CLASSIFICATION OF INTEGRAL MODULAR DATA UP TO RANK 13

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Abstract. This paper classifies all integral modular data up to rank 13 (all can be categorified). Furthermore, it also classifies all integral half-Frobenius fusion rings up to rank 12. We find that each perfect integral modular fusion category up to rank 13, as well as every perfect integral half-Frobenius fusion ring up to rank 12, is trivial. We have also refined the categorifiable non-pointed odd-dimensional modular data at ranks below 25 to three possible items, all of rank 17, FPdim 225, and type [[1,3],[3,8],[5,6]], filling potential gaps in the literature. For rank 25, we have narrowed down the perfect case to 4 types.

Our initial key insight is that the Egyptian fractions, which are typically employed to list possible types, can be chosen with squared denominators. We then develop several type criteria as initial filters. In particular, we establish that the number of distinct basic FPdims in a non-trivial perfect fusion ring must be at least 4. From rank 13 onward, the types were further restricted by additional properties unique to the modular case, which involved the universal grading, congruence representations of the modular group and Galois action, leading to critical arithmetic constraints. To obtain the fusion rings, we solve the dimension and associativity equations using an updated version of Normaliz designed specifically for this purpose. The S-matrices (if they exist) are obtained by self-transposing the character table, while the T-matrices are derived by solving the Anderson-Moore-Vafa equations. Finally, we verify the extended axioms of modular data.

1. Introduction

In this paper, we assume that all fusion categories are defined over the complex field. The concept of an integral modular fusion category has been extensively studied, as detailed in the references at the beginning of [12]. In [4], they have been classified up to rank 6 (all pointed), with Egyptian fractions playing a crucial role. The approach that enables us to extend this classification up to rank 13 in our work hinges on the observation that it is sufficient to consider Egyptian fractions with squared denominators. This restriction significantly reduces the combinatorial complexity. To illustrate this point, consider that the number of Egyptian fractions (summing to 1) of length $n = 1, 2, \dots, 8$ is $1, 1, 3, 14, 147, 3462, 294314, 159330691$, respectively (as per [41]). In contrast, when limited to squared denominators, the counts are $1, 0, 0, 1, 0, 1, 1, 4$, respectively (refer to [1]).

We begin by recalling the concept of a fusion ring and its fundamental results in §2.1, with reference to [17, Chapter 3]. As defined in [19], a fusion ring $\mathcal{F}$ is termed *s-Frobenius* if for every basic element b, the ratio $\mathrm{FPdim}(\mathcal{F})^s/\mathrm{FPdim}(b)$ is an algebraic integer. According to [17, Proposition 8.14.6], the Grothendieck ring of a modular fusion category is 1/2-Frobenius (denoted *half-Frobenius* in the rest of the paper). Consider $\mathcal{F}$ to be an integral half-Frobenius fusion ring with a basis $\{b_1, \dots, b_r\}$, FPdim D, and type $[d_1, \dots, d_r]$, where $1 = d_1 \leq d_2 \leq \cdots \leq d_r$ and $d_i = \mathrm{FPdim}(b_i)$. Thus d_i^2 is a divisor of D, for all i. There exists a unique square-free integer q such that $D = qs^2$, implying that each d_i is a divisor of s. Let s_i denote the positive integer s/d_i. Given that $D = \sum_{i=1}^r d_i^2$, we arrive at the following representation of q as an Egyptian fraction with squared denominators:

$$q = \sum_{i=1}^{r} \frac{1}{s_i^2}.$$

We have classified all such Egyptian fractions up to $r = 13$ using SageMath, as will be discussed in §4, where a method to constrain to $q \leq r/4$ is also described. Since $s_1 = s$, we have $d_i = s_1/s_i$, and we may assume that s_i is a divisor of s_1, for all i. As detailed in §4, this leads us to consider only 9025 types up to rank 13.

The subsequent phase entails implementing new criteria for identifying a type that emerges from a fusion ring, as delineated in §5. Particularly, we establish that the minimum number of distinct basic FPdims in a non-trivial perfect fusion ring is four (Theorem 5.1). The proof of these criteria predominantly relies on modular arithmetic and serves to rule out approximately 62% of the types up to rank 13.

To address the remaining types, we classify all possible fusion data $(N_{i,j}^k)$, as defined in §2.1, for each type (d_i), utilizing our fusion ring solver described in §6. We begin by reducing the number of variables, leveraging the Unit axiom of fusion data and the Frobenius reciprocity. The main challenge, denoted as "patching", involves integrating the associativity equations $\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t$ (which are non-linear) as efficiently as possible into the (linear) solving process of the dimension equations $d_i d_j = \sum_k N_{i,j}^k d_k$, which are positive linear Diophantine equations. This approach was implemented using Normaliz [8], on which we developed new features dedicated to the classification of fusion rings, as detailed in Appendix H of its manual [9].

This step culminates in a classification of all the half-Frobenius integral fusion rings up to rank 12, tallying exactly 10628 instances derived from 71 types, and proves the absence of any non-trivial perfect integral half-Frobenius fusion rings up to rank 12 (see §7). The rank 13 will be mentioned later in this introduction.

Our objective now is to classify all possible modular data related to these fusion rings. The definition of modular data we employ (refer to §2.2) is informed by the key attributes of a modular fusion category, specifically a pseudo-unitary one, as our research is centered on the integral case (see [17, Proposition 9.6.5]). We can limit our attention to commutative fusion rings since a modular fusion category, being braided, possesses a commutative Grothendieck ring (although our classification encompasses 213 noncommutative fusion rings as well; see §7).

First, we examine the S-matrices: for a given commutative fusion ring, we take its eigentable (as defined in Definition 2.7) and consider it as a matrix, retaining only those with cyclotomic elements—such fusion rings are termed *cyclotomic*. If suitable renormalization and permutation yield a self-transpose matrix (detailed in §3.1), we call the fusion ring as *self-transposable*; if not, it is dismissed. From this, we infer that there are precisely 69 self-transposable, cyclotomic, half-Frobenius, integral fusion rings up to rank 12, originating from 27 types, which is fewer than 0.7% of the 10628 identified in the initial stage.

From rank 13 onward, the types were further restricted by additional properties coming from more advanced results on modular fusion categories §8. This adjustment was necessary because we encountered computational limits for classifying all half-Frobenius fusion rings. Consequently, the result became less general at the fusion ring level compared what we get up to rank 12. In the non-perfect case, we applied specific *universal grading* techniques, as discussed in §8.1 and based on [28, Proposition VI.2]. Additionally, we explored congruence representations of the modular group (see §8.2) and Galois actions (see §8.3). Notably, we provide two proofs of the following folklore result, the shorter one uses [16, Theorem II (iii)].

Theorem 1.1. *For any prime number p that divides the global dimension norm of a modular fusion category with rank r, then $p \leq 2r+1$.*

This inequality is optimal, and the examples for which the equality holds are classified in [31]. Regarding the integral case, discussions with Eric Rowell and Andrew Schopieray indicated that the rank r can be substituted with the multiplicity m of a certain basic FPdim, leading to the inequality $p \leq 2m+1$. An even stronger integral version is introduced in Theorem 8.3, which was crucial in demonstrating that for all ranks $r < 22$, then the inequality $p \leq r$ holds, as detailed in Corollary 8.8. This latter inequality is conjectured to hold in general (integral), and has been proven for $\mathcal{Z}(\mathrm{Rep}(G))$, across all finite groups G.

Moving on to the T-matrices: for the fusion rings that remain, we solve the Anderson-Moore-Vafa equations (see §2.2) in the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. We preserve only those S- and T-matrices that satisfy all the conditions of Definition 2.11. The use of our so-called *magic criterion* was pivotal for several big cases and could lead to an interesting theoretical reformulation, see Question 3.2. Ultimately, we arrive at $19+64$ modular data, derived from $5+18$ fusion rings and $3+13$ types (non-pointed + pointed).

Remark 1.2. *Every pointed modular fusion category corresponds to a metric group (G,q)—a finite abelian group G equipped with a non-degenerate quadratic form $q : G \to \mathbb{C}^*$, represented by the T-matrix, as described in* [17, §8.4].

The modular data (MD) mentioned in §11 encompass S- and T-matrices, central charge, fusion data, and second Frobenius-Schur indicators for the non-pointed case. For the pointed case, however, it includes only the T-matrices. The following theorem provides a concise overview:

Theorem 1.3. *There are* 19 *MD of non-pointed integral modular fusion categories up to rank* 13*, given by:*

- *Rank* 8*,* FPdim 36*, type* $[1,1,2,2,2,2,3,3]$*:*
  - 6 *MD with central charge $c=0$ from $\mathcal{Z}(\mathrm{Vec}_{S_3}^{\omega})$, see* [21]*,*
  - 2 *MD with $c=4$ from $(C_3^2+0)^{C_2}$, see* [20, point (b) on page 983]*.*
- *Rank* 10*,* FPdim 36*, type* $[1,1,1,2,2,2,2,2,2,3]$*:*
  - 3 *MD with $c=4$ from $SU(3)_3$, its complex conjugate and a zesting, see* [14, §6.3.1]*.*
- *Rank* 11*,* FPdim 32*, type* $[1,1,1,1,2,2,2,2,2,2,2]$*:*
  - 8 *MD with $c=\pm1$ from $SO(8)_2$, conjugates and zestings, see §3.3.*

There are 64 *MD of pointed modular fusion categories up to rank* 12*; here are their number per group G:*

G	C_1	C_2	C_3	C_2^2	C_4	C_5	C_6	C_7	C_2^3	$C_2 \times C_4$	C_8	C_3^2	C_9	C_{10}	C_{11}	$C_2 \times C_6$	C_{12}	C_{13}
$\#MD$	1	2	2	5	4	2	4	2	4	4	4	2	2	4	2	10	8	2

There is no other integral modular data up to rank 13 *(i.e. all categorifiable as above).*

Question 1.4. *Is there a modular data without categorification?*

It should be noted that [28] provides an interesting (non-integral) candidate of rank 11 (in its introduction) relevant to Question 1.4, and it also recovers Theorem 1.3 up to rank 12. This theorem yields the following consequence:

Corollary 1.5. *Every perfect integral modular fusion category up to rank* 13 *is trivial.*

In fact, we obtained the following more general result within the context of fusion rings up to rank 12:

Theorem 1.6. *Every perfect integral half-Frobenius fusion ring up to rank* 12 *is trivial.*

The proof of Theorem 1.6 for ranks up to 9 is straightforward, following the list provided in §4.1 combined with an extended version of the Nichols-Richmond theorem applied to fusion rings, as detailed in the proof of [32, Theorem 11]. This is due to the consistent presence of a non-trivial basic element with FPdim ≤ 2. However, proving the theorem for ranks up to 12 necessitates the employment of type criteria, as discussed in §5, and the use of a fusion ring solver, elaborated in §6.

It should be noted that the Drinfeld center of the representation category of any non-abelian finite simple group G—and, more broadly, any centerless perfect group—is a perfect (though not simple) integral modular fusion category denoted as $\mathcal{Z}(\mathrm{Rep}(G))$ with FPdim $= |G|^2$. For further information, see [10, §11.1]. Thus, the Grothendieck ring of $\mathcal{Z}(\mathrm{Rep}(A_5))$, of rank 22 and type $[[1,1],[3,2],[4,1],[5,1],[12,10],[15,4],[20,3]]$, constitutes a perfect integral half-Frobenius fusion ring. Consequently, Theorem 1.6 cannot be extended to all ranks; however, it remains an open question whether its simple version can be:

Question 1.7. *Is there a non-pointed simple integral half-Frobenius fusion ring?*

A negative response to Question 1.7 would imply a negative answer to the renowned [18, Question 2] in the simple case, due to a result in [24], which states that every simple integral fusion category is weakly group-theoretical if and only if every simple integral modular fusion category is pointed. With this in mind, we propose the following question:

Question 1.8. *Is there a non-pointed simple integral modular fusion category?*

For further insights into Question 1.8 at the fusion ring level, [33, Corollary 6.16] adds a constraint: the absence of any basic elements with a prime-power FPdim. It is worth noting that Theorem 1.6 cannot be generalized to all ranks, even with this added constraint. This is because the Grothendieck ring of $\mathcal{Z}(\mathrm{Rep}(A_7))$, which is a perfect integral half-Frobenius fusion ring of rank 74 and type

$$[[1,1],[6,1],[10,2],[14,2],[15,1],[21,1],[35,1],[70,9],[105,4],[210,20],[280,9],[360,14],[504,5],[630,4]],$$

satisfies this constraint (but consider Question 9.2). If necessary, Question 1.7 could be refined to include this constraint and the property of commutativity, and even the more advanced constraints mentioned above.

Employing similar techniques, along with [12, Remark 4.3] and [33, Corollary 6.16], we are able to prove the following (see §10):

Theorem 1.9. *The possibly categorifiable non-pointed odd-dimensional modular data at ranks below* 25 *reduce to:*

- *Rank* 17*,* FPdim 225*, type* $[1,1,1,3,3,3,3,3,3,3,3,5,5,5,5,5,5]$*:*
  - 3 *MD with central charge* $c = 4$ *(see §12 for the details) from* 2 *fusion rings.*

Remark 1.10. *A categorification of a MD mentioned in Theorem 1.9 would not align with* [2, Theorem 4.2, proof of Case (viii) $\mathrm{FPdim}(\mathcal{C}_{pt}) = p$] *as well as* [12, Theorem 6.3 (b), proof of Case $|\mathcal{G}(\mathcal{C})| = 3$]*. But Remark 10.3 points out gaps in these proofs. Should these gaps be filled, these MD would affirmatively address Question 1.4. Otherwise, consider the candidates discussed in Remark 10.6.*

Finally, this paper narrows down the possible rank 25 odd-dimensional perfect types to 4 ones, see Proposition 10.7.

Contents

2. Fusion Data and Modular Data

In this section, we review the concepts of fusion data and modular data, along with the essential results. For further details, we refer the reader to [17].

2.1. Fusion Data.

The concept of fusion data expands upon the idea of a finite group.

Definition 2.1. Fusion data *consist of a finite set* $\{1, 2, ..., r\}$ *with an involution* $i \mapsto i^*$*, and nonnegative integers* $N_{i,j}^k$ *satisfying the following conditions for all* i, j, k, t*:*

- *(Associativity)* $\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t$,
- *(Unit)* $N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$,
- *(Dual)* $N_{i^*,j}^1 = N_{j,i^*}^1 = \delta_{i,j}$,
- *(Anti-involution)* $N_{i,j}^k = N_{j^*,i^*}^{k^*}$.

Note that $1^* = 1$*. We may represent the fusion data simply as* $(N_{i,j}^k)$*.*

Proposition 2.2 (Frobenius Reciprocity)**.** *For all* i, j, k*,* $N_{i,j}^k = N_{k,j^*}^i = N_{k^*,i}^{j^*} = N_{j^*,i^*}^{k^*} = N_{j,k^*}^{i^*} = N_{i^*,k}^j$.

Proof. Starting with (Associativity) and setting $t = 1$, we have $\sum_s N_{i,j}^s N_{s,k}^1 = \sum_s N_{j,k}^s N_{i,s}^1$. Applying (Dual), we get $\sum_s N_{i,j}^s \delta_{s,k^*} = \sum_s N_{j,k}^s \delta_{s,i^*}$. Consequently, $N_{i,j}^{k^*} = N_{j,k}^{i^*}$. Substituting k^* with k, we obtain $N_{i,j}^k = N_{j,k^*}^{i^*}$, which equals N_{k,j^*}^i by (Anti-involution). The proposition follows by iterating the equality $N_{i,j}^k = N_{k,j^*}^i$. □

Remark 2.3. *We can construct data that satisfy the first three axioms of Definition 2.1 but not the fourth, proving it is not superfluous. However, (Unit) is redundant when combined with the other axioms, as it is not utilized in the proof of Proposition 2.2. Taken together, (Dual) and (Frobenius Reciprocity) trivially imply (Unit).*

A *fusion ring* $\mathcal{R}$ is a free $\mathbb{Z}$-module equipped with a finite basis $\mathcal{B} = \{b_1, \ldots, b_r\}$ and a fusion product defined by

$$b_i b_j = \sum_k N_{i,j}^k b_k,$$

where $(N_{i,j}^k)$ constitutes fusion data, and a $*$-structure given by $b_i^* := b_{i^*}$. The four axioms for fusion data translate to the following for all i, j, k:

- $(b_i b_j) b_k = b_i (b_j b_k)$,
- $b_1 b_i = b_i b_1 = b_i$,
- $\tau(b_i b_j^*) = \delta_{i,j}$,
- $(b_i b_j)^* = b_j^* b_i^*$,

where $\tau(x)$ is the coefficient of b_1 in the decomposition of $x \in \mathcal{R}$. Consequently, $\mathcal{R}_{\mathbb{C}} := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{C}$ becomes a finite-dimensional unital $*$-algebra, with τ extending linearly to a trace (i.e., $\tau(xy) = \tau(yx)$) and an inner product defined by $\langle x, y \rangle := \tau(xy^*)$. Here, $\langle x, b_i \rangle$ is the coefficient of b_i in the decomposition of x.

Theorem 2.4 (Frobenius-Perron Theorem)**.** *Given a fusion ring $\mathcal{R}$ with basis $\mathcal{B}$ and the corresponding finite-dimensional unital $*$-algebra $\mathcal{R}_{\mathbb{C}}$ as defined above, there exists a unique $*$-homomorphism $d : \mathcal{R}_{\mathbb{C}} \to \mathbb{C}$ such that $d(\mathcal{B}) \subset \mathbb{R}_{>0}$.*

The value $d(b_i)$ is termed the *Frobenius-Perron dimension* of b_i, denoted as $\mathrm{FPdim}(b_i)$ or simply d_i. The sum $\sum_i d_i^2$ is referred to as the Frobenius-Perron dimension of $\mathcal{R}$, denoted $\mathrm{FPdim}(\mathcal{R})$. The sequence $[d_1, d_2, \ldots, d_r]$ is called the *type* of $\mathcal{R}$. A fusion ring $\mathcal{R}$ is described as:

- *Frobenius* (or 1-Frobenius, or of Frobenius type) if $\frac{\mathrm{FPdim}(\mathcal{R})}{\mathrm{FPdim}(b_i)}$ is an algebraic integer for all i,
- *integral* if $\mathrm{FPdim}(b_i)$ is an integer for all i,
- *pointed* if $\mathrm{FPdim}(b_i) = 1$ for all i,
- *commutative* if $b_i b_j = b_j b_i$ for all i, j, meaning $N_{i,j}^k = N_{j,i}^k$.

The *multiplicity* of $\mathcal{R}$ is the maximum value among $N_{i,j}^k$, and its *rank* is r, the size of the basis.

Remark 2.5. *Fusion data enable a representation of its corresponding fusion ring. Consider the matrices $M_i = (N_{i,j}^k)_{k,j}$. By the Associativity axiom in Definition 2.1, we verify that $M_i M_j = \sum_k N_{i,j}^k M_k$. Additionally, M_1 is the identity matrix, and Frobenius Reciprocity ensures that the adjoint matrix M_i^* is M_{i^*}. According to Frobenius-Perron Theorem, the operator norm $\|M_i\|$ equals* $\mathrm{FPdim}(b_i)$.

Remark 2.6. *The concept of fusion data is a combinatorial reformulation of the fusion ring notion, so any property applicable to a fusion ring is also applicable to its fusion data.*

Definition 2.7 (Eigentable)**.** *Given commutative fusion data $(N_{i,j}^k)$, consider the corresponding fusion matrices $M_i = (N_{i,j}^k)_{k,j}$. The commutativity and the property that $M_i^* = M_{i^*}$ render these matrices normal and thus simultaneously diagonalizable. Let (D_i) denote their simultaneous diagonalization, where $D_i = \mathrm{diag}(\lambda_{i,j})$. We can select $\lambda_{i,1} = \|M_i\| = d_i$. The matrix $(\lambda_{i,j})$ is termed the* eigentable *(or character table) of the fusion data, and the values $c_j := \sum_i |\lambda_{i,j}|^2$ are called the* formal codegrees*.*

Lemma 2.8. *Let $M \in M_n(\mathbb{Z}_{\geq 0})$. The matrix M is a permutation matrix if and only if $\|M\| = 1$.*

Proof. Consider an orthonormal basis $\{e_1, \ldots, e_n\}$ for which the entries of M are non-negative integers. If M is not a permutation matrix, then one of the following cases must occur:

(0) there exists i for which $Me_i = 0$,
(1) there exist i, j such that $\langle Me_i, e_j \rangle > 1$,
(2) there exist i, j, k with $j \neq k$, such that $\langle Me_i, e_j \rangle = \langle Me_i, e_k \rangle = 1$,
(3) there exist i, j, k with $i \neq j$, such that $Me_i = Me_j = e_k$.

However, case (0) implies $\|Me_i\|/\|e_i\| = 0$, while case (1) leads to $\|Me_i\|/\|e_i\| > 1$. In case (2), it follows that $\|Me_i\|/\|e_i\| \geq \sqrt{2}$. Likewise, case (3) implies $\|M(e_i + e_j)\|/\|e_i + e_j\| = \sqrt{2}$. Each of these cases indicates that $\|M\| > 1$. Conversely, if M is a permutation matrix, it trivially follows that $\|M\| = 1$. □

Corollary 2.9. *For two basic elements x, y of a fusion ring with* $\mathrm{FPdim}(x) = 1$*, both xy and yx are basic elements, and* $\mathrm{FPdim}(xy) = \mathrm{FPdim}(yx) = \mathrm{FPdim}(y)$*.*

Proof. This follows directly from Remark 2.5, Lemma 2.8, and the fact that FPdim is a ring homomorphism. □

Corollary 2.10. *A fusion ring is pointed if and only if its basis forms a finite group under the fusion product.*

2.2. **Modular Data.** Broadly speaking, modular data refers to a fusion data together with two matrices, S and $T = (t_{i,j})$, that generate a projective representation of the modular group $\mathrm{SL}(2, \mathbb{Z})$. To provide a more detailed description, we draw upon [27, Theorem 2.1] and [17, §8.13, §8.18]. Let $\mathbf{i}$ be the imaginary unit.

Definition 2.11. *Given a fusion ring $\mathcal{R}$ of rank r, type $[d_1, \ldots, d_r]$, and fusion data $(N_{i,j}^k)$, let $\mathsf{d} := \mathrm{FPdim}(\mathcal{R})$ and $\zeta_n := \exp(2\pi\mathbf{i}/n)$. A (pseudounitary)* modular data *for $\mathcal{R}$ consists of two matrices $S, T \in M_r(\mathbb{C})$ satisfying:*

- *S and T are symmetric, T is unitary and diagonal with $T_{1,1}=1$, $S_{1,i}=d_i$ for all i, and $SS^*=\mathsf{d}\mathbf{1}$.*
- *Verlinde formula: $N_{i,j}^k = \frac{1}{\mathsf{d}}\sum_l \frac{S_{li}S_{lj}\overline{S_{lk}}}{d_l}$.*
- *Twist: let θ_i be $T_{i,i}$, then $\sum_k N_{i,j}^k\theta_k d_k = \theta_i\theta_j S_{i,j}$.*
- *Ribbon structure: $\theta_i = \theta_{i^*}$ (see Remark 2.13).*
- *Central charge: $p_\pm := \sum_{i=1}^r d_i^2(\theta_i)^{\pm 1}$. The ratio p_+/p_- is a root of unity, and $p_+ = \sqrt{\mathsf{d}}\zeta_8^c$ for some rational number c, referred to as the **central charge**, determined modulo 8.*
- *The matrices S and T afford a projective representation of $\mathrm{SL}(2,\mathbb{Z})$: we have $(ST)^3 = p_+S^2$, $\frac{S^2}{\mathsf{d}} = C$, $C^2=\mathbf{1}$, where C is the permutation matrix associated with the involution $i\to i^*$ and satisfies $\mathrm{Tr}(C)>0$.*
- *Cauchy theorem: the set of distinct prime factors of $\mathrm{ord}(T)$ is identical to the distinct prime factors of $\mathrm{norm}(\mathsf{d})$, where $\mathrm{norm}(x)$ denotes the product of the distinct Galois conjugates of the algebraic number x.*
- *Cyclotomic integers: for all i,j, the elements $S_{i,j}$, $S_{i,j}/d_j$ and $T_{i,i}$ are cyclotomic integers. The conductor of $S_{i,j}$ divides $\mathrm{ord}(T)$, which in turn divides $\mathsf{d}^{5/2}$, and there exists j such that $S_{i,j}/d_j \in \mathbb{R}_{\geq 1}$, for all i.*
- *Frobenius-Schur indicators: for every i and for all $n\geq 1$, the sum $\nu_n(i) := \frac{1}{\mathsf{d}}\sum_{j,k} N_{j,k}^i (d_j\theta_j^n)\overline{(d_k\theta_k^n)}$ is a cyclotomic integer with a conductor that divides both n and $\mathrm{ord}(T)$. Additionally, $\nu_1(i)=\delta_{i,1}$ and $\nu_2(i) = \pm\delta_{i,i^*}$.*
- *Anderson-Moore-Vafa equations: $T_{i,i} = e^{2\pi \mathbf{i} t_i}$, and $\forall i,j,k,l$, the following equation holds in the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$:*
$$\left(\sum_{p=1}^r N_{i,j}^p N_{p,k}^l\right)(t_i+t_j+t_k+t_l) = \sum_{p=1}^r \left(N_{i,j}^pN_{p,k}^l + N_{i,k}^pN_{j,p}^l + N_{j,k}^pN_{i,p}^l\right)t_p.$$
*The **topological spin** of the i-th basic element is the representative $s_i\in(-1/2,1/2]$ of $t_i\in\mathbb{Q}/\mathbb{Z}$.*

We could question the necessity of each component in Definition 2.11, particularly whether the Anderson-Moore-Vafa equations can be inferred from the other assumptions.

Remark 2.12. *The Verlinde formula, in conjunction with results from* [24, §2]*, implies that the fusion ring $\mathcal{R}$ is commutative. Together with S symmetric and the identity $SS^* = \mathrm{FPdim}(\mathcal{R})\mathbf{1}$, it can be deduced that $\mathcal{R}$ is self-transposable (as discussed in §3.1). Moreover, according to the proof presented in* [17, Proposition 8.14.6]*, $\mathcal{R}$ is also half-Frobenius.*

Remark 2.13. *A modular tensor category $\mathcal{C}$ possesses a ribbon structure, which means that the twist $\theta\in\mathrm{Aut}(\mathrm{id}_\mathcal{C})$ satisfies the condition $(\theta_X)^* = \theta_{X^*}$ for every object X within $\mathcal{C}$. Let (X_i) represent the set of simple objects (up to isomorphism) within $\mathcal{C}$. Schur's Lemma guarantees that $\theta_{X_i} = \theta_i\mathrm{id}_{X_i}$, where the scalar θ_i is consistent with the one described in Definition 2.11. Owing to the ribbon structure, we deduce the following:*
$$\theta_{i^*}\mathrm{id}_{X_{i^*}} = \theta_{X_{i^*}} = (\theta_{X_i})^* = (\theta_i\mathrm{id}_{X_i})^* = \theta_i(\mathrm{id}_{X_i})^* = \theta_i\mathrm{id}_{X_{i^*}}.$$
From this, it follows that $\theta_{i^*}=\theta_i$ for all simple objects X_i.

This paper primarily addresses integral fusion categories, implying that d is an integer and $\mathrm{norm}(\mathsf{d})=\mathsf{d}$. Such categories are pseudounitary and, consequently, spherical as well as pivotal (see [17]). In contexts that are not pseudounitary, Definition 2.11 would require modifications (as suggested in [27, Theorem 2.1]) because the equality $S_{1,i} = \mathrm{FPdim}(b_i)$ may not be valid.

It should be noted that the definition of modular data provided here is so stringent that, as of now, no instances exist that lack a categorification, leading to Question 1.4.

3. From Fusion Data to Modular Data

This section elucidates the classification of all potential modular data associated with a given set of fusion data. Initially, we may consider the fusion data to be commutative and half-Frobenius (refer to Remark 2.12). For ranks up to 12, there are exactly 10628 half-Frobenius integral fusion rings, of which 213 are noncommutative. The details by rank are provided in §7. About the rank 13, there is nothing non-pointed, see §9.

3.1. **S-matrix.** Consider a commutative fusion data $(N_{i,j}^k)$ of rank r, eigentable $(\lambda_{i,j})$, and formal codegrees (c_j) as defined in Definition 2.7. The objective here is to identify all permutations q of the set $\{1,\dots,r\}$ such that:

- $q(1)=1$,
- $d_{q(i)} = d_i$ for all i,
- The matrix $S = (\sqrt{c_1/c_j}\lambda_{i,q(j)})$ is symmetric (i.e. self-transpose).

Remark 3.1. *The symmetric requirement implies that*
$$\sqrt{c_1/c_j} = \sqrt{c_1/c_j}\lambda_{1,q(j)} = \sqrt{c_1/c_1}\lambda_{j,q(1)} = d_j,$$
hence we can infer that $c_1/c_j = d_j^2$ for all j, as shown in [35, Example 2.9].

If such a permutation q exists (Remark 3.1 can serve as an effective necessary condition), the fusion data are referred to as *self-transposable*. This property is exceedingly rare, rendering this step a potent sieve. Using the Verlinde formula, one can reconstruct the fusion data from S. It is important to note that we need only consider *cyclotomic* fusion data, i.e. whose eigentable entries are all cyclotomic. Incorporating the self-transposable and cyclotomic prerequisites allows us to exclude over 99.3% of the commutative half-Frobenius integral fusion rings up to rank 12 discovered in §7. This leaves 69 fusion rings; their distribution by type and rank is as follows:

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Types	1	1	1	1	1	1	2	2	2	4	5	6
#Fusion Rings	1	1	1	2	1	1	3	7	4	11	13	24

The types mentioned above, restricted to the non-pointed ones, are listed below:

- Rank 7: [1,1,1,1,2,2,2],
- Rank 8: [1,1,2,2,2,2,3,3],
- Rank 9: [1,1,1,1,4,4,6,6,6]
- Rank 10:[1,1,1,1,2,2,2,4,4,4],[1,1,1,2,2,2,2,2,2,3],[1,1,2,3,3,4,4,4,6,6],
- Rank 11:[1,1,1,1,2,2,2,2,2,2,2],[1,1,1,1,2,6,6,8,12,12,12],[1,1,1,3,4,4,4,4,4,4,6],[1,1,1,1,4,4,12,12,18,18,18],
- Rank 12:[1,1,1,1,2,8,18,18,24,36,36,36],[1,1,1,3,6,8,8,8,8,8,8,12], [1,1,2,2,2,2,6,6,6,6,9,9], [1,1,2,3,3,6,6,8,8,8,12,12], [1,1,2,6,6,6,6,10,10,10,15,15].

The list of fusion rings referenced above can be found in [38]. They were classified utilizing the list from §7 in conjunction with the function `preSmatrix` contained within the file `ModularData.sage`, also available at [38].

3.2. **T-matrix.** For the remaining fusion rings $\mathcal{R}$ with fusion data $(N_{i,j}^k)$, we address the Anderson-Moore-Vafa equations:

$$\left(\sum_{p=1}^{r} N_{i,j}^p N_{p,k}^l\right)(t_i+t_j+t_k+t_l) = \sum_{p=1}^{r}\left(N_{i,j}^p N_{p,k}^l + N_{i,k}^p N_{j,p}^l + N_{j,k}^p N_{i,p}^l\right) t_p$$

within the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. For each valid solution $t=(t_i)\in(\mathbb{Q}/\mathbb{Z})^r$, if any, the corresponding T-matrix is $\mathrm{diag}(e^{2\pi \mathbf{i} t_i})$.

The solutions to the aforementioned equations are determined using the following method: Initially, the matrix reformulation is represented as $At=0$, where A is an $m\times n$ matrix over $\mathbb{Z}$ with $m=r^4$ and $n=r$. Subsequently, the Smith normal form is employed, denoted as $D=UAV$, in which U and V are invertible matrices over $\mathbb{Z}$ of sizes $m\times m$ and $n\times n$, respectively, and D is a diagonal $m\times n$ matrix $(\alpha_i\delta_{i,j})$, where the integer α_i is divisible by α_{i+1} for all $i<r$, and $\delta_{i,j}$ is the Kronecker delta. The solutions to $Dx=0$ are precisely represented by the vectors (k_i/α_i), where $0\le k_i<\alpha_i$. Consequently, we have $U^{-1}DV^{-1}t=0$, which simplifies to $DV^{-1}t=0$. Therefore, the solutions can be expressed as $t=Vx$.

A complete listing of potential T-matrices requires considering all vectors (k_i/α_i), where $0\le k_i<\alpha_i$. As a result, there are $p=\prod_i \alpha_i$ possible combinations. This task remains manageable up to rank 11. However, in certain rank 12 cases, the value of p becomes too large. But a miraculous circumstance arises (referred to as the **magic criterion**): for all such cases, if one abstractly considers the T-matrix with variables (k_i), then for every determined S-matrix S in §3.1, the abstract product $(ST)^3$ consistently exhibits a zero where it should not, specifically at an entry (i,i^*) for some i. This is because $(ST)^3/p_+ = S^2 = \mathtt{d}C$, where C is the duality matrix (realizing the involution $i\to i^*$, and thus $C_{i,i^*}=1$, non-zero), as defined in Definition 2.11. The function `MagicCriterion` (also covered by the function `STmatrix`) can verify this. It effectively excludes exactly 33 fusion rings (8 distinct types) out of the 52 non-pointed ones (15 distinct types) resulting from §3.1. The remaining 7 types are [1,1,1,1,2,2,2], [1,1,2,2,2,2,3,3], [1,1,1,1,2,2,2,4,4,4], [1,1,1,2,2,2,2,2,2,3], [1,1,1,1,2,2,2,2,2,2,2], [1,1,2,2,2,2,6,6,6,6,9,9], [1,1,2,6,6,6,6,10,10,10,15,15]. For these types, some (but not all) of their possible S-matrices can still be excluded by this method.

Question 3.2. *Can the aforementioned* magic criterion *be reformulated at the level of fusion data?*

In conclusion, we retain only those S- and T-matrices that meet all criteria outlined in Definition 2.11. This process results in $19+64$ distinct modular data originating from $5+18$ fusion rings of $3+13$ types (non-pointed + pointed), up to rank 13, thereby substantiating Theorem 1.3. This classification was executed by applying the function `STmatrix` or `STmatrix2` to the list of fusion rings discussed in §3.1, all of which can be found at [38]. When considering isomorphism classes, it is appropriate to adopt a *normal form* by sorting the basic elements according to their FPdim *and* spin, and limit basis permutations to those preserving both.

3.3. **Model by Zesting.** This subsection seeks to model certain modular data indicated in Theorem 1.3, predominantly through the process of zesting as delineated in [14]. The ensuing proposition is attributed to Eric C. Rowell.

Proposition 3.3. *The eight modular data delineated in §11.1.5 are derived from $SO(8)_2$, its conjugates, and zestings.*

Proof sketch. Commencing with $SO(8)_2$, one identifies that it is graded by the group $G = C_2 \times C_2$. This enables to twist the braiding by a bicharacter: the braiding is altered to $B(\deg(X), \deg(Y))c_{X,Y}$, where B represents the bicharacter. Correspondingly, the twists must be adjusted. This action exemplifies a specialized instance of braided (or ribbon) zesting. The resultant effect is the multiplication of specific rows and columns of the S-matrix by a sign. Upon inspecting the S-matrices itemized in §11.1.5, the rationale behind these variations should become apparent. Complex conjugation preserves the S-matrix while altering the T-matrix, thus providing a comprehensive explanation (notably, complex conjugation modifies the underlying fusion category). □

4. Egyptian Fractions with Squared Denominators

A (q, r)-Egyptian fraction with squared denominators is defined as a sum of the form:

$$q = \sum_{i=1}^{r} \frac{1}{s_i^2},$$

where $q, r, s_i \in \mathbb{Z}_{\geq 1}$ and the sequence satisfies $s_1 \geq s_2 \geq \cdots \geq s_r \geq 1$. Additionally, in the context of classifying potential types of Grothendieck rings for modular integral fusion categories (or more broadly, half-Frobenius integral fusion rings), we can assume that each s_i is a divisor of s_1 for all i. By repeatedly subtracting 1 from both q and r as necessary, we can further assume that $s_i \geq 2$ for all i (and so q is no more assumed square-free). Subsequently, we can augment the list of (q, r)-Egyptian fractions with squared denominators by including the $(q - k, r - k)$ variations, which are achieved by adding the number 1 to the sum k times. Using this technique, we can assume that $q \leq r/4$.

The following steps outline our methodology:

- Employ the function `ModularRep` provided in §4.2 for $1 \leq r \leq 13$ and $1 \leq q \leq r/4$.
- Refine the classification by incorporating additional 1s as described previously.
- Construct all possible types using $d_i = s_1/s_i$. The resulting list is presented in §4.1.

4.1. **List of Types Up to Rank 13.**

Below is the count of possible types up to rank 13, based solely on Egyptian fractions with squared denominators. It also includes the count of perfect types (refer to §5):

Rank	1	2	3	4	5	6	7	8	9	10	11	12	13
#Types	1	1	1	1	2	3	3	7	11	42	144	812	7997
#Perfect Types	1	0	0	0	0	1	1	2	2	24	88	591	6517

The ratio of perfect types exhibits an increasing trend, e.g. 18% for rank 9, but 81% for rank 13. This leads us to question whether this ratio tends to 1 as the rank goes to infinity.

The list of all such (non-pointed) types up to rank 10 is provided (those up to rank 13 can be found online at [38]):

- Rank 5: [[1, 1, 1, 1, 2]],
- Rank 6: [[1, 1, 1, 1, 2, 2], [1, 2, 2, 3, 3, 3]],
- Rank 7: [[1, 1, 1, 1, 2, 2, 2], [1, 2, 2, 3, 3, 3, 6]],
- Rank 8: [[1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4], [1, 1, 2, 2, 2, 2, 3, 3], [1, 1, 3, 3, 4, 6, 6, 6], [1, 2, 2, 3, 3, 3, 6, 6], [1, 2, 2, 6, 6, 9, 9, 9]],
- Rank 9: [[1, 1, 1, 1, 1, 1, 1, 1, 2], [1, 1, 1, 1, 1, 2, 3, 3, 3], [1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6], [1, 1, 3, 3, 4, 6, 6, 6, 12], [1, 1, 4, 9, 9, 12, 18, 18, 18], [1, 2, 2, 3, 3, 3, 6, 6, 6], [1, 2, 2, 6, 6, 9, 9, 9, 18]],
- Rank 10: [[1, 1, 1, 1, 1, 1, 1, 1, 1, 3], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2], [1, 1, 1, 1, 1, 2, 3, 3, 3, 6], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [1, 1, 1, 2, 2, 3, 4, 6, 6, 6], [1, 1, 1, 2, 3, 8, 8, 12, 12, 12], [1, 1, 1, 7, 12, 28, 28, 42, 42, 42], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6], [1, 1, 3, 3, 4, 6, 6, 6, 12, 12], [1, 1, 3, 3, 4, 12, 12, 18, 18, 18], [1, 1, 4, 4, 4, 5, 5, 10, 10, 10], [1, 1, 4, 9, 9, 12, 18, 18, 18, 36], [1, 1, 4, 12, 27, 27, 36, 54, 54, 54], [1, 2, 2, 2, 2, 2, 2, 5, 5, 5], [1, 2, 2, 2, 6, 14, 14, 21, 21, 21], [1, 2, 2, 3, 3, 3, 3, 3, 3, 3], [1, 2, 2, 3, 3, 3, 6, 6, 6, 6], [1, 2, 2, 3, 3, 3, 6, 6, 6, 12], [1, 2, 2, 3, 6, 6, 6, 6, 9, 9], [1, 2, 2, 3, 9, 9, 12, 18, 18, 18], [1, 2, 2, 4, 5, 5, 5, 10, 10, 10], [1, 2, 2, 6, 6, 9, 9, 9, 18, 18], [1, 2, 2, 6, 6, 18, 18, 27, 27, 27], [1, 2, 3, 6, 10, 10, 10, 10, 15, 15], [1, 2, 3, 6, 15, 15, 20, 30, 30, 30], [1, 3, 3, 3, 4, 4, 4, 4, 4, 6], [1, 3, 3, 3, 4, 6, 8, 12, 12, 12], [1, 3, 3, 3, 6, 16, 16, 24, 24, 24], [1, 4, 4, 4, 7, 7, 7, 14, 14, 14], [1, 4, 9, 28, 63, 63, 84, 126, 126, 126], [1, 5, 7, 35, 60, 140, 140, 210, 210, 210], [1, 5, 10, 18, 30, 30, 30, 30, 45, 45], [1, 5, 10, 18, 45, 45, 60, 90, 90, 90], [1, 6, 18, 38, 38, 114, 114, 171, 171, 171], [1, 12, 12, 17, 51, 51, 68, 102, 102, 102], [1, 18, 30, 70, 70, 210, 210, 315, 315, 315], [1, 70, 130, 182, 390, 910, 910, 1365, 1365, 1365]].

4.2. **SageMath Code.**

```
def ModularRep(q,r):
    L=all_rep(q, r)
    P=[]
    for l in L:
        if l[0]!=1: # those starting with 1 should be considered with q-1.
            k=0
            for ll in l:
                if l[-1]%ll!=0:
                    k=1
                    break
            if k==0:
                lll=[l[-1]/ll for ll in l]
```

```
                lll.sort()
                Di=sum([i^2 for i in lll])
                P.append(lll+[[sqrt(Di)]])
    return P

def res_rep(s, N):
    def succ(t):
        s0, m = t
        if s0==0 or len(m)>=N:
            return []
        p = numerator(s0)
        q = denominator(s0)
        if len(m)==N-1:
            if p==1 and is_square(q):
                r = q.isqrt()
                if r>=m[-1]:
                    return [(0,m+(r,))]
            return []
        L = max(m[-1], ((q-1)//p).isqrt()+1)
        U = floor((N-len(m))/s0).isqrt()
        if len(m)==N-2:
            S = []
            try:
                two_squares(p)
                two_squares(q)
            except:
                return S
            q2 = q^2
            for r in (L..U):
                d = p*r^2-q
                if d>0 and q2%d==0:
                    r2 = (q2//d + q)//p
                    if is_square(r2):
                        S.append( (0,m+(r,r2.isqrt())) )
            return S
        if len(m)==N-3:
            t = p*q
            a = valuation(t,2)
            if a%2==0 and (t>>a)%8==7:
                return []
        return ( (s0-1/r^2, m+(r,)) for r in (L..U) )
    return RecursivelyEnumeratedSet(seeds=[(s-1/r^2,(r,)) for r in range(1,floor(N/s).isqrt()+1)], \
    successors=succ, structure='forest')

def all_rep(s, N):
    return res_rep(s,N).map_reduce(lambda t: {t[1]} if t[0]==0 and len(t[1])==N else set(), set.union, \
    set() )

def count_rep(s, N):
    return res_rep(s,N).map_reduce(lambda t: int(t[0]==0 and len(t[1])==N))
```

5. Type Criteria

In this section, we delineate criteria that were employed to exclude certain candidates from being the type of a fusion ring. A *type* refers to a list denoted by $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$, where the conditions $1 = d_1 < d_2 < \cdots < d_s$ and $m_i \geq 1$ for all indices i are satisfied. Such a type is characterized as:

- *trivial* if $t = [[1, 1]]$,
- *pointed* if $t = [[1, m]]$ for some m,

- *perfect* if $m_1 = 1$,
- *integral* if each d_i is an integer.

A type $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$ may sometimes be represented simply as

$$[d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_s, \dots, d_s],$$

where each d_i appears m_i times. Thus, we can rephrase the notation for a type of rank r as $[d_1, \dots, d_r]$ with the condition $1 = d_1 \leq d_2 \leq \cdots \leq d_r$.

The criteria described herein are proved using modular arithmetic, and arranged in order of increasing computational complexity. They permit to exclude about 62% of the types presented in §4. Here is their counting:

Rank	1	2	3	4	5	6	7	8	9	10	11	12	13
# Types	1	1	1	1	2	3	3	7	11	42	144	812	7997
# Excluded Types	0	0	0	0	0	1	1	3	5	26	85	520	4970

For the remaining types, we will utilize the fusion ring solver, as elaborated in §6.

5.1. **Small Perfect Type.**

Theorem 5.1. *A perfect integral fusion ring of the type $[[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$, with $s \leq 3$, is trivial.*

Proof. If $s = 1$, then the type $t = [[1, 1]]$ is trivial. If $s = 2$, then we have a type $t = [[1, 1], [d, n]]$ with $d > 1$ and $n \geq 1$. However, should a fusion ring of type t exist, let b be a basic element with $\mathrm{FPdim}(b) = d$. When applying FPdim to the decomposition of bb^*, we obtain that $d^2 = 1 + kd$ for some integer $k \geq 1$. Reducing this equation modulo d yields $0 = 1 \mod d$, which is contradictory to $d > 1$. Lastly, if $s = 3$, then the type is $t = [[1, 1], [a, m], [b, n]]$, with $1 < a < b$ and $m, n \geq 1$. Suppose $\mathcal{R}$ is a fusion ring of this type, with basic elements $1, a_1, \dots, a_m, b_1, \dots, b_n$.

Claim 5.2. *The greatest common divisor of a and b, denoted $a \wedge b$, is 1.*

Proof. Let $d = a \wedge b$. Then $\mathrm{FPdim}(a_i a_i^*) = a^2 = 1 + \alpha a + \beta b$, but since d divides both a and b, we have $0 = 1 \mod d$, which implies $d = 1$. □

Claim 5.3. *For every i, there exists j such that $\langle a_i a_i^*, b_j \rangle \neq 0$.*

Proof. If this were not the case, then $a^2 = 1 + \alpha a$, leading to $1 = 0 \mod a$, which contradicts the fact that $a > 1$. □

Claim 5.4. *If $k \neq i^*$, then $\langle a_i a_k, b_j \rangle = 0$.*

Proof. If the claim were false, then $a^2 = \alpha a + \beta b$ with $\beta \neq 0$, which would mean that $\beta b = 0 \mod a$. However, since $a \wedge b = 1$ (indicating that b is invertible modulo a), we get $\beta = b^{-1} \times 0 = 0 \mod a$. Therefore, $\beta = ka$ for some $k \geq 1$. Now, since $a^2 = \alpha a + \beta b \geq \beta b = kab \geq ab$, we would have $a^2 \geq ab$, which contradicts the fact that $b > a$. □

Claim 5.5. $a_{i^*} b_j = b a_{i^*}$.

Proof. By Frobenius reciprocity and Claim 5.4, if $k \neq i^*$ then $\langle a_{i^*} b_j, a_k \rangle = 0$. Claim 5.3 ensures that $\langle a_{i^*} b_j, a_{i^*} \rangle \neq 0$. We know $\mathrm{FPdim}(a_{i^*} b_j) = ab = \alpha a + \beta b$, with $\alpha \geq 1$, leading to the conclusion that $\beta = 0 \mod a$. Hence, $\beta = ka$ for some $k \geq 0$. As a result, $ab = \alpha a + kab$, which simplifies to $(1 - k)ab = \alpha a > 0$. This implies $(1 - k) > 0$ and thus $k < 1$. Therefore, $k = 0$ and $\beta = 0$. Combining the initial part of this proof with $\beta = 0$ indicates that $a_{i^*} b_j = \alpha a_{i^*}$, where α must equal b (determined by applying FPdim). □

Claim 5.5, together with Frobenius reciprocity, leads us to deduce that $\langle a_i a_{i^*}, b_j \rangle = b$, which means that $a^2 \geq b^2$. This is in contradiction with $a < b$. □

Remark 5.6. *Theorem 5.1 is not extendable to $s = 4$ because the representation category of the alternating group A_5, denoted $\mathrm{Rep}(A_5)$, is of type $[[1, 1], [3, 2], [4, 1], [5, 1]]$.*

By applying Theorem 5.1 to the list presented in §4, we can exclude the following four types (up to rank 13): $[[1, 1], [2, 2], [3, 3]]$, $[[1, 1], [2, 6], [5, 3]]$, $[[1, 1], [2, 2], [3, 7]]$, $[[1, 1], [3, 7], [4, 5]]$.

Corollary 5.7. *A non-trivial perfect integral fusion ring has a rank of at least 4.*

Proof. Suppose there is a perfect integral fusion ring with a rank less than 4. Then its type would be $[[d_1, m_1], \dots, [d_s, m_s]]$ with $s \leq r = \sum_i m_i \leq 3$, which contradicts Theorem 5.1. □

5.2. **Gcd Criterion.**

Lemma 5.8. *Consider a non-pointed fusion ring of type* $[d_1, d_2, \dots, d_r]$. *For all* i *such that* $d_i > 1$, *let* Z_i *be the set of indices* $j \neq 1$ *for which* N_{i,i^*}^j *is nonzero, and let* g_i *be* $\gcd(d_j \mid j \in Z_i)$. *Then it holds that* $d_i^2 \equiv 1 \pmod{g_i}$ *and* $\gcd(d_i, g_i) = 1$.

Proof. First, note that Z_i is non-empty, which implies that $g_i \neq 0$. According to the Frobenius-Perron theorem, the dimension equation, and the Dual axiom, we have

$$d_i^2 = d_i d_{i^*} = \sum_k d_k N_{i,i^*}^k = 1 + \sum_{j \in Z_i} d_j N_{i,i^*}^j = 1 + K g_i,$$

where K is some integer. Consequently, $d_i^2 \equiv 1 \pmod{g_i}$, and $0 \equiv 1 \pmod{\gcd(d_i, g_i)}$. The lemma follows. □

Proposition 5.9. *Consider a non-trivial perfect fusion ring of type* $[d_1, d_2, \dots, d_r]$. *Take* $i > 1$, *let* Z'_i *be the set of indices* $j \neq 1$ *for which* $d_j < d_i^2$, *and let* g'_i *be* $\gcd(d_j \mid j \in Z'_i)$. *Then* $g'_i = 1$. *In particular,* $\gcd(d_2, \dots, d_r) = 1$.

Proof. Note that if N_{i,i^*}^j is nonzero, then $d_i^2 \geq d_j$. Hence, following the notation in Lemma 5.8, Z_i is included in Z'_i, and as a result, g'_i divides g_i. Due to perfectness, we have $d_i > 1$, implying $d_i^2 > d_i$ and therefore i belongs to Z'_i. Consequently, g'_i divides d_i. However, according to Lemma 5.8, $g'_i = 1$. For the final assertion, note that $\gcd(d_2, \dots, d_r)$ is a divisor of $g'_2 = 1$. □

Note that Proposition 5.9 excludes more than 37% of the perfect types listed in §4, for example, $[1, 2, 2, 6, 6, 9, 9, 9]$. Here is the count per rank:

Rank	8	9	10	11	12	13
# Excluded Perfect Types	1	1	7	19	212	2474

5.3. **Type Test.** Let's consider a type $t = [d_1, \dots, d_r]$ with $1 = d_1 \leq \dots \leq d_r$ and $d_2 > 1$ (signifying that it is perfect). If there is an index i and $g_i > 1$ such that g_i divides every d_j not equal to 1 or d_i, and d_i is coprime with g_i, then assume a fusion ring of this type exists with a basis $\{b_1, \dots, b_r\}$ where $d_k = \mathrm{FPdim}(b_k)$.

Lemma 5.10. *For every* j *with* $d_j \neq 1$ *and* $d_j \neq d_i$, *the following equation holds:*

$$\sum_{k;\ d_k = d_i} N_{j,j^*}^k \equiv -1/d_i \mod g_i.$$

Proof. For each j with $d_j \neq 1$ and $d_j \neq d_i$, we have:

$$b_j b_{j^*} = b_1 + \sum_{k;\ d_k = d_i} N_{j,j^*}^k b_k + \sum_{k;\ d_k \neq 1, d_i} N_{j,j^*}^k b_k.$$

By applying FPdim and reducing modulo g_i, we obtain:

$$0 = 1 + x d_i \mod g_i,$$

where d_i has a multiplicative inverse modulo g_i. Therefore, $x \equiv -1/d_i \mod g_i$. □

Given an integer a_{d_i} such that $0 \leq a_{d_i} < g_i$ and $a_{d_i} \equiv -1/d_i \mod g_i$, let S be the set containing all such d_i. From Lemma 5.10, for every $j \neq 1$, the inequality below must hold:

$$d_j^2 \geq 1 + \sum_{d \in S \setminus \{d_j\}} a_d d,$$

thus if the inequality does not hold, t cannot be a type of a fusion ring. Furthermore, if the set $\{k \mid d_k = d_j\}$ is a singleton, we can use a stronger inequality:

$$d_j^2 \geq 1 + b_j d_j + \sum_{d \in S \setminus \{d_j\}} a_d d,$$

with $0 \leq b_j < g_j^2$ and $b_j \equiv d_j - \frac{1}{d_j} \mod g_j^2$.

The SageMath code implementing this criterion can be found in the function `TypeTest` within the file `TypeCriteria.sage`, available at [38]. This criterion helped to exclude a certain number of perfect types per rank in the list from §4, as shown in the table below:

Rank	8	9	10	11	12	13
#Excluded Perfect Types	1	1	12	37	249	2380

5.4. **Local Criterion.** Consider a type $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$. Assume the existence of $g, i_0 > 1$ such that g divides each d_i for all indices i not in the set $\{1, i_0\}$, and d_{i_0} is coprime with g. Let $(d, m) := (d_{i_0}, m_{i_0})$. If t corresponds to a fusion ring with a basis $\{b_{1-m_1}, \dots, b_0, b_1, \dots, b_{r-1}\}$, where b_0 is the unit, $\mathrm{FPdim}(b_i) = 1$ for $i \le 0$, and $\mathrm{FPdim}(b_j) = d$ for $j \in \{1, \dots, m\}$, then the following lemma applies:

Lemma 5.11. *For each $i \in \{1, \dots, m\}$, the equation below is valid:*

$$\sum_{j,k=1}^{m} N_{i,j}^k \equiv md - \frac{m_1}{d} \mod g^2,$$

and for all $j > m$, the integer g divides $\sum_{k=1}^m N_{i,j}^k$.

Proof. For any $i \in \{1, \dots, m\}$ and $j > m$, since $\mathrm{FPdim}(b_i) \neq \mathrm{FPdim}(b_j)$, by Corollary 2.9 and Frobenius reciprocity, we have:

$$b_i b_j = \sum_{k \ge 1} N_{i,j}^k b_k = \sum_{k=1}^{m} N_{i,j}^k b_k + \dots,$$

Applying FPdim and reducing modulo g, we conclude that:

$$d \sum_{k=1}^{m} N_{i,j}^k \equiv 0 \mod g,$$

which implies that g divides $\sum_{k=1}^m N_{i,j}^k$. For each $i \in \{1, \dots, m\}$, the sum over the basis elements yields:

$$b_{i^*} \sum_{k=1}^{m} b_k = \sum_{s \le 0} b_s + \sum_{j=1}^{m} \left(\sum_{k=1}^{m} N_{i,j}^k \right) b_j + \sum_{j > m} \left(\sum_{k=1}^{m} N_{i,j}^k \right) b_j.$$

After applying FPdim, we obtain $md^2 = m_1 + xd + yg^2$, hence $x \equiv md - \frac{m_1}{d} \mod g^2$. □

For a type t, we can analyze the partitions of $md^2 - xd - m_1$ in the form $\sum_{i \notin \{1, i_0\}} a_i d_i$, with $x \equiv md - \frac{1}{d} \mod g^2$ and $a_i \equiv 0 \mod g$. The SageMath code performing this analysis can be found in the function `LocalCriterion` within the file specified earlier, also available at [38]. This criterion, which can rule out types when no suitable partitions are found, is further detailed in the following example.

Example 5.12. *Consider the type $t = [[1, 1], [1295, 2], [3990, 1], [4218, 1], [24605, 1], [42180, 1], [98420, 2], [147630, 3]]$. We can apply Lemma 5.11 to the triples $(d, m, g) = (1295, 2, 19), (3990, 1, 37), (4218, 1, 5)$. Subsequently, we obtain $md - \frac{1}{d} \equiv 126, 1135, 11 \pmod{g^2}$ for each respective triple. The application of the function* `LocalCriterion` *to the triple $(d, m, g) = (1295, 2, 19)$ enables us to eliminate the type t in less than one second.*

```
sage: %time LocalCriterion(T, 1295, 2, 19)
CPU times: user 640 ms, sys: 0 ns, total: 640 ms
Wall time: 982 ms
[]
```

However, we cannot employ the triple $(d, m, g) = (3990, 1, 37)$, as it yields 55 solutions.

```
sage: L = LocalCriterion(T, 3990, 1, 37)
sage: len(L)
55
```

The application of LocalCriterion to the list in §4 led to the exclusion of several types per rank, as summarized in the following table:

Rank	6	7	8	9	10	11	12	13
# Excluded Types	1	1	3	5	21	63	344	2852
# Excluded Perfect Types	1	1	2	2	14	37	238	2173

It is noteworthy that this criterion alone suffices to eliminate all perfect types up to rank 9. Therefore, it can be stated conclusively that no non-trivial perfect integral half-Frobenius fusion rings, and thus no non-trivial perfect modular integral fusion categories, exist up to rank 9. The use of a fusion ring solver as detailed in §6 can extend this conclusion to rank 12, as discussed in §7.

6. Enhanced Fusion Ring Solver Using Normaliz

A fusion ring solver is a computational tool designed to receive a particular type as input and output all corresponding fusion rings of that type. In this section, we introduce two versions of a fusion ring solver: the full version, which is discussed in §6.3 and addresses both dimension equations and associativity equations, and the intermediate version, which is detailed in §6.4 and focuses on a simplified set of dimension equations through the implementation of a partition. The software employed in this process is Normaliz [8]. Initially, §6.1 provides a brief introduction to the highly intuitive user interface, and §6.2 offers an overview of Normaliz's goals and outlines the adjustments made to support the unique linear and polynomial constraints specific to fusion rings.

6.1. **Normaliz user interface for fusion rings.** Starting from version 3.10.2, Normaliz [8] offers a streamlined user interface for computing fusion rings. This is illustrated through the input file named `bracket_4.in`, found in the `example` directory of the Normaliz distribution:

```
amb_space auto
fusion_type
[1,1,2,3,3,6,6,8,8,8,12,12]
fusion_duality
[0,1,2,3,4,5,6,7,8,9,11,10]
```

,2,3,4,5,6,7,8,9,11,10]

It's important to note that in the duality, indices start from 0. From this input, Normaliz generates the linear and quadratic equations defining the fusion data. The default computational goal for this input is `FusionRings`.

To run this command on Linux or MacOS, use the following command line syntax, assuming `example` is the current directory. For some progress information on the terminal, you can add the `-c` flag. On a modern laptop, the computation typically takes less than 10 seconds and requires about 2.4 GB of RAM.

```
path/to/normaliz bracket_4
```

A brief informal explanation of the algorithm used to solve the system of equations is presented in §6.2.

The results are detailed in the `bracket_4.out` file, beginning with a preamble:

```
148 fusion rings up to isomorphism
0 simple fusion rings up to isomorphism
148 nonsimple fusion rings up to isomorphism
Embedding dimension 231
dehomogenization
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ... 0 0 0 0 0 1
```

The 148 fusion rings correspond to the orbits of the set of lattice points with respect to the symmetries of the equation system. These symmetries are observed under permutations of the type vector that adhere to the Frobenius-Perron equations and are compatible with the duality. Put simply, we have identified 148 pairwise nonisomprphic fusion rings, classified according to their type and duality. They are automatically categorized into simple and nonsimple fusion rings. To limit the computation exclusively to simple fusion rings, one can modify the input file by including the `SimpleFusionRings` option.

The term *embedding dimension* refers to the number of coordinates utilized during the computation. The rationale behind the selection of these coordinates is discussed in §6.3. The final component, represented by the number 1 in the dehomogenization process, signifies that the equations' right-hand side corresponds to the last coordinate of the solutions. This component is not included in the fusion data.

The latter part of the output file details the fusion rings represented by lattice points:

```
0 simple fusion rings up to isomorphism:

148 nonsimple fusion rings up to isomorphism:
0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 ... 1 3 3 1 1 1
...
```

To generate fusion data from the lattice points, add the `FusionData` option in the input file. The result includes a list of fusion data for each fusion ring, presented as a series of matrices M_i, where $i = 1, \ldots, r$ and r is the rank of the fusion ring ($r = 12$ in our example). The matrix M_i comprises the elements N_{ij}^k, with row index k and column index j.

The input file for solving the dimension partition version (see §6.4), for instance, `bracket_3_part.in`, is as follows:

```
amb_space auto
fusion_type_for_partition
[1,1,2,3,3,6,6,8,8,8,12,12]
```

Here, the default computation goal is `SingleLatticePoint`, focusing on the solvability of the system.

For additional information and further options, refer to Appendix H of the Normaliz manual (`Normaliz.pdf`), available in the `doc` directory of the Normaliz distribution or online at [9]. This includes details on restricting computations to fusion rings that meet certain criteria for modular categorification.

6.2. **Normaliz and its approach to fusion rings.** Normaliz [8] is an open source software for discrete convex geometry and its algebraic aspects. Readers are referred to Bruns and Gubeladze [6] for detailed terminology and a comprehensive discussion. Normaliz is designed to solve Diophantine systems of linear inequalities, equations, and congruences with integer coefficients. Additionally, it calculates enumerative information such as multiplicities (which correspond to geometric volumes) and Hilbert series. Objects in Normaliz can be defined either by generators, such as the extreme rays of cones, bases of lattices, and vertices of polytopes, or by constraints like inequalities, equations, and congruences. For systems with coefficients in real algebraic number fields, Normaliz can execute fundamental operations like convex hull computation and its dual, vertex enumeration. Moreover, it is capable of computing lattice points within (bounded) polytopes over real algebraic number fields, facilitating applications to non-integral fusion rings. In the context of fusion rings, it is crucial that lattice points within polytopes can be subjected to constraints imposed by polynomial equations and inequalities. Each release of Normaliz includes source code, comprehensive documentation, sample examples, a testing suite, and pre-compiled binaries for Linux, Mac OS, and MS Windows systems.

For lattice points in generic polytopes denoted by P, Normaliz employs the project-and-lift algorithm. It sequentially projects P onto coordinate hyperplanes until reaching zero dimensions and then lifts the lattice points back up. If P' is a projection of P onto a coordinate hyperplane, then the lattice points of P are projected to lattice points in P', and if $x \in P'$ is a lattice point within P', its preimages are the lattice points in a line segment. Polynomial constraints can be introduced as soon as the lifting process reaches the highest coordinate present in the constraint.

In its standard form, the project-and-lift method is suitable for only minor cases of fusion rings. For satisfactory performance, the algorithm has been tailored to the special linear and polynomial constraint structure specific to fusion rings. Each linear equation is inhomogeneous with nonnegative coefficients and a positive right-hand side. We can refer to the set of coordinates that appear in the equation with positive coefficients as a "patch". These patches encompass the entire set of coordinates, and thus the linear equations, when restricted to the nonnegative orthant, delineate a polytope P. Solutions to a linear equation, confined to its patch, are ascertained using the project-and-lift technique previously outlined, and the lattice points in P are derived by combining these local solutions along matching components. In essence, we begin with the solutions of one of the equations and progressively extend them patch by patch. The sequence in which patches are integrated into the extension process is pivotal. Normaliz includes options that allow alteration of the sequence, as detailed later on.

The input files only include linear equations in their partition versions (XXX). Particularly for these cases, it is critical to recognize a secondary, implicit constraint type: congruences extracted from the linear equations by taking successive residue classes modulo their coefficients. By default, each congruence involves only the coordinates pertaining to the patch of its originating equation. Nonetheless, since congruences only involve a subset of these coordinates, they frequently pertain to other patches or combinations thereof, potentially significantly limiting their number of solutions. The simplification of rank 13 to just two unresolved cases (see §9.2) would not have been achievable without meticulous utilization of the congruences.

When polynomial equations of degree two or higher are in play, Normaliz endeavors to determine an optimal patch extension order that allows these equations to be applied as early as feasible. Users can influence this order by either insisting on the "linear" input order or by directing Normaliz to employ "weights" that gauge the anticipated solution count for each patch and prioritize those with lower weight. Regardless of whether polynomial equations are present, users can request an order based on the applicability of congruences. This order can also be weight-dependent.

Some computations for simple rank 13 were executed on the high-performance cluster (HPC) at Osnabrück by early splitting of partial solutions into parts, which were then processed separately. Despite the rather basic approach of using a static subdivision without intercommunication between running instances of Normaliz, the HPC proved to be advantageous.

6.3. **Full Version.**

Remark 6.1. *All the processes outlined in this subsection have been fully automated in the recently released Normaliz 3.10.2* [8]*, see §6.1. Appendix H of its manual* [9] *specifically addresses the computation of the fusion rings for a specified type.*

Consider a fusion ring with the basis $\{b_1, \dots, b_r\}$. As described in §2.1, for all indices i, j:

$$b_i b_j = \sum_k N_{i,j}^k b_k,$$

and by applying FPdim, we obtain the type $[d_1, \dots, d_r]$ and the corresponding *dimension equations*:

$$d_i d_j = \sum_k N_{i,j}^k d_k.$$

The objective is to resolve these r^2 linear positive Diophantine equations, where (d_i) are specified and $(N_{i,j}^k)$ represent r^3 variables, using Normaliz. Now, we can decrease the variable count to roughly $(r-1)^3/6$ by invoking the Unit axiom ($N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$) from the Definition 2.1 of fusion data, as well as the Frobenius reciprocity (Proposition 2.2).

A critical factor in accelerating computation is the strategic use of associativity equations (non-linear)

$$\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t,$$

in the most effective manner possible during the solving process of the aforementioned linear Diophantine equations. While the optimal approach is not confirmed, the method we employ is highly efficient (refer to §6.2 for further details).

In practice, for a given type $L = [d_1, d_2, \dots, d_r]$, utilize the function `TypeToNormaliz`, the SageMath code for which can be found at [38]. This function generates input files (.in), one for each potential duality map $i \to i^*$. Place these files in a directory alongside the normaliz.exe and run_normaliz.bat files available at [38], and execute run_normaliz (note the existence of a more recent and faster Linux version used for our latest computations). This process yields output files (.out) containing all potential solutions (if any exist). The remaining task is to convert these solutions into fusion data, considering isomorphism. We demonstrate how this can be done with the following example. Take the type $L = [1, 1, 2]$ of the character ring of S_3. When `TypeToNormaliz` is applied, it generates the file [1,1,2][0,1,2].in with the content as follows:

```
amb_space 4
inhom_equations 4
1 2 0 0 0
0 1 2 0 -2
0 1 2 0 -2
0 0 1 2 -3
LatticePoints
convert_equations
nonnegative
polynomial_equations 2
x[2]^2 - x[1]*x[3] + x[3]^2 - x[2]*x[4] - 1;
-x[2]^2 + x[1]*x[3] - x[3]^2 + x[2]*x[4] + 1;
```

The upper part encodes the linear Diophantine equations, and the lower part lists the associativity equations. Following the execution of run_normaliz, the file [1,1,2][0,1,2].out is produced, containing:

```
1 lattice points in polytope (module generators) satisfying polynomial constraints:
 0 0 1 1 1
```

Here, we encounter a single solution, but there could be multiple in general (as seen in the subsequent example). Next, remove the final '1' from each line of the solution and convert it into a list of lists:

```
sage: LL=[[0,0,1,1]]
```

Collect the lists for the type and the duality map:

```
sage: L=[1,1,2]
sage: d=[0,1,2]
```

Finally, to obtain all the fusion data up to isomorphism, apply the function `ListToFusion`:

```
sage: ListToFusion(LL,L,d)
[[[[1, 0, 0], [0, 1, 0], [0, 0, 1]],
  [[0, 1, 0], [1, 0, 0], [0, 0, 1]],
  [[0, 0, 1], [0, 0, 1], [1, 1, 1]]]]
```

The result is the fusion data of $\mathrm{ch}(S_3)$, which can ultimately be formatted in TeX as follows:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}, \quad \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}, \quad \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}$$

Now, applying the same procedure with the type $L = [1, 5, 5, 5, 6, 7, 7]$, we obtain four input files. Only the file corresponding to the trivial duality map yields solutions, with its output file containing:

```
6 lattice points in polytope (module generators) satisfying polynomial constraints:
 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 0 3 1 2 1
 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 2 1 1
 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 1 3 0 1
 1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 0 3 1 2 1
 1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 2 1 1
 1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 1 3 0 1
```

All files can be accessed at [38]. Ultimately, we acquire the following two sets of fusion data, up to isomorphism:

$$
\begin{smallmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{smallmatrix},\;
\begin{smallmatrix}0&1&0&0&0&0&0\\1&1&0&1&0&1&1\\0&0&1&0&1&1&1\\0&1&0&0&1&1&1\\0&0&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\;
\begin{smallmatrix}0&0&1&0&0&0&0\\0&0&1&0&1&1&1\\1&1&1&0&0&1&1\\0&0&0&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\;
\begin{smallmatrix}0&0&0&1&0&0&0\\0&1&0&0&1&1&1\\0&0&0&1&1&1&1\\1&0&1&1&0&1&1\\0&1&1&0&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\;
\begin{smallmatrix}0&0&0&0&1&0&0\\0&0&1&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&0&1&1&1\\1&1&1&1&1&1&1\\0&1&1&1&1&2&1\\0&1&1&1&1&1&2\end{smallmatrix},\;
\begin{smallmatrix}0&0&0&0&0&1&0\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&2&1\\1&1&1&1&2&0&3\\0&1&1&1&1&3&1\end{smallmatrix},\;
\begin{smallmatrix}0&0&0&0&0&0&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&2\\0&1&1&1&1&3&1\\1&1&1&1&2&1&2\end{smallmatrix}
$$

$$
\begin{smallmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{smallmatrix},\;
\begin{smallmatrix}0&1&0&0&0&0&0\\1&1&0&1&0&1&1\\0&0&1&0&1&1&1\\0&1&0&0&1&1&1\\0&0&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\;
\begin{smallmatrix}0&0&1&0&0&0&0\\0&0&1&0&1&1&1\\1&1&1&0&0&1&1\\0&0&0&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\;
\begin{smallmatrix}0&0&0&1&0&0&0\\0&1&0&0&1&1&1\\0&0&0&1&1&1&1\\1&0&1&1&0&1&1\\0&1&1&0&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{smallmatrix},\;
\begin{smallmatrix}0&0&0&0&1&0&0\\0&0&1&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&0&1&1&1\\1&1&1&1&1&1&1\\0&1&1&1&1&2&1\\0&1&1&1&1&1&2\end{smallmatrix},\;
\begin{smallmatrix}0&0&0&0&0&1&0\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&2&1\\1&1&1&1&2&1&2\\0&1&1&1&1&2&2\end{smallmatrix},\;
\begin{smallmatrix}0&0&0&0&0&0&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&2\\0&1&1&1&1&2&2\\1&1&1&1&2&2&1\end{smallmatrix}
$$

6.4. **Dimension Partition Version.** This method is applicable primarily for types denoted by

$$T = [[1, m_1], [d_2, m_2], \dots, [d_s, m_s]],$$

where s is not exceedingly large. This is because we can streamline the dimension equations by grouping elements that share the same dimension (i.e. dimension partition). However, the conversion of the associativity equations remains an open challenge. This version is intended to serve as an intermediary step to the full version for suitable types. Its utility lies in its ability to circumvent certain computational complexities by breaking symmetries. For the time being, it functions as a criterion; that is, if this version fails to yield a solution, the full version will similarly lack a solution.

We can reframe the type as $[1, d_{1,1}, \dots, d_{1,n_1}, d_{2,1}, \dots, d_{2,n_2}, \dots, d_{s,1}, d_{s,n_s}]$, where $d_{i,a} = d_i$, $d_1 = 1 = d_{0,1}$, and $n_i = m_i - \delta_{1,i}$. The dimension equations are then expressed as follows:

$$d_{i,a} d_{j,b} = \sum_{k,c} N_{i,a,j,b}^{k,c} d_{k,c}.$$

Let us define $D_i := \sum_{a=1}^{n_i} d_{i,a} = n_i d_i$ and $M_{i,j}^k := \sum_{a,b,c} N_{i,a,j,b}^{k,c}$, which simplifies the equations to:

$$D_i D_j = \sum_{a,b} \sum_{k,c} N_{i,a,j,b}^{k,c} d_{k,c} = \sum_k \Big(\sum_{a,b,c} N_{i,a,j,b}^{k,c}\Big) d_k = \sum_k M_{i,j}^k d_k.$$

Consequently, we are tasked with solving the linear positive Diophantine equations:

$$n_i d_i n_j d_j = \sum_k M_{i,j}^k d_k,$$

where (d_i, n_i) are predetermined, and the variables $(M_{i,j}^k)$ are reduced to roughly $s^3/6$ by employing the dimension partition variant of the Unit axiom and Frobenius reciprocity. After grouping by dimension, the duality map becomes straightforward (that is, $i^* = i$). Note that we have not yet derived a satisfactory dimension partition version of the associativity axiom, but about the other ones:

Lemma 6.2. *The following equalities hold:*

- *(Unit)* $M_{i,0}^j = M_{0,i}^j = \delta_{i,j} m_i$
- *(Dual)* $M_{i,j}^0 = M_{j,i}^0 = \delta_{i,j} m_i$
- *(Frobenius reciprocity)* $M_{i,j}^k = M_{i,k}^j = M_{j,k}^i = M_{j,i}^k = M_{k,i}^j = M_{k,j}^i$.

Proof. The proof is straightforward. □

In practice, one should follow the procedure outlined in §6.3 up to the generation of output files but replace the function `TypeToNormaliz` with `TypeToPreNormaliz`. For instance, consider the type $L = [1, 6, 12, 12, 15, 15, 15, 20, 20, 30, 30, 60]$. The corresponding input and output files can be found in the reference [38]. This dimension partition version is sufficiently robust to demonstrate Theorem 1.6 in instances without prime-power basic FPdim (refer to §7 immediately following Theorem 9.1), with L being the first of 24 types to be excluded at rank 12.

Remark 6.3. *While this version utilizes the dimension partition of the type, alternative versions could explore other pertinent partitions.*

7. Half-Frobenius Integral Fusion Rings up to rank 12

Here is the count of half-Frobenius integral fusion rings and noncommutative ones up to rank 12.

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Fusion Rings	1	1	1	2	3	6	9	23	105	158	1218	9101
#Noncommutative	0	0	0	0	0	1	0	4	5	7	38	158

The comprehensive list of these fusion rings has been compiled and made accessible online as indicated in [38]. These rings were identified by implementing the type criteria detailed in §5 and utilizing the fusion ring solver discussed in §6 on the list of types referenced in §4. The count of types for each rank is relatively modest when contrasted with the table in §4.1.

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Types	1	1	1	1	2	2	2	4	5	9	15	28

Below is the list of (non-pointed) types for each rank, arranged in lexicographic order:

- Rank 5: [[1, 1, 1, 1, 2]],
- Rank 6: [[1, 1, 1, 1, 2, 2]],
- Rank 7: [[1, 1, 1, 1, 2, 2, 2]],
- Rank 8: [[1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4], [1, 1, 2, 2, 2, 2, 3, 3]],
- Rank 9: [[1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6]],
- Rank 10:[[1, 1, 1, 1, 1, 1, 1, 1, 1, 3], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6]],
- Rank 11:[[1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 4], [1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 6], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8], [1, 1, 1, 1, 2, 4, 4, 4, 4, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 12], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12, 12], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18]],
- Rank 12:[[1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 4, 4], [1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 6], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 4], [1, 1, 1, 1, 2, 2, 2, 2, 4, 6, 6, 6], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4, 4], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8, 8], [1, 1, 1, 1, 2, 2, 2, 8, 8, 12, 12, 12], [1, 1, 1, 1, 2, 4, 4, 4, 4, 6, 6, 12], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 12, 24], [1, 1, 1, 1, 2, 8, 18, 18, 24, 36, 36, 36], [1, 1, 1, 1, 3, 3, 3, 3, 4, 4, 6, 6], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12, 12, 12], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18, 36], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 6, 6], [1, 1, 1, 2, 2, 2, 3, 4, 4, 4, 6, 6], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6, 12], [1, 1, 1, 3, 6, 8, 8, 8, 8, 8, 8, 12], [1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 12], [1, 1, 2, 2, 2, 2, 6, 6, 6, 6, 9, 9], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12, 12], [1, 1, 2, 3, 3, 6, 6, 8, 8, 8, 12, 12], [1, 1, 2, 6, 6, 6, 6, 10, 10, 10, 15, 15]].

Observe that the list above contains no non-trivial perfect types, which proves Theorem 1.6. It is noteworthy that the perfect integral modular fusion category (and therefore half-Frobenius) $\mathcal{Z}(\mathrm{Rep}(A_5))$ has an FPdim of $60^2 = 3600$, a rank of 22, and a type of $[[1,1],[3,2],[4,1],[5,1],[12,10],[15,4],[20,3]]$. This category was calculated using GAP, as described in [17, §8.5].

Question 7.1. *Is there a perfect integral half-Frobenius fusion ring/category with a rank less than* 22*?*

8. Advanced results on modular fusion categories

From rank 13 onward, the types were further restricted by additional properties coming from more advanced results on modular fusion categories §8, which involved the universal grading §8.1, congruence representations of the modular group §8.2 and Galois action §8.3.

8.1. **Universal Grading.** Let G de a finite group. A *G-grading* of a fusion ring R is given by a partiton $B = \sqcup_{g\in G} B_g$ of its basis such that:

- For any $x \in B_g$ and any $y \in B_{g'}$, the basic components of xy belong to $B_{gg'}$.
- For any $x \in B_g$, x^* is in $B_{g^{-1}}$.

A G-grading is called *faithful* if B_g is non-empty for all $g \in G$. Consequently, by [17, Theorem 3.5.2], $\mathrm{FPdim}(B_g) := \sum_{x\in B_g} \mathrm{FPdim}(x)^2$ is constant in g. The faithful grading with the largest group is called the *universal grading.* By [17, Lemma 8.22.9], the universal grading group of the Grothendieck ring of modular fusion category is $G = B_{pt}$, the group of basic element with $\mathrm{FPdim} = 1$ (see Corollary 2.10). The classification of fusion rings constrained to such a grading is a coming Normaliz feature, already available for the C_2-grading, see §6.

Theorem 8.1. *Let $\mathcal{C}$ be an integral modular fusion category. Let R be its Grothendieck ring with basis B. Let $G = B_{pt}$ be the universal grading group. Let $T_g := (\mathrm{FPdim}(x))_{x\in B_g}$. Let $\mathcal{C}_e$ be the fusion subcategory corresponding to B_e. Then:*

(0) If $\mathcal{C}_e$ is perfect then it is modular,
(1) If $B_{pt} \subset B_e$ and T_e has an entry with odd multiplicity, then $\forall g \neq e$, every entry of T_g has multiplicity ≥ 2,
(2) If (1) holds, if $p := |B_{pt}|$ is prime, and if an entry d appears with multiplicity one in T_e, then p divides d.

Proof. Immediate from [28, Proposition VI.2] as a group of prime order must be cyclic. □

Here are three examples of exclusion using each point of Theorem 8.1 (fully automated in §9.4):

(0) In the partitioned type $[[1,2,2,3,3,3,3,3],[1,2,2,3,6]]$, the neutral component is perfect, but we already know that there is no perfect integral modular fusion category of rank 8,
(1) In the partitioned type $[[1,1,1,1,2,2,2,3,10,10],[15],[15],[15]]$, the pointed part is in the neutral component T_e, and the entry 2 has multiplicity three (odd) in T_e, but 15 appears with multiplicity one in some non-neutral components.
(2) In the partitioned type $[[1,1,2,2,3,3,5,6,6,10,15],[15,15]]$, the pointed part is in the neutral component T_e, and the entry 5 has multiplicity one (odd) in T_e, but 5 is not divisible by the prime $2 = |B_{pt}|$.

8.2. **Congruence Representation.** This subsection reviews some applications of congruence representations of the modular group to modular fusion categories, leading to a proof of the folklore Theorem 1.1. Although a more concise proof is presented later in §8.3, the current exposition is meant to be informative and to serve for future research.

As discussed in [27, Section 3], a modular fusion category $\mathcal{C}$ is associated with modular data (S, T), which gives a projective representation of

$$\mathrm{SL}(2,\mathbb{Z}) = \langle s, t \mid (st)^3 = s^2, s^4 = e\rangle.$$

This representation can be lifted to a usual (linear) representation ρ by utilizing the linear characters (i.e. one-dimensional representations), forming a cyclic group of order 12. This representation is r-dimensional—where r represents the rank of $\mathcal{C}$—and is *congruence*. This means it factors through $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$, for some n whose smallest one is called the *level*. The level is determined as $\mathrm{ord}(\rho(t))$, and as previously described, satisfies

$$\mathrm{ord}(T) \mid \mathrm{ord}(\rho(t)) \mid 12\mathrm{ord}(T).$$

A finite-dimensional congruence representation ρ of level n is completely reducible, hence it can be broken down into a direct sum of irreducible representations of $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$. It's important to note that this includes only those irreducible representations that do not further factor through $\mathrm{SL}(2,\mathbb{Z}/d\mathbb{Z})$ for any proper divisor d of n. Nevertheless, if $n = \prod_i p_i^{n_i}$ represents the prime factorization of n, then $\rho = \bigotimes_i \rho_i$ with each ρ_i being a congruence representation of level $p_i^{n_i}$.

For deeper applications, note that [29] proves that the finite-dimensional congruence representations are equivalent to *symmetric* ones, which are classified in [30].

The dimensions d of the irreducible finite-dimensional congruence representations at level $n = p^a$ are provided in the table at the end of [34]. Observe that $d \geq (n-1)/2$ [with equality only if $a = 1$], leading to $p \leq p^a = n \leq 2d+1$. Given that the rank r of the modular fusion category is the sum of dimensions d of such irreducible representations, it follows that $d \leq r$ and therefore $p \leq 2r+1$.

According to the Cauchy theorem in [3], the set S of prime factors of $\mathrm{ord}(T)$ coincides with the prime factors of the norm N of the global dimension of the modular fusion category (of rank r). The prime numbers p lastly mentioned (satisfying $p \leq 2r+1$) constitute the set S' of prime factors of the level n of the congruence representation. Thus, $S \subseteq S' \subseteq S \cup \{2,3\}$, since $\mathrm{ord}(T) \mid n \mid 12\mathrm{ord}(T)$ and $12 = 2^2 \cdot 3$. Hence, for all prime factors $p \neq 2, 3$ of N, it follows that $p \leq 2r+1$. The inequality trivially holds for $p = 2, 3$. This end the proof of Theorem 1.1. □

8.3. **Galois Action.** Let (s, t) be a *normalized* modular data. A Galois automorphism σ induces a permutation $X \to \sigma(X)$ on the simple objects, and acts as follows on dim, s and t:

(1) $\sigma(\dim(X)^2) = \frac{\sigma(\dim(\mathcal{C}))}{\dim(\mathcal{C})}\dim(\sigma(X))^2$, see [17],
(2) $\sigma(s_{X,Y}^2) = s_{X,\sigma(Y)}^2$, see [17],
(3) $\sigma^2(t_X) = t_{\sigma(X)}$, see [16, Theorem II (iii)].

See for example [39, Section 2] for an explicit normalization of the modular data.

Shorter proof of Theorem 1.1:

Proof. If $p = 2, 3$ then $p \leq 2r+1$ trivially as $r \geq 1$. Let $p \neq 2, 3$ be a prime factor of the global dimension norm. By Cauchy's theorem in [BNRW], p divides ord(t). So there must be a simple object X such that p divides the conductor of t_X, thus the orbit of $(\sigma^2(t_X))$ has at least $(p-1)/2$ distinct elements, because the group of units in $\mathbb{Z}/p\mathbb{Z}$ is cyclic of order $p-1$, so it has an element g with $\mathrm{ord}(g^2) = (p-1)/2$. So by (3), $r \geq (p-1)/2$, i.e., $p \leq 2r+1$. □

Here is an example of type of rank $r = 13$ and $\mathrm{FPdim} = 2^4 3^2 5^2 7^2 19^2 37^2$ excluded Theorem 1.1:

$$[1, 777, 1036, 1295, 3990, 4218, 24605, 42180, 98420, 98420, 147630, 147630, 147630],$$

because $p = 37 > 2r+1 = 27$.

Here is a stronger version in the integer case (shared by Eric Rowell and Andrew Schopieray):

Theorem 8.2. *For an integral modular fusion category, for every prime p dividing the global* FPdim*, there is a basic* FPdim *of multiplicity m such that $p \leq 2m+1$.*

Proof. Consider the orbit $(\sigma^2(t_X))$ with at least $(p-1)/2$ distinct elements from above proof of Theorem 1.1. By (3), the orbit $(\sigma(X))$ has also at least $(p-1)/2$ distinct elements. By applying (1) on the (weakly) integral case, we get that $\sigma(\mathrm{FPdim}(X)) = \mathrm{FPdim}(X)$. Thus all simple objects in the orbit $(\sigma(X))$ has the same FPdim, so the multiplicity m of this basic FPdim satisfies $m \geq (p-1)/2$, i.e., $p \leq 2m+1$. □

Here is an example of type of rank 13 and $\mathrm{FPdim} = 2^4 3^6 5^2 7^2 17^2$ excluded by Theorem 8.2 (but not Theorem 1.1):

$$[1, 238, 459, 540, 595, 918, 5355, 9180, 21420, 21420, 32130, 32130, 32130],$$

because $p = 17 > 2m+1 = 7$, where $m = 3$ is the largest multiplicty of a basic FPdim.

Here is en even stronger version:

Theorem 8.3. *For an integral modular fusion category, let S be the set of odd prime factors of the global* FPdim*. There is a partition (S_i) of S, and multiplicities (m_i) of* some *distinct basic* FPdim*s such that*

$$m_i \geq \frac{1}{2}\mathrm{lcm}_{p\in S_i}(p-1).$$

Proof. Let S_X be the set of odd prime divisors of t_X. By Cauchy theorem in [3], the union of the sets S_X over all the simple objects X is exactly S. Let λ be the *Carmichael function*, i.e. the exponent of the multiplicative group of integers modulo n. It is well-known that if $n = \prod_i p_i^{n_i}$ is the prime factorization of n, then $\lambda(n) = \mathrm{lcm}_i \lambda(p_i^{n_i})$, whereas for p_i odd then $\lambda(p_i^{n_i}) = \varphi(\lambda(p_i^{n_i})) = (p_i - 1)p_i^{n_i-1}$, where φ is the Euler totient function. The orbit $(\sigma(X))$ has at least $\frac{1}{2}\lambda(\prod_{p\in S_X} p)$ distinct elements. Therefore, by above and the proof of Theorem 8.2, the multiplicity of $\mathrm{FPdim}(X)$ is at least $\frac{1}{2}\mathrm{lcm}_{p\in S_X}(p-1)$. The result follows. □

Here is an example of type of rank 25 and $\mathrm{FPdim} = 3^4 5^2 7^2 11^2 13^2$ excluded by Theorem 8.3 (but not Theorem 8.2):

$$[[1,1],[39,2],[231,2],[273,2],[1001,2],[1287,2],[3465,2],[4095,2],[9009,2],[15015,8]],$$

because the largest prime, $p = 13$, is insufficient for the largest multiplicity of 8 since $(13-1)/2 = 6$, which is less than 8. Therefore, the subset S_i that includes 13 must also include another prime to increase $\mathrm{lcm}_{pi\in S_i}(p-1)$. Primes 3, 5, and 7 do not contribute to an increase; although they could be part of S_i, another prime is needed. Thus, 11 must be included in S_i. Now, $\mathrm{lcm}(13-1, 11-1)/2 = 60$, which is greater than 8, presenting a contradiction.

The checking of Theorems 8.2 and 8.3 are automated by the function `Theorem3Check` and `Theorem4Check` in `Equipartition.sage` in [38].

The following conjecture generalizes Corollary 8.8 to any integral modular fusion category.

Conjecture 8.4. *For any prime number p that divides the global dimension of a integral modular fusion category with rank r, then $p \leq r$.*

If Conjecture 8.4 is true then it is optimal as demonstrated by the pointed examples of prime rank. Thus, we could expect better for the non-pointed case.

Proposition 8.5. *The statement of Conjecture 8.4 is true up to rank* 21*.*

Proof. The proof is computer-assisted. We get that there is no possible types satisfying Theorem 8.3 but contradicting Conjecture 8.4, up to rank 14. Then we found only $1, 4, 22, 2, 28, 0, 8, 122$ possible such types of rank $14, \dots, 21$ respectively, all available in `ConjUpToRank21.txt` in [38]. The smallest one, of rank 14 is $[[1,1],[6,8],[34,2],[51,3]]$ with $\mathrm{FPdim} = 2^2 3^2 17^2$, where we observe that $p = 17 > 14$, but it can easily be excluded by hand: if b is a basic element with an FPdim of 6, then the equation $bb^* = b_1 + X$, where $\mathrm{FPdim}(X) = 35$, holds. The only feasible way to decompose bb^* into basic elements is through the partition $36 = 1 + 1 + 34$. However, for such a perfect type, the multiplicity of an $\mathrm{FPdim} = 1$ for the product of two basic elements must be no greater than 1, leading to a contradiction. In fact none of the possible types mentioned above come from a fusion ring, as they can all quickly be excluded using first the type criteria in §5 and then the partition version of our fusion ring solver in §6.4. □

Remark 8.6. *The extension of Proposition 8.5 to rank* 22 *is in progress. About the rank* 23*, we can reduced to the following type of* $\mathrm{FPdim} = 2^4 3^8 5^4 29^2$ *which requires deeper examination:*

$$[[1,1],[540,1],[725,1],[1450,1],[2610,1],[3132,1],[8100,1],[26100,1],[58725,14],[78300,1]].$$

8.4. Rank of the Drinfeld center of Rep(G).

This subsection is inspired from comments by Goeff Robinson and Dave Benson in [36]. The goal is to simplify the group theoretic way to express the rank of $\mathcal{Z}(\mathrm{Rep}(G))$, for any finite group G, and to provide some upper bounds. Consequently, this class of integral modular fusion categories satisfies the statement of Conjecture 8.4.

Theorem 8.7. *Let G be a finite group. Let Γ be a complete set of conjugacy class representatives. Let c_G be the number of conjugacy classes (i.e. $|\Gamma|$). Let r be the rank of $\mathcal{Z}(\mathrm{Rep}(G))$. Let $Z(g)$ be the center of G. Then*

$$r = \sum_{a\in\Gamma_G} c_{C_G(a)} \geq |Z(G)|c_G + \sum_{g\in\Gamma_G\setminus Z(G)} \mathrm{ord(g)} \geq \sum_{g\in\Gamma_G} \mathrm{ord(g)}.$$

Proof. According to [11] or [26, Section 3], the rank r is determined by the number of irreducible characters within the centralizers of class representatives of G, which is precisely the equality. In general, we have

$$c_{C_G(a)} \geq |Z(C_G(a))| \geq \mathrm{ord}(a),$$

but if $a \in Z(G)$ then $C_G(a) = G$ and so $c_{C_G(a)} = c_G$. The two inequalities follows. □

Corollary 8.8. *Let G be a finite group. Let r be the rank of the Drinfeld center $\mathcal{Z}(\mathrm{Rep}(G))$. For every prime p dividing its $|G|^2$, then $p \le r$.*

Proof. By Theorem 8.7, $\sum_{g \in \Gamma_G} \mathrm{ord(g)} \le r$, in particular, for all g in G then $\mathrm{ord(g)} \le r$. By Cauchy's theorem, for all prime p dividing $|G|$, there is g in G such that $\mathrm{ord(g)} = p$, therefore $p \le r$. □

Finally, let us simplify how to express the rank r of $\mathcal{Z}(\mathrm{Rep}(G))$.

Proposition 8.9. *Let G be a finite group. The rank of the Drinfeld center $\mathcal{Z}(\mathrm{Rep}(G))$ is the number of conjugacy classes of pairs of commuting elements of G, i.e. the cardinality of the set*

$$A_G := \{c(a_1, a_2) \mid a_1, a_2 \in G \text{ with } a_1 a_2 = a_2 a_1\},$$

where

$$c(a_1, a_2) := \{(g a_1 g^{-1}, g a_2 g^{-1}) \mid g \in G\}.$$

Proof. By the equality in Theorem 8.7, it suffices to establish a bijection between the set A_G and the set

$$B_G := \{(a, \beta) \mid a \in \Gamma_G \text{ and } \beta \text{ is a conjugacy class within } C_G(a)\}.$$

Given $c(a_1, a_2) \in A_G$, we associate the element $(a_1, \{h a_2 h^{-1} \mid h \in C_G(a_1)\}) \in B_G$. We merely need to confirm that if $a_1 = g a_1 g^{-1}$, then a_2 and $g a_2 g^{-1}$ are conjugates in $C_G(a_1)$, which is apparent since $a_1 = g a_1 g^{-1}$ means that $g \in C_G(a_1)$. Given $(a, \beta) \in B_G$, we associate the element $c(a, b) \in A_G$, where $b \in \beta$. We only need to verify that if $b' \in \beta$, then $c(a, b') = c(a, b)$. Note that $b' = h b h^{-1}$, where $h \in C_G(a)$. Therefore, $c(a, b) = c(h a h^{-1}, h b h^{-1}) = c(a, b')$ because $h a h^{-1} = a$, given that $h \in C_G(a)$. □

The number of conjugacy classes of pairs of commuting elements in the alternating group A_n is $1, 1, 9, 14, 22, 44, 74$ for $n = 1, \dots, 7$, respectively, see [37]. Note that we can recover Corollary 8.8 using Proposition 8.9, because the pair of commuting elements (g, g^i) for $0 \le i < \mathrm{ord(g)}$ are all in distinct conjugacy classes.

9. Rank 13

From rank 13 onwards, it becomes impractical to classify all half-Frobenius integral fusion rings using our current technology. Therefore, when necessary, we will either assume the commutativity or apply more advanced modular criteria mentioned in §8. This approach yields results that are less general at the fusion ring level but still sufficiently comprehensive for general modular data.

9.1. **General.** The process commenced, as previously, with 7997 half-Frobenius types derived from Egyptian fractions with squared denominators, as described in §4. Initially, more than 60% were dismissed based on type criteria outlined in §5. Subsequently, over 90% were eliminated using the partition version in §6.4, restricted to a quick use, leaving only 212 types for further consideration. Next, the application of Theorems 8.1 and 8.2, via the functions `AllCriteria` and `Theorem3Check` in `Equipartition.sage` in [38], reduces to 56 types. Among them, 11 are identified as perfect without any prime-power entries (see §9.2 for further examination), 22 are perfect but include a prime-power entry (see §9.3), and the remaining 23 are non-perfect (see §9.4); see `Rank13ReducedList` in [38].

9.2. **Simple.** We turn our attention to the study of simple integral modular fusion categories of rank 13. The following theorem provides a significant constraint on the types occuring in such categories.

Theorem 9.1 (Corollary 6.16 in [33])**.** *Let $\mathcal{C}$ be an integral modular fusion category. If $\mathcal{C}$ contains a simple object whose* FPdim *is a prime-power, then it must have a nontrivial symmetric subcategory.*

Consequently, our analysis is narrowed down to perfect integral half-Frobenius types that lack any prime-power entries. From §4 and the list in [38], precisely 2044 such types have been classified, but as explained in §9.1, we can quickly reduce to 11 types only as explained in §9.1. A more extensive use (involving HPC) of §6.4, and then §6.3 assuming the commutativity, excluded all these 11 types.

Note that the perfect integral modular fusion category $\mathcal{Z}(\mathrm{Rep}(A_7))$, with FPdim $(7!/2)^2$, rank 74, and type:

$$[[1,1],[6,1],[10,2],[14,2],[15,1],[21,1],[35,1],[70,9],[105,4],[210,20],[280,9],[360,14],[504,5],[630,4]],$$

notably lacks any basic elements whose FPdim is a prime-power.

Question 9.2. *Is there a perfect integral half-Frobenius fusion ring/category, without any basic elements of prime-power* FPdim*, that has a rank lower than 74?*

9.3. **Perfect Non-Simple.** According to §9.2, a perfect type that includes a prime-power entry cannot characterize a simple integral modular fusion category. From §4 and the list in [38], exactly 4473 such types have been identified. However, but as explained in §9.1, we can quickly reduce to 22 types. Subsequently, all of these types can be ruled out, by §6.3, using relatively short computations. The exception is the type [1,20,20,27,27,30,45,54,180,180,270,270,270], which requires a longer, yet still reasonable, computation time.

9.4. **Non-Perfect.** From §4 and the list in [38], precisely 1480 non-perfect types have been classified, but as explained in §9.1, we can quickly reduce to 23 types only. A more extensive use of §6.3 reduces the remaining 23 types to the following 15 types having fusion rings, only 6 ones have cyclotomic self-transposable fusion rings (see §3.1):

1. $[1,1,1,1,1,1,1,1,1,1,1,1,1]$,
2. $[1,1,1,1,4,4,4,4,4,4,10,10,10]$,
3. $[1,1,1,1,4,4,12,12,36,36,54,54,54]$,
4. $[1,1,1,3,6,12,16,16,16,16,16,16,24]$,
5. $[1,1,1,3,12,12,20,20,20,20,20,20,30]$,
6. $[1,1,2,3,3,24,30,30,40,40,40,60,60]$.

Finally, all non-pointed ones are quickly excluded by the magic criterion (see §3.2), which completes the proof of Theorem 1.3 at rank 13.

10. The Odd-Dimensional Case

For an overview of the current state of knowledge on odd-dimensional modular fusion categories, we refer the reader to [12, 13]. A foundational result in this area establishes that an odd-dimensional modular fusion category $\mathcal{C}$ is equivalent to being maximally non self-dual (MNSD), meaning that its only self-dual simple object is the unit object. Let $(d_i)_{i\in I}$ represent the FPdim of the simple objects in $\mathcal{C}$, considered up to isomorphism. Since d_i^2 is a divisor of the odd $\mathrm{FPdim}(\mathcal{C})$, each d_i must be odd. Furthermore, the equation $\sum_{i\in I} d_i^2 = \mathrm{FPdim}(\mathcal{C})$ implies that the rank $r = |I|$ must also be odd. This reduces our investigation to Egyptian fractions of the form $q = \sum_{i=1}^{r} \frac{1}{s_i^2}$, where $q, r, s_i \in \mathbb{Z}_{\geq 1}$, $s_1 \geq \cdots \geq s_r \geq 1$, and both r and s_i are odd. Additionally, s_i divides s_1 for all i, and $s_{2k} = s_{2k+1}$. This yields the expression

$$q = \frac{1}{s_1^2} + \sum_{k=1}^{(r-1)/2} \frac{2}{s_{2k}^2}.$$

Since each s_i is odd, we have $s_i^2 \equiv 1 \pmod 8$, which implies $q \equiv r \pmod 8$ and that q is odd as well. Utilizing a similar technique as in §4, we can assume $s_i > 1$ (hence $s_i \geq 3$), by completing the classification with additional 1s if necessary. Consequently, we can assume $q \leq r/9$. For $r < 27$, this allows us to deduce that $q = 1$, and therefore $r \equiv 1 \pmod 8$, which narrows the possibilities for r to $1, 9, 17, 25$ (up to completing by 1s).

Remark 10.1. *This strategy can be extended. For instance, by adding eighteen 3s to complete the classification, we may assume that $s_i = 5$ for $i + 16 \leq r$, which leads to $q \leq 16/9 + (r-16)/25$. If $r < 47$ (which becomes 51 because $q \equiv r \pmod 8$), we can assume that $q = 1$. However, this extended strategy will not be applied in this paper.*

Consequently, for all $r < 25$, we have compiled the following list of all possible non-pointed types (as for §4):

- [[1, 9], [3, 8], [81, 2a]],
- [[1, 7], [3, 2], [5, 8], [225, 2a]],
- [[1, 3], [3, 8], [5, 6], [225, 2a]],
- [[1, 1], [3, 2], [7, 2], [9, 4], [21, 8], [3969, 2a]],
- [[1, 1], [9, 4], [25, 2], [45, 2], [75, 8], [50625, 2a]],

where $a \geq 0$ represents the number of 1s added for completion. It is noteworthy that these ranks are $17 + 2a$, which corroborates a result from [12] stating that any odd-dimensional modular fusion category with rank less than 17 is pointed. Further, [12, Remark 4.3] states that any perfect odd-dimensional modular fusion category is a Deligne product of simple categories. From the preceding analysis, a non-pointed one must have a rank of at least 17, meaning a perfect non-simple one must have a rank of at least 289 ($= 17^2$). Therefore, a perfect one with a rank less than 289 must be simple and cannot have non-trivial simple objects of prime-power FPdim, as shown in [33, Corollary 6.16]. Consequently, the previously mentioned perfect types are excluded. It follows that:

Theorem 10.2. *Every perfect odd-dimensional modular fusion category of rank less than 25 is trivial, and so everyone of rank less than 625 ($= 25^2$) is simple.*

Proof of Theorem 1.9

Proof. By Theorem 10.2, there remain to address the non-perfect types above. Their rank is always $17 + 2a < 25$, which implies $a < 4$. As outlined in §8.1, the modular grading results in a partition indexed by the pointed part, with each component having the same FPdim, in particular the FPdim of the pointed part divides the global FPdim.

- First, let's examine the type $[[1,9],[3,8],[81,2a]]$. The FPdim for this type is $81(1+2a81)$. Consequently, each partition component must have $\mathrm{FPdim} = 9(1+2a81)$. If $a > 0$, a component with 81 must have $\mathrm{FPdim} \geq 81^2$. This leads to $81^2 \leq 9(1+2a81)$, resulting in $a > 4$, a contradiction. Therefore, $a = 0$. The modular partition then must be $[[1,1,1,1,1,1,1,1,1],[3],[3],[3],[3],[3],[3],[3],[3]]$, which contradicts Theorem 8.1 (1).
- Regarding the second type $[[1,7],[3,2],[5,8],[225,2a]]$, the FPdim of the pointed part equaling 7 is not a divisor of the global $\mathrm{FPdim} = 225(1+2a225)$, for $0 \leq a < 4$, except $a = 3$. Similar to the first type, we must have $225^2 \leq 225(1+6\times 225)/7 = 225 \times 193$, contradiction.
- Lastly, for the third type $[[1,3],[3,8],[5,6],[225,2a]]$, its FPdim is also $225(1+2a225)$. Therefore, each partition component must have $\mathrm{FPdim} = 75(1+2a225)$. Similar to the first type, if $a > 0$ then $2 \leq a < 4$, leading to $a = 2,3$. This gives $75(1+2a225) = 2 \times 225^2 + b$ with $b = -33675, 75$. If $a = 2$, a component can contain at most one entry equal to 225. However, with only three components for four entries of 225, this results in a contradiction. Next, if $a = 3$ then the modular partition must be $[[1,1,1,3,3,3,3,3,3,3,3,225,225],[5,5,5,225,225],[5,5,5,225,225]]$, so the neutral component would be a braided integral fusion category of type $[[1,3],[3,8],[225,2]]$. Now integral implies spherical [17, Propositions 9.5.1 and 9.6.5] and braided spherical means pre-modular [17, Definition 8.13.1], but pre-modular implies 1-Frobenius [17, Proposition 8.13.11(ii)], whereas $225 = 3\times 75$ does not divide $\mathrm{FPdim} = 75\times 1351$, contradiction. Hence, $a = 0$. Following this, applying the fusion ring solver outlined in §6.3 to the type $[[1,3],[3,8],[5,6]]$ yields two fusion rings. Applying `STmatrix2` to these fusion rings provides 3 modular data, detailed in §12. ☐

Remark 10.3. *As highlighted in Remark 1.10, potential gaps have been identified in the literature:*

(1) *In* [2, Theorem 4.2, proof of Case (viii) $\mathrm{FPdim}(\mathcal{C}_{pt}) = p$], *on page 727, the assertion that the anomaly-freeness (as defined in their reference* [16], *see Definition 10.4) necessarily leads to $p_+ = pq$ is incorrect, see Lemma 10.5, it may also be $-pq$, as for the MD described in §12, thus also allowing $p|(q+1)$.*
(2) *In* [12, Theorem 6.3 (b), proof of Case $|\mathcal{G}(\mathcal{C})| = 3$], *on page 1936, the deduction "Hence $l \leq 24$" in the seventh last line is accurate, except when $c_{X_1} = 1$, which permits $l = 5$, thereby accommodating the type* $[[1,3],[3,8],[5,6]]$.

Definition 10.4. *A modular data is called* anomaly-free *if its Gauss sums are equal, i.e. $p_+ = p_-$, see Definition 2.11.*

Lemma 10.5. *A modular data is anomaly-free if and only if $p_+ = \pm\sqrt{\dim}$, if and only if the central charge $c = 0$ or* 4.

Proof. Recall from Definition 2.11 that $p_\pm := \sum_{i=1}^r d_i^2(\theta_i)^{\pm 1}$. Thus p_+ and p_- are complex-conjugate. So anomaly-free is equivalent to p_+ real. Now, $p_+ = \sqrt{\dim}\zeta_8^c$, thus it is real if and only if $\zeta_8^c = \pm 1$, if and only if $c = 0$ or 4. ☐

Remark 10.6. *Regarding the categorification of a MD in §12, as a modular fusion category $\mathcal{C}$, a discussion with Sebastian Burciu revealed that the adjoint fusion subring, which is of rank* 11, FPdim 75, *type* $[[1,3],[3,8]]$, *and basis $\{b_g\}_{g\in C_3} \cup \{x_i, x_i^*\}_{i\in\{1,2,3,4\}}$, is isomorphic to $\mathrm{ch}(G)$, where $G = C_5^2 \rtimes C_3$ is the unique non-abelian finite group of order* 75. *In fact, §6.3 shows a unique MNSD cyclotomic fusion ring of this type. Furthermore, the braided adjoint fusion subcategory $\mathcal{D} = \mathcal{C}_{ad}$ would not be symmetric, as indicated by the S-matrix mentioned in §12. In fact, the Müger center $\mathcal{Z}_2(\mathcal{D})$ would be $\mathcal{C}_{pt}$, pointed of rank* 3. *Therefore, $\mathcal{D}$ could be* $\mathrm{Rep}(G)$, *albeit with an unusual braiding (see* [7]*), or, more broadly, a Jordan-Larson category* [23] *with an* FPdim *of* 3×5^2. *Finally, according to the S-matrix again, $\mathcal{C}$ would clearly be a minimal modular extension of $\mathcal{D}$, see* [25] *and* [22, §1.1].

There are non-pointed and non-perfect odd-dimensional modular fusion categories of rank 25, exemplified by $\mathcal{Z}(\mathrm{Vec}^\omega_{C_7\rtimes C_3})$. Furthermore, [13] demonstrates that, up to equivalence, no additional such examples exist. Consequently, our attention must now turn to the examination of the perfect case (which is simple and have $q = 1$). We have identified precisely 91 possible types (listed in [38]) using the aforementioned method combined with [33, Corollary 6.16]. Subsequent application of the type criteria from §5 reduces this to 29 types, and the fusion ring solver in §6.4 quickly eliminates 8 more types. Among the remaining 21 types, just 4 listed in Proposition 10.7, pass Theorem 8.3.

Proposition 10.7. *A perfect odd-dimensional modular fusion category of rank* 25, *if any, must have one of the following* 4 *types:*

1. $[[1,1],[39,4],[65,2],[189,2],[315,2],[585,2],[1365,2],[2457,2],[4095,8]]$,
2. $[[1,1],[75,2],[91,4],[175,2],[585,2],[975,2],[2275,2],[4095,2],[6825,8]]$,
3. $[[1,1],[75,2],[91,4],[175,2],[975,2],[2275,2],[2925,4],[6825,8]]$,
4. $[[1,1],[99,2],[231,2],[385,2],[675,2],[10395,4],[28875,2],[51975,2],[86625,8]]$.

11. Catalogue of Integral Modular Data Up To Rank 13

This section presents a comprehensive catalogue of integral modular data (MD) up to rank 13. Notably, all the MD listed here are categorifiable, meaning each corresponds to the MD of an integral modular fusion category. These corresponding categorical models are identified in Theorem 1.3. We will begin by listing the non-pointed MD in §11.1, followed by the pointed MD in §11.2. Additionally, these data can be accessed in a machine-readable format in [38].

11.1. Non-Pointed Modular Data Up To Rank 13.

The data are organized in the following sequence:

- Rank,
- Type,
- Fusion coefficients,
- List of topological spins (s_i) giving the T-matrix (see Definition 2.11),
- Cyclotomic conductor of the S-matrix,
- Cyclotomic conductor of the T-matrix (its order),
- Central charge,
- List of 2nd Frobenius-Schur indicators,
- S-matrix.

11.1.1. *Rank* 8, *type* $[1,1,2,2,2,2,3,3]$, *first fusion ring.*

$$\begin{matrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{matrix},\quad \begin{matrix}0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{matrix},\quad \begin{matrix}0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&0&1&0&0\\0&0&0&1&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&1&0&1&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{matrix},\quad \begin{matrix}0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&1&1&0&0&0\\1&1&1&0&0&0&0&0\\0&0&1&0&0&1&0&0\\0&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{matrix},$$

$$\begin{matrix}0&0&0&0&1&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{matrix},\quad \begin{matrix}0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&1&0&0&0\\0&0&0&0&1&1&0&0\\0&0&1&1&0&0&0&0\\1&1&0&1&0&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{matrix},\quad \begin{matrix}0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\1&0&1&1&1&1&0&0\\0&1&1&1&1&1&0&0\end{matrix},\quad \begin{matrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&1&1&1&1&1&0&0\\1&0&1&1&1&1&0&0\end{matrix}$$

- $[0,0,-2/9,4/9,0,1/9,1/2,0]$, 9, 18, 0, $[1,1,1,1,1,1,1,1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&2\zeta_9^4+2\zeta_9^5&2\zeta_9^2+2\zeta_9^7&-2&2\zeta_9+2\zeta_9^8&0&0\\2&2&2\zeta_9^2+2\zeta_9^7&2\zeta_9+2\zeta_9^8&-2&2\zeta_9^4+2\zeta_9^5&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&2\zeta_9+2\zeta_9^8&2\zeta_9^4+2\zeta_9^5&-2&2\zeta_9^2+2\zeta_9^7&0&0\\3&-3&0&0&0&0&3&-3\\3&-3&0&0&0&0&-3&3\end{bmatrix},$$

- $[0,0,-4/9,-1/9,0,2/9,1/2,0]$, 9, 18, 0, $[1,1,1,1,1,1,1,1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&2\zeta_9+2\zeta_9^8&2\zeta_9^4+2\zeta_9^5&-2&2\zeta_9^2+2\zeta_9^7&0&0\\2&2&2\zeta_9^4+2\zeta_9^5&2\zeta_9^2+2\zeta_9^7&-2&2\zeta_9+2\zeta_9^8&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&2\zeta_9^2+2\zeta_9^7&2\zeta_9+2\zeta_9^8&-2&2\zeta_9^4+2\zeta_9^5&0&0\\3&-3&0&0&0&0&3&-3\\3&-3&0&0&0&0&-3&3\end{bmatrix},$$

- $[0,0,-2/9,4/9,0,1/9,-1/4,1/4]$, 9, 36, 0, $[1,1,1,1,1,1,-1,-1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&2\zeta_9^4+2\zeta_9^5&2\zeta_9^2+2\zeta_9^7&-2&2\zeta_9+2\zeta_9^8&0&0\\2&2&2\zeta_9^2+2\zeta_9^7&2\zeta_9+2\zeta_9^8&-2&2\zeta_9^4+2\zeta_9^5&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&2\zeta_9+2\zeta_9^8&2\zeta_9^4+2\zeta_9^5&-2&2\zeta_9^2+2\zeta_9^7&0&0\\3&-3&0&0&0&0&-3&3\\3&-3&0&0&0&0&3&-3\end{bmatrix}$$

- $[0, 0, -4/9, -1/9, 0, 2/9, -1/4, 1/4]$, 9, 36, 0, $[1, 1, 1, 1, 1, 1, -1, -1]$,

$$\begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\
2 & 2 & 2\zeta_9 + 2\zeta_9^8 & 2\zeta_9^4 + 2\zeta_9^5 & -2 & 2\zeta_9^2 + 2\zeta_9^7 & 0 & 0 \\
2 & 2 & 2\zeta_9^4 + 2\zeta_9^5 & 2\zeta_9^2 + 2\zeta_9^7 & -2 & 2\zeta_9 + 2\zeta_9^8 & 0 & 0 \\
2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\
2 & 2 & 2\zeta_9^2 + 2\zeta_9^7 & 2\zeta_9 + 2\zeta_9^8 & -2 & 2\zeta_9^4 + 2\zeta_9^5 & 0 & 0 \\
3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \\
3 & -3 & 0 & 0 & 0 & 0 & 3 & -3
\end{bmatrix}$$

11.1.2. *Rank* 8*, type* $[1, 1, 2, 2, 2, 2, 3, 3]$*, second fusion ring.*

$$\begin{matrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{matrix}, \quad
\begin{matrix} 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{matrix}, \quad
\begin{matrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1 \end{matrix}, \quad
\begin{matrix} 0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 1&1&0&1&0&0&0&0\\ 0&0&1&0&0&1&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1 \end{matrix}, \quad
\begin{matrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&1&0&0&1&0&0\\ 1&1&0&0&1&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1 \end{matrix}, \quad
\begin{matrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1 \end{matrix}, \quad
\begin{matrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1\\ 1&0&1&1&1&1&0&0\\ 0&1&1&1&1&1&0&0 \end{matrix}, \quad
\begin{matrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1\\ 0&1&1&1&1&1&0&0\\ 1&0&1&1&1&1&0&0 \end{matrix}$$

- $[0, 0, -1/3, 1/3, 0, 0, 1/2, 0]$, 1, 6, 0, $[1, 1, 1, 1, 1, 1, 1, 1]$,

$$\begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\
2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\
2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\
2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\
2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\
3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \\
3 & -3 & 0 & 0 & 0 & 0 & -3 & 3
\end{bmatrix}$$

- $[0, 0, -1/3, -1/3, 1/3, 1/3, 1/2, 0]$, 1, 6, 4, $[1, 1, 1, 1, 1, 1, -1, -1]$,

$$\begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\
2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\
2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\
2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\
2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\
3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \\
3 & -3 & 0 & 0 & 0 & 0 & 3 & -3
\end{bmatrix}$$

- $[0, 0, -1/3, 1/3, 0, 0, -1/4, 1/4]$, 1, 12, 0, $[1, 1, 1, 1, 1, 1, -1, -1]$,

$$\begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\
2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\
2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\
2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\
2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\
3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \\
3 & -3 & 0 & 0 & 0 & 0 & 3 & -3
\end{bmatrix}$$

- $[0, 0, -1/3, -1/3, 1/3, 1/3, -1/4, 1/4]$, 1, 12, 4, $[1, 1, 1, 1, 1, 1, 1, 1]$,

$$\begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\
2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\
2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\
2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\
2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\
3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \\
3 & -3 & 0 & 0 & 0 & 0 & -3 & 3
\end{bmatrix}$$

11.1.3. *Rank* 10*, type* [1, 1, 1, 2, 2, 2, 2, 2, 2, 3]*, first fusion ring.*

$$\begin{matrix}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\end{matrix}, \quad \begin{matrix}0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\end{matrix}, \quad \begin{matrix}0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\end{matrix}, \quad \begin{matrix}0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0&1&0\\0&0&0&0&1&1&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix}, \quad \begin{matrix}0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&1&0&0\\0&0&0&1&0&0&1&0&0&0\\0&1&0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix},$$

$$\begin{matrix}0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&0&0&1&1&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&1&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix}, \quad \begin{matrix}0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&1&1&0&0&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix}, \quad \begin{matrix}0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&1&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&1&0\\0&0&0&0&1&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix}, \quad \begin{matrix}0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1&0&0\\0&0&0&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix}, \quad \begin{matrix}0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\\0&0&0&0&1&1&0&0&1&0\\0&0&0&1&0&0&1&1&0&0\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\\1&1&1&0&0&0&0&0&0&2\end{matrix}$$

- $[0, 0, 0, 1/9, 4/9, -2/9, -2/9, 4/9, 1/9, 1/2]$, 9, 18, 4, $[1, 0, 0, 0, 0, 0, 0, 0, 0, 1]$,

$$\begin{bmatrix}1&1&1&2&2&2&2&2&2&3\\1&1&1&2\zeta_3^2&2\zeta_3&2\zeta_3&2\zeta_3^2&2\zeta_3^2&2\zeta_3&3\\1&1&1&2\zeta_3&2\zeta_3^2&2\zeta_3^2&2\zeta_3&2\zeta_3&2\zeta_3^2&3\\2&2\zeta_3^2&2\zeta_3&-2\zeta_9^7&-2\zeta_9^5&2\zeta_9^2+2\zeta_9^5&2\zeta_9^4+2\zeta_9^7&-2\zeta_9^4&-2\zeta_9^2&0\\2&2\zeta_3&2\zeta_3^2&-2\zeta_9^5&2\zeta_9^4+2\zeta_9^7&-2\zeta_9^7&-2\zeta_9^2&2\zeta_9^2+2\zeta_9^5&-2\zeta_9^4&0\\2&2\zeta_3&2\zeta_3^2&2\zeta_9^2+2\zeta_9^5&-2\zeta_9^7&-2\zeta_9^4&-2\zeta_9^5&-2\zeta_9^2&2\zeta_9^4+2\zeta_9^7&0\\2&2\zeta_3^2&2\zeta_3&2\zeta_9^4+2\zeta_9^7&-2\zeta_9^2&-2\zeta_9^5&-2\zeta_9^4&-2\zeta_9^7&2\zeta_9^2+2\zeta_9^5&0\\2&2\zeta_3^2&2\zeta_3&-2\zeta_9^4&2\zeta_9^2+2\zeta_9^5&-2\zeta_9^2&-2\zeta_9^7&2\zeta_9^4+2\zeta_9^7&-2\zeta_9^5&0\\2&2\zeta_3&2\zeta_3^2&-2\zeta_9^2&-2\zeta_9^4&2\zeta_9^4+2\zeta_9^7&2\zeta_9^2+2\zeta_9^5&-2\zeta_9^5&-2\zeta_9^7&0\\3&3&3&0&0&0&0&0&0&-3\end{bmatrix}$$

- $[0, 0, 0, 2/9, -1/9, -4/9, -4/9, -1/9, 2/9, 1/2]$, 9, 18, 4, $[1, 0, 0, 0, 0, 0, 0, 0, 0, 1]$,

$$\begin{bmatrix}1&1&1&2&2&2&2&2&2&3\\1&1&1&2\zeta_3&2\zeta_3^2&2\zeta_3^2&2\zeta_3&2\zeta_3&2\zeta_3^2&3\\1&1&1&2\zeta_3^2&2\zeta_3&2\zeta_3&2\zeta_3^2&2\zeta_3^2&2\zeta_3&3\\2&2\zeta_3&2\zeta_3^2&-2\zeta_9^5&2\zeta_9^4+2\zeta_9^7&-2\zeta_9^7&-2\zeta_9^2&2\zeta_9^2+2\zeta_9^5&-2\zeta_9^4&0\\2&2\zeta_3^2&2\zeta_3&2\zeta_9^4+2\zeta_9^7&-2\zeta_9^2&-2\zeta_9^5&-2\zeta_9^4&-2\zeta_9^7&2\zeta_9^2+2\zeta_9^5&0\\2&2\zeta_3^2&2\zeta_3&-2\zeta_9^7&-2\zeta_9^5&2\zeta_9^2+2\zeta_9^5&2\zeta_9^4+2\zeta_9^7&-2\zeta_9^4&-2\zeta_9^2&0\\2&2\zeta_3&2\zeta_3^2&-2\zeta_9^2&-2\zeta_9^4&2\zeta_9^4+2\zeta_9^7&2\zeta_9^2+2\zeta_9^5&-2\zeta_9^5&-2\zeta_9^7&0\\2&2\zeta_3&2\zeta_3^2&2\zeta_9^2+2\zeta_9^5&-2\zeta_9^7&-2\zeta_9^4&-2\zeta_9^5&-2\zeta_9^2&2\zeta_9^4+2\zeta_9^7&0\\2&2\zeta_3^2&2\zeta_3&-2\zeta_9^4&2\zeta_9^2+2\zeta_9^5&-2\zeta_9^2&-2\zeta_9^7&2\zeta_9^4+2\zeta_9^7&-2\zeta_9^5&0\\3&3&3&0&0&0&0&0&0&-3\end{bmatrix}$$

11.1.4. *Rank* 10*, type* [1, 1, 1, 2, 2, 2, 2, 2, 2, 3]*, second fusion ring.*

$$\begin{matrix}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\end{matrix}, \quad \begin{matrix}0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\end{matrix}, \quad \begin{matrix}0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\end{matrix}, \quad \begin{matrix}0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0&1&0\\0&0&0&0&1&1&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix}, \quad \begin{matrix}0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&1&0&0\\0&0&0&1&0&0&1&0&0&0\\0&1&0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix},$$

$$\begin{matrix}0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&0&0&1&1&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&1&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix}, \quad \begin{matrix}0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&1&1&0&0&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix}, \quad \begin{matrix}0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&1&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&1&0\\0&0&0&0&1&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix}, \quad \begin{matrix}0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1&0&0\\0&0&0&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix}, \quad \begin{matrix}0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\\0&0&0&0&1&1&0&0&1&0\\0&0&0&1&0&0&1&1&0&0\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\\1&1&1&0&0&0&0&0&0&2\end{matrix}$$

- $[0,0,0,0,1/3,-1/3,-1/3,1/3,0,1/2]$, 3, 6, 4, $[1,0,0,0,0,0,0,0,0,1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\
1 & 1 & 1 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_3^2 & 2\zeta_3 & 3 \\
1 & 1 & 1 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_3 & 2\zeta_3^2 & 3 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2 & -2\zeta_3 & -2\zeta_3^2 & -2\zeta_3 & -2\zeta_3^2 & -2 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2\zeta_3 & -2\zeta_3 & -2 & -2 & -2\zeta_3^2 & -2\zeta_3^2 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2\zeta_3^2 & -2 & -2\zeta_3^2 & -2\zeta_3 & -2 & -2\zeta_3 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2\zeta_3 & -2 & -2\zeta_3 & -2\zeta_3^2 & -2 & -2\zeta_3^2 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2\zeta_3^2 & -2\zeta_3^2 & -2 & -2 & -2\zeta_3 & -2\zeta_3 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2 & -2\zeta_3^2 & -2\zeta_3 & -2\zeta_3^2 & -2\zeta_3 & -2 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{bmatrix}$$

11.1.5. *Rank* 11, $[1,1,1,1,2,2,2,2,2,2,2]$.

$$\begin{matrix}
1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1
\end{matrix},\quad
\begin{matrix}
0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0
\end{matrix},\quad
\begin{matrix}
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0
\end{matrix},\quad
\begin{matrix}
0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1
\end{matrix},\quad
\begin{matrix}
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&1&0&1&0&0\\
0&1&1&0&0&0&0&1&0&0&0
\end{matrix},$$

$$\begin{matrix}
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&1&0&0\\
1&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&1&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0
\end{matrix},\quad
\begin{matrix}
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&1\\
1&1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0\\
0&0&1&1&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0
\end{matrix},\quad
\begin{matrix}
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&1&0&1&0&0\\
1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&1
\end{matrix},\quad
\begin{matrix}
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&0&1&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0\\
1&1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0
\end{matrix},\quad
\begin{matrix}
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0\\
0&1&0&1&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&1\\
1&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0
\end{matrix},\quad
\begin{matrix}
0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1\\
0&1&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&1&0&1&0&0\\
1&0&0&1&0&0&0&1&0&0&0
\end{matrix}$$

- $[0,0,0,0,-7/16,-3/16,-3/16,1/4,5/16,5/16,1/16]$, 8, 16, 1, $[1,1,1,1,1,-1,-1,1,-1,-1,1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\
2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 \\
2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\
2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\
2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3
\end{bmatrix}$$

- $[0,0,0,0,-7/16,-3/16,1/16,1/4,-7/16,5/16,1/16]$, 8, 16, 1, $[1,1,1,1,-1,1,-1,1,-1,1,-1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\
2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 \\
2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\
2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\
2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3
\end{bmatrix}$$

- $[0, 0, 0, 0, -7/16, 1/16, 1/16, 1/4, -7/16, -7/16, 1/16]$, 8, 16, 1, $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -3/16, 5/16, 5/16, 1/4, -3/16, -3/16, 5/16]$, 8, 16, 1, $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, 7/16, 7/16, -1/4, -1/16, -1/16, 3/16]$, 8, 16, -1, $[1, 1, 1, 1, 1, -1, -1, 1, -1, -1, 1]$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, 7/16, -5/16, -1/4, 3/16, -1/16, 3/16]$, 8, 16, -1, $[1, 1, 1, 1, -1, 1, -1, 1, -1, 1, -1]$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, -5/16, -5/16, -1/4, 3/16, 3/16, 3/16]$, 8, 16, -1, $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\
2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \\
2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\
2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\
2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3
\end{bmatrix}$$

- $[0, 0, 0, 0, -1/16, -1/16, -1/16, -1/4, 7/16, 7/16, 7/16]$, 8, 16, -1, $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\
2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \\
2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\
2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\
2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3
\end{bmatrix}$$

11.2. **Pointed Modular Data Up To Rank 13.** In light of Remark 1.2, it suffices to specify only the abelian groups and their corresponding T-matrices, which encapsulate the topological spins in this context. Let us denote by C_n the cyclic group of order n. The complete data set, akin to the format presented in §11.1, is available at [38].

- C_1: [0],
- C_2: [0, 1/4], [0, -1/4],
- C_3: [0, 1/3, 1/3], [0, -1/3, -1/3],
- C_2^2: [0, 1/2, 0, 0], [0, -1/4, 1/4, 0], [0, 1/2, 1/4, 1/4], [0, 1/2, 1/2, 1/2], [0, -1/4, -1/4, 1/2],
- C_4: [0, 1/2, 1/8, 1/8], [0, 1/2, 3/8, 3/8], [0, 1/2, -3/8, -3/8], [0, 1/2, -1/8, -1/8],
- C_5: [0, 1/5, -1/5, -1/5, 1/5], [0, 2/5, -2/5, -2/5, 2/5],
- C_6: [0, -1/4, 1/12, 1/3, 1/3, 1/12], [0, -1/4, 5/12, -1/3, -1/3, 5/12], [0, 1/4, -5/12, 1/3, 1/3, -5/12], [0, 1/4, -1/12, -1/3, -1/3, -1/12],
- C_7: [0, 1/7, -3/7, 2/7, 2/7, -3/7, 1/7], [0, 3/7, -2/7, -1/7, -1/7, -2/7, 3/7],
- C_2^3: [0, -1/4, 1/4, 1/4, 1/4, 1/2, 0, 0], [0, -1/4, -1/4, -1/4, 1/4, 1/2, 0, 0], [0, -1/4, 1/2, 1/2, 1/4, 1/2, 1/4, 1/4], [0, -1/4, 1/2, -1/4, 1/2, -1/4, 1/2, 1/4],
- $C_2 \times C_4$: [0, 1/2, -1/4, 1/4, 1/8, 3/8, 3/8, 1/8], [0, 1/2, 1/4, -1/4, 1/8, -1/8, -1/8, 1/8], [0, 1/2, -1/4, 1/4, 3/8, -3/8, -3/8, 3/8], [0, 1/2, -1/4, 1/4, -3/8, -1/8, -1/8, -3/8],
- C_8: [0, 0, 1/4, -7/16, 1/16, 1/16, -7/16, 1/4], [0, 0, 1/4, -3/16, 5/16, 5/16, -3/16, 1/4], [0, 0, -1/4, -5/16, 3/16, 3/16, -5/16, -1/4], [0, 0, -1/4, -1/16, 7/16, 7/16, -1/16, -1/4],
- C_3^2: [0, 0, 0, 1/3, -1/3, -1/3, 1/3, 0, 0], [0, 1/3, 1/3, -1/3, -1/3, -1/3, -1/3, 1/3, 1/3],
- C_9: [0, 0, 1/9, -2/9, 4/9, 4/9, -2/9, 1/9, 0], [0, 0, 2/9, -4/9, -1/9, -1/9, -4/9, 2/9, 0],
- C_{10}: [0, 1/4, 1/20, -1/5, 1/5, 9/20, 9/20, 1/5, -1/5, 1/20], [0, -1/4, 3/20, 2/5, -2/5, 7/20, 7/20, -2/5, 2/5, 3/20], [0, -1/4, -9/20, -1/5, 1/5, -1/20, -1/20, 1/5, -1/5, -9/20], [0, 1/4, -7/20, 2/5, -2/5, -3/20, -3/20, -2/5, 2/5, -7/20],
- C_{11}: [0, 1/11, 4/11, -2/11, 5/11, 3/11, 3/11, 5/11, -2/11, 4/11, 1/11], [0, 2/11, -3/11, -4/11, -1/11, -5/11, -5/11, -1/11, -4/11, -3/11, 2/11],
- $C_2 \times C_6$: [0, -1/4, 0, 1/4, -1/3, 5/12, -1/3, -1/12, -1/12, -1/3, 5/12, -1/3],[0, 1/2, -1/4, -1/4, -1/3, 1/6, 5/12, 5/12, 5/12, 5/12, 1/6, -1/3], [0, 1/2, 1/2, 1/2, -1/3, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, -1/3], [0, 1/2, 1/4, 1/4, -1/3, 1/6, -1/12, -1/12, -1/12, -1/12, 1/6, -1/3], [0, 1/2, 0, 0, -1/3, 1/6, -1/3, -1/3, -1/3, -1/3, 1/6, -1/3], [0, 1/2, 1/2, 1/2, 1/3, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, 1/3], [0, 1/4, 1/2, 1/4, 1/3, -5/12, -1/6, -5/12, -5/12, -1/6, -5/12, 1/3],[0, 0, 1/2, 0, 1/3, 1/3, -1/6, 1/3, 1/3, -1/6, 1/3, 1/3],[0, -1/4, 0, 1/4, 1/3, 1/12, 1/3, -5/12, -5/12, 1/3, 1/12, 1/3], [0, -1/4, 1/2, -1/4, 1/3, 1/12, -1/6, 1/12, 1/12, -1/6, 1/12, 1/3],
- C_{12}: [0, 1/2, -1/8, -1/3, 1/6, -11/24, -11/24, -11/24, -11/24, 1/6, -1/3, -1/8], [0, 1/2, -1/8, 1/3, -1/6, 5/24, 5/24, 5/24, 5/24, -1/6, 1/3, -1/8], [0, 1/2, -3/8, -1/3, 1/6, 7/24, 7/24, 7/24, 7/24, 1/6, -1/3, -3/8], [0, 1/2, -3/8, 1/3, -1/6, -1/24, -1/24, -1/24, -1/24, -1/6, 1/3, -3/8], [0, 1/2, 3/8, -1/3, 1/6, 1/24, 1/24, 1/24, 1/24, 1/6, -1/3, 3/8], [0, 1/2, 3/8, 1/3, -1/6, -7/24, -7/24, -7/24, -7/24, -1/6, 1/3, 3/8], [0, 1/2, 1/8, -1/3, 1/6, -5/24, -5/24, -5/24, -5/24, 1/6, -1/3, 1/8], [0, 1/2, 1/8, 1/3, -1/6, 11/24, 11/24, 11/24, 11/24, -1/6, 1/3, 1/8],
- C_{13}: [0, 1/13, 4/13, -4/13, 3/13, -1/13, -3/13, -3/13, -1/13, 3/13, -4/13, 4/13, 1/13], [0, 2/13, -5/13, 5/13, 6/13, -2/13, -6/13, -6/13, -2/13, 6/13, 5/13, -5/13, 2/13].

12. Non-Pointed Odd-Dimensional Modular Data of Rank 17

The data are organized as in §11.1. Additionally, it is accessible in a format compatible with computers within the file `Rank17MNSD.txt` in [38]. About their categorification, see Remark 1.10, 10.3 and 10.6.

12.1. **Type [[1,3],[3,8],[5,6]], First Fusion Ring.** The whole fusion data would be too big to be entirely displayed here. So here is a compressed version. Consider its basis

$$B = \{b_g\}_{g\in C_3} \cup \{x_i, x_i^*\}_{i\in\{1,2,3,4\}} \cup \{y_g, y_g^*\}_{g\in C_3},$$

where $\mathrm{FPdim}(b_g) = 1$, $\mathrm{FPdim}(x_i) = 3$ and $\mathrm{FPdim}(y_g) = 5$. Here are the fusion rules:

- $b_{g_1} b_{g_2} = b_{g_1 g_2}$, $b_g^* = b_{g^{-1}}$,
- $b_g x_i = x_i$,
- $b_{g_1} y_{g_2} = y_{g_1 g_2}$,
- $x_i x_j$ and $x_i x_j^* - \delta_{i,j} \sum_{g\in C_3} b_g$ are sum of x_k or x_k^*, with multiplicites written below (see also Remark 10.6):

$$\begin{matrix} 0&2&0&0&0&0&0&1\\ 0&0&1&1&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 1&0&0&0&0&1&1&0\\ 0&0&1&0&1&0&0&1\\ 0&0&0&1&0&1&0&1\\ 0&1&0&0&1&1&0&0\\ 0&0&1&1&0&0&1&0 \end{matrix},\quad \begin{matrix} 0&0&1&1&0&0&0&0\\ 2&0&0&0&0&0&1&0\\ 0&1&0&0&1&0&0&1\\ 0&1&0&0&0&1&0&1\\ 0&0&1&0&1&0&1&0\\ 0&0&0&1&0&1&1&0\\ 0&0&1&1&0&0&0&1\\ 1&0&0&0&1&1&0&0 \end{matrix},\quad \begin{matrix} 1&0&0&0&1&0&1&0\\ 0&1&0&0&1&0&0&1\\ 0&0&0&2&1&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&1&1&1\\ 1&1&0&1&0&0&0&0\\ 1&0&1&0&0&1&0&0\\ 0&1&1&0&0&1&0&0 \end{matrix},\quad \begin{matrix} 1&0&0&0&0&1&1&0\\ 0&1&0&0&0&1&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&2&0&0&1&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&0&1&0&1&1\\ 1&0&0&1&1&0&0&0\\ 0&1&0&1&1&0&0&0 \end{matrix},$$

$$\begin{matrix} 0&0&1&0&1&0&0&1\\ 0&0&1&0&1&0&1&0\\ 0&0&0&0&0&1&1&1\\ 1&1&1&0&0&0&0&0\\ 0&0&0&1&0&2&0&0\\ 1&1&0&0&0&0&0&0\\ 0&1&0&1&0&0&1&0\\ 1&0&0&1&0&0&0&1 \end{matrix},\quad \begin{matrix} 0&0&0&1&0&1&0&1\\ 0&0&0&1&0&1&1&0\\ 1&1&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&1\\ 1&1&0&0&0&0&0&0\\ 0&0&1&0&2&0&0&0\\ 0&1&1&0&0&0&1&0\\ 1&0&1&0&0&0&0&1 \end{matrix},\quad \begin{matrix} 0&1&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1\\ 1&0&1&0&0&1&0&0\\ 1&0&0&1&1&0&0&0\\ 0&1&0&1&0&0&1&0\\ 0&1&1&0&0&0&1&0\\ 1&0&0&0&0&0&0&2\\ 0&0&0&0&1&1&0&0 \end{matrix},\quad \begin{matrix} 0&0&1&1&0&0&1&0\\ 1&0&0&0&1&1&0&0\\ 0&1&1&0&0&1&0&0\\ 0&1&0&1&1&0&0&0\\ 1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&0&1\\ 0&0&0&0&1&1&0&0\\ 0&1&0&0&0&0&2&0 \end{matrix}$$

- $x_i y_g = x_i^* y_g = \sum_{g\in C_3} y_g$,
- $y_{g_1} y_{g_2} = y^*_{(g_1 g_2)^{-1}} + 2\sum_{g_1 g_2 g \neq e} y_g^*$,
- $y_{g_1} y_{g_2}^* = b_{g_1 g_2^{-1}} + \sum_i (x_i + x_i^*)$,

the other rules follows by commutativity and duality. Now we write the rest of the modular data as in §11.1:

- $[0, 0, 0, -2/5, -2/5, -1/5, -1/5, 1/5, 1/5, 2/5, 2/5, -1/3, -1/3, 0, 0, 1/3, 1/3]$, 15, 15, 4, $[1, 0, \ldots, 0]$,

$$\begin{bmatrix}
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 5 & 5\\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3\\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2\\
3 & 3 & 3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 0 & 0 & 0 & 0 & 0 & 0\\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_3^2 & -5\zeta_3 & -5\zeta_3 & -5\zeta_3^2 & -5 & -5\\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_3 & -5\zeta_3^2 & -5\zeta_3^2 & -5\zeta_3 & -5 & -5\\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_3 & -5\zeta_3^2 & -5 & -5 & -5\zeta_3^2 & -5\zeta_3\\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_3^2 & -5\zeta_3 & -5 & -5 & -5\zeta_3 & -5\zeta_3^2\\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & -5 & -5\zeta_3^2 & -5\zeta_3 & -5\zeta_3 & -5\zeta_3^2\\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & -5 & -5\zeta_3 & -5\zeta_3^2 & -5\zeta_3^2 & -5\zeta_3
\end{bmatrix}$$

12.2. **Type [[1,3],[3,8],[5,6]], Second Fusion Ring.** The basis and fusion rules are the same as in §12.1, except that there is $h \in C_3 \setminus \{e\}$ such that

- $y_{g_1} y_{g_2} = b_h (y^*_{(g_1 g_2)^{-1}} + 2\sum_{g_1 g_2 g \neq e} y_g^*)$,

which sounds like a zesting. The element b_h can equivalently be the second or the third basic element (here we chose the second one).

- $[0, 0, 0, -2/5, -2/5, -1/5, -1/5, 1/5, 1/5, 2/5, 2/5, -2/9, -2/9, 1/9, 1/9, 4/9, 4/9]$, 45, 45, 4, $[1, 0, \ldots, 0]$,

$$\begin{bmatrix}
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 5 & 5\\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2\\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3\\
3 & 3 & 3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 3 & 3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 0 & 0 & 0 & 0 & 0 & 0\\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^4 & -5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & -5\zeta_9^7 & -5\zeta_9^2\\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^5 & -5\zeta_9^4 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & -5\zeta_9^2 & -5\zeta_9^7\\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & -5\zeta_9^7 & -5\zeta_9^2 & -5\zeta_9^4 & -5\zeta_9^5\\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & -5\zeta_9^2 & -5\zeta_9^7 & -5\zeta_9^5 & -5\zeta_9^4\\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^7 & -5\zeta_9^2 & -5\zeta_9^4 & -5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5\\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^2 & -5\zeta_9^7 & -5\zeta_9^5 & -5\zeta_9^4 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7
\end{bmatrix}$$

- $[0, 0, 0, -2/5, -2/5, -1/5, -1/5, 1/5, 1/5, 2/5, 2/5, -4/9, -4/9, -1/9, -1/9, 2/9, 2/9]$, 45, 45, 4, $[1, 0, \ldots, 0]$,

$$\begin{bmatrix}
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 5 & 5 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 \\
3 & 3 & 3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & -5\zeta_9^5 & -5\zeta_9^4 & -5\zeta_9^2 & -5\zeta_9^7 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & -5\zeta_9^4 & -5\zeta_9^5 & -5\zeta_9^7 & -5\zeta_9^2 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^5 & -5\zeta_9^4 & -5\zeta_9^2 & -5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^4 & -5\zeta_9^5 & -5\zeta_9^7 & -5\zeta_9^2 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^2 & -5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & -5\zeta_9^5 & -5\zeta_9^4 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^7 & -5\zeta_9^2 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & -5\zeta_9^4 & -5\zeta_9^5
\end{bmatrix}$$

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Availability of data and materials. Data for the computations in this paper are available on reasonable request from the authors. The softwares used for the computations can be downloaded from the URLs listed in the references.

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