

CLASSIFICATION OF MODULAR DATA OF INTEGRAL MODULAR FUSION CATEGORIES UP TO RANK 12

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ABSTRACT. In this paper, we obtain a classification of all integral modular fusion categories' modular data and, more broadly, all integral half-Frobenius fusion rings, up to rank 12. We get that each perfect integral modular fusion category, as well as every perfect integral half-Frobenius fusion ring, up to rank 12, is trivial. Furthermore, we establish that every integral modular fusion category of rank 13 is pointed, up to 2 types in progress. We also prove that there are three non-pointed odd-dimensional modular data of rank less than 25, all of rank 17, FPdim 225 and type $[[1,3],[3,8],[5,6]]$, and their categorification would be in contradiction with the literature; and finally, we narrow down the rank 25 perfect case to 21 possible types.

Our initial key insight is that the Egyptian fractions, which are typically employed to list possible types, can be chosen with squared denominators. We then develop several type criteria as initial filters. Particularly, we establish that the number of distinct basic FPdims in a non-trivial perfect fusion ring must be at least 4. To obtain the fusion rings, we solve the dimension and associativity equations using an updated version of Normaliz designed specifically for this purpose. The S-matrices (if they exist) are obtained by self-transposing the character table, while the T-matrices are derived by solving the Anderson-Moore-Vafa equations. Finally, we verify the extended axioms of modular data.

1. INTRODUCTION

In this paper, we assume that all fusion categories are defined over the complex field. The concept of an integral modular fusion category has been extensively studied, as detailed in the references at the beginning of [9]. In [3], they have been classified up to rank 6 (all pointed), with Egyptian fractions playing a crucial role. The approach that enables us to extend this classification up to rank 12 in our work hinges on the observation that it is sufficient to consider Egyptian fractions with squared denominators. This restriction significantly reduces the combinatorial complexity. To illustrate this point, consider that the number of Egyptian fractions (summing to 1) of length $n = 1, 2, \dots, 8$ is 1, 1, 3, 14, 147, 3462, 294314, 159330691, respectively (as per [25]). In contrast, when limited to squared denominators, the counts are 1, 0, 0, 1, 0, 1, 1, 4, respectively (refer to [1]).

We begin by recalling the concept of a fusion ring and its fundamental results in §2.1, with reference to [12, Chapter 3]. As defined in [14], a fusion ring \mathcal{F} is termed *s-Frobenius* if for every basic element b , the ratio $\text{FPdim}(\mathcal{F})^s / \text{FPdim}(b)$ is an algebraic integer. According to [12, Proposition 8.14.6], the Grothendieck ring of a modular fusion category is 1/2-Frobenius (denoted *half-Frobenius* in the rest of the paper). Consider \mathcal{F} to be an integral half-Frobenius fusion ring with a basis $\{b_1, \dots, b_r\}$, FPdim D , and type $[d_1, \dots, d_r]$, where $1 = d_1 \leq d_2 \leq \dots \leq d_r$ and $d_i = \text{FPdim}(b_i)$. It is assumed that d_i^2 is a divisor of D for all i . There exists a unique square-free positive integer q such that $D = qs^2$, implying that each d_i is a divisor of s . Let s_i denote the positive integer s/d_i . Given that $D = \sum_{i=1}^r d_i^2$, we arrive at the following representation of q as an Egyptian fraction with squared denominators:

$$q = \sum_{i=1}^r \frac{1}{s_i^2}.$$

We have classified all such Egyptian fractions up to $r = 13$ using SageMath, as will be discussed in §4, where the method to constrain q to be less than or equal to $r/4$ is also described. Since $s_1 = s$, we have $d_i = s_1/s_i$, and we may assume that s_i is a divisor of s_1 for all i . As detailed in §4, this leads us to consider only 1028 types up to rank 12 (and 9025 types up to rank 13).

The subsequent phase entails implementing new criteria for identifying a type that emerges from a fusion ring, as delineated in §5. Particularly, we establish that the minimum number of distinct basic FPdims in a non-trivial perfect fusion ring is four (Theorem 5.1). The proof of these criteria predominantly relies on modular arithmetic and serves to rule out approximately 62% of the types with a rank up to 13. To address the types that remain, we utilize our fusion ring solver, detailed in §6. The core objective is to resolve the dimension equations $d_i d_j = \sum_k N_{i,j}^k d_k$, which are positive linear Diophantine equations. We employ Normaliz [6] for this purpose, having first reduced the number of variables by invoking the Unit axiom of fusion data ($N_{i,j}^k$) and applying Frobenius reciprocity (see §2.1). Additionally, we efficiently integrate the associativity equations (non-linear) during the linear solving process, a challenge referred to as "patching" in §6. This step culminates in a classification of all the half-Frobenius integral fusion rings up to rank 12,

tallying exactly 10628 instances derived from 71 types, and proves the absence of any non-trivial perfect fusion rings at ranks up to 12 (see §7).

Our objective is to classify all possible modular data related to these fusion rings. The definition of modular data we employ (refer to §2.2) is informed by the key attributes of a modular fusion category, specifically a pseudo-unitary one, as our research is centered on the integral case (see [12, Proposition 9.6.5]). We can limit our attention to commutative fusion rings since a modular fusion category, being braided, possesses a commutative Grothendieck ring (although our classification encompasses 213 noncommutative fusion rings as well; see §7).

First, we examine the S -matrices: for a given commutative fusion ring, we take its eigentable (as defined in Definition 2.7) and consider it as a matrix, retaining only those with cyclotomic elements—such fusion rings are termed *cyclotomic*. If suitable renormalization and permutation yield a self-transpose matrix (detailed in §3.1), we call the fusion ring as *self-transposable*; if not, it is dismissed. From this, we infer that there are precisely 69 self-transposable, cyclotomic, half-Frobenius, integral fusion rings up to rank 12, originating from 27 types, which is fewer than 0.7% of the 10628 identified in the initial stage.

Moving on to the T -matrices: for the fusion rings that remain, we solve the Anderson-Moore-Vafa equations (see §2.2) in the \mathbb{Z} -module \mathbb{Q}/\mathbb{Z} . We preserve only those S - and T -matrices that satisfy all the conditions of Definition 2.11. The use of our so-called *magic criterion* was pivotal for several big cases and could lead to an interesting theoretical reformulation, see Question 3.2. Ultimately, we arrive at $19 + 62$ modular data, derived from $5 + 17$ fusion rings and $3 + 12$ types (non-pointed + pointed).

Remark 1.1. *Every pointed modular fusion category corresponds to a metric group (G, q) —a finite abelian group G equipped with a non-degenerate quadratic form $q : G \rightarrow \mathbb{C}^*$, represented by the T -matrix, as described in [12, §8.4].*

The modular data (MD) mentioned in §10 encompass S - and T -matrices, central charge, fusion data, and second Frobenius-Schur indicators for the non-pointed case. For the pointed case, however, it includes only the T -matrices. The following theorem provides a concise overview:

Theorem 1.2. *There are 19 MD of non-pointed integral modular fusion categories up to rank 12, given by:*

- Rank 8, FPdim 36, type $[1, 1, 2, 2, 2, 2, 3, 3]$:
 - 6 MD with central charge $c = 0$ from $\mathcal{Z}(\text{Vec}_{S_3}^\omega)$, see [16],
 - 2 MD with $c = 4$ from $(C_3^2 + 0)^{C_2}$, see [15, point (b) on page 983].
- Rank 10, FPdim 36, type $[1, 1, 1, 2, 2, 2, 2, 2, 3]$:
 - 3 MD with $c = 4$ from $SU(3)_3$, its complex conjugate and a zesting, see [11, §6.3.1].
- Rank 11, FPdim 32, type $[1, 1, 1, 1, 2, 2, 2, 2, 2, 2]$:
 - 8 MD with $c = \pm 1$ from $SO(8)_2$, conjugates and zestings, see §3.3.

There are 62 MD of pointed modular fusion categories up to rank 12; here are their number per group G :

G	C_1	C_2	C_3	C_2^2	C_4	C_5	C_6	C_7	C_3^2	$C_2 \times C_4$	C_8	C_3^2	C_9	C_{10}	C_{11}	$C_2 \times C_6$	C_{12}
#MD	1	2	2	5	4	2	4	2	4	4	4	2	2	4	2	10	8

There is no other integral modular data up to rank 12 (i.e. all categorifiable as above).

Question 1.3. *Is there a modular data without categorification?*

It should be noted that [19] provides an interesting (non-integral) candidate of rank 11 (in its introduction) relevant to Question 1.3, and it also recovers Theorem 1.2. This theorem yields the following consequence:

Corollary 1.4. *Every perfect integral modular fusion category up to rank 12 is trivial.*

In fact, as mentioned above, we obtained the following more general result within the context of fusion rings:

Theorem 1.5. *Every perfect integral half-Frobenius fusion ring up to rank 12 is trivial.*

The proof of Theorem 1.5 for ranks up to 9 is straightforward, following the list provided in §4.1 combined with an extended version of the Nichols-Richmond theorem applied to fusion rings, as detailed in the proof of [20, Theorem 11]. This is due to the consistent presence of a non-trivial basic element with $\text{FPdim} \leq 2$. However, proving the theorem for ranks up to 12 necessitates the employment of type criteria, as discussed in §5, and the use of a fusion ring solver, elaborated in §6.

It should be noted that the Drinfeld center of the representation category of any non-abelian finite simple group G —and, more broadly, any centerless perfect group—is a perfect (though not simple) integral modular fusion category denoted as $\mathcal{Z}(\text{Rep}(G))$ with $\text{FPdim} = |G|^2$. For further information, see [8, §11.1]. Thus, the Grothendieck ring of $\mathcal{Z}(\text{Rep}(A_5))$, of rank 22 and type $[[1, 1], [3, 2], [4, 1], [5, 1], [12, 10], [15, 4], [20, 3]]$, constitutes a perfect integral half-Frobenius fusion ring. Consequently, Theorem 1.5 cannot be extended to all ranks; however, it remains an open question whether its simple version can be:

Question 1.6. *Is there a non-pointed simple integral half-Frobenius fusion ring?*

A negative response to Question 1.6 would imply a negative answer to the renowned [13, Question 2] in the simple case, due to a result in [17], which states that every simple integral fusion category is weakly group-theoretical if and only if every simple integral modular fusion category is pointed. With this in mind, we propose the following question:

Question 1.7. *Is there a non-pointed simple integral modular fusion category?*

For further insights into Question 1.7 at the fusion ring level, [21, Corollary 6.16] adds a constraint: the absence of any basic elements with a prime-power FPdim. It is worth noting that Theorem 1.5 cannot be generalized to all ranks, even with this added constraint. This is because the Grothendieck ring of $\mathcal{Z}(\text{Rep}(A_7))$, which is a perfect integral half-Frobenius fusion ring of rank 74 and type

$$[[1, 1], [6, 1], [10, 2], [14, 2], [15, 1], [21, 1], [35, 1], [70, 9], [105, 4], [210, 20], [280, 9], [360, 14], [504, 5], [630, 4]],$$

satisfies this constraint. If necessary, Question 1.6 could be refined to include this constraint and the property of commutativity. This refinement almost allows for a negative resolution of Question 1.7 at rank 13. In fact, there remain only two types to be examined, and nothing else *in general* (see §8):

Theorem 1.8. *A non-pointed integral modular fusion category of rank 13, if any, has one of the following types:*

1. $[1, 238, 459, 540, 595, 918, 5355, 9180, 21420, 21420, 32130, 32130, 32130]$,
2. $[1, 777, 1036, 1295, 3990, 4218, 24605, 42180, 98420, 98420, 147630, 147630, 147630]$.

In addition to the methods we described earlier, the proof of Theorem 1.8 in the non-perfect case employed certain *universal grading* tricks specific to modular fusion categories, found in [19, Proposition VI.2]. Consequently, it is somewhat less general at the fusion ring level compared to what we did up to rank 12. Employing similar techniques, along with [9, Remark 4.3] and [21, Corollary 6.16], we are able to prove the following (see §9):

Theorem 1.9. *There are three non-pointed odd-dimensional modular data of rank less than 25, listed below:*

- Rank 17, FPdim 225, type $[1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5]$:
– 3 MD with central charge $c = 4$ (see §11 for the details) from 2 fusion rings.

Remark 1.10. *A categorification of a MD mentioned in Theorem 1.9 is in contradiction with [2, Theorem 4.2, proof of Case (viii) FPdim(C_{pt}) = p] as well as [9, Theorem 6.3 (b), proof of Case $|\mathcal{G}(C)| = 3]$. But Remark 9.3 points out potential gaps in their proofs. Should these gaps be filled, these MD would affirmatively address Question 1.3.*

Finally, this paper narrows down the possible rank 25 odd-dimensional perfect types to 21 ones, as detailed in §9.

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2. FUSION DATA AND MODULAR DATA

In this section, we review the concepts of fusion data and modular data, along with the essential results. For further details, we refer the reader to [12].

2.1. Fusion Data. The concept of fusion data expands upon the idea of a finite group.

Definition 2.1. Fusion data consist of a finite set $\{1, 2, \dots, r\}$ with an involution $i \mapsto i^*$, and nonnegative integers $N_{i,j}^k$ satisfying the following conditions for all i, j, k, t :

- (Associativity) $\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t$,
- (Unit) $N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$,
- (Dual) $N_{i^*,j}^1 = N_{j,i^*}^1 = \delta_{i,j}$,
- (Anti-involution) $N_{i,j}^k = N_{j^*,i^*}^{k^*}$.

Note that $1^* = 1$. We may represent the fusion data simply as $(N_{i,j}^k)$.

Proposition 2.2 (Frobenius Reciprocity). For all i, j, k , $N_{i,j}^k = N_{k,j^*}^i = N_{k^*,i}^{j^*} = N_{j^*,i^*}^{k^*} = N_{j,k^*}^{i^*} = N_{i^*,k}^j$.

Proof. Starting with (Associativity) and setting $t = 1$, we have $\sum_s N_{i,j}^s N_{s,k}^1 = \sum_s N_{j,k}^s N_{i,s}^1$. Applying (Dual), we get $\sum_s N_{i,j}^s \delta_{s,k^*} = \sum_s N_{j,k}^s \delta_{s,i^*}$. Consequently, $N_{i,j}^{k^*} = N_{j,k}^{i^*}$. Substituting k^* with k , we obtain $N_{i,j}^k = N_{j,k^*}^{i^*}$, which equals N_{k,j^*}^i by (Anti-involution). The proposition follows by iterating the equality $N_{i,j}^k = N_{k,j^*}^i$. \square

Remark 2.3. We can construct data that satisfy the first three axioms of Definition 2.1 but not the fourth, proving it is not superfluous. However, (Unit) is redundant when combined with the other axioms, as it is not utilized in the proof of Proposition 2.2. Taken together, (Dual) and (Frobenius Reciprocity) trivially imply (Unit).

A fusion ring \mathcal{R} is a free \mathbb{Z} -module equipped with a finite basis $\mathcal{B} = \{b_1, \dots, b_r\}$ and a fusion product defined by

$$b_i b_j = \sum_k N_{i,j}^k b_k,$$

where $(N_{i,j}^k)$ constitutes fusion data, and a $*$ -structure given by $b_i^* := b_{i^*}$. The four axioms for fusion data translate to the following for all i, j, k :

- $(b_i b_j) b_k = b_i (b_j b_k)$,
- $b_1 b_i = b_i b_1 = b_i$,
- $\tau(b_i b_j^*) = \delta_{i,j}$,
- $(b_i b_j)^* = b_j^* b_i^*$,

where $\tau(x)$ is the coefficient of b_1 in the decomposition of $x \in \mathcal{R}$. Consequently, $\mathcal{R}_{\mathbb{C}} := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{C}$ becomes a finite-dimensional unital $*$ -algebra, with τ extending linearly to a trace (i.e., $\tau(xy) = \tau(yx)$) and an inner product defined by $\langle x, y \rangle := \tau(xy^*)$. Here, $\langle x, b_i \rangle$ is the coefficient of b_i in the decomposition of x .

Theorem 2.4 (Frobenius-Perron Theorem). Given a fusion ring \mathcal{R} with basis \mathcal{B} and the corresponding finite-dimensional unital $*$ -algebra $\mathcal{R}_{\mathbb{C}}$ as defined above, there exists a unique $*$ -homomorphism $d : \mathcal{R}_{\mathbb{C}} \rightarrow \mathbb{C}$ such that $d(\mathcal{B}) \subset \mathbb{R}_{>0}$.

The value $d(b_i)$ is termed the *Frobenius-Perron dimension* of b_i , denoted as $\text{FPdim}(b_i)$ or simply d_i . The sum $\sum_i d_i^2$ is referred to as the Frobenius-Perron dimension of \mathcal{R} , denoted $\text{FPdim}(\mathcal{R})$. The sequence $[d_1, d_2, \dots, d_r]$ is called the *type* of \mathcal{R} . A fusion ring \mathcal{R} is described as:

- *Frobenius* (or 1-Frobenius, or of Frobenius type) if $\frac{\text{FPdim}(\mathcal{R})}{\text{FPdim}(b_i)}$ is an algebraic integer for all i ,

- *integral* if $\text{FPdim}(b_i)$ is an integer for all i ,
- *pointed* if $\text{FPdim}(b_i) = 1$ for all i ,
- *commutative* if $b_i b_j = b_j b_i$ for all i, j , meaning $N_{i,j}^k = N_{j,i}^k$.

The *multiplicity* of \mathcal{R} is the maximum value among $N_{i,j}^k$, and its *rank* is r , the size of the basis.

Remark 2.5. *Fusion data enable a representation of its corresponding fusion ring. Consider the matrices $M_i = (N_{i,j}^k)_{k,j}$. By the Associativity axiom in Definition 2.1, we verify that $M_i M_j = \sum_k N_{i,j}^k M_k$. Additionally, M_1 is the identity matrix, and Frobenius Reciprocity ensures that the adjoint matrix M_i^* is M_{i^*} . According to Frobenius-Perron Theorem, the operator norm $\|M_i\|$ equals $\text{FPdim}(b_i)$.*

Remark 2.6. *The concept of fusion data is a combinatorial reformulation of the fusion ring notion, so any property applicable to a fusion ring is also applicable to its fusion data.*

Definition 2.7 (Eigentable). *Given commutative fusion data $(N_{i,j}^k)$, consider the corresponding fusion matrices $M_i = (N_{i,j}^k)_{k,j}$. The commutativity and the property that $M_i^* = M_{i^*}$ render these matrices normal and thus simultaneously diagonalizable. Let (D_i) denote their simultaneous diagonalization, where $D_i = \text{diag}(\lambda_{i,j})$. We can select $\lambda_{i,1} = \|M_i\| = d_i$. The matrix $(\lambda_{i,j})$ is termed the eigentable (or character table) of the fusion data, and the values $c_j := \sum_i |\lambda_{i,j}|^2$ are called the formal codegrees.*

Lemma 2.8. *Let $M \in M_n(\mathbb{Z}_{\geq 0})$. The matrix M is a permutation matrix if and only if $\|M\| = 1$.*

Proof. Consider an orthonormal basis $\{e_1, \dots, e_n\}$ for which the entries of M are non-negative integers. If M is not a permutation matrix, then one of the following cases must occur:

- (0) there exists i for which $Me_i = 0$,
- (1) there exist i, j such that $\langle Me_i, e_j \rangle > 1$,
- (2) there exist i, j, k with $j \neq k$, such that $\langle Me_i, e_j \rangle = \langle Me_i, e_k \rangle = 1$,
- (3) there exist i, j, k with $i \neq j$, such that $Me_i = Me_j = e_k$.

However, case (0) implies $\|Me_i\|/\|e_i\| = 0$, while case (1) leads to $\|Me_i\|/\|e_i\| > 1$. In case (2), it follows that $\|Me_i\|/\|e_i\| \geq \sqrt{2}$. Likewise, case (3) implies $\|M(e_i + e_j)\|/\|e_i + e_j\| = \sqrt{2}$. Each of these cases indicates that $\|M\| > 1$. Conversely, if M is a permutation matrix, it trivially follows that $\|M\| = 1$. \square

Corollary 2.9. *For two basic elements x, y of a fusion ring with $\text{FPdim}(x) = 1$, both xy and yx are basic elements, and $\text{FPdim}(xy) = \text{FPdim}(yx) = \text{FPdim}(y)$.*

Proof. This follows directly from Remark 2.5, Lemma 2.8, and the fact that FPdim is a ring homomorphism. \square

Corollary 2.10. *A fusion ring is pointed if and only if its basis forms a finite group under the fusion product.*

2.2. Modular Data. Broadly speaking, modular data refers to a fusion data together with two matrices, S and $T = (t_{i,j})$, that generate a projective representation of the modular group $\text{SL}(2, \mathbb{Z})$. To provide a more detailed description, we draw upon [18, Theorem 2.1] and [12, §8.13, §8.18]. Let \mathbf{i} be the imaginary unit.

Definition 2.11. *Given a fusion ring \mathcal{R} of rank r , type $[d_1, \dots, d_r]$, and fusion data $(N_{i,j}^k)$, let $\mathbf{d} := \text{FPdim}(\mathcal{R})$ and $\zeta_n := \exp(2\pi\mathbf{i}/n)$. A (pseudounitary) modular data for \mathcal{R} consists of two matrices $S, T \in M_r(\mathbb{C})$ satisfying:*

- S and T are symmetric, T is unitary and diagonal with $T_{1,1} = 1$, $S_{1,i} = d_i$ for all i , and $SS^* = \mathbf{d}\mathbf{1}$.
- Verlinde formula: $N_{i,j}^k = \frac{1}{\mathbf{d}} \sum_l \frac{S_{il} S_{lj} \overline{S_{lk}}}{d_l}$.
- Twist: let θ_i be $T_{i,i}$, then $\sum_k N_{i,j}^k \theta_k d_k = \theta_i \theta_j S_{i,j}$.
- Ribbon structure: $\theta_i = \theta_{i^*}$ (see Remark 2.13).
- Central charge: $p_{\pm} := \sum_{i=1}^r d_i^2 (\theta_i)^{\pm 1}$. The ratio p_+/p_- is a root of unity, and $p_+ = \sqrt{\mathbf{d}} \zeta_8^c$ for some rational number c , referred to as the **central charge**, determined modulo 8.
- The matrices S and T afford a projective representation of $\text{SL}(2, \mathbb{Z})$: we have $(ST)^3 = p_+ S^2$, $\frac{S^2}{\mathbf{d}} = C$, $C^2 = \mathbf{1}$, where C is the permutation matrix associated with the involution $i \rightarrow i^*$ and satisfies $\text{Tr}(C) > 0$.
- Cauchy theorem: the set of distinct prime factors of $\text{ord}(T)$ is identical to the distinct prime factors of $\text{norm}(\mathbf{d})$, where $\text{norm}(x)$ denotes the product of the distinct Galois conjugates of the algebraic number x .
- Cyclotomic integers: for all i, j , the elements $S_{i,j}$, $S_{i,j}/d_j$ and $T_{i,i}$ are cyclotomic integers. The conductor of $S_{i,j}$ divides $\text{ord}(T)$, which in turn divides $\mathbf{d}^{5/2}$, and there exists j such that $S_{i,j}/d_j \in \mathbb{R}_{\geq 1}$, for all i .
- Frobenius-Schur indicators: for every i and for all $n \geq 1$, the sum $\nu_n(i) := \frac{1}{\mathbf{d}} \sum_{j,k} N_{j,k}^i (d_j \theta_j^n) \overline{(d_k \theta_k^n)}$ is a cyclotomic integer with a conductor that divides both n and $\text{ord}(T)$. Additionally, $\nu_1(i) = \delta_{i,1}$ and $\nu_2(i) = \pm \delta_{i,i^*}$.

- *Anderson-Moore-Vafa equations:* $T_{i,i} = e^{2\pi i t_i}$, and $\forall i, j, k, l$, the following equation holds in the \mathbb{Z} -module \mathbb{Q}/\mathbb{Z} :

$$\left(\sum_{p=1}^r N_{i,j}^p N_{p,k}^l \right) (t_i + t_j + t_k + t_l) = \sum_{p=1}^r \left(N_{i,j}^p N_{p,k}^l + N_{i,k}^p N_{j,p}^l + N_{j,k}^p N_{i,p}^l \right) t_p.$$

The **topological spin** of the i -th basic element is the representative $s_i \in (-1/2, 1/2]$ of $t_i \in \mathbb{Q}/\mathbb{Z}$.

We could question the necessity of each component in Definition 2.11, particularly whether the Anderson-Moore-Vafa equations can be inferred from the other assumptions.

Remark 2.12. The Verlinde formula, in conjunction with results from [17, §2], implies that the fusion ring \mathcal{R} is commutative. Together with S symmetric and the identity $SS^* = \text{FPdim}(\mathcal{R})\mathbf{1}$, it can be deduced that \mathcal{R} is self-transposable (as discussed in §3.1). Moreover, according to the proof presented in [12, Proposition 8.14.6], \mathcal{R} is also half-Frobenius.

Remark 2.13. A modular tensor category \mathcal{C} possesses a ribbon structure, which means that the twist $\theta \in \text{Aut}(\text{id}_{\mathcal{C}})$ satisfies the condition $(\theta_X)^* = \theta_{X^*}$ for every object X within \mathcal{C} . Let (X_i) represent the set of simple objects (up to isomorphism) within \mathcal{C} . Schur's Lemma guarantees that $\theta_{X_i} = \theta_i \text{id}_{X_i}$, where the scalar θ_i is consistent with the one described in Definition 2.11. Owing to the ribbon structure, we deduce the following:

$$\theta_{i^*} \text{id}_{X_{i^*}} = \theta_{X_{i^*}} = (\theta_{X_i})^* = (\theta_i \text{id}_{X_i})^* = \theta_i (\text{id}_{X_i})^* = \theta_i \text{id}_{X_{i^*}}.$$

From this, it follows that $\theta_{i^*} = \theta_i$ for all simple objects X_i .

This paper primarily addresses integral fusion categories, implying that \mathbf{d} is an integer and $\text{norm}(\mathbf{d}) = \mathbf{d}$. Such categories are pseudounitary and, consequently, spherical as well as pivotal (see [12]). In contexts that are not pseudounitary, Definition 2.11 would require modifications (as suggested in [18, Theorem 2.1]) because the equality $S_{1,i} = \text{FPdim}(b_i)$ may not be valid.

It should be noted that the definition of modular data provided here is so stringent that, as of now, no instances exist that lack a categorification, leading to Question 1.3.

3. FROM FUSION DATA TO MODULAR DATA

This section elucidates the classification of all potential modular data associated with a given set of fusion data. Initially, we may consider the fusion data to be commutative and half-Frobenius (refer to Remark 2.12). For ranks up to 12, there are exactly 10628 half-Frobenius integral fusion rings, of which 213 are noncommutative. The details by rank are provided in §7.

3.1. S-matrix. Consider a commutative fusion data $(N_{i,j}^k)$ of rank r , eigentable $(\lambda_{i,j})$, and formal codegrees (c_j) as defined in Definition 2.7. The objective here is to identify all permutations q of the set $\{1, \dots, r\}$ such that:

- $q(1) = 1$,
- $d_{q(i)} = d_i$ for all i ,
- The matrix $S = (\sqrt{c_1/c_j} \lambda_{i,q(j)})$ is symmetric (i.e. self-transpose).

Remark 3.1. The symmetric requirement implies that

$$\sqrt{c_1/c_j} = \sqrt{c_1/c_j} \lambda_{1,q(j)} = \sqrt{c_1/c_1} \lambda_{j,q(1)} = d_j,$$

hence we can infer that $c_1/c_j = d_j^2$ for all j , as shown in [22, Example 2.9].

If such a permutation q exists (Remark 3.1 can serve as an effective necessary condition), the fusion data are referred to as *self-transposable*. This property is exceedingly rare, rendering this step a potent sieve. Using the Verlinde formula, one can reconstruct the fusion data from S . It is important to note that we need only consider *cyclotomic* fusion data, i.e. whose eigentable entries are all cyclotomic. Incorporating the self-transposable and cyclotomic prerequisites allows us to exclude over 99.3% of the commutative half-Frobenius integral fusion rings up to rank 12 discovered in §7. This leaves 69 fusion rings; their distribution by type and rank is as follows:

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Types	1	1	1	1	1	1	2	2	2	4	5	6
#Fusion Rings	1	1	1	2	1	1	3	7	4	11	13	24

The types mentioned above, restricted to the non-pointed ones, are listed below:

- Rank 7: $[1, 1, 1, 1, 2, 2, 2]$,
- Rank 8: $[1, 1, 2, 2, 2, 2, 3, 3]$,
- Rank 9: $[1, 1, 1, 1, 4, 4, 6, 6, 6]$

- Rank 10: [1,1,1,1,2,2,2,4,4,4], [1,1,1,2,2,2,2,2,3], [1,1,2,3,3,4,4,4,6,6],
- Rank 11: [1,1,1,1,2,2,2,2,2,2,2], [1,1,1,1,2,6,6,8,12,12,12], [1,1,1,3,4,4,4,4,4,6], [1,1,1,1,4,4,12,12,18,18,18],
- Rank 12: [1,1,1,1,2,8,18,18,24,36,36,36], [1,1,1,3,6,8,8,8,8,8,12], [1,1,2,2,2,6,6,6,9,9], [1,1,2,3,3,6,6,8,8,12,12], [1,1,2,6,6,6,10,10,10,15,15].

The list of fusion rings referenced above can be found in [23]. They were classified utilizing the list from §7 in conjunction with the function `preSmatrix` contained within the file `ModularData.sage`, also available at [23].

3.2. T-matrix. For the remaining fusion rings \mathcal{R} with fusion data $(N_{i,j}^k)$, we address the Anderson-Moore-Vafa equations:

$$\left(\sum_{p=1}^r N_{i,j}^p N_{p,k}^l \right) (t_i + t_j + t_k + t_l) = \sum_{p=1}^r \left(N_{i,j}^p N_{p,k}^l + N_{i,k}^p N_{j,p}^l + N_{j,k}^p N_{i,p}^l \right) t_p$$

within the \mathbb{Z} -module \mathbb{Q}/\mathbb{Z} . For each valid solution $t = (t_i) \in (\mathbb{Q}/\mathbb{Z})^r$, if any, the corresponding T -matrix is $\text{diag}(e^{2\pi i t_i})$.

The solutions to the aforementioned equations are determined using the following method: Initially, the matrix reformulation is represented as $At = 0$, where A is an $m \times n$ matrix over \mathbb{Z} with $m = r^4$ and $n = r$. Subsequently, the Smith normal form is employed, denoted as $D = UAV$, in which U and V are invertible matrices over \mathbb{Z} of sizes $m \times m$ and $n \times n$, respectively, and D is a diagonal $m \times n$ matrix $(\alpha_i \delta_{i,j})$, where the integer α_i is divisible by α_{i+1} for all $i < r$, and $\delta_{i,j}$ is the Kronecker delta. The solutions to $Dx = 0$ are precisely represented by the vectors (k_i/α_i) , where $0 \leq k_i < \alpha_i$. Consequently, we have $U^{-1}DV^{-1}t = 0$, which simplifies to $DV^{-1}t = 0$. Therefore, the solutions can be expressed as $t = Vx$.

A complete listing of potential T -matrices requires considering all vectors (k_i/α_i) , where $0 \leq k_i < \alpha_i$. As a result, there are $p = \prod_i \alpha_i$ possible combinations. This task remains manageable up to rank 11. However, in certain rank 12 cases, the value of p becomes too large. But a miraculous circumstance arises (referred to as the **magic criterion**): for all such cases, if one abstractly considers the T -matrix with variables (k_i) , then for every determined S -matrix S in §3.1, the abstract product $(ST)^3$ consistently exhibits a zero where it should not, specifically at an entry (i, i^*) for some i . This is because $(ST)^3/p_+ = S^2 = dC$, where C is the duality matrix (realizing the involution $i \rightarrow i^*$, and thus $C_{i,i^*} = 1$, non-zero), as defined in Definition 2.11. The function `MagicCriterion` (also covered by the function `STmatrix`) can verify this. It effectively excludes exactly 33 fusion rings (8 distinct types) out of the 52 non-pointed ones (15 distinct types) resulting from §3.1. The remaining 7 types are [1,1,1,1,2,2,2], [1,1,2,2,2,2,3,3], [1,1,1,1,2,2,2,4,4,4], [1,1,1,2,2,2,2,2,3], [1,1,1,1,2,2,2,2,2,2], [1,1,2,2,2,2,6,6,6,9,9], [1,1,2,6,6,6,10,10,10,15,15]. For these types, some (but not all) of their possible S -matrices can still be excluded by this method.

Question 3.2. *Can the aforementioned magic criterion be reformulated at the level of fusion data?*

In conclusion, we retain only those S - and T -matrices that meet all criteria outlined in Definition 2.11. This process results in $19 + 62$ distinct modular data originating from $5 + 17$ fusion rings of $3 + 12$ types (non-pointed + pointed), thereby substantiating Theorem 1.2. This classification was executed by applying the function `STmatrix` or `STmatrix2` to the list of fusion rings discussed in §3.1, all of which can be found at [23]. When considering isomorphism classes, it is appropriate to adopt a *normal form* by sorting the basic elements according to their FPdim and spin, and limit basis permutations to those preserving both.

3.3. Model by Zesting. This subsection seeks to model certain modular data indicated in Theorem 1.2, predominantly through the process of zesting as delineated in [11]. The ensuing proposition is attributed to Eric C. Rowell.

Proposition 3.3. *The eight modular data delineated in §10.1.5 are derived from $SO(8)_2$, its conjugates, and zestings.*

Proof sketch. Commencing with $SO(8)_2$, one identifies that it is graded by the group $G = C_2 \times C_2$. This enables to twist the braiding by a bicharacter: the braiding is altered to $B(\deg(X), \deg(Y))c_{X,Y}$, where B represents the bicharacter. Correspondingly, the twists must be adjusted. This action exemplifies a specialized instance of braided (or ribbon) zesting. The resultant effect is the multiplication of specific rows and columns of the S -matrix by a sign. Upon inspecting the S -matrices itemized in §10.1.5, the rationale behind these variations should become apparent. Complex conjugation preserves the S -matrix while altering the T -matrix, thus providing a comprehensive explanation (notably, complex conjugation modifies the underlying fusion category). \square

4. EGYPTIAN FRACTIONS WITH SQUARED DENOMINATORS

A (q, r) -Egyptian fraction with squared denominators is defined as a sum of the form:

$$q = \sum_{i=1}^r \frac{1}{s_i^2},$$

where $q, r, s_i \in \mathbb{Z}_{\geq 1}$ and the sequence satisfies $s_1 \geq s_2 \geq \dots \geq s_r \geq 1$. Additionally, in the context of classifying potential types of Grothendieck rings for modular integral fusion categories (or more broadly, half-Frobenius integral fusion rings), we can assume that each s_i is a divisor of s_1 for all i . By repeatedly subtracting 1 from both q and r as necessary, we can further assume that $s_i \geq 2$ for all i (and so q is no more assumed square-free). Subsequently, we can augment the list of (q, r) -Egyptian fractions with squared denominators by including the $(q - k, r - k)$ variations, which are achieved by adding the number 1 to the sum k times. Using this technique, we can assume that $q \leq r/4$.

The following steps outline our methodology:

- Employ the function `ModularRep` provided in §4.2 for $1 \leq r \leq 12$ and $1 \leq q \leq r/4$.
- Refine the classification by incorporating additional 1s as described previously.
- Construct all possible types using $d_i = s_1/s_i$. The resulting list is presented in §4.1.

4.1. List of Types Up to Rank 13. Below is the count of possible types up to rank 13, based solely on Egyptian fractions with squared denominators. It also includes the count of perfect types (refer to §5):

Rank	1	2	3	4	5	6	7	8	9	10	11	12	13
#Types	1	1	1	1	2	3	3	7	11	42	144	812	7997
#Perfect Types	1	0	0	0	0	1	1	2	2	24	88	591	6517

The ratio of perfect types exhibits an increasing trend, e.g. 18% for rank 9, but 81% for rank 13. This leads us to question whether this ratio tends to 1 as the rank goes to infinity.

The list of all such (non-pointed) types up to rank 10 is provided (those up to rank 13 can be found online at [23]):

- Rank 5: $[[1, 1, 1, 1, 2]]$,
- Rank 6: $[[1, 1, 1, 1, 2, 2], [1, 2, 2, 3, 3, 3]]$,
- Rank 7: $[[1, 1, 1, 1, 2, 2, 2], [1, 2, 2, 3, 3, 3, 6]]$,
- Rank 8: $[[1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4], [1, 1, 2, 2, 2, 2, 3, 3], [1, 1, 3, 3, 4, 6, 6, 6], [1, 2, 2, 3, 3, 3, 6, 6], [1, 2, 2, 6, 6, 9, 9, 9]]$,
- Rank 9: $[[1, 1, 1, 1, 1, 1, 1, 2], [1, 1, 1, 1, 1, 2, 3, 3, 3], [1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6], [1, 1, 2, 2, 2, 3, 3, 6], [1, 1, 3, 3, 4, 6, 6, 6, 12], [1, 1, 4, 9, 9, 12, 18, 18, 18], [1, 2, 2, 3, 3, 3, 6, 6, 6], [1, 2, 2, 6, 6, 9, 9, 9, 18]]$,
- Rank 10: $[[1, 1, 1, 1, 1, 1, 1, 1, 3], [1, 1, 1, 1, 1, 1, 1, 2, 2], [1, 1, 1, 1, 1, 2, 3, 3, 3, 6], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [1, 1, 1, 2, 2, 3, 4, 6, 6, 6], [1, 1, 1, 2, 3, 8, 8, 12, 12, 12], [1, 1, 1, 7, 12, 28, 28, 42, 42, 42], [1, 1, 2, 2, 2, 3, 3, 6, 6], [1, 1, 2, 3, 4, 4, 4, 6, 6], [1, 1, 3, 3, 4, 6, 6, 6, 12, 12], [1, 1, 3, 3, 4, 12, 12, 18, 18, 18], [1, 1, 4, 4, 4, 5, 5, 10, 10, 10], [1, 1, 4, 9, 9, 12, 18, 18, 18, 36], [1, 1, 4, 12, 27, 27, 36, 54, 54, 54], [1, 2, 2, 2, 2, 2, 5, 5, 5], [1, 2, 2, 2, 6, 14, 14, 21, 21, 21], [1, 2, 2, 3, 3, 3, 3, 3, 3, 3], [1, 2, 2, 3, 3, 3, 6, 6, 6, 6], [1, 2, 2, 3, 3, 3, 6, 6, 6, 12], [1, 2, 2, 3, 6, 6, 6, 6, 9, 9], [1, 2, 2, 3, 9, 9, 12, 18, 18, 18], [1, 2, 2, 4, 5, 5, 5, 10, 10, 10], [1, 2, 2, 6, 6, 9, 9, 9, 18, 18], [1, 2, 2, 6, 6, 18, 18, 27, 27, 27], [1, 2, 3, 6, 10, 10, 10, 10, 15, 15], [1, 2, 3, 6, 15, 20, 30, 30, 30], [1, 3, 3, 3, 4, 4, 4, 4, 4, 6], [1, 3, 3, 3, 4, 6, 8, 12, 12, 12], [1, 3, 3, 3, 6, 16, 16, 24, 24, 24], [1, 4, 4, 4, 7, 7, 7, 14, 14, 14], [1, 4, 9, 28, 63, 63, 84, 126, 126, 126], [1, 5, 7, 35, 60, 140, 140, 210, 210, 210], [1, 5, 10, 18, 30, 30, 30, 30, 45, 45], [1, 5, 10, 18, 45, 45, 60, 90, 90, 90], [1, 6, 18, 38, 38, 114, 114, 171, 171, 171], [1, 12, 12, 17, 51, 51, 68, 102, 102, 102], [1, 18, 30, 70, 70, 210, 210, 315, 315, 315], [1, 70, 130, 182, 390, 910, 910, 1365, 1365, 1365]]$.

4.2. SageMath Code.

```
def ModularRep(q,r):
    L=all_rep(q, r)
    P=[]
    for l in L:
        if l[0]!=1: # those starting with 1 should be considered with q-1.
            k=0
            for ll in l:
                if l[-1]%ll!=0:
                    k=1
                    break
            if k==0:
                lll=[l[-1]/ll for ll in l]
                lll.sort()
                Di=sum([i^2 for i in lll])
                P.append(lll+[[sqrt(Di)]])
    return P

def res_rep(s, N):
    def succ(t):
        s0, m = t
        if s0==0 or len(m)>=N:
            return []
        p = numerator(s0)
        q = denominator(s0)
        if len(m)==N-1:
```



```

    if p==1 and is_square(q):
        r = q.isqrt()
        if r>=m[-1]:
            return [(0,m+(r,))]
    return []
L = max(m[-1], ((q-1)//p).isqrt()+1)
U = floor((N-len(m))/s0).isqrt()
if len(m)==N-2:
    S = []
    try:
        two_squares(p)
        two_squares(q)
    except:
        return S
    q2 = q^2
    for r in (L..U):
        d = p*r^2-q
        if d>0 and q2%d==0:
            r2 = (q2//d + q)//p
            if is_square(r2):
                S.append( (0,m+(r,r2.isqrt())) )
    return S
if len(m)==N-3:
    t = p*q
    a = valuation(t,2)
    if a%2==0 and (t>>a)%8==7:
        return []
    return ( (s0-1/r^2, m+(r,)) for r in (L..U) )
return RecursivelyEnumeratedSet(seeds=[(s-1/r^2,(r,)) for r in range(1,floor(N/s).isqrt()+1)], \
    successors=succ, structure='forest')

def all_rep(s, N):
    return res_rep(s,N).map_reduce(lambda t: {t[1]} if t[0]==0 and len(t[1])==N else set(), set.union, \
    set() )

def count_rep(s, N):
    return res_rep(s,N).map_reduce(lambda t: int(t[0]==0 and len(t[1])==N))

```

5. TYPE CRITERIA

In this section, we delineate criteria that were employed to exclude certain candidates from being the type of a fusion ring. A *type* refers to a list denoted by $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$, where the conditions $1 = d_1 < d_2 < \dots < d_s$ and $m_i \geq 1$ for all indices i are satisfied. Such a type is characterized as:

- *trivial* if $t = [[1, 1]]$,
- *pointed* if $t = [[1, m]]$ for some m ,
- *perfect* if $m_1 = 1$,
- *integral* if each d_i is an integer.

A type $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$ may sometimes be represented simply as

$$[d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_s, \dots, d_s],$$

where each d_i appears m_i times. Thus, we can rephrase the notation for a type of rank r as $[d_1, \dots, d_r]$ with the condition $1 = d_1 \leq d_2 \leq \dots \leq d_r$.

The criteria described herein are proved using modular arithmetic, and arranged in order of increasing computational complexity. They permit to exclude about 62% of the types presented in §4. Here is their counting:

Rank	1	2	3	4	5	6	7	8	9	10	11	12	13
# Types	1	1	1	1	2	3	3	7	11	42	144	812	7997
# Excluded Types	0	0	0	0	0	1	1	3	5	26	85	520	4970

For the remaining types, we will utilize the fusion ring solver, as elaborated in §6.

5.1. Small Perfect Type.

Theorem 5.1. *A perfect integral fusion ring of the type $[[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$, with $s \leq 3$, is trivial.*

Proof. If $s = 1$, then the type $t = [[1, 1]]$ is trivial. If $s = 2$, then we have a type $t = [[1, 1], [d, n]]$ with $d > 1$ and $n \geq 1$. However, should a fusion ring of type t exist, let b be a basic element with $\text{FPdim}(b) = d$. When applying FPdim to the decomposition of bb^* , we obtain that $d^2 = 1 + kd$ for some integer $k \geq 1$. Reducing this equation modulo d yields $0 = 1 \pmod{d}$, which is contradictory to $d > 1$. Lastly, if $s = 3$, then the type is $t = [[1, 1], [a, m], [b, n]]$, with $1 < a < b$ and $m, n \geq 1$. Suppose \mathcal{R} is a fusion ring of this type, with basic elements $1, a_1, \dots, a_m, b_1, \dots, b_n$.

Claim 5.2. *The greatest common divisor of a and b , denoted $a \wedge b$, is 1.*

Proof. Let $d = a \wedge b$. Then $\text{FPdim}(a_i a_i^*) = a^2 = 1 + \alpha a + \beta b$, but since d divides both a and b , we have $0 = 1 \pmod{d}$, which implies $d = 1$. \square

Claim 5.3. *For every i , there exists j such that $\langle a_i a_i^*, b_j \rangle \neq 0$.*

Proof. If this were not the case, then $a^2 = 1 + \alpha a$, leading to $1 = 0 \pmod{a}$, which contradicts the fact that $a > 1$. \square

Claim 5.4. *If $k \neq i^*$, then $\langle a_i a_k, b_j \rangle = 0$.*

Proof. If the claim were false, then $a^2 = \alpha a + \beta b$ with $\beta \neq 0$, which would mean that $\beta b = 0 \pmod{a}$. However, since $a \wedge b = 1$ (indicating that b is invertible modulo a), we get $\beta = b^{-1} \times 0 = 0 \pmod{a}$. Therefore, $\beta = ka$ for some $k \geq 1$. Now, since $a^2 = \alpha a + \beta b \geq \beta b = kab \geq ab$, we would have $a^2 \geq ab$, which contradicts the fact that $b > a$. \square

Claim 5.5. $a_{i^*} b_j = b a_{i^*}$.

Proof. By Frobenius reciprocity and Claim 5.4, if $k \neq i^*$ then $\langle a_{i^*} b_j, a_k \rangle = 0$. Claim 5.3 ensures that $\langle a_{i^*} b_j, a_{i^*} \rangle \neq 0$. We know $\text{FPdim}(a_{i^*} b_j) = ab = \alpha a + \beta b$, with $\alpha \geq 1$, leading to the conclusion that $\beta = 0 \pmod{a}$. Hence, $\beta = ka$ for some $k \geq 0$. As a result, $ab = \alpha a + kab$, which simplifies to $(1 - k)ab = \alpha a > 0$. This implies $(1 - k) > 0$ and thus $k < 1$. Therefore, $k = 0$ and $\beta = 0$. Combining the initial part of this proof with $\beta = 0$ indicates that $a_{i^*} b_j = \alpha a_{i^*}$, where α must equal b (determined by applying FPdim). \square

Claim 5.5, together with Frobenius reciprocity, leads us to deduce that $\langle a_i a_{i^*}, b_j \rangle = b$, which means that $a^2 \geq b^2$. This is in contradiction with $a < b$. \square

Remark 5.6. *Theorem 5.1 is not extendable to $s = 4$ because the representation category of the alternating group A_5 , denoted $\text{Rep}(A_5)$, is of type $[[1, 1], [3, 2], [4, 1], [5, 1]]$.*

By applying Theorem 5.1 to the list presented in §4, we can exclude the following four types (up to rank 13): $[[1, 1], [2, 2], [3, 3]]$, $[[1, 1], [2, 6], [5, 3]]$, $[[1, 1], [2, 2], [3, 7]]$, $[[1, 1], [3, 7], [4, 5]]$.

Corollary 5.7. *A non-trivial perfect integral fusion ring has a rank of at least 4.*

Proof. Suppose there is a perfect integral fusion ring with a rank less than 4. Then its type would be $[[d_1, m_1], \dots, [d_s, m_s]]$ with $s \leq r = \sum_i m_i \leq 3$, which contradicts Theorem 5.1. \square

5.2. Gcd Criterion.

Lemma 5.8. *Consider a non-pointed fusion ring of type $[d_1, d_2, \dots, d_r]$. For all i such that $d_i > 1$, let Z_i be the set of indices $j \neq 1$ for which N_{i, i^*}^j is nonzero, and let g_i be $\gcd(d_j \mid j \in Z_i)$. Then it holds that $d_i^2 \equiv 1 \pmod{g_i}$ and $\gcd(d_i, g_i) = 1$.*

Proof. First, note that Z_i is non-empty, which implies that $g_i \neq 0$. According to the Frobenius-Perron theorem, the dimension equation, and the Dual axiom, we have

$$d_i^2 = d_i d_{i^*} = \sum_k d_k N_{i, i^*}^k = 1 + \sum_{j \in Z_i} d_j N_{i, i^*}^j = 1 + K g_i,$$

where K is some integer. Consequently, $d_i^2 \equiv 1 \pmod{g_i}$, and $0 \equiv 1 \pmod{\gcd(d_i, g_i)}$. The lemma follows. \square

Proposition 5.9. *Consider a non-trivial perfect fusion ring of type $[d_1, d_2, \dots, d_r]$. Take $i > 1$, let Z'_i be the set of indices $j \neq 1$ for which $d_j < d_i^2$, and let g'_i be $\gcd(d_j \mid j \in Z'_i)$. Then $g'_i = 1$. In particular, $\gcd(d_2, \dots, d_r) = 1$.*

Proof. Note that if N_{i, i^*}^j is nonzero, then $d_i^2 \geq d_j$. Hence, following the notation in Lemma 5.8, Z_i is included in Z'_i , and as a result, g'_i divides g_i . Due to perfectness, we have $d_i > 1$, implying $d_i^2 > d_i$ and therefore i belongs to Z'_i . Consequently, g'_i divides d_i . However, according to Lemma 5.8, $g'_i = 1$. For the final assertion, note that $\gcd(d_2, \dots, d_r)$ is a divisor of $g'_2 = 1$. \square

Note that Proposition 5.9 excludes more than 37% of the perfect types listed in §4, for example, $[1, 2, 2, 6, 6, 9, 9, 9]$. Here is the count per rank:

Rank	8	9	10	11	12	13
# Excluded Perfect Types	1	1	7	19	212	2474

5.3. Type Test. Let's consider a type $t = [d_1, \dots, d_r]$ with $1 = d_1 \leq \dots \leq d_r$ and $d_2 > 1$ (signifying that it is perfect). If there is an index i and $g_i > 1$ such that g_i divides every d_j not equal to 1 or d_i , and d_i is coprime with g_i , then assume a fusion ring of this type exists with a basis $\{b_1, \dots, b_r\}$ where $d_k = \text{FPdim}(b_k)$.

Lemma 5.10. *For every j with $d_j \neq 1$ and $d_j \neq d_i$, the following equation holds:*

$$\sum_{k; d_k = d_i} N_{j,j^*}^k \equiv -1/d_i \pmod{g_i}.$$

Proof. For each j with $d_j \neq 1$ and $d_j \neq d_i$, we have:

$$b_j b_{j^*} = b_1 + \sum_{k; d_k = d_i} N_{j,j^*}^k b_k + \sum_{k; d_k \neq 1, d_i} N_{j,j^*}^k b_k.$$

By applying FPdim and reducing modulo g_i , we obtain:

$$0 = 1 + x d_i \pmod{g_i},$$

where d_i has a multiplicative inverse modulo g_i . Therefore, $x \equiv -1/d_i \pmod{g_i}$. \square

Given an integer a_{d_i} such that $0 \leq a_{d_i} < g_i$ and $a_{d_i} \equiv -1/d_i \pmod{g_i}$, let S be the set containing all such d_i . From Lemma 5.10, for every $j \neq 1$, the inequality below must hold:

$$d_j^2 \geq 1 + \sum_{d \in S \setminus \{d_j\}} a_d d,$$

thus if the inequality does not hold, t cannot be a type of a fusion ring. Furthermore, if the set $\{k \mid d_k = d_j\}$ is a singleton, we can use a stronger inequality:

$$d_j^2 \geq 1 + b_j d_j + \sum_{d \in S \setminus \{d_j\}} a_d d,$$

with $0 \leq b_j < g_j^2$ and $b_j \equiv d_j - \frac{1}{d_j} \pmod{g_j^2}$.

The SageMath code implementing this criterion can be found in the function `TypeTest` within the file `TypeCriteria.sage`, available at [23]. This criterion helped to exclude a certain number of perfect types per rank in the list from §4, as shown in the table below:

Rank	8	9	10	11	12	13
#Excluded Perfect Types	1	1	12	37	249	2380

5.4. Local Criterion. Consider a type $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$. Assume the existence of $g, i_0 > 1$ such that g divides each d_i for all indices i not in the set $\{1, i_0\}$, and d_{i_0} is coprime with g . Let $(d, m) := (d_{i_0}, m_{i_0})$. If t corresponds to a fusion ring with a basis $\{b_{1-m_1}, \dots, b_0, b_1, \dots, b_{r-1}\}$, where b_0 is the unit, $\text{FPdim}(b_i) = 1$ for $i \leq 0$, and $\text{FPdim}(b_j) = d$ for $j \in \{1, \dots, m\}$, then the following lemma applies:

Lemma 5.11. *For each $i \in \{1, \dots, m\}$, the equation below is valid:*

$$\sum_{j,k=1}^m N_{i,j}^k \equiv md - \frac{m_1}{d} \pmod{g^2},$$

and for all $j > m$, the integer g divides $\sum_{k=1}^m N_{i,j}^k$.

Proof. For any $i \in \{1, \dots, m\}$ and $j > m$, since $\text{FPdim}(b_i) \neq \text{FPdim}(b_j)$, by Corollary 2.9 and Frobenius reciprocity, we have:

$$b_i b_j = \sum_{k \geq 1} N_{i,j}^k b_k = \sum_{k=1}^m N_{i,j}^k b_k + \dots,$$

Applying FPdim and reducing modulo g , we conclude that:

$$d \sum_{k=1}^m N_{i,j}^k \equiv 0 \pmod{g},$$

which implies that g divides $\sum_{k=1}^m N_{i,j}^k$. For each $i \in \{1, \dots, m\}$, the sum over the basis elements yields:

$$b_{i*} \sum_{k=1}^m b_k = \sum_{s \leq 0} b_s + \sum_{j=1}^m \left(\sum_{k=1}^m N_{i,j}^k \right) b_j + \sum_{j>m} \left(\sum_{k=1}^m N_{i,j}^k \right) b_j.$$

After applying FPdim , we obtain $md^2 = m_1 + xd + yg^2$, hence $x \equiv md - \frac{m_1}{d} \pmod{g^2}$. \square

For a type t , we can analyze the partitions of $md^2 - xd - m_1$ in the form $\sum_{i \notin \{1, i_0\}} a_i d_i$, with $x \equiv md - \frac{1}{d} \pmod{g^2}$ and $a_i \equiv 0 \pmod{g}$. The SageMath code performing this analysis can be found in the function `LocalCriterion` within the file specified earlier, also available at [23]. This criterion, which can rule out types when no suitable partitions are found, is further detailed in the following example.

Example 5.12. Consider the type $t = [[1, 1], [1295, 2], [3990, 1], [4218, 1], [24605, 1], [42180, 1], [98420, 2], [147630, 3]]$. We can apply Lemma 5.11 to the triples $(d, m, g) = (1295, 2, 19), (3990, 1, 37), (4218, 1, 5)$. Subsequently, we obtain $md - \frac{1}{d} \equiv 126, 1135, 11 \pmod{g^2}$ for each respective triple. The application of the function `LocalCriterion` to the triple $(d, m, g) = (1295, 2, 19)$ enables us to eliminate the type t in less than one second.

```
sage: %time LocalCriterion(T, 1295, 2, 19)
CPU times: user 640 ms, sys: 0 ns, total: 640 ms
Wall time: 982 ms
[]
```

However, we cannot employ the triple $(d, m, g) = (3990, 1, 37)$, as it yields 55 solutions.

```
sage: L = LocalCriterion(T, 3990, 1, 37)
sage: len(L)
55
```

The application of `LocalCriterion` to the list in §4 led to the exclusion of several types per rank, as summarized in the following table:

Rank	6	7	8	9	10	11	12	13
# Excluded Types	1	1	3	5	21	63	344	2852
# Excluded Perfect Types	1	1	2	2	14	37	238	2173

It is noteworthy that this criterion alone suffices to eliminate all perfect types up to rank 9. Therefore, it can be stated conclusively that no non-trivial perfect integral half-Frobenius fusion rings, and thus no non-trivial perfect modular integral fusion categories, exist up to rank 9. The use of a fusion ring solver as detailed in §6 can extend this conclusion to rank 12, as discussed in §7.

6. ENHANCED FUSION RING SOLVER USING NORMALIZ

A fusion ring solver is a computational tool that accepts a specific type as input and returns all corresponding fusion rings of that type. In this section, we present two variations of a fusion ring solver: a full version discussed in §6.2 which handles dimension equations and associativity equations, and an intermediate version (involving a partition) covered in §6.3 that deals only with a simplified set of dimension equations. Initially, §6.1 introduces `Normaliz` [6], the software utilized in this process, and describes the customizations made to accommodate the special linear and polynomial constraints of fusion rings.

6.1. Introduction to Normaliz. `Normaliz` [6] is an open source software for discrete convex geometry and its algebraic aspects. Readers are referred to Bruns and Gubeladze [5] for detailed terminology and a comprehensive discussion. `Normaliz` is designed to solve Diophantine systems of linear inequalities, equations, and congruences with integer coefficients. Additionally, it calculates enumerative information such as multiplicities (which correspond to geometric volumes) and Hilbert series. Objects in `Normaliz` can be defined either by generators, such as the extreme rays of cones, bases of lattices, and vertices of polytopes, or by constraints like inequalities, equations, and congruences. For systems with coefficients in real algebraic number fields, `Normaliz` can execute fundamental operations like convex hull computation and its dual, vertex enumeration. Moreover, it is capable of computing lattice points within (bounded) polytopes over real algebraic number fields, facilitating applications to non-integral fusion rings. In the context of fusion rings, it is crucial that lattice points within polytopes can be subjected to constraints imposed by polynomial equations and inequalities. Each release of `Normaliz` includes source code, comprehensive documentation, sample examples, a testing suite, and pre-compiled binaries for Linux, Mac OS, and MS Windows systems.

For lattice points in generic polytopes denoted by P , `Normaliz` employs the project-and-lift algorithm. It sequentially projects P onto coordinate hyperplanes until reaching zero dimensions and then lifts the lattice points back up. If P' is a projection of P onto a coordinate hyperplane, then the lattice points of P are projected to lattice points in P' ,

and if $x \in P'$ is a lattice point within P' , its preimages are the lattice points in a line segment. Polynomial constraints can be introduced as soon as the lifting process reaches the highest coordinate present in the constraint.

In its standard form, the project-and-lift method is suitable for only minor cases of fusion rings. For satisfactory performance, the algorithm has been tailored to the special linear and polynomial constraint structure specific to fusion rings. Each linear equation is inhomogeneous with nonnegative coefficients and a positive right-hand side. We can refer to the set of coordinates that appear in the equation with positive coefficients as a "patch". These patches encompass the entire set of coordinates, and thus the linear equations, when restricted to the nonnegative orthant, delineate a polytope P . Solutions to a linear equation, confined to its patch, are ascertained using the project-and-lift technique previously outlined, and the lattice points in P are derived by combining these local solutions along matching components. In essence, we begin with the solutions of one of the equations and progressively extend them patch by patch. The sequence in which patches are integrated into the extension process is pivotal. Normaliz includes options that allow alteration of the sequence, as detailed later on.

The input files only include linear equations in their partition versions. Particularly for these cases, it is critical to recognize a secondary, implicit constraint type: congruences extracted from the linear equations by taking successive residue classes modulo their coefficients. By default, each congruence involves only the coordinates pertaining to the patch of its originating equation. Nonetheless, since congruences only involve a subset of these coordinates, they frequently pertain to other patches or combinations thereof, potentially significantly limiting their number of solutions. The simplification of rank 13 to just two unresolved cases (see §8.2) would not have been achievable without meticulous utilization of the congruences.

When polynomial equations of degree two or higher are in play, Normaliz endeavors to determine an optimal patch extension order that allows these equations to be applied as early as feasible. Users can influence this order by either insisting on the "linear" input order or by directing Normaliz to employ "weights" that gauge the anticipated solution count for each patch and prioritize those with lower weight. Regardless of whether polynomial equations are present, users can request an order based on the applicability of congruences. This order can also be weight-dependent.

Some computations for simple rank 13 were executed on the high-performance cluster (HPC) at Osnabrück by early splitting of partial solutions into parts, which were then processed separately. Despite the rather basic approach of using a static subdivision without intercommunication between running instances of Normaliz, the HPC proved to be advantageous.

6.2. Full Version. Consider a fusion ring with the basis $\{b_1, \dots, b_r\}$. As described in §2.1, for all indices i, j :

$$b_i b_j = \sum_k N_{i,j}^k b_k,$$

and by applying FPdim, we obtain the type $[d_1, \dots, d_r]$ and the corresponding *dimension equations*:

$$d_i d_j = \sum_k N_{i,j}^k d_k.$$

The objective is to resolve these r^2 linear positive Diophantine equations, where (d_i) are specified and $(N_{i,j}^k)$ represent r^3 variables, using Normaliz. Now, we can decrease the variable count to roughly $(r-1)^3/6$ by invoking the Unit axiom $(N_{1,i}^j = N_{i,1}^j = \delta_{i,j})$ from the Definition 2.1 of fusion data, as well as the Frobenius reciprocity (Proposition 2.2).

A critical factor in accelerating computation is the strategic use of associativity equations (non-linear)

$$\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t,$$

in the most effective manner possible during the solving process of the aforementioned linear Diophantine equations. While the optimal approach is not confirmed, the method we employ is highly efficient (refer to §6.1 for further details).

In practice, for a given type $L = [d_1, d_2, \dots, d_r]$, utilize the function `TypeToNormaliz`, the SageMath code for which can be found at [23]. This function generates input files (.in), one for each potential duality map $i \rightarrow i^*$. Place these files in a directory alongside the `normaliz.exe` and `run_normaliz.bat` files available at [23], and execute `run_normaliz` (note the existence of a more recent and faster Linux version used for our latest computations). This process yields output files (.out) containing all potential solutions (if any exist). The remaining task is to convert these solutions into fusion data, considering isomorphism. We demonstrate how this can be done with the following example. Take the type $L = [1, 1, 2]$ of the character ring of S_3 . When `TypeToNormaliz` is applied, it generates the file `[1,1,2][0,1,2].in` with the content as follows:

```
amb_space 4
inhom_equations 4
1 2 0 0 0
0 1 2 0 -2
0 1 2 0 -2
```

where s is not exceedingly large. This is because we can streamline the dimension equations by grouping elements that share the same dimension (i.e. dimension partition). However, the conversion of the associativity equations remains an open challenge. This version is intended to serve as an intermediary step to the full version for suitable types. Its utility lies in its ability to circumvent certain computational complexities by breaking symmetries. For the time being, it functions as a criterion; that is, if this version fails to yield a solution, the full version will similarly lack a solution.

We can reframe the type as $[1, d_{1,1}, \dots, d_{1,n_1}, d_{2,1}, \dots, d_{2,n_2}, \dots, d_{s,1}, d_{s,n_s}]$, where $d_{i,a} = d_i$, $d_1 = 1 = d_{0,1}$, and $n_i = m_i - \delta_{1,i}$. The dimension equations are then expressed as follows:

$$d_{i,a}d_{j,b} = \sum_{k,c} N_{i,a,j,b}^{k,c} d_{k,c}.$$

Let us define $D_i := \sum_{a=1}^{n_i} d_{i,a} = n_i d_i$ and $M_{i,j}^k := \sum_{a,b,c} N_{i,a,j,b}^{k,c}$, which simplifies the equations to:

$$D_i D_j = \sum_{a,b} \sum_{k,c} N_{i,a,j,b}^{k,c} d_{k,c} = \sum_k \left(\sum_{a,b,c} N_{i,a,j,b}^{k,c} \right) d_k = \sum_k M_{i,j}^k d_k.$$

Consequently, we are tasked with solving the linear positive Diophantine equations:

$$n_i d_i n_j d_j = \sum_k M_{i,j}^k d_k,$$

where (d_i, n_i) are predetermined, and the variables $(M_{i,j}^k)$ are reduced to roughly $s^3/6$ by employing the dimension partition variant of the Unit axiom and Frobenius reciprocity. After grouping by dimension, the duality map becomes straightforward (that is, $i^* = i$). Note that we have not yet derived a satisfactory dimension partition version of the associativity axiom, but about the other ones:

Lemma 6.2. *The following equalities hold:*

- (Unit) $M_{i,0}^j = M_{0,i}^j = \delta_{i,j} m_i$
- (Dual) $M_{i,j}^0 = M_{j,i}^0 = \delta_{i,j} m_i$
- (Frobenius reciprocity) $M_{i,j}^k = M_{i,k}^j = M_{j,k}^i = M_{j,i}^k = M_{k,i}^j = M_{k,j}^i$.

Proof. The proof is straightforward. □

In practice, one should follow the procedure outlined in §6.2 up to the generation of output files but replace the function `TypeToNormaliz` with `TypeToPreNormaliz`. For instance, consider the type $L = [1, 6, 12, 12, 15, 15, 15, 20, 20, 30, 30, 60]$. The corresponding input and output files can be found in the reference [23]. This dimension partition version is sufficiently robust to demonstrate Theorem 1.5 in instances without prime-power basic FPdim (refer to §7 immediately following Theorem 8.1), with L being the first of 24 types to be excluded at rank 12.

Remark 6.3. *While this version utilizes the dimension partition of the type, alternative versions could explore other pertinent partitions.*

7. HALF-FROBENIUS INTEGRAL FUSION RINGS UP TO RANK 12

Here is the count of half-Frobenius integral fusion rings and noncommutative ones up to rank 12.

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Fusion Rings	1	1	1	2	3	6	9	23	105	158	1218	9101
#Noncommutative	0	0	0	0	0	1	0	4	5	7	38	158

The comprehensive list of these fusion rings has been compiled and made accessible online as indicated in [23]. These rings were identified by implementing the type criteria detailed in §5 and utilizing the fusion ring solver discussed in §6 on the list of types referenced in §4. The count of types for each rank is relatively modest when contrasted with the table in §4.1.

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Types	1	1	1	1	2	2	2	4	5	9	15	28

Below is the list of (non-pointed) types for each rank, arranged in lexicographic order:

- Rank 5: $[[1, 1, 1, 1, 2]]$,
- Rank 6: $[[1, 1, 1, 1, 2, 2]]$,
- Rank 7: $[[1, 1, 1, 1, 2, 2, 2]]$,
- Rank 8: $[[1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4], [1, 1, 2, 2, 2, 2, 3, 3]]$,
- Rank 9: $[[1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6]]$,
- Rank 10: $[[1, 1, 1, 1, 1, 1, 1, 1, 3], [1, 1, 1, 1, 1, 1, 1, 2, 2], [1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 12], [1, 1, 1, 2, 2, 2, 2, 2, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6]]$,
- Rank 11: $[[1, 1, 1, 1, 1, 1, 1, 1, 3, 3], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 1, 1, 1, 2, 2, 4], [1, 1, 1, 1, 1, 1, 2, 2, 3, 3], [1, 1, 1, 2, 2, 2, 2, 3, 6], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8], [1, 1, 1, 1, 2, 4, 4, 4, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 4, 6, 6, 12], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 12], [1, 1, 1, 1, 4, 4, 6, 6, 6, 6, 12, 12], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18]]$,

- Rank 12: $[[1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3], [1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 1, 1, 1, 2, 2, 4, 4], [1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 6], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 4], [1, 1, 1, 1, 2, 2, 2, 4, 6, 6, 6], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4, 4], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8, 8], [1, 1, 1, 1, 2, 2, 2, 8, 8, 12, 12, 12], [1, 1, 1, 1, 2, 4, 4, 4, 4, 6, 6, 12], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 24], [1, 1, 1, 1, 2, 8, 18, 18, 24, 36, 36, 36], [1, 1, 1, 1, 3, 3, 3, 3, 4, 4, 6, 6], [1, 1, 1, 1, 4, 4, 6, 6, 12, 12, 12], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18, 36], [1, 1, 1, 2, 2, 2, 2, 2, 3, 6, 6], [1, 1, 1, 2, 2, 2, 3, 4, 4, 4, 6, 6], [1, 1, 1, 3, 4, 4, 4, 4, 4, 6, 12], [1, 1, 1, 3, 6, 8, 8, 8, 8, 8, 12], [1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3], [1, 1, 2, 2, 2, 3, 3, 6, 6, 6, 6], [1, 1, 2, 2, 2, 3, 3, 6, 6, 6, 12], [1, 1, 2, 2, 2, 6, 6, 6, 6, 9, 9], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12, 12], [1, 1, 2, 3, 3, 6, 6, 8, 8, 8, 12, 12], [1, 1, 2, 6, 6, 6, 6, 10, 10, 10, 15, 15]]$.

Observe that the list above contains no non-trivial perfect types, which proves Theorem 1.5. It is noteworthy that the perfect integral modular fusion category (and therefore half-Frobenius) $\mathcal{Z}(\text{Rep}(A_5))$ has an FPdim of $60^2 = 3600$, a rank of 22, and a type of $[[1, 1], [3, 2], [4, 1], [5, 1], [12, 10], [15, 4], [20, 3]]$. This category was calculated using GAP, as described in [12, §8.5].

Question 7.1. *Is there a perfect integral half-Frobenius fusion ring/category with a rank less than 22?*

8. RANK 13

From rank 13 onwards, it becomes impractical to classify all half-Frobenius integral fusion rings using our current technology. Therefore, when necessary, we will either assume the commutativity of the ring or directly apply certain modular criteria (see [19, Proposition VI.2] and [4]). This approach yields results that are less general at the fusion ring level but still sufficiently comprehensive for general modular data.

8.1. General. The process commenced, as previously, with 7997 half-Frobenius types derived from Egyptian fractions with squared denominators, as described in §4. Initially, more than 60% were dismissed based on type criteria outlined in §5. Subsequently, over 90% were eliminated using the partition version in §6.3, restricted to a quick use, leaving only 212 types for further consideration. Among these, 53 are identified as perfect without any prime-power entries (§8.2), 24 are perfect but include a prime-power entry (§8.3), and the remaining 135 are non-perfect (§8.4); see `Rank13ReducedList` in [23].

8.2. Simple. We turn our attention to the study of simple integral modular fusion categories of rank 13. The following theorem provides a significant constraint on the types occurring in such categories.

Theorem 8.1 (Corollary 6.16 in [21]). *Let \mathcal{C} be an integral modular fusion category. If \mathcal{C} contains a simple object whose FPdim is a prime-power, then it must have a nontrivial symmetric subcategory.*

Consequently, our analysis is narrowed down to perfect integral half-Frobenius types that lack any prime-power entries. From §4 and the list in [23], precisely 2044 such types have been classified, but as explained in §8.1, we can quickly reduce to 35 types only by using §5 and §6.3. A more extensive use (involving HPC) of §6.3, and then §6.2 assuming the commutativity, reduces to 11, and then 2 types below in progress (mentioned in Theorem 1.8):

1. $[1, 238, 459, 540, 595, 918, 5355, 9180, 21420, 21420, 32130, 32130, 32130]$,
2. $[1, 777, 1036, 1295, 3990, 4218, 24605, 42180, 98420, 98420, 147630, 147630, 147630]$.

Note that the perfect integral modular fusion category $\mathcal{Z}(\text{Rep}(A_7))$, with FPdim $(7!/2)^2$, rank 74, and type:

$$[[1, 1], [6, 1], [10, 2], [14, 2], [15, 1], [21, 1], [35, 1], [70, 9], [105, 4], [210, 20], [280, 9], [360, 14], [504, 5], [630, 4]],$$

notably lacks any basic elements whose FPdim is a prime-power.

Question 8.2. *Is there a perfect integral half-Frobenius fusion ring/category, without any basic elements of prime-power FPdim, that has a rank lower than 74?*

8.3. Perfect Non-Simple. According to §8.2, a perfect type that includes a prime-power entry cannot characterize a simple integral modular fusion category. From §4 and the list in [23], exactly 4473 such types have been identified. However, as discussed in §8.1, employing the tools from §5 and §6.3 allows us to rapidly narrow this down to only 24 types. Subsequently, all of these types can be ruled out, by §6.2, using relatively short computations. The exception is the type $[1, 20, 20, 27, 27, 30, 45, 54, 180, 180, 270, 270, 270]$, which requires a longer, yet still reasonable, computation time.

8.4. Non-Perfect. From §4 and the list in [23], precisely 1480 non-perfect types have been classified, but as explained in §8.1, we can quickly reduce to 135 types only by using §5 and §6.3. Before applying the fusion ring solver in §6.2, we will filter our list of types by criteria specific to the modular case involving the grading structure. Let G be a finite group. A G -grading of a fusion ring R is given by a partition $B = \sqcup_{g \in G} B_g$ of its basis such that:

- For any $x \in B_g$ and any $y \in B_{g'}$, the basic components of xy belong to $B_{gg'}$.
- For any $x \in B_g$, x^* is in $B_{g^{-1}}$.

A G -grading is called *faithful* if B_g is non-empty for all $g \in G$. Consequently, by [12, Theorem 3.5.2], $\text{FPdim}(B_g) := \sum_{x \in B_g} \text{FPdim}(x)^2$ is constant in g . The faithful grading with the largest group is called the *universal grading*. By [12, Lemma 8.22.9], the universal grading group of the Grothendieck ring of modular fusion category is $G = B_{pt}$, the group of basic element with $\text{FPdim} = 1$ (see Corollary 2.10).

Theorem 8.3. *Let \mathcal{C} be an integral modular fusion category. Let R be its Grothendieck ring with basis B . Let $G = B_{pt}$ be the universal grading group. Let $T_g := (\text{FPdim}(x))_{x \in B_g}$. Let \mathcal{C}_e be the fusion subcategory corresponding to B_e . Then:*

- (0) *If \mathcal{C}_e is perfect then it is modular,*
- (1) *If $B_{pt} \subset B_e$ and T_e has an entry with odd multiplicity, then $\forall g \neq e$, every entry of T_g has multiplicity ≥ 2 ,*
- (2) *If (1) holds, if $p := |B_{pt}|$ is prime, and if an entry d appears with multiplicity one in T_e , then p divides d .*

Proof. Immediate from [19, Proposition VI.2] as a group of prime order must be cyclic. \square

The application of Theorem 8.3 reduces our list of 135 types to 25 types. For so, we used the function `squareaquipartitiontype` in `EquiPartition.sage` in [23] classifying all the possible *modular partitions* of a type, i.e. corresponding to the universal grading of a modular fusion category. Here are three examples of exclusion using each point of Theorem 8.3:

- (0) In the partitioned type $[[1, 2, 2, 3, 3, 3, 3, 3], [1, 2, 2, 3, 6]]$, the neutral component is perfect, but we already know that there is no perfect integral modular fusion category of rank 8,
- (1) In the partitioned type $[[1, 1, 1, 1, 2, 2, 2, 3, 10, 10], [15], [15], [15]]$, the pointed part is in the neutral component T_e , and the entry 2 has multiplicity three (odd) in T_e , but 15 appears with multiplicity one in some non-neutral components.
- (2) In the partitioned type $[[1, 1, 2, 2, 3, 3, 5, 6, 6, 10, 15], [15, 15]]$, the pointed part is in the neutral component T_e , and the entry 5 has multiplicity one (odd) in T_e , but 5 is not divisible by the prime $2 = |B_{pt}|$.

The details of the computation are available in `Rank13NonPerfectGradingReduction.txt` in [23].

Next, the application of §6.2 reduces the remaining 25 types to the following 15 types having fusion rings:

1. $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$,
2. $[1, 1, 1, 1, 2, 2, 4, 4, 4, 4, 4, 4, 6]$,
3. $[1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 10, 10, 10]$,
4. $[1, 1, 1, 1, 4, 4, 6, 6, 6, 6, 6, 6, 6]$,
5. $[1, 1, 1, 1, 4, 4, 6, 12, 12, 12, 12, 18, 18]$,
6. $[1, 1, 1, 1, 4, 4, 12, 12, 36, 36, 54, 54, 54]$,
7. $[1, 1, 1, 3, 6, 12, 16, 16, 16, 16, 16, 16, 24]$,
8. $[1, 1, 1, 3, 12, 12, 20, 20, 20, 20, 20, 20, 30]$,
9. $[1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 6, 6]$,
10. $[1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 6]$,
11. $[1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 6]$,
12. $[1, 1, 2, 2, 2, 2, 3, 3, 12, 12, 18, 18, 18]$,
13. $[1, 1, 2, 2, 2, 2, 6, 6, 6, 6, 9, 9, 18]$,
14. $[1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12, 12, 12]$,
15. $[1, 1, 2, 3, 3, 24, 30, 30, 40, 40, 40, 60, 60]$.

plus the following one for which the existence of a fusion ring is open.

$$[1, 1, 1020, 1292, 1710, 11628, 14535, 14535, 19380, 19380, 19380, 29070, 29070]$$

So let us first exclude it from being the type of a modular category. It would admit a unique modular partition:

$$[[1, 1, 1020, 1292, 1710, 11628, 14535, 14535, 19380, 19380, 19380], [29070, 29070]],$$

and by [19, Proposition VI.2 (e)] citing [4], the *modularization* of the neutral component \mathcal{C}_e would be of type in:

- $[1, 510, 510, 646, 646, 855, 855, 5814, 5814, 9690, 9690, 14535, 19380]$,
- $[1, 510, 510, 646, 646, 855, 855, 5814, 5814, 9690, 9690, 9690, 9690, 9690, 14535]$,

but the first one is not half-Frobenius, so cannot be the type of a modular fusion category, and the second is excluded instantaneously by the partition version of the fusion ring solver outlined in §6.3.

Next, among the 15 types above, only 6 ones have cyclotomic self-transposable fusion rings (see §3.1), namely 1. 3. 6. 7. 8. and 15. Some of them could also be excluded by considering the modularization of the neutral component, but it was not required here. Finally, all except 1. are quickly excluded by the magic criterion (see §3.2), which reduces to the pointed case, and so completes the proof of Theorem 1.8.

9. THE ODD-DIMENSIONAL CASE

For an overview of the current state of knowledge on odd-dimensional modular fusion categories, we refer the reader to [9, 10]. A foundational result in this area establishes that an odd-dimensional modular fusion category \mathcal{C} is equivalent to being maximally non self-dual (MNSD), meaning that its only self-dual simple object is the unit object. Let $(d_i)_{i \in I}$ represent the FPdim of the simple objects in \mathcal{C} , considered up to isomorphism. Since d_i^2 is a divisor of the odd FPdim(\mathcal{C}), each d_i must be odd. Furthermore, the equation $\sum_{i \in I} d_i^2 = \text{FPdim}(\mathcal{C})$ implies that the rank $r = |I|$ must also be odd. This reduces our investigation to Egyptian fractions of the form $q = \sum_{i=1}^r \frac{1}{s_i^2}$, where $q, r, s_i \in \mathbb{Z}_{\geq 1}$, $s_1 \geq \dots \geq s_r \geq 1$, and both r and s_i are odd. Additionally, s_i divides s_1 for all i , and $s_{2k} = s_{2k+1}$. This yields the expression

$$q = \frac{1}{s_1^2} + \sum_{k=1}^{(r-1)/2} \frac{2}{s_{2k}^2}.$$

Since each s_i is odd, we have $s_i^2 \equiv 1 \pmod{8}$, which implies $q \equiv r \pmod{8}$ and that q is odd as well. Utilizing a similar technique as in §4, we can assume $s_i > 1$ (hence $s_i \geq 3$), by completing the classification with additional 1s if necessary. Consequently, we can assume $q \leq r/9$. For $r < 27$, this allows us to deduce that $q = 1$, and therefore $r \equiv 1 \pmod{8}$, which narrows the possibilities for r to 1, 9, 17, 25 (up to completing by 1s).

Remark 9.1. *This strategy can be extended. For instance, by adding eighteen 3s to complete the classification, we may assume that $s_i = 5$ for $i + 16 \leq r$, which leads to $q \leq 16/9 + (r - 16)/25$. If $r < 47$ (which becomes 51 because $q \equiv r \pmod{8}$), we can assume that $q = 1$. However, this extended strategy will not be applied in this paper.*

Consequently, for all $r < 25$, we have compiled the following list of all possible non-pointed types (as for §4):

- $[[1, 9], [3, 8], [81, 2a]]$,
- $[[1, 7], [3, 2], [5, 8], [225, 2a]]$,
- $[[1, 3], [3, 8], [5, 6], [225, 2a]]$,
- $[[1, 1], [3, 2], [7, 2], [9, 4], [21, 8], [3969, 2a]]$,
- $[[1, 1], [9, 4], [25, 2], [45, 2], [75, 8], [50625, 2a]]$,

where $a \geq 0$ represents the number of 1s added for completion. It is noteworthy that these ranks are $17 + 2a$, which corroborates a result from [9] stating that any odd-dimensional modular fusion category with rank less than 17 is pointed. Further, [9] establishes that any perfect odd-dimensional modular fusion category is a Deligne product of simple categories. From the preceding analysis, a non-pointed one must have a rank of at least 17, meaning a perfect non-simple one must have a rank of at least 289 ($= 17^2$). Therefore, a perfect one with a rank less than 289 must be simple and cannot have non-trivial simple objects of prime-power FPdim, as shown in [21, Corollary 6.16]. Consequently, the previously mentioned perfect types are excluded. It follows that any perfect odd-dimensional modular fusion category with a rank under 25 must be trivial, and so:

Theorem 9.2. *Every perfect odd-dimensional modular fusion category of rank less than 625 ($= 25^2$) is simple.*

Now, we complete the proof of Theorem 1.9. We need to address the non-perfect types above. As outlined in §8.4, the modular grading results in a partition indexed by the pointed part, with each component having the same FPdim.

- First, let's examine the type $[[1, 9], [3, 8], [81, 2a]]$. Its rank is $17 + 2a < 25$, which implies $a < 4$. The FPdim for this type is $81(1 + 81a)$. Consequently, each partition component must have FPdim $= 9(1 + 81a)$. If $a > 0$, a component with 81 must have FPdim $\geq 81^2$. This leads to $81^2 \leq 9(1 + 81a)$, resulting in $a > 8$, a contradiction. Therefore, $a = 0$. The modular partition then must be $[[1, 1, 1, 1, 1, 1, 1, 1], [3], [3], [3], [3], [3], [3], [3], [3]]$, which contradicts Theorem 8.3 (1).
- Regarding the second type $[[1, 7], [3, 2], [5, 8], [225, 2a]]$, its FPdim is $225(1 + 225a)$. However, 7 is not a divisor of 225, which is a contradiction.
- Lastly, for the third type $[[1, 3], [3, 8], [5, 6], [225, 2a]]$, its FPdim is also $225(1 + 225a)$. Therefore, each partition component must have FPdim $= 75(1 + 225a)$. Similar to the first type, if $a > 0$ then $2 < a < 4$, leading to $a = 3$. This gives $75(1 + 225a) = 225^2 + 75$. Thus, a component can contain at most one entry equal to 225. However, with only three components for six entries of 225, this results in a contradiction. Hence, $a = 0$. Following this, applying the fusion ring solver outlined in §6.2 to the type $[[1, 3], [3, 8], [5, 6]]$ yields two fusion rings. Applying `STmatrix2` to these fusion rings provides three modular data, detailed in §11.

Remark 9.3. *As highlighted in Remark 1.10, potential gaps have been identified in the literature:*

- (1) In [2, Theorem 4.2, proof of Case (viii) FPdim(\mathcal{C}_{pt}) = p], on page 727, the assertion that the anomaly-free nature of \mathcal{C} , meaning its Gauss sums are equal ($p_+ = p_-$), necessarily leads to $p_+ = pq$ is unclear. The MD described in §11, which align with this case, satisfy the equation on top of page 727, and are anomaly-free,

yet exhibit a Gauss sum $p_+ = -pq$. Similarly, the two (categorifiable) MD discussed in §10.1.2 with a central charge $c = 4$, have $p_+ = -pq$ for one and pq for the other, as also verified in [19].

- (2) In [9, Theorem 6.3 (b), proof of Case $|\mathcal{G}(C)| = 3$], on page 1936, the deduction “Hence $l \leq 24$ ” in the seventh last line is accurate, except when $c_{X_1} = 1$, which permits $l = 5$, thereby accommodating the type $[[1, 3], [3, 8], [5, 6]]$.

There are non-pointed and non-perfect odd-dimensional modular fusion categories of rank 25, exemplified by $\mathcal{Z}(\text{Vec}_{C_7 \rtimes C_3}^\omega)$. Furthermore, [10] demonstrates that, up to equivalence, no additional such examples exist. Consequently, our attention must now turn to the examination of the perfect case (which is simple and have $q = 1$). We have identified precisely 91 possible types (listed in [23]) using the aforementioned method combined with [21, Corollary 6.16]. Subsequent application of the type criteria from §5 reduces this to 29 types, and the fusion ring solver in §6.3 quickly eliminates 8 more types. The remaining 21 types (currently intractable) are:

1. $[[1, 1], [15, 2], [63, 2], [115, 2], [161, 4], [315, 2], [805, 2], [1449, 2], [2415, 8]]$,
2. $[[1, 1], [21, 2], [35, 2], [63, 2], [85, 2], [119, 2], [255, 2], [595, 2], [1071, 2], [1785, 8]]$,
3. $[[1, 1], [39, 4], [65, 2], [189, 2], [315, 2], [585, 2], [1365, 2], [2457, 2], [4095, 8]]$,
4. $[[1, 1], [39, 2], [231, 2], [273, 2], [1001, 2], [1287, 2], [3465, 2], [4095, 2], [9009, 2], [15015, 8]]$,
5. $[[1, 1], [39, 2], [231, 2], [273, 2], [1001, 2], [3465, 2], [4095, 2], [6435, 4], [15015, 8]]$,
6. $[[1, 1], [45, 2], [95, 6], [99, 2], [495, 2], [1045, 2], [1881, 2], [3135, 8]]$,
7. $[[1, 1], [75, 2], [91, 4], [175, 2], [585, 2], [975, 2], [2275, 2], [4095, 2], [6825, 8]]$,
8. $[[1, 1], [75, 2], [91, 4], [175, 2], [975, 2], [2275, 2], [2925, 4], [6825, 8]]$,
9. $[[1, 1], [99, 2], [195, 2], [385, 2], [405, 2], [15015, 2], [31185, 2], [36855, 2], [81081, 2], [135135, 8]]$,
10. $[[1, 1], [99, 2], [231, 2], [385, 2], [675, 2], [10395, 4], [28875, 2], [51975, 2], [86625, 8]]$,
11. $[[1, 1], [99, 2], [273, 2], [1001, 2], [1365, 2], [2145, 2], [3465, 4], [9009, 2], [15015, 8]]$,
12. $[[1, 1], [135, 4], [165, 2], [189, 2], [315, 2], [385, 2], [1155, 2], [2079, 2], [3465, 8]]$,
13. $[[1, 1], [195, 4], [231, 2], [1001, 2], [1287, 2], [3465, 2], [4095, 2], [9009, 2], [15015, 8]]$,
14. $[[1, 1], [195, 4], [231, 2], [1001, 2], [3465, 2], [4095, 2], [6435, 4], [15015, 8]]$,
15. $[[1, 1], [205, 2], [369, 2], [805, 2], [6601, 2], [8487, 2], [12915, 2], [33005, 2], [59409, 2], [99015, 8]]$,
16. $[[1, 1], [205, 2], [369, 2], [805, 2], [6601, 2], [12915, 2], [33005, 2], [42435, 4], [99015, 8]]$,
17. $[[1, 1], [231, 2], [777, 2], [1155, 2], [1221, 2], [3465, 2], [6105, 2], [14245, 2], [25641, 2], [42735, 8]]$,
18. $[[1, 1], [297, 2], [385, 2], [3465, 2], [5859, 2], [7161, 2], [15345, 2], [35805, 2], [64449, 2], [107415, 8]]$,
19. $[[1, 1], [693, 2], [1771, 2], [4301, 2], [11781, 2], [24633, 2], [30107, 4], [38709, 2], [90321, 8]]$,
20. $[[1, 1], [693, 2], [1771, 2], [11781, 2], [12903, 4], [24633, 2], [38709, 4], [90321, 8]]$,
21. $[[1, 1], [3485, 2], [6273, 2], [33825, 2], [34425, 2], [621027, 4], [1725075, 2], [3105135, 2], [5175225, 8]]$.

10. CATALOGUE OF INTEGRAL MODULAR DATA UP TO RANK 12

This section presents a comprehensive catalogue of integral modular data (MD) up to rank 12. Notably, all the MD listed here are categorifiable, meaning each corresponds to the MD of an integral modular fusion category. These corresponding categorical models are identified in Theorem 1.2. We will begin by listing the non-pointed MD in §10.1, followed by the pointed MD in §10.2. Additionally, these data can be accessed in a machine-readable format in [23].

10.1. Non-Pointed Modular Data Up To Rank 12. The data are organized in the following sequence:

- Rank,
- Type,
- Fusion coefficients,
- List of topological spins (s_i) giving the T-matrix (see Definition 2.11),
- Cyclotomic conductor of the S -matrix,
- Cyclotomic conductor of the T -matrix (its order),
- Central charge,
- List of 2nd Frobenius-Schur indicators,
- S -matrix.

10.1.1. Rank 8, type $[1, 1, 2, 2, 2, 3, 3]$, first fusion ring.

1 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0	0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0	0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0	0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0	0 0 1 0 0 0 0 0	1 1 0 0 0 1 0 0	0 0 0 1 1 0 0 0	0 0 0 1 0 1 0 0	0 0 1 0 1 0 0 0	0 0 0 0 0 0 1 1	0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0	0 0 0 1 0 0 0 0	0 0 0 1 1 0 0 0	1 1 1 0 0 0 0 0	0 0 1 0 0 0 0 0	0 0 1 0 0 1 0 0	0 0 0 0 0 0 1 1	0 0 0 0 0 0 0 1
0 0 0 0 1 0 0 0	0 0 0 0 1 0 0 0	0 0 0 1 0 1 0 0	0 0 0 1 0 0 1 0	1 1 0 0 1 0 0 0	0 0 1 1 0 0 0 0	0 0 0 0 0 0 1 1	0 0 0 0 0 0 0 1
0 0 0 0 0 1 0 0	0 0 0 0 0 1 0 0	0 0 1 0 1 0 0 0	0 0 0 1 0 1 0 0	0 0 1 1 0 0 0 0	1 1 0 1 0 0 0 0	0 0 0 0 0 0 1 1	0 0 0 0 0 0 0 1
0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 1	0 0 0 0 0 0 1 1	0 0 0 0 0 0 1 1	0 0 0 0 0 0 1 1	0 0 0 0 0 0 1 1	0 0 0 0 0 0 1 1
0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 1	0 0 0 0 0 0 1 1	0 0 0 0 0 0 1 1	0 0 0 0 0 0 1 1	0 0 0 0 0 0 1 1	0 1 1 1 1 1 0 0	0 1 1 1 1 1 0 0

- $[0, 0, -1/3, -1/3, 1/3, 1/3, 1/2, 0], 1, 6, 4, [1, 1, 1, 1, 1, 1, -1, -1].$

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \\ 3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \end{bmatrix}$$

- $[0, 0, -1/3, 1/3, 0, 0, -1/4, 1/4], 1, 12, 0, [1, 1, 1, 1, 1, 1, -1, -1].$

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \\ 3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \end{bmatrix}$$

- $[0, 0, -1/3, -1/3, 1/3, 1/3, -1/4, 1/4], 1, 12, 4, [1, 1, 1, 1, 1, 1, 1, 1],$

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \end{bmatrix}$$

10.1.3. *Rank 10, type $[1, 1, 1, 2, 2, 2, 2, 2, 2, 3]$, first fusion ring.*

[illegible]

- $[0, 0, 0, 1/9, 4/9, -2/9, -2/9, 4/9, 1/9, 1/2], 9, 18, 4, [1, 0, 0, 0, 0, 0, 0, 0, 0, 1].$

1	1	1	2	2	2	2	2	2	3
1	1	1	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_3^2$	$2\zeta_3$	3
1	1	1	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_3$	$2\zeta_3^2$	3
2	$2\zeta_3^2$	$2\zeta_3$	$-2\zeta_9^7$	$-2\zeta_9^5$	$2\zeta_9^2 + 2\zeta_9^5$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^4$	$-2\zeta_9^2$	0
2	$2\zeta_3$	$2\zeta_3^2$	$-2\zeta_9^5$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^7$	$-2\zeta_9^2$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^4$	0
2	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^7$	$-2\zeta_9^4$	$-2\zeta_9^5$	$-2\zeta_9^2$	$2\zeta_9^4 + 2\zeta_9^7$	0
2	$2\zeta_3^2$	$2\zeta_3$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^2$	$-2\zeta_9^5$	$-2\zeta_9^4$	$-2\zeta_9^7$	$2\zeta_9^2 + 2\zeta_9^5$	0
2	$2\zeta_3^2$	$2\zeta_3$	$-2\zeta_9^4$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^2$	$-2\zeta_9^7$	$2\zeta_9^4 + 2\zeta_9^7$	$-2\zeta_9^5$	0
2	$2\zeta_3$	$2\zeta_3^2$	$-2\zeta_9^2$	$-2\zeta_9^4$	$2\zeta_9^4 + 2\zeta_9^7$	$2\zeta_9^2 + 2\zeta_9^5$	$-2\zeta_9^5$	$-2\zeta_9^7$	0
3	3	3	0	0	0	0	0	0	-3

- $[0, 0, 0, 0, -5/16, 7/16, 7/16, -1/4, -1/16, -1/16, 3/16], 8, 16, -1, [1, 1, 1, 1, 1, -1, -1, 1, -1, -1, 1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, 7/16, -5/16, -1/4, 3/16, -1/16, 3/16], 8, 16, -1, [1, 1, 1, 1, -1, 1, -1, 1, -1, 1, -1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, -5/16, -5/16, -1/4, 3/16, 3/16, 3/16], 8, 16, -1, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -1/16, -1/16, -1/16, -1/4, 7/16, 7/16, 7/16], 8, 16, -1, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 \\ 2 & -2 & 2 & -2 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 & 0 \\ 2 & -2 & 2 & -2 & 0 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 & 0 \\ 2 & -2 & -2 & 2 & -2\zeta_8 + 2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8 - 2\zeta_8^3 \end{bmatrix}$$

10.2. Pointed Modular Data Up To Rank 12. In light of Remark 1.1, it suffices to specify only the abelian groups and their corresponding T-matrices, which encapsulate the topological spins in this context. Let us denote by C_n the cyclic group of order n . The complete data set, akin to the format presented in §10.1, is available at [23].

- C_1 : $[0]$,
- C_2 : $[0, 1/4], [0, -1/4]$,
- C_3 : $[0, 1/3, 1/3], [0, -1/3, -1/3]$,
- C_2^2 : $[0, 1/2, 0, 0], [0, -1/4, 1/4, 0], [0, 1/2, 1/4, 1/4], [0, 1/2, 1/2, 1/2], [0, -1/4, -1/4, 1/2]$,

- C_4 : $[0, 1/2, 1/8, 1/8], [0, 1/2, 3/8, 3/8], [0, 1/2, -3/8, -3/8], [0, 1/2, -1/8, -1/8]$,
- C_5 : $[0, 1/5, -1/5, -1/5, 1/5], [0, 2/5, -2/5, -2/5, 2/5]$,
- C_6 : $[0, -1/4, 1/12, 1/3, 1/3, 1/12], [0, -1/4, 5/12, -1/3, -1/3, 5/12], [0, 1/4, -5/12, 1/3, 1/3, -5/12], [0, 1/4, -1/12, -1/3, -1/3, -1/12]$,
- C_7 : $[0, 1/7, -3/7, 2/7, 2/7, -3/7, 1/7], [0, 3/7, -2/7, -1/7, -1/7, -2/7, 3/7]$,
- C_8^2 : $[0, -1/4, 1/4, 1/4, 1/4, 1/2, 0, 0], [0, -1/4, -1/4, -1/4, 1/4, 1/2, 0, 0], [0, -1/4, 1/2, 1/2, 1/4, 1/2, 1/4, 1/4], [0, -1/4, 1/2, -1/4, 1/2, -1/4, 1/4]$,
- $C_2 \times C_4$: $[0, 1/2, -1/4, 1/4, 1/8, 3/8, 3/8, 1/8], [0, 1/2, 1/4, -1/4, 1/8, -1/8, -1/8, 1/8], [0, 1/2, -1/4, 1/4, 3/8, -3/8, -3/8, 3/8], [0, 1/2, -1/4, 1/4, -3/8, -1/8, -1/8, -3/8]$,
- C_8 : $[0, 0, 1/4, -7/16, 1/16, 1/16, -7/16, 1/4], [0, 0, 1/4, -3/16, 5/16, 5/16, -3/16, 1/4], [0, 0, -1/4, -5/16, 3/16, 3/16, -5/16, -1/4], [0, 0, -1/4, -1/16, 7/16, 7/16, -1/16, -1/4]$,
- C_3^2 : $[0, 0, 0, 1/3, -1/3, -1/3, -1/3, 0, 0], [0, 1/3, 1/3, -1/3, -1/3, -1/3, -1/3, 1/3, 1/3]$,
- C_9 : $[0, 0, 1/9, -2/9, 4/9, 4/9, -2/9, 1/9, 0], [0, 0, 2/9, -4/9, -1/9, -1/9, -4/9, 2/9, 0]$,
- C_{10} : $[0, 1/4, 1/20, -1/5, 1/5, 9/20, 9/20, 1/5, -1/5, 1/20], [0, -1/4, 3/20, 2/5, -2/5, 7/20, 7/20, -2/5, 2/5, 3/20], [0, -1/4, -9/20, -1/5, 1/5, -1/20, -1/20, 1/5, -1/5, -9/20], [0, 1/4, -7/20, 2/5, -2/5, -3/20, -3/20, -2/5, 2/5, -7/20]$,
- C_{11} : $[0, 1/11, 4/11, -2/11, 5/11, 3/11, 3/11, 5/11, -2/11, 4/11, 1/11], [0, 2/11, -3/11, -4/11, -1/11, -5/11, -5/11, -1/11, -4/11, -3/11, 2/11]$,
- $C_2 \times C_6$: $[0, -1/4, 0, 1/4, -1/3, 5/12, -1/3, -1/12, -1/12, -1/3, 5/12, -1/3], [0, 1/2, -1/4, -1/4, -1/3, 1/6, 5/12, 5/12, 5/12, 5/12, 1/6, -1/3], [0, 1/2, 1/2, 1/2, -1/3, 1/6, 1/6, 1/6, 1/6, 1/6, -1/3], [0, 1/2, 1/4, 1/4, -1/3, 1/6, -1/12, -1/12, -1/12, -1/12, 1/6, -1/3], [0, 1/2, 0, 0, -1/3, 1/6, -1/3, -1/3, -1/3, 1/6, -1/3], [0, 1/2, 1/2, 1/2, 1/3, -1/6, -1/6, -1/6, -1/6, -1/6, 1/3], [0, 1/4, 1/2, 1/4, 1/3, -5/12, -1/6, -5/12, -5/12, -1/6, -5/12, 1/3], [0, 0, 1/2, 0, 1/3, 1/3, -1/6, 1/3, 1/3, -1/6, 1/3, 1/3], [0, -1/4, 0, 1/4, 1/3, 1/12, 1/3, -5/12, -5/12, 1/3, 1/12, 1/3], [0, -1/4, 1/2, -1/4, 1/3, 1/12, -1/6, 1/12, 1/12, -1/6, 1/12, 1/3]$,
- C_{12} : $[0, 1/2, -1/8, -1/3, 1/6, -11/24, -11/24, -11/24, -11/24, 1/6, -1/3, -1/8], [0, 1/2, -1/8, 1/3, -1/6, 5/24, 5/24, 5/24, 5/24, -1/6, 1/3, -1/8], [0, 1/2, -3/8, -1/3, 1/6, 7/24, 7/24, 7/24, 7/24, 1/6, -1/3, -3/8], [0, 1/2, -3/8, 1/3, -1/6, -1/24, -1/24, -1/24, -1/24, -1/6, 1/3, -3/8], [0, 1/2, 3/8, -1/3, 1/6, 1/24, 1/24, 1/24, 1/24, 1/6, -1/3, 3/8], [0, 1/2, 3/8, 1/3, -1/6, -7/24, -7/24, -7/24, -7/24, -1/6, 1/3, 3/8], [0, 1/2, 1/8, -1/3, 1/6, -5/24, -5/24, -5/24, -5/24, 1/6, -1/3, 1/8], [0, 1/2, 1/8, 1/3, -1/6, 11/24, 11/24, 11/24, 11/24, -1/6, 1/3, 1/8]$.

11. NON-POINTED ODD-DIMENSIONAL MODULAR DATA OF RANK 17

The data are organized as in §10.1. Additionally, it is accessible in a format compatible with computers within the file `Rank17MNSD.txt` in [23]. About their categorification, see Remark 1.10 and 9.3.

11.1. **Type $[[1,3],[3,8],[5,6]]$, First Fusion Ring.** The whole fusion data would be too big to be entirely displayed here. So here is a compressed version. Consider its basis

$$B = \{b_g\}_{g \in C_3} \cup \{x_i, x_i^*\}_{i \in \{1,2,3,4\}} \cup \{y_g, y_g^*\}_{g \in C_3},$$

where $\text{FPdim}(b_q) = 1$, $\text{FPdim}(x_i) = 3$ and $\text{FPdim}(y_q) = 5$. Here are the fusion rules:

- $b_{g_1} b_{g_2} = b_{g_1 g_2}$, $b_g^* = b_{g^{-1}}$,
- $b_g x_i = x_i$,
- $b_{g_1} y_{g_2} = y_{g_1 g_2}$,
- $x_i x_j$ and $x_i x_j^* - \delta_{i,j} \sum_{q \in C_3} b_q$ are sum of x_k of x_k^* , where the multiplicites are written below:

[illegible]

- $x_i y_g = x_i^* y_g = \sum_{g \in C_3} y_g,$
- $y_{g_1} y_{g_2} = y_{(g_1 g_2)}^* + 2 \sum_{g_1 g_2 g \neq e} y_g^*,$
- $y_{g_1} y_{g_2}^* = b_{g_1 g_2}^{-1} + \sum_i (x_i + x_i^*),$

the other rules follows by commutativity and duality. Now we write the rest of the modular data as in §10.1:

- $[0, 0, 0, -2/5, -2/5, -1/5, -1/5, 1/5, 1/5, 2/5, 2/5, -1/3, -1/3, 0, 0, 1/3, 1/3], 15, 15, 4, [1, 0, \dots, 0],$

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