,
- rank 10:  $[1,1,1,1,1,1,1,1,1,1], [1,1,1,1,2,2,2,4,4,4], [1,1,1,2,2,2,2,2,3], [1,1,2,3,3,4,4,4,6,6]$ ,
- rank 11:  $[1,1,1,1,1,1,1,1,1,1,1], [1,1,1,1,2,2,2,2,2,2,2], [1,1,1,1,2,6,6,8,12,12,12], [1,1,1,3,4,4,4,4,4,6], [1,1,1,1,4,4,12,12,18,18,18]$ ,
- rank 12:  $[1,1,1,1,1,1,1,1,1,1,1,1], [1,1,1,1,2,8,18,18,24,36,36,36], [1,1,1,3,6,8,8,8,8,8,12], [1,1,2,2,2,2,6,6,6,9,9], [1,1,2,3,3,6,6,8,8,12,12], [1,1,2,6,6,6,6,10,10,10,15,15]$ .

The list of fusion rings counted above is available at [19]. They were classified using the list from §5 together with the function `preSMatrix` in the file `ModularData.sage` at [19].

**2.3.2.  $T$ -matrix.** For the remaining fusion rings  $\mathcal{R}$  with fusion data  $(N_{i,j}^k)$ , we solve the Anderson-Moore-Vafa equations

$$\left( \sum_{p=1}^r N_{i,j}^p N_{p,k}^l \right) (s_i + s_j + s_k + s_l) = \sum_{p=1}^r \left( N_{i,j}^p N_{p,k}^l + N_{i,k}^p N_{j,p}^l + N_{j,k}^p N_{i,p}^l \right) s_p$$

on the finite ring  $\mathbb{Z}/m\mathbb{Z}$ , where  $m$  is the biggest integer dividing  $\text{FPdim}(\mathcal{R})^{5/2}$ . For each solution  $(s_i) \in (\mathbb{Z}/m\mathbb{Z})^r$ , if any, take the representative  $(v_i) \in \mathbb{Z}^r$ , with  $0 \leq v_i < m$ . The  $T$ -matrix is  $\text{diag}(\zeta_m^{v_i})$ .

Finally, we keep only the  $S$ - and  $T$ -matrices satisfying all the assumptions of Definition 2.10. We end up with  $19 + 62$  modular data, from  $5 + 17$  fusion rings and  $3 + 12$  types (non-pointed + pointed), which prove Theorem 1.2. This classification was done by applying the function `STmatrix` (or `STmatrix2` according to relevance) in `ModularData.sage` to the list of fusion rings mentioned in §2.3.1, all in [19].

**2.4. Model by zesting.** This subsection is dedicated to make a model of some modular data mentioned in Theorem 1.2, mainly by zesting [8]. The following proposition is due to Eric C. Rowell.

**Proposition 2.13.** *The eight modular data listed in §8.1.5 come from  $SO(8)_2$ , conjugates and zestings.*

*Proof sketch.* Starting with  $SO(8)_2$  one observes that this is  $G = C_2 \times C_2$  graded. Therefore one can twist the braiding by a bicharacter: change the braiding to  $B(\deg(X), \deg(Y))c_{X,Y}$  where  $B$  is the bicharacter. One has to change the twists accordingly (of course). This is a special case of braided zesting (or rather ribbon zesting). The effect is to multiply certain rows/columns of  $S$  by a sign. Glancing at the  $S$ -matrices listed in §8.1.5, this should explain it all. Complex conjugation fixes  $S$  but changes  $T$ , so that should be the whole story (of course complex conjugation changes the underlying fusion category).  $\square$

### 3. TYPE CRITERIA

In this section, we explain some criteria we used to exclude some types from being the type of a fusion ring. A *type* is a list of the form  $T = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$  such that  $1 = d_1 < d_2 < \dots < d_s$ ,  $m_i \geq 1$  for all  $i$ . It is called:

- *trivial* if  $T = [[1, 1]]$ ,
- *pointed* if  $T = [[1, m]]$ ,
- *perfect* if  $m_1 = 1$ ,
- *integral* if  $d_i \in \mathbb{Z}$ , for all  $i$ .

The idea is to apply the criteria explained in this section to our list of possible types in §6, in ascending order of computation time. For the remaining types, we will apply our fusion ring solver explained in §4.

**Notation 3.1.** *We warn the reader that what is denoted  $T$  in this section is a type, not a  $T$ -matrix. Moreover, a type  $T = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$  can sometimes be noted just  $[d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_s, \dots, d_s]$  where  $d_i$  appears  $m_i$  times. Then we reformulate the notation of a type of rank  $r$  as  $[d_1, \dots, d_r]$  with  $1 = d_1 \leq d_2 \leq \dots \leq d_r$ .*

#### 3.1. Small perfect type.

**Proposition 3.2.** *There is no non-trivial perfect integral fusion ring with  $s \leq 3$ .*

*Proof.* If  $s = 1$  then  $T = [[1, 1]]$  is trivial. If  $s = 2$  then  $T = [[1, 1], [d, n]]$  with  $d > 1$  and  $n \geq 1$ , but if there is a fusion ring of type  $T$ , let  $b$  be a basis element with  $\text{FPdim}(b) = d$ . By applying  $\text{FPdim}$  to the decomposition of  $bb^*$  we get that  $d^2 = 1 + kd$ , for some integer  $k \geq 1$ , and by reducing modulo  $d$ , we get that  $0 = 1 \pmod{d}$ , contradiction with  $d > 1$ . Finally, if  $s = 3$ , then  $T = [[1, 1], [a, m], [b, n]]$ , with  $1 < a < b$  and  $m, n \geq 1$ . Let  $\mathcal{R}$  be a fusion ring of this type. Let us call the basis elements  $1, a_1, \dots, a_m, b_1, \dots, b_n$ .

**Claim 3.3.**  $a \wedge b = 1$ .

*Proof.* Let  $d = a \wedge b$ . Then  $\text{FPdim}(a_i a_i^*) = a^2 = 1 + \alpha a + \beta b$ , but  $d$  divides  $a$  and  $b$ , so  $0 = 1 \pmod{d}$ , i.e.  $d = 1$ .  $\square$

**Claim 3.4.** For all  $i$ , there is  $j$  such that  $\langle a_i a_i^*, b_j \rangle \neq 0$ .

*Proof.* If not, then  $a^2 = 1 + \alpha a$ , so  $1 = 0 \pmod{a}$ , contradiction with  $a > 1$ .  $\square$

**Claim 3.5.** If  $k \neq i^*$  then  $\langle a_i a_k, b_j \rangle = 0$ .

*Proof.* If not then  $a^2 = \alpha a + \beta b$  with  $\beta \neq 0$ , so  $\beta b = 0 \pmod{a}$ , but  $a \wedge b = 1$  (i.e.  $b$  is invertible modulo  $a$ ), thus  $\beta = b^{-1}0 = 0 \pmod{a}$ , i.e.  $\beta = ka$  with  $k \geq 1$ . Now,  $a^2 = \alpha a + \beta b \geq \beta b = kab \geq ab$ . So  $a^2 \geq ab$ , contradiction with  $b > a$ .  $\square$

**Claim 3.6.**  $a_{i^*} b_j = b a_{i^*}$ .

*Proof.* By Frobenius reciprocity and Claim 3.5, if  $k \neq i^*$  then  $\langle a_{i^*} b_j, a_k \rangle = 0$ , and by Claim 3.4,  $\langle a_{i^*} b_j, a_{i^*} \rangle \neq 0$ . Now,  $\text{FPdim}(a_{i^*} b_j) = ab = \alpha a + \beta b$ , with  $\alpha \geq 1$ , so  $\beta = 0 \pmod{a}$ , and  $\beta = ka$  with  $k \geq 0$ . Thus,  $ab = \alpha a + kab$ , so  $(1 - k)ab = \alpha a > 0$ , then  $(1 - k) > 0$  and  $k < 1$ . Thus,  $k \geq 0$  and  $k < 1$ , i.e.  $k = 0$  and  $\beta = 0$ . By combining the first sentence of the proof with  $\beta = 0$ , we get that  $a_{i^*} b_j = \alpha a_{i^*}$ , where  $\alpha$  must be  $b$  (by applying  $\text{FPdim}$ ).  $\square$

By Claim 3.6 and Frobenius reciprocity,  $\langle a_i a_{i^*}, b_j \rangle = b$ , so  $a^2 \geq b^2$ , contradiction with  $a < b$ .  $\square$

**Remark 3.7.** Proposition 3.2 cannot be extended to  $s = 4$  because  $\text{Rep}(A_5)$  is of type  $[[1, 1], [3, 2], [4, 1], [5, 1]]$ .

The application of Proposition 3.2 to the list in §6 permits to exclude the following 4 types (up to rank 13):  $[[1, 1], [2, 2], [3, 3]]$ ,  $[[1, 1], [2, 6], [5, 3]]$ ,  $[[1, 1], [2, 2], [3, 7]]$ ,  $[[1, 1], [3, 7], [4, 5]]$ .

### 3.2. Gcd criterion.

**Proposition 3.8.** There is no perfect integral fusion ring of type  $T = [d_1, d_2, \dots, d_r]$  such that  $\gcd(d_2, \dots, d_r) > 1$ .

*Proof.* Let  $d = \gcd(d_2, \dots, d_r)$ . Assume that  $T$  is the type of a fusion ring. Let  $i > 1$ , then

$$d_i d_{i^*} = \sum_k d_k N_{i, i^*}^k,$$

but by Definition 2.1 and Frobenius reciprocity (Proposition 2.2),  $N_{i, i^*}^1 = 1$ , so  $d_i d_{i^*} = 1 + d \times K$ , where  $K = \sum_{k>1} (d_k/d) N_{i, i^*}^k$  is an integer, so  $d_i d_{i^*} = 1 \pmod{d}$ , contradiction with  $i > 1$  and  $d = \gcd(d_2, \dots, d_r) > 1$ .  $\square$

**3.3. TypeTest.** Consider a type  $T = [d_1, \dots, d_r]$  with  $1 = d_1 \leq \dots \leq d_r$  and  $d_2 > 1$  (i.e. perfect). Assume that there is  $i, g_i > 1$  such that  $g_i$  divides  $d_j$  for all  $d_j \notin \{1, d_i\}$ , and  $d_i \wedge g_i = 1$ . Assume that there is a fusion ring of this type, with basis  $\{b_1, \dots, b_r\}$  and  $d_k = \text{FPdim}(b_k)$ .

**Lemma 3.9.** For all  $j$  such that  $d_j \notin \{1, d_i\}$  then

$$\sum_{k; d_k = d_i} N_{j, j^*}^k = -1/d_i \pmod{g_i}.$$

*Proof.* For all  $j$  such that  $d_j \notin \{1, d_i\}$  then

$$b_j b_{j^*} = b_1 + \sum_{k; d_k = d_i} N_{j, j^*}^k b_k + \sum_{k; d_k \notin \{1, d_i\}} N_{j, j^*}^k b_k.$$

Next by applying  $\text{FPdim}$  modulo  $g_i$ , we get that

$$0 = 1 + x d_i \pmod{g_i},$$

but  $d_i$  is invertible modulo  $g_i$ , it follows that  $x = -1/d_i \pmod{g_i}$ .  $\square$

Let  $a_{d_i}$  such that  $0 \leq a_{d_i} < g_i$  and  $a_{d_i} = -1/d_i \pmod{g_i}$ . Let  $S$  be the set of all such  $d_i$ . From the proof of Lemma 3.9, we get that for all  $j \neq 1$ ,

$$d_j^2 \geq 1 + \sum_{d \in S \setminus \{d_j\}} a_d d,$$

so if above inequality does not hold then  $T$  cannot be the type of a fusion ring. Moreover, by Lemma 3.10, if the set  $\{k \mid d_k = d_j\}$  has cardinal one, then we can use the following stronger inequality

$$d_j^2 \geq 1 + b_j d_j + \sum_{d \in S \setminus \{d_j\}} a_d d,$$

with  $0 \leq b_j < g_j^2$  and  $b_j = d_j - \frac{1}{d_j} \pmod{g_j^2}$ .

The Sagemath code of this criterion is the function `TypeTest` in the file `TypeCriteria.sage` at [19]. The application of this criterion to the list in §6 permits to exclude the following number of perfect types (per rank):

rank	8	9	10	11	12	13
#perfect types	1	1	12	37	249	2380

**3.4. LocalCriterion.** Consider a type  $T = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$  as defined above. Assume that there is  $g, i_0 > 1$  such that  $g$  divides  $d_i$  for all  $i \notin \{1, i_0\}$ , and  $d_{i_0} \wedge g = 1$ . Let  $(d, m) := (d_{i_0}, m_{i_0})$ . Assume that  $T$  admits a fusion ring of basis  $\{b_{1-m_1}, \dots, b_0, b_1, \dots, b_{r-1}\}$  such that,  $b_0 = 1$ ,  $\text{FPdim}(b_i) = 1$  for all  $i \leq 0$ ,  $\text{FPdim}(b_j) = d$  for all  $j \in \{1, \dots, m\}$ .

**Lemma 3.10.** *For all  $i \in \{1, \dots, m\}$  we have that*

$$\sum_{j,k=1}^m N_{i,j}^k = md - \frac{m_1}{d} \pmod{g^2},$$

and for all  $j > m$ , we have that  $g$  divides  $\sum_{k=1}^m N_{i,j}^k$ .

*Proof.* For all  $i \in \{1, \dots, m\}$  and for all  $j > m$ ,  $\text{FPdim}(b_i) \neq \text{FPdim}(b_j)$ , so by Corollary 2.8 and Frobenius reciprocity,

$$b_i b_j = \sum_{k \geq 1} N_{i,j}^k b_k = \sum_{k=1}^m N_{i,j}^k b_k + \dots,$$

Apply  $\text{FPdim}$  and reduce modulo  $g$ , we get that

$$d \sum_{k=1}^m N_{i,j}^k = 0 \pmod{g},$$

but  $d$  is invertible modulo  $g$ , so  $g$  divides  $\sum_{k=1}^m N_{i,j}^k$ . By Corollary 2.8, for all  $i \in \{1, \dots, m\}$  and for all  $s \leq 0$ , then  $b_i b_s = b_k$  with  $k \in \{1, \dots, m\}$ , so by Frobenius reciprocity,

$$b_{i^*} \sum_{k=1}^m b_k = \sum_{s \leq 0} b_s + \sum_{j=1}^m \left( \sum_{k=1}^m N_{i,j}^k \right) b_j + \sum_{j>m} \left( \sum_{k=1}^m N_{i,j}^k \right) b_j.$$

Again apply  $\text{FPdim}$ , we get that  $md^2 = m_1 + xd + yg^2$ , so by reducing modulo  $g^2$ , we get that  $x = md - \frac{m_1}{d} \pmod{g^2}$ .  $\square$

Then we can consider the partitions of  $md^2 - xd - m_1$  of the form  $\sum_{i \notin \{1, i_0\}} a_i d_i$ , with  $x = md - \frac{1}{d} \pmod{g^2}$  and  $a_i = 0 \pmod{g}$ , see the SageMath code in §3.4.1, also available at [19]. It is a criterion because for some types, we can find  $d$  such that there is no partition as above.

**Example 3.11.** *For  $T = [[1, 1], [1295, 2], [3990, 1], [4218, 1], [24605, 1], [42180, 1], [98420, 2], [147630, 3]]$ , we can apply Lemma 3.10 to  $(d, m, g) = (1295, 2, 19), (3990, 1, 37), (4218, 1, 5)$ . Then  $md - \frac{1}{d} = 126, 1135, 11 \pmod{g^2}$ , respectively. The use of the function `LocalCriterion` (in §3.4.1) to  $(d, m, g) = (1295, 2, 19)$  permits to exclude  $T$  in less than 1s.*

```
sage: %time LocalCriterion(T, 1295, 2, 19)
CPU times: user 640 ms, sys: 0 ns, total: 640 ms
Wall time: 982 ms
[]
```

*Note that we cannot use  $(d, m, g) = (3990, 1, 37)$ , because it provides 55 solutions.*

```
sage: L=LocalCriterion(T, 3990, 1, 37)
sage: len(L)
55
```

The application of this criterion to the list in §6 (with less than 2s for each possible  $(d, m, g)$ ) permits to exclude the following number of types (per rank):

rank	1	2	3	4	5	6	7	8	9	10	11	12	13
#types	0	0	0	0	0	1	1	3	5	21	63	344	2852
#perfect types	0	0	0	0	0	1	1	2	2	14	37	238	2173

Observe that this critrion alone is enough to exclude all the perfect types up to rank 9, so enough to prove that there is no (non-trivial) perfect integral  $\frac{1}{2}$ -Frobenius fusion ring, and so no (non-trivial) perfect modular integral fusion category, up to rank 9, but the use of our fusion ring solver in §4 permits to prove up to rank 12, see §5.



3.4.1. *SageMath code.*

```

def LocalCriterion(T, d, m, g):
    global SSL
    SSL=[]
    L=[]
    L.append(T[0])
    m1=T[0][1]
    mm=m*d**2-m1
    dim=[t[0] for t in T[1:]]
    dim.remove(d)
    a=(m*d-m1/d)%(g**2)
    q=(mm-a)/(d*(g**2))
    for i in range(q+1):
        LL=copy.deepcopy(L)
        LL.append([d,a+i*(g**2)])
        LocalCriterionInter(dim,mm-d*(a+i*(g**2)),g,LL)
    SSL.sort()
    return SSL

def LocalCriterionInter(dim, m, g, L):
    global SSL
    if m>=0 and len(dim)>0:
        d=dim[-1]
        ddim=copy.deepcopy(dim)
        ddim.remove(d)
        q=m/(d*g)
        for i in range(q+1):
            LL=copy.deepcopy(L)
            LL.append([d,i*g])
            LocalCriterionInter(ddim,m-i*d*g,g,LL)
    elif m==0:
        sL=copy.deepcopy(L)
        sL.sort()
        SSL.append(sL)

```

## 4. FUSION RING SOLVER WITH NORMALIZ

A fusion ring solver is a program which takes a type as input and gives all the fusion rings of this type as output. This section introduces two versions of a fusion ring solver, a full one in §4.2 involving the dimension equations and the associativity equations, and an intermediate one in §4.3 involving only a simplified set of dimension equations. But first, §4.1 introduces the software Normaliz [4] used, and explains how it was adapted to the special structure of the linear and polynomial constraints for fusion rings.

**4.1. About Normaliz.** Normaliz [4] is an open source software tool for discrete convex geometry and related algebra. See Bruns and Gubeladze [3] for terminology and an extensive treatment. Normaliz solves diophantine systems of linear inequalities, equations and congruences with integral coefficients. In addition it computes enumerative data such as multiplicities (geometrically: volumes) and Hilbert series. Its objects can be defined by generators, for example extreme rays of cones, bases of lattices and vertices of polytopes, or by constraints, inequalities, equations and congruences. For systems with coefficients in real algebraic number fields, Normaliz performs the basic tasks of convex hull computation and its dual, vertex enumeration. Lattice points in (bounded) polytopes defined over real algebraic number fields can also be computed, which allows applications to nonintegral fusion rings. For fusion rings it is important that lattice points in polytopes can be constrained by polynomial equations and inequalities. Normaliz releases contain source files, documentation, examples and a test suite, together with binaries for Linux, Mac OS and MS Windows.

For lattice points in general polytopes  $P$  Normaliz uses the project-and-lift algorithm. It projects  $P$  successively to coordinate hyperplanes until dimension 0 and then lifts the lattice points back. If  $P'$  is a projection of  $P$  to a coordinate hyperplane, then the lattice points of  $P$  are projected to lattice points in  $P'$ , and if  $x \in P'$  is a lattice

point in  $P'$ , then its preimages are the lattice points in a line segment. Polynomial constraints can be applied as soon as the lifting has reached the highest coordinate appearing in the constraint.

In this generic form, project-and-lift can do only very small cases of fusion rings. For reasonable performance the algorithm has been adapted to the special structure of the linear and polynomial constraints for fusion rings. Each linear equation is inhomogeneous with nonnegative coefficients and a positive right hand side. Let us call the set of coordinates appearing in it with positive coefficients a *patch*. The patches cover the full set of coordinates, and therefore the linear equations, restricted to the nonnegative orthant, define a polytope  $P$ . The solutions to a linear equation, restricted to its patch, are computed by the project-and-lift algorithm sketched above, and the lattice points in  $P$  are obtained by patching these local solutions along coinciding components. In other words, we start from the solutions of one of the equations, and then extend them patch by patch. The order in which the patches are inserted into the extension process is crucial. Normaliz has some options by which the order can be varied, as discussed below.

There are only linear equations in the partition versions of the input files. Especially for them it is important that there is a second, hidden type of constraints: congruences derived from the linear equations by successively taking residue classes modulo their coefficients. A priori each such a congruence involves only coordinates that belong to the patch of the equation from which it is derived. However, since the congruences involve only a subset of these coordinates, they often apply to other patches or unions of these as well, and can dramatically restrict their number of solutions. The reduction of simple rank 13 to 12 open cases (see 5.2) would have been impossible without a careful exploitation of the congruences.

If polynomial equations of degree  $\geq 2$  are present, then Normaliz tries to find an extension order for the patches that make these equations applicable as soon as possible. The user can modify it by insisting on the “linear” input order or by asking Normaliz to apply “weights” that measure the expected number of solutions of the patches and give preference to low weight. With or without polynomial equations, the user can ask for an order depending on the applicability of congruences. Again this order can be made dependent on weights.

Some of the computations for simple rank 13 were performed on the high performance cluster (HPC) at Osnabrück by splitting the partial solutions early into parts whose extensions are then treated independently. Despite of the very rudimentary approach using a static subdivision without communication between the running instance of Normaliz, the HPC turned out useful.

**4.2. Full version.** Consider a fusion ring with basis  $\{b_1, \dots, b_r\}$ . Following §2.1 we have that for all  $i, j$ :

$$b_i b_j = \sum_k N_{i,j}^k b_k,$$

and by applying FPdim, we get the type  $[d_1, \dots, d_r]$  and its *dimension equations*:

$$d_i d_j = \sum_k N_{i,j}^k d_k.$$

The idea is to solve these  $r^2$  linear positive diophantine equations where  $(d_i)$  are fixed and  $(N_{i,j}^k)$  are  $r^3$  variables, using Normaliz. Now, we can reduce the number of variables to about  $(r-1)^3/6$  using the Unit axiom ( $N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$ ) from Definition 2.1 of a fusion data, and the Frobenius reciprocity (Proposition 2.2) which can be developed as:

$$N_{i,j}^k = N_{i^*,k}^j = N_{j,k^*}^{i^*} = N_{j^*,i^*}^{k^*} = N_{k^*,i}^{j^*} = N_{k,j^*}^i.$$

The key that makes the computation much faster is to exploit the associativity equations (non-linear)

$$\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t,$$

in the most relevant way possible during the solving of above linear diophantine equations. We don't know whether the way we found is the best possible, but it is already very efficient (see §4.1 for more details).

In practice, on a given type  $L = [d_1, d_2, \dots, d_r]$ , apply the function TypeToNormaliz whose SageMath code is available at [19]. That will produce input files (.in), one for each possible duality map  $i \rightarrow i^*$ . Put these files in a folder together with the files normaliz.exe and run\_normaliz.bat available at [19], and click on run\_normaliz (note that there is also a more recent and much quicker version for Linux, on which our last computations were done). That will produce output files (.out) containing all the possible solutions (if any). It remains to convert these solutions into fusion data, up to isomorphism. This last step is not yet completely automated, it still requires manual manipulation, but we will see how in the following example. Take the type  $L = [1, 1, 2]$  of the Grothendieck ring of  $\text{Rep}(\mathcal{S}_3)$ . The application of TypeToNormaliz produces the file  $[1,1,2][0,1,2].in$  with the following content:

```
amb_space 4
inhom_equations 4
1 2 0 0 0
0 1 2 0 -2
0 1 2 0 -2
```

```

0 0 1 2 -3
LatticePoints
convert_equations
nonnegative
polynomial_equations 2
x[2]^2 - x[1]*x[3] + x[3]^2 - x[2]*x[4] - 1;
-x[2]^2 + x[1]*x[3] - x[3]^2 + x[2]*x[4] + 1;

```

The top part encodes the linear diophantine equations, and the bottom part, the associativity equations. Next the application of `run_normaliz` produces the file `[1,1,2][0,1,2].out` containing the following:

```

1 lattice points in polytope (module generators) satisfying polynomial constraints:
0 0 1 1 1

```

Here there is a single solution, but in general there can be several ones (see the next example). Then remove the last 1 of each line of solution and convert into a list of lists:

```
sage: LL=[[0,0,1,1]]
```

Take the list for the type and for the duality map:

```
sage: L=[1,1,2]
```

```
sage: d=[0,1,2]
```

Apply the function `System` as follows:

```

sage: [A,V,M,Assoc]=System(L,d)
[x1, x2, x3, x4]
sage: M
[[[1, 0, 0], [0, 1, 0], [0, 0, 1]],
 [[0, 1, 0], [1, x1, x2], [0, x2, x3]],
 [[0, 0, 1], [0, x2, x3], [1, x3, x4]]]

```

That prints the list of variables, and `M` is the fusion data with these variables. Next make the function `mat` as follows:

```

sage: def mat(l):
....:     [x1, x2, x3, x4]=l
....:     return [[[1, 0, 0], [0, 1, 0], [0, 0, 1]],
....:     [[0, 1, 0], [1, x1, x2], [0, x2, x3]],
....:     [[0, 0, 1], [0, x2, x3], [1, x3, x4]]]

```

Finally, apply the function `ListToFusion` as follows to get all the fusion data up to isomorphism:

```

sage: ListToFusion(mat,LL,L,d)
[[[[1, 0, 0], [0, 1, 0], [0, 0, 1]],
 [[0, 1, 0], [1, 0, 0], [0, 0, 1]],
 [[0, 0, 1], [0, 0, 1], [1, 1, 1]]]

```

It is the fusion data of  $\text{Rep}(S_3)$ , which can ultimately be written in a Tex form as follows:

$$\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

Now if we apply the same procedure, with the type  $L = [1, 5, 5, 5, 6, 7, 7]$ , we get four input files. Only the one for the trivial duality map admits solutions, its output file contains the following:

```

6 lattice points in polytope (module generators) satisfying polynomial constraints:
1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 0 3 1 2 1
1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 2 1 1
1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 1 3 0 1
1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 0 3 1 2 1
1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 2 1 1
1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 1 3 0 1

```

All the files are available at [19]. Ultimately, we get the following two fusion data:

$$\begin{array}{ccccccc}
10000000 & 01000000 & 00100000 & 00010000 & 00001000 & 00000010 & 00000001 \\
01000000 & 11010111 & 00101011 & 01001111 & 00111111 & 01111111 & 01111111 \\
00100000 & 00101111 & 11100111 & 00011111 & 01011111 & 01111111 & 01111111 \\
00010000 & 01001111 & 00011111 & 10110111 & 01101111 & 01111111 & 01111111 \\
00001000 & 00111111 & 01011111 & 01101111 & 11111111 & 01111121 & 01111131 \\
00000100 & 01111111 & 01111111 & 01111111 & 01111121 & 11111203 & 01111131 \\
00000010 & 01111111 & 01111111 & 01111111 & 01111112 & 01111131 & 11111212 \\
00000001 & 01111111 & 01111111 & 01111111 & 01111112 & 01111131 & 11111212
\end{array}$$

**4.3. Dimension partition version.** This approach is relevant only for the types  $T = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$  with  $s$  not too large, because we can simplify the dimension equations by grouping the elements of same dimension (i.e. dimension partition), but we don't know (yet) how to convert the associativity equations. This version should be used as an intermediate of the full version for the appropriate types, in the sense that it can avoid some computational explosions by breaking some symmetries, but for now, we use it only as a criterion, i.e. if it has no solution then neither the full one.

Let us reformulate the type as  $[1, d_{1,1}, \dots, d_{1,n_1}, d_{2,1}, \dots, d_{2,n_2}, \dots, d_{s,1}, d_{s,n_s}]$ , where  $d_{i,a} = d_i$ ,  $d_1 \geq 1 = d_{0,1}$  and  $n_i = m_i - \delta_{1,d_1} \delta_{1,i}$ . Then the dimension equations write as:

$$d_{i,a} d_{j,b} = \sum_{k,c} N_{i,a,j,b}^{k,c} d_{k,c}.$$

Now let  $D_i := \sum_{a=1}^{n_i} d_{i,a} = n_i d_i$  and  $M_{i,j}^k := \sum_{a,b,c} N_{i,a,j,b}^{k,c}$  then

$$D_i D_j = \sum_{a,b} \sum_{k,c} N_{i,a,j,b}^{k,c} d_{k,c} = \sum_k \left( \sum_{a,b,c} N_{i,a,j,b}^{k,c} \right) d_k = \sum_k M_{i,j}^k d_k.$$

So we are reduced to solve the linear positive Diophantine equations

$$n_i d_i n_j d_j = \sum_k M_{i,j}^k d_k,$$

where  $(d_i, n_i)$  are fixed, and the variables  $(M_{i,j}^k)$  reduce to about  $s^3/6$  ones by the following dimension partition version of the Unit axiom and of the Frobenius reciprocity. Note that after the dimension grouping, the duality map becomes trivial (i.e.  $i^* = i$ ).

**Lemma 4.1.** *Here is the dimension partition version of:*

- (Unit)  $M_{i,0}^j = M_{0,i}^j = \delta_{i,j} m_i$
- (Dual)  $M_{i,j}^0 = M_{j,i}^0 = \delta_{i,j} m_i$
- (Frobenius reciprocity)  $M_{i,j}^k = M_{i,k}^j = M_{j,k}^i = M_{j,i}^k = M_{k,i}^j = M_{k,j}^i$ .

*Proof.* Straightforward. □

Note that we did not (yet) find a satisfying dimension partition version of the associativity axiom. In practice, apply the procedure in §4.2 up to the output files, but replace the function `TypeToNormaliz` by `TypeToPreNormaliz`. For example, consider the type  $L = [1, 6, 12, 12, 15, 15, 15, 20, 20, 30, 30, 60]$ . You will find the input and output files at [19]. Note that this dimension partition version was enough to prove Theorem 1.4, see §5 just after Theorem 5.2, where  $L$  is the first type (among 24 ones) to exclude at rank 12.

**Remark 4.2.** *This version uses the dimension partition of the type, but we could consider other versions exploiting other relevant partitions.*

## 5. $\frac{1}{2}$ -FROBENIUS INTEGRAL FUSION RINGS

**5.1. Up to rank 12.** Here is the number of  $\frac{1}{2}$ -Frobenius integral fusion rings (FR) up to rank 12, and the number of noncommutative (NC) ones.

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#FR	1	1	1	2	3	6	9	23	105	158	1218	9101
#NC	0	0	0	0	0	1	0	4	5	7	38	158

The list of these fusion rings is quite big but made available online at [19]. They were obtained by applying the type criteria in §3 and the fusion ring solver in §4 to the list of types mentioned in §6. The number of types per rank is quite small compared to the table in §6.1.

rank	1	2	3	4	5	6	7	8	9	10	11	12
#types	1	1	1	1	2	2	2	4	5	9	15	28

Here is this list of types for each rank and with lexicographic order:

- rank 1:  $[[1]]$ ,
- rank 2:  $[[1, 1]]$ ,
- rank 3:  $[[1, 1, 1]]$ ,
- rank 4:  $[[1, 1, 1, 1]]$ ,
- rank 5:  $[[1, 1, 1, 1, 1], [1, 1, 1, 1, 2]]$ ,
- rank 6:  $[[1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 2, 2]]$ ,

- rank 7:  $[[1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 2, 2, 2]]$ ,
- rank 8:  $[[1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4], [1, 1, 2, 2, 2, 2, 3, 3]]$ ,
- rank 9:  $[[1, 1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6]]$ ,
- rank 10:  $[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1, 1, 3], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6]]$ ,
- rank 11:  $[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 4], [1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 6], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8], [1, 1, 1, 1, 2, 4, 4, 4, 4, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 12], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12, 12], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18]]$ ,
- rank 12:  $[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 1, 1, 1, 2, 2, 4, 4], [1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 6], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 4], [1, 1, 1, 1, 2, 2, 2, 2, 4, 6, 6, 6], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4, 4], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8, 8], [1, 1, 1, 1, 2, 2, 8, 8, 12, 12, 12, 12], [1, 1, 1, 1, 2, 4, 4, 4, 4, 6, 6, 12], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 12, 24], [1, 1, 1, 1, 2, 8, 18, 18, 24, 36, 36, 36], [1, 1, 1, 1, 3, 3, 3, 3, 4, 4, 6, 6], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12, 12, 12], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18, 36], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 6, 6], [1, 1, 1, 2, 2, 2, 3, 4, 4, 4, 6, 6], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6, 12], [1, 1, 1, 3, 6, 8, 8, 8, 8, 8, 12], [1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 12], [1, 1, 2, 2, 2, 2, 6, 6, 6, 9, 9], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12, 12], [1, 1, 2, 3, 3, 6, 6, 8, 8, 8, 12, 12], [1, 1, 2, 6, 6, 6, 6, 10, 10, 10, 15, 15]]$ .

Observe that there is no non-trivial perfect types in the above list, which proves Theorem 1.8 up to rank 12.

Note that there is the perfect integral modular (so  $\frac{1}{2}$ -Frobenius) fusion category  $\mathcal{Z}(\text{Rep}(A_5))$  of  $\text{FPdim } 60^2 = 3600$ , rank 22 and type  $[[1, 1], [3, 2], [4, 1], [5, 1], [12, 10], [15, 4], [20, 3]]$ , computed with GAP using [10, §8.5].

**Question 5.1.** *Is there a perfect integral  $\frac{1}{2}$ -Frobenius fusion ring/category of rank  $< 22$ ?*

**5.2. Simple Rank 13.** Now, about the simple integral modular fusion category of rank 13, we will use the following result constraining the types in the simple case.

**Theorem 5.2** (Corollary 6.16 in [17]). *Let  $\mathcal{C}$  be an integral modular fusion category. If there is a simple object of prime-power  $\text{FPdim}$  then there is a nontrivial symmetric subcategory.*

So, we are reduced to consider perfect integral  $\frac{1}{2}$ -Frobenius types without prime-power element. There are exactly 2044 ones classified from §6, see [19]. Then, we applied the type criteria from §3. Next, we completed the fusion ring solver with dimension partition in §4.3 to all the remaining ones, except 12 ones (in progress) mentioned below. Surprisingly, none of the completed ones admit a solution, except

$$[1, 91, 104, 117, 117, 126, 504, 936, 2184, 2184, 3276, 3276, 3276]$$

requiring the use of the full fusion ring solver in §4.2 to be excluded (in fact, it is the first of this kind up to rank 13).

Here is the list of the 12 remaining types (in progress):

1.  $[1, 60, 75, 110, 132, 150, 300, 825, 825, 1100, 1650, 1650, 1650]$ ,
2.  $[1, 91, 104, 117, 126, 468, 504, 819, 2184, 2184, 3276, 3276, 3276]$ ,
3.  $[1, 91, 117, 126, 312, 364, 504, 819, 2184, 2184, 3276, 3276, 3276]$ ,
4.  $[1, 105, 175, 182, 300, 390, 2275, 3900, 9100, 9100, 13650, 13650, 13650]$ ,
5.  $[1, 105, 175, 357, 595, 850, 1050, 2550, 5950, 5950, 8925, 8925, 8925]$ ,
6.  $[1, 105, 180, 252, 476, 612, 2380, 5355, 5355, 7140, 10710, 10710, 10710]$ ,
7.  $[1, 174, 348, 882, 1044, 1421, 5684, 12789, 12789, 17052, 25578, 25578, 25578]$ ,
8.  $[1, 210, 495, 550, 693, 5775, 6300, 7700, 23100, 23100, 34650, 34650, 34650]$ ,
9.  $[1, 238, 459, 540, 595, 918, 5355, 9180, 21420, 21420, 32130, 32130, 32130]$ ,
10.  $[1, 777, 1036, 1295, 3990, 4218, 24605, 42180, 98420, 98420, 147630, 147630, 147630]$ ,
11.  $[1, 1300, 1620, 2457, 9100, 27300, 81900, 184275, 184275, 245700, 368550, 368550, 368550]$ ,
12.  $[1, 1885, 5005, 6699, 47502, 87087, 200970, 373230, 870870, 870870, 1306305, 1306305, 1306305]$ .

Note that there is the perfect integral modular fusion category  $\mathcal{Z}(\text{Rep}(A_7))$  of  $\text{FPdim } (7!/2)^2$ , rank 74 and type

$$[[1, 1], [6, 1], [10, 2], [14, 2], [15, 1], [21, 1], [35, 1], [70, 9], [105, 4], [210, 20], [280, 9], [360, 14], [504, 5], [630, 4]],$$

so without basis element of prime-power  $\text{FPdim}$ .

**Question 5.3.** *Is there a perfect integral  $\frac{1}{2}$ -Frobenius fusion ring/category without basis element of prime-power  $\text{FPdim}$  and of rank  $< 74$ ?*

## 6. EGYPTIAN FRACTION WITH SQUARED DENOMINATORS

A  $(q, r)$ -Egyptian fraction with squared denominators will denote a sum of the following form:

$$q = \sum_{i=1}^r \frac{1}{s_i^2},$$

with  $q, r, s_i \in \mathbb{Z}_{\geq 1}$  and  $s_1 \geq s_2 \geq \dots \geq s_r \geq 1$ . Moreover, for our need of classification of possible types of Grothendieck rings of modular integral fusion categories (more generally, of  $\frac{1}{2}$ -Frobenius integral fusion rings), we can assume that  $s_i$  divides  $s_1$ , for all  $i$ . Up to subtracting 1 to  $q$  and  $r$  as much as necessary, we can assume that  $s_i \geq 2$ , for all  $i$  ( $q$  is not assumed square-free) and then complete the list of  $(q, r)$ -Egyptian fraction with squared denominators with the  $(q - k, r - k)$  ones on which we add  $k$  times the number 1 in the sum. Under this trick, we can assume that  $q \leq r/4$ .

Here is what we did:

- Use the function `ModularRep` in §6.2 for  $1 \leq r \leq 12$  and  $1 \leq q \leq r/4$ ,
- Complete the classification by adding some 1 as explained above,
- Make all the possible types by  $d_i = s_1/s_i$ , the list is displayed in §6.1.

**6.1. List of types up to rank 13.** Here is the number (per rank) of possible types up to rank 13, deduced only from the use of Egyptian fractions with squared denominators, and also the number of perfect types (see §3):

rank	1	2	3	4	5	6	7	8	9	10	11	12	13
#types	1	1	1	1	2	3	3	7	11	42	144	812	7997
#perfect types	1	0	0	0	0	1	1	2	2	24	88	591	6517

Observe that the ratio of perfect types increases:  $\simeq 18\%$ ,  $57\%$ ,  $61\%$ ,  $73\%$ ,  $81\%$ , at rank 9,  $\dots$ , 13, respectively.

**Question 6.1.** *Does the ratio of perfect types converge to 1 when the rank goes to infinity?*

Let us display (lexicographically) all these types up to rank 10 (those up to rank 13 are available online at [19]):

- rank 1:  $[[1]]$ ,
- rank 2:  $[[1, 1]]$ ,
- rank 3:  $[[1, 1, 1]]$ ,
- rank 4:  $[[1, 1, 1, 1]]$ ,
- rank 5:  $[[1, 1, 1, 1, 1], [1, 1, 1, 1, 2]]$ ,
- rank 6:  $[[1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 2, 2], [1, 2, 2, 3, 3, 3]]$ ,
- rank 7:  $[[1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 2, 2, 2], [1, 2, 2, 3, 3, 3, 6]]$ ,
- rank 8:  $[[1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4], [1, 1, 2, 2, 2, 2, 3, 3], [1, 1, 3, 3, 4, 6, 6, 6], [1, 2, 2, 3, 3, 6, 6], [1, 2, 2, 6, 6, 9, 9, 9]]$ ,
- rank 9:  $[[1, 1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1, 2], [1, 1, 1, 1, 1, 2, 3, 3, 3], [1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 6, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6], [1, 1, 3, 3, 4, 6, 6, 6, 12], [1, 1, 4, 9, 9, 12, 18, 18, 18], [1, 2, 2, 3, 3, 3, 6, 6, 6], [1, 2, 2, 6, 6, 9, 9, 9, 18]]$ ,
- rank 10:  $[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1, 1, 3], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2], [1, 1, 1, 1, 1, 2, 3, 3, 3, 6], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [1, 1, 1, 2, 2, 3, 4, 6, 6, 6], [1, 1, 1, 2, 3, 8, 8, 12, 12, 12], [1, 1, 1, 7, 12, 28, 28, 42, 42, 42], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6], [1, 1, 3, 3, 4, 6, 6, 6, 12, 12], [1, 1, 3, 3, 4, 12, 18, 18, 18], [1, 1, 4, 4, 4, 5, 10, 10, 10], [1, 1, 4, 9, 9, 12, 18, 18, 36], [1, 1, 4, 12, 27, 27, 36, 54, 54], [1, 2, 2, 2, 2, 2, 5, 5, 5], [1, 2, 2, 6, 14, 14, 21, 21, 21], [1, 2, 2, 3, 3, 3, 3, 3, 3], [1, 2, 2, 3, 3, 3, 6, 6, 6], [1, 2, 2, 3, 3, 3, 6, 6, 12], [1, 2, 2, 3, 6, 6, 6, 9, 9], [1, 2, 2, 3, 9, 9, 12, 18, 18, 18], [1, 2, 2, 4, 5, 5, 10, 10, 10], [1, 2, 2, 6, 6, 9, 9, 9, 18, 18], [1, 2, 2, 6, 6, 18, 18, 27, 27, 27], [1, 2, 3, 6, 10, 10, 10, 15, 15], [1, 2, 3, 6, 15, 15, 20, 30, 30, 30], [1, 3, 3, 3, 4, 4, 4, 4, 6], [1, 3, 3, 3, 4, 6, 8, 12, 12, 12], [1, 3, 3, 3, 6, 16, 16, 24, 24, 24], [1, 4, 4, 4, 7, 7, 7, 14, 14, 14], [1, 4, 9, 28, 63, 63, 84, 126, 126, 126], [1, 5, 7, 35, 60, 140, 140, 210, 210, 210], [1, 5, 10, 18, 30, 30, 30, 45, 45], [1, 5, 10, 18, 45, 45, 60, 90, 90, 90], [1, 6, 18, 38, 38, 114, 171, 171, 171], [1, 12, 12, 17, 51, 51, 68, 102, 102, 102], [1, 18, 30, 70, 70, 210, 210, 315, 315, 315], [1, 70, 130, 182, 390, 910, 1365, 1365, 1365]]$ .

## 6.2. SageMath code.

```
def ModularRep(q,r):
    L=all_rep(q, r)
    P=[]
    for l in L:
        if l[0]!=1: # those starting with 1 should be considered with q-1.
            k=0
            for ll in l:
                if l[-1]%ll!=0:
                    k=1
                    break
```

```

        if k==0:
            lll=[l[-1]/ll for ll in l]
            lll.sort()
            Di=sum([i^2 for i in lll])
            P.append(lll+[[sqrt(Di)]])
    return P

def res_rep(s, N):
    def succ(t):
        s0, m = t
        if s0==0 or len(m)>=N:
            return []
        p = numerator(s0)
        q = denominator(s0)
        if len(m)==N-1:
            if p==1 and is_square(q):
                r = q.isqrt()
                if r>=m[-1]:
                    return [(0,m+(r,))]
            return []
        L = max(m[-1], ((q-1)//p).isqrt()+1)
        U = floor((N-len(m))/s0).isqrt()
        if len(m)==N-2:
            S = []
            try:
                two_squares(p)
                two_squares(q)
            except:
                return S
            q2 = q^2
            for r in (L..U):
                d = p*r^2-q
                if d>0 and q2%d==0:
                    r2 = (q2//d + q)//p
                    if is_square(r2):
                        S.append( (0,m+(r,r2.isqrt())) )
            return S
        if len(m)==N-3:
            t = p*q
            a = valuation(t,2)
            if a%2==0 and (t>>a)%8==7:
                return []
            return ( (s0-1/r^2, m+(r,)) for r in (L..U) )
    return RecursivelyEnumeratedSet(seeds=[(s-1/r^2,(r,)) for r in range(1,floor(N/s).isqrt()+1)], \
    successors=succ, structure='forest')

def all_rep(s, N):
    return res_rep(s,N).map_reduce(lambda t: {t[1]} if t[0]==0 and len(t[1])==N else set(), set.union, \
    set() )

def count_rep(s, N):
    return res_rep(s,N).map_reduce(lambda t: int(t[0]==0 and len(t[1])==N))

```

## 7. ODD-DIMENSIONAL CASE

We will refer to [6] and [7] for the state of the art about the odd-dimensional modular fusion categories. The first important result is that a modular fusion category  $\mathcal{C}$  is odd-dimensional if and only if it is maximally nonself dual (MNSD), i.e., the only self-dual simple object is the unit object. Let  $(d_i)_{i \in I}$  be the FPdim of the simple objects

(up to isomorphism). Then  $d_i^2$  divides  $\text{FPdim}(\mathcal{C})$  odd, so  $d_i$  must be odd, moreover  $\sum_{i \in I} d_i^2 = \text{FPdim}(\mathcal{C})$ , so modulo 2, we get that the rank  $r = |I|$  must be odd too. So we are reduced to considering Egyptian fractions of the form  $q = \sum_{i=1}^r \frac{1}{s_i^2}$ , with  $q, r, s_i \in \mathbb{Z}_{\geq 1}$ ,  $s_1 \geq \dots \geq s_r \geq 1$ ,  $r$  and  $s_i$  odd,  $s_i$  divides  $s_1$  and  $s_{2k} = s_{2k+1}$ , so that

$$q = \frac{1}{s_1^2} + \sum_{k=1}^{(r-1)/2} \frac{2}{s_{2k}^2}.$$

But  $s_i$  is odd, so  $s_i^2 \equiv 1 \pmod{8}$ , which implies that  $q \equiv r \pmod{8}$  and  $q$  odd also. We can use the same trick as for §6, and assume that  $s_i > 1$  (so  $s_i \geq 3$ ), up to completing the classification by adding some 1s. So we can assume  $q \leq r/9$ . Then, as long as  $r < 27$ , we can assume that  $q = 1$ , and so  $r \equiv 1 \pmod{8}$ , i.e.  $r = 1, 9, 17, 25$  (up to completing by 1s).

**Remark 7.1.** *This trick can be extended. For example, up to also completing the classification by adding eighteen 3s, we can assume that  $s_i = 5$  for  $i + 16 \leq r$ , so that  $q \leq 16/9 + (r - 16)/25$ . Then, as long as  $r < 47$  (so 51, because  $q \equiv r \pmod{8}$ ), we can assume that  $q = 1$ . This extended trick will not be used here.*

For  $r < 25$ , we obtained the following list for the possible non-pointed types:

- $[[1, 9], [3, 8], [81, 2a]]$ ,
- $[[1, 7], [3, 2], [5, 8], [225, 2a]]$ ,
- $[[1, 3], [3, 8], [5, 6], [225, 2a]]$ ,
- $[[1, 1], [3, 2], [7, 2], [9, 4], [21, 8], [3969, 2a]]$ ,
- $[[1, 1], [9, 4], [25, 2], [45, 2], [75, 8], [50625, 2a]]$ ,

where  $a \geq 0$  corresponds to the completion. Note that there are of rank  $17 + 2a$ , which immediately recovers a result from [6] stating that every odd-dimensional modular fusion category of rank less than 17 is pointed. Again, by [6], every perfect odd-dimensional modular fusion category is a Deligne product of simple ones. But according to above, a non-pointed one must be of rank at least 17, so a perfect non-simple one of rank at least  $17^2 = 289$ . Then, a perfect one of rank less than 289 must be simple, so cannot have a non-trivial simple object of prime-power  $\text{FPdim}$  by [17, Corollary 6.16]. Thus, the perfect types written above are excluded. It follows that a perfect odd-dimensional modular fusion category of rank less than 25 is trivial. Now, by [6], every odd-dimensional modular fusion category of rank less than 25 is pointed or perfect, so by above:

**Corollary 7.2.** *Every odd-dimensional modular fusion category of rank less than 25 is pointed. Moreover, every perfect one of rank less than  $25^2 = 625$  is simple.*

Now, there are non-pointed and non-perfect odd-dimensional modular fusion categories at rank 25 given by  $\mathcal{Z}(\text{Vec}_{C_7 \rtimes C_3}^\omega)$ , and [7] shows that there is no other one (up to equivalence). Thus, it remains to consider the perfect ones (so simple and  $q = 1$ ). We found exactly 91 possible types (available at [19]) by above method together with [17, Corollary 6.16]. Next, the application of the type criteria from §3 reduces to 29 types, and the solver from §4.3 quickly excludes 8 more (the other ones being currently out of reach). There remain the following 21 types:

1.  $[[1, 1], [15, 2], [63, 2], [115, 2], [161, 4], [315, 2], [805, 2], [1449, 2], [2415, 8]]$ ,
2.  $[[1, 1], [21, 2], [35, 2], [63, 2], [85, 2], [119, 2], [255, 2], [595, 2], [1071, 2], [1785, 8]]$ ,
3.  $[[1, 1], [39, 4], [65, 2], [189, 2], [315, 2], [585, 2], [1365, 2], [2457, 2], [4095, 8]]$ ,
4.  $[[1, 1], [39, 2], [231, 2], [273, 2], [1001, 2], [1287, 2], [3465, 2], [4095, 2], [9009, 2], [15015, 8]]$ ,
5.  $[[1, 1], [39, 2], [231, 2], [273, 2], [1001, 2], [3465, 2], [4095, 2], [6435, 4], [15015, 8]]$ ,
6.  $[[1, 1], [45, 2], [95, 6], [99, 2], [495, 2], [1045, 2], [1881, 2], [3135, 8]]$ ,
7.  $[[1, 1], [75, 2], [91, 4], [175, 2], [585, 2], [975, 2], [2275, 2], [4095, 2], [6825, 8]]$ ,
8.  $[[1, 1], [75, 2], [91, 4], [175, 2], [975, 2], [2275, 2], [2925, 4], [6825, 8]]$ ,
9.  $[[1, 1], [99, 2], [195, 2], [385, 2], [405, 2], [15015, 2], [31185, 2], [36855, 2], [81081, 2], [135135, 8]]$ ,
10.  $[[1, 1], [99, 2], [231, 2], [385, 2], [675, 2], [10395, 4], [28875, 2], [51975, 2], [86625, 8]]$ ,
11.  $[[1, 1], [99, 2], [273, 2], [1001, 2], [1365, 2], [2145, 2], [3465, 4], [9009, 2], [15015, 8]]$ ,
12.  $[[1, 1], [135, 4], [165, 2], [189, 2], [315, 2], [385, 2], [1155, 2], [2079, 2], [3465, 8]]$ ,
13.  $[[1, 1], [195, 4], [231, 2], [1001, 2], [1287, 2], [3465, 2], [4095, 2], [9009, 2], [15015, 8]]$ ,
14.  $[[1, 1], [195, 4], [231, 2], [1001, 2], [3465, 2], [4095, 2], [6435, 4], [15015, 8]]$ ,
15.  $[[1, 1], [205, 2], [369, 2], [805, 2], [6601, 2], [8487, 2], [12915, 2], [33005, 2], [59409, 2], [99015, 8]]$ ,
16.  $[[1, 1], [205, 2], [369, 2], [805, 2], [6601, 2], [12915, 2], [33005, 2], [42435, 4], [99015, 8]]$ ,
17.  $[[1, 1], [231, 2], [777, 2], [1155, 2], [1221, 2], [3465, 2], [6105, 2], [14245, 2], [25641, 2], [42735, 8]]$ ,
18.  $[[1, 1], [297, 2], [385, 2], [3465, 2], [5859, 2], [7161, 2], [15345, 2], [35805, 2], [64449, 2], [107415, 8]]$ ,
19.  $[[1, 1], [693, 2], [1771, 2], [4301, 2], [11781, 2], [24633, 2], [30107, 4], [38709, 2], [90321, 8]]$ ,
20.  $[[1, 1], [693, 2], [1771, 2], [11781, 2], [12903, 4], [24633, 2], [38709, 4], [90321, 8]]$ ,
21.  $[[1, 1], [3485, 2], [6273, 2], [33825, 2], [34425, 2], [621027, 4], [1725075, 2], [3105135, 2], [5175225, 8]]$ ,



$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\ 2 & 2 & 2\zeta_9^4 + 2\zeta_9^5 & 2\zeta_9^2 + 2\zeta_9^7 & -2 & -2\zeta_9^2 - 2\zeta_9^4 - 2\zeta_9^5 - 2\zeta_9^7 & 0 & 0 \\ 2 & 2 & 2\zeta_9^2 + 2\zeta_9^7 & -2\zeta_9^2 - 2\zeta_9^4 - 2\zeta_9^5 - 2\zeta_9^7 & -2 & 2\zeta_9^4 + 2\zeta_9^5 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 2 & 2 & -2\zeta_9^2 - 2\zeta_9^4 - 2\zeta_9^5 - 2\zeta_9^7 & 2\zeta_9^4 + 2\zeta_9^5 & -2 & 2\zeta_9^2 + 2\zeta_9^7 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \\ 3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \end{bmatrix},$$



8.1.3. *Rank 10, type  $[1, 1, 1, 2, 2, 2, 2, 2, 2, 3]$ , first fusion ring.*

[illegible][illegible]

- $[0, 0, 0, 1/9, 4/9, -2/9, -2/9, 4/9, 1/9, 1/2], 9, 18, 4, [1, 0, 0, 0, 0, 0, 0, 0, 0, 1],$

1	1	1	2	2	2	2	2	2	3
1	1	1	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_3^2$	$2\zeta_3$	3
1	1	1	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_3$	$2\zeta_3^2$	3
2	$2\zeta_3^2$	$2\zeta_3$	$-2\zeta_9^7$	$-2\zeta_9^5$	$2\zeta_9^2 + 2\zeta_9^5$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^4$	$-2\zeta_9^2$	0
2	$2\zeta_3$	$2\zeta_3^2$	$-2\zeta_9^5$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^7$	$-2\zeta_9^2$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^4$	0
2	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^7$	$-2\zeta_9^4$	$-2\zeta_9^5$	$-2\zeta_9^2$	$2\zeta_9^4 + 2\zeta_9^7$	0
2	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^2$	$-2\zeta_9^5$	$-2\zeta_9^4$	$-2\zeta_9^7$	$2\zeta_9^2 + 2\zeta_9^5$	0
2	$2\zeta_3^2$	$2\zeta_3$	$-2\zeta_9^4$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^2$	$-2\zeta_9^7$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^5$	0
2	$2\zeta_3$	$2\zeta_3^2$	$-2\zeta_9^2$	$-2\zeta_9^4$	$2\zeta_9^4 + 2\zeta_9^7$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^5$	$-2\zeta_9^7$	0
3	3	3	0	0	0	0	0	0	-3

- $[0, 0, 0, 2/9, -1/9, -4/9, -4/9, -1/9, 2/9, 1/2], 9, 18, 4, [1, 0, 0, 0, 0, 0, 0, 0, 0, 1],$

1	1	1	2	2	2	2	2	2	3
1	1	1	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_3$	$2\zeta_3^2$	3
1	1	1	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_3^2$	$2\zeta_3$	3
2	$2\zeta_3$	$2\zeta_3^2$	$-2\zeta_9^5$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^7$	$-2\zeta_9^2$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^4$	0
2	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^2$	$-2\zeta_9^5$	$-2\zeta_9^4$	$-2\zeta_9^7$	$2\zeta_9^2 + 2\zeta_9^5$	0
2	$2\zeta_3^2$	$2\zeta_3$	$-2\zeta_9^7$	$-2\zeta_9^5$	$2\zeta_9^2 + 2\zeta_9^5$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^4$	$-2\zeta_9^2$	0
2	$2\zeta_3$	$2\zeta_3^2$	$-2\zeta_9^2$	$-2\zeta_9^4$	$2\zeta_9^4 + 2\zeta_9^7$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^5$	$-2\zeta_9^7$	0
2	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^7$	$-2\zeta_9^4$	$-2\zeta_9^5$	$-2\zeta_9^2$	$2\zeta_9^4 + 2\zeta_9^7$	0
2	$2\zeta_3^2$	$2\zeta_3$	$-2\zeta_9^4$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^2$	$-2\zeta_9^7$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^5$	0
3	3	3	0	0	0	0	0	0	-3

8.1.4. Rank 10, type  $[1, 1, 1, 2, 2, 2, 2, 2, 2, 3]$ , second fusion ring.

[illegible][illegible]



- $[0, 0, 0, 0, -7/16, 1/16, 1/16, 1/4, -7/16, -7/16, 1/16], 8, 16, 1, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -3/16, 5/16, 5/16, 1/4, -3/16, -3/16, 5/16], 8, 16, 1, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, 7/16, 7/16, -1/4, -1/16, -1/16, 3/16], 8, 16, -1, [1, 1, 1, 1, 1, -1, -1, 1, -1, -1, 1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, 7/16, -5/16, -1/4, 3/16, -1/16, 3/16], 8, 16, -1, [1, 1, 1, 1, -1, 1, -1, 1, -1, 1, -1],$

1	1	1	1	2	2	2	2	2	2	2
1	1	1	1	-2	-2	2	2	2	-2	-2
1	1	1	1	-2	2	-2	2	-2	2	-2
1	1	1	1	2	-2	-2	2	-2	-2	2
2	-2	-2	2	$-2\zeta_8 + 2\zeta_8^3$	0	0	0	0	0	$2\zeta_8 - 2\zeta_8^3$
2	-2	2	-2	0	$2\zeta_8 - 2\zeta_8^3$	0	0	0	$-2\zeta_8 + 2\zeta_8^3$	0
2	2	-2	-2	0	0	$-2\zeta_8 + 2\zeta_8^3$	0	$2\zeta_8 - 2\zeta_8^3$	0	0
2	2	2	2	0	0	0	-4	0	0	0
2	2	-2	-2	0	0	$2\zeta_8 - 2\zeta_8^3$	0	$-2\zeta_8 + 2\zeta_8^3$	0	0
2	-2	2	-2	0	$-2\zeta_8 + 2\zeta_8^3$	0	0	0	$2\zeta_8 - 2\zeta_8^3$	0
2	-2	-2	2	$2\zeta_8 - 2\zeta_8^3$	0	0	0	0	0	$-2\zeta_8 + 2\zeta_8^3$

- $[0, 0, 0, 0, -5/16, -5/16, -5/16, -1/4, 3/16, 3/16, 3/16], 8, 16, -1, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -1/16, -1/16, -1/16, -1/4, 7/16, 7/16, 7/16], 8, 16, -1, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \end{bmatrix}$$

**8.2. Pointed case.** Following Remark 1.1, we only need to write the abelian groups and its T-matrices (here the topological spins). Recall that  $C_n$  denotes the cyclic group of order  $n$ . The full data (as in §8.1) are available at [19].

- $C_1$ :  $[0]$ ,
- $C_2$ :  $[0, 1/4], [0, -1/4]$ ,
- $C_3$ :  $[0, 1/3, 1/3], [0, -1/3, -1/3]$ ,
- $C_2^2$ :  $[0, 1/2, 0, 0], [0, -1/4, 1/4, 0], [0, 1/2, 1/4, 1/4], [0, 1/2, 1/2, 1/2], [0, -1/4, -1/4, 1/2]$ ,
- $C_4$ :  $[0, 1/2, 1/8, 1/8], [0, 1/2, 3/8, 3/8], [0, 1/2, -3/8, -3/8], [0, 1/2, -1/8, -1/8]$ ,
- $C_5$ :  $[0, 1/5, -1/5, -1/5, 1/5], [0, 2/5, -2/5, -2/5, 2/5]$ ,
- $C_6$ :  $[0, -1/4, 1/12, 1/3, 1/3, 1/12], [0, -1/4, 5/12, -1/3, -1/3, 5/12], [0, 1/4, -5/12, 1/3, 1/3, -5/12], [0, 1/4, -1/12, -1/3, -1/3, -1/12]$ ,
- $C_7$ :  $[0, 1/7, -3/7, 2/7, 2/7, -3/7, 1/7], [0, 3/7, -2/7, -1/7, -1/7, -2/7, 3/7]$ ,
- $C_2^3$ :  $[0, -1/4, 1/4, 1/4, 1/4, 1/2, 0, 0], [0, -1/4, -1/4, -1/4, 1/4, 1/2, 0, 0], [0, -1/4, 1/2, 1/2, 1/4, 1/2, 1/4, 1/4], [0, -1/4, 1/2, -1/4, 1/2, -1/4, 1/2, 1/4]$ ,
- $C_2 \times C_4$ :  $[0, 1/2, -1/4, 1/4, 1/8, 3/8, 3/8, 1/8], [0, 1/2, 1/4, -1/4, 1/8, -1/8, -1/8, 1/8], [0, 1/2, -1/4, 1/4, 3/8, -3/8, -3/8, 3/8], [0, 1/2, -1/4, 1/4, -3/8, -1/8, -1/8, -3/8]$ ,
- $C_8$ :  $[0, 0, 1/4, -7/16, 1/16, 1/16, -7/16, 1/4], [0, 0, 1/4, -3/16, 5/16, 5/16, -3/16, 1/4], [0, 0, -1/4, -5/16, 3/16, 3/16, -5/16, -1/4], [0, 0, -1/4, -1/16, 7/16, 7/16, -1/16, -1/4]$ ,
- $C_2^2$ :  $[0, 0, 0, 1/3, -1/3, -1/3, 1/3, 0, 0], [0, 1/3, 1/3, -1/3, -1/3, -1/3, -1/3, 1/3, 1/3]$ ,
- $C_9$ :  $[0, 0, 1/9, -2/9, 4/9, 4/9, -2/9, 1/9, 0], [0, 0, 2/9, -4/9, -1/9, -1/9, -4/9, 2/9, 0]$ ,
- $C_{10}$ :  $[0, 1/4, 1/20, -1/5, 1/5, 9/20, 9/20, 1/5, -1/5, 1/20], [0, -1/4, 3/20, 2/5, -2/5, 7/20, 7/20, -2/5, 2/5, 3/20], [0, -1/4, -9/20, -1/5, 1/5, -1/20, -1/20, 1/5, -1/5, -9/20], [0, 1/4, -7/20, 2/5, -2/5, -3/20, -3/20, -2/5, 2/5, -7/20]$ ,
- $C_{11}$ :  $[0, 1/11, 4/11, -2/11, 5/11, 3/11, 3/11, 5/11, -2/11, 4/11, 1/11], [0, 2/11, -3/11, -4/11, -1/11, -5/11, -5/11, -1/11, -4/11, -3/11, 2/11]$ ,
- $C_2 \times C_6$ :  $[0, -1/4, 0, 1/4, -1/3, 5/12, -1/3, -1/12, -1/12, -1/3, 5/12, -1/3], [0, 1/2, -1/4, -1/4, -1/3, 1/6, 5/12, 5/12, 5/12, 1/6, -1/3], [0, 1/2, 1/2, 1/2, -1/3, 1/6, 1/6, 1/6, 1/6, 1/6, -1/3], [0, 1/2, 1/4, 1/4, -1/3, 1/6, -1/12, -1/12, -1/12, -1/12, 1/6, -1/3], [0, 1/2, 0, 0, -1/3, 1/6, -1/3, -1/3, -1/3, -1/3, 1/6, -1/3], [0, 1/2, 1/2, 1/2, 1/3, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, 1/3], [0, 1/4, 1/2, 1/4, 1/3, -5/12, -1/6, -5/12, -5/12, -1/6, -5/12, 1/3], [0, 0, 1/2, 0, 1/3, 1/3, -1/6, 1/3, 1/3, -1/6, 1/3, 1/3], [0, -1/4, 0, 1/4, 1/3, 1/12, 1/3, -5/12, -5/12, 1/3, 1/12, 1/3], [0, -1/4, 1/2, -1/4, 1/3, 1/12, -1/6, 1/12, -1/6, 1/12, 1/12, 1/3]$ ,

- $C_{12}$ :  $[0, 1/2, -1/8, -1/3, 1/6, -11/24, -11/24, -11/24, -11/24, 1/6, -1/3, -1/8]$ ,  $[0, 1/2, -1/8, 1/3, -1/6, 5/24, 5/24, 5/24, -1/6, 1/3, -1/8]$ ,  $[0, 1/2, -3/8, -1/3, 1/6, 7/24, 7/24, 7/24, 7/24, 1/6, -1/3, -3/8]$ ,  $[0, 1/2, -3/8, 1/3, -1/6, -1/24, -1/24, -1/24, -1/24, -1/6, 1/3, -3/8]$ ,  $[0, 1/2, 3/8, -1/3, 1/6, 1/24, 1/24, 1/24, 1/24, 1/6, -1/3, 3/8]$ ,  $[0, 1/2, 3/8, 1/3, -1/6, -7/24, -7/24, -7/24, -7/24, -1/6, 1/3, 3/8]$ ,  $[0, 1/2, 1/8, -1/3, 1/6, -5/24, -5/24, -5/24, 1/6, -1/3, 1/8]$ ,  $[0, 1/2, 1/8, 1/3, -1/6, 11/24, 11/24, 11/24, 11/24, -1/6, 1/3, 1/8]$ .

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