

APPENDICES OF CLASSIFYING INTEGRAL GROTHENDIECK RINGS UP TO RANK 5 AND BEYOND

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Appendix A. INTEGRAL DRINFELD RINGS

A.1. Up to rank 5. This section presents the comprehensive list of integral Drinfeld rings up to rank 5—including their global FPdim, type, duality, formal codegrees, and fusion data. Copy-pastable data can be found in the file `GeneralUpToRank5DataOnly.txt`, located in the `Data/General` directory of [7]. For each case, either an explicit categorification is provided, or a reference is given to a theoretical result ruling out its existence.

A.1.1. *Rank 1.* Trivial case

A.1.2. *Rank 2.*

(1) FPdim 2, type $[1, 1]$, duality $[0, 1]$, fusion data:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Formal codegrees: $[2, 2]$,
- Property: simple,
- Categorification: $\text{Rep}(C_2)$.

A.1.3. *Rank 3.*

(1) FPdim 3, type $[1, 1, 1]$, duality $[0, 2, 1]$, fusion data:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Formal codegrees: $[3, 3, 3]$,
- Property: simple,
- Categorification: $\text{Rep}(C_3)$.

(2) FPdim 6, type $[1, 1, 2]$, duality $[0, 1, 2]$, fusion data:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Formal codegrees: $[2, 3, 6]$,
- Categorification: $\text{Rep}(S_3)$.

A.1.4. *Rank 4.*

(1) FPdim 4, type $[1, 1, 1, 1]$, duality $[0, 1, 2, 3]$, fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Formal codegrees: $[4, 4, 4, 4]$,
- Categorification: $\text{Rep}(C_2^2)$.

(2) FPdim 4, type $[1, 1, 1, 1]$, duality $[0, 1, 2, 3]$, fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Formal codegrees: $[4, 4, 4, 4]$,
- Categorification: $\text{Rep}(C_4)$.

(3) FPdim 10, type $[1, 1, 2, 2]$, duality $[0, 1, 2, 3]$, fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- Formal codegrees: $[2, 5, 5, 10]$,
- Categorification: $\text{Rep}(D_5)$.

- Formal codegrees: $[2, 3, 9, 39, 39, 234]$,
- Property: non-1-Frobenius, non-3-positive, extension of $\text{ch}(S_3)$.

- Formal codegrees: $[2, 3, 11, 14, 232, 53592]$,
- Property: non-1-Frobenius,
- Categorification: excluded by the non-Drinfeld fusion subring of type $[1, 1, 2, 9, 12]$.

- Formal codegrees: $[3, 3, 6, 8, 42, 57, 3192]$,
- Property: simple, non-1-Frobenius, non-3-positive
- Categorification: open, non-braided, non-unitary

(5) FPdim 4284, type $[1, 17, 17, 20, 28, 35, 36]$, duality $[0, 2, 1, 3, 4, 5, 6]$, fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 & 3 & 3 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 3 & 2 & 3 & 4 & 5 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 3 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 3 & 3 & 4 & 5 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 3 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 & 6 & 4 & 4 \\ 0 & 3 & 3 & 3 & 4 & 6 & 6 \\ 0 & 3 & 3 & 4 & 4 & 6 & 6 \\ 0 & 3 & 3 & 4 & 4 & 6 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 2 & 6 & 4 & 4 \\ 1 & 3 & 3 & 6 & 1 & 7 & 8 \\ 0 & 4 & 4 & 4 & 7 & 8 & 8 \\ 0 & 4 & 4 & 4 & 7 & 8 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 3 & 3 & 4 & 6 & 6 & 6 \\ 0 & 4 & 4 & 4 & 7 & 8 & 8 \\ 1 & 5 & 5 & 6 & 8 & 10 & 10 \\ 0 & 5 & 5 & 6 & 8 & 10 & 10 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 3 & 3 & 4 & 5 & 5 \\ 0 & 3 & 3 & 4 & 4 & 6 & 6 \\ 0 & 4 & 4 & 4 & 8 & 8 & 8 \\ 0 & 5 & 5 & 6 & 8 & 10 & 11 \\ 1 & 5 & 5 & 6 & 8 & 10 & 11 \end{bmatrix}$$

- Formal codegrees: $[3, 4, 7, 7, 9, 51, 4284]$,
- Property: simple, non-1-Frobenius, non-3-positive
- Categorification: open, non-braided, non-unitary

These fusion rings also invite questions in the spirit of Question 1.13.

A.4. Ranks 8 and 9. There are 792 integral 1-Frobenius Drinfeld rings of rank 8 with $\text{FPdim} \leq 25000$, and 1292 such rings of rank 9 with $\text{FPdim} \leq 2000$. Copy-pastable data can be found in the files `1FrobR8d25000.txt` and `1FrobR9d2000.txt`, located in the `Data/General` directory of [7].

A.5. Addressing divisibility. The classification at rank r begins with the list of length r Egyptian fractions representing the possible

$$\sum_V \sum_{i=1}^{n_V^2} \frac{1}{n_V f_V} = 1,$$

where the (f_V) are the formal codegrees of the Drinfeld ring; see §2.2. Under the Drinfeld assumption, each formal codegree f_V divides f_1 . Moreover, the Drinfeld ring is commutative if and only if $n_V = 1$ for all V . In this case, we can restrict to Egyptian fractions of length r satisfying the divisibility condition $f_V \mid f_1$ for all V .

In the noncommutative setting, it may happen that $n_V f_V$ does not divide f_1 for some V . However, as verified in Appendix C, no such exceptions occur up to rank 8 (but they do appear at rank 9; see §2.2). Therefore, for ranks $r \leq 8$, it is safe to restrict to Egyptian fractions of length r that satisfy the divisibility condition.

At rank 9, Lemma 8.3 shows that the complexified noncommutative Drinfeld ring must be isomorphic to either $\mathbb{C} \oplus M_2(\mathbb{C})^2$ or $\mathbb{C}^5 \oplus M_2(\mathbb{C})$. The exception corresponds to Egyptian fractions of length 5 or 7 with one or two terms having $n_V = 2$, and at least one violating the divisibility condition $n_V f_V \mid f_1$.

We verified that for $\text{FPdim} \leq 32000$, the values arising from these exceptional cases are already covered by those for the Egyptian fractions of length 9 satisfying the divisibility condition. Details of this computation can be found in the file `InvestNCRank9Except.txt` of the `Data/EgyptianFractionsDiv/Except` folder of [7].

Appendix B. MNSD DRINFELD RINGS

As established in §7, the Grothendieck ring of any odd-dimensional integral fusion category over \mathbb{C} is an MNSD integral Drinfeld ring (Definition 7.6). This section provides a complete classification of such rings up to rank 9.

B.1. Up to rank 5. There are four MNSD integral Drinfeld rings up to rank 5, contained in §A.1, namely the Grothendieck rings of $\text{Rep}(G)$, with $G = C_1, C_3, C_5, C_7 \rtimes C_3$.

B.2. Rank 7. Here is the complete list of 4 MNSD integral Drinfeld ring of rank 7. Copy-pastable data can be found in the file `MNSDRank7DataOnly.txt.txt`, located in the `Data/Odd` directory of [7].

(1) FPdim 7, type $[1, 1, 1, 1, 1, 1, 1]$, duality $[0, 6, 5, 4, 3, 2, 1]$, fusion data: see §A.3.1.

(2) FPdim 39, type $[1, 1, 1, 3, 3, 3, 3]$, duality $[0, 2, 1, 6, 5, 4, 3]$, fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Formal codegrees: $[3, 3, 13, 13, 13, 13, 39]$,
- Categorification: $\text{Rep}(C_{13} \rtimes C_3)$.

(3) FPdim 55, type $[1, 1, 1, 1, 1, 5, 5]$, duality $[0, 4, 3, 2, 1, 6, 5]$, fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Formal codegrees: $[5, 5, 5, 5, 11, 11, 55]$,
- Categorification: $\text{Rep}(C_{11} \rtimes C_5)$.

[illegible]

(16) FPdim 960, type $[1, 1, 1, 3, 4, 4, 30]$, duality $[0, 2, 1, 3, 4, 5, 6, 7]$, fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 1 0
0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 1 0
0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	1 1 1 2 0 0 0 0 0	0 0 0 0 1 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 0 0 0 3
0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 0	1 0 0 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 1 0 1 1 1 1 1 0	0 0 0 0 0 0 0 0 4
0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 1 1 1 1 0	1 0 0 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 0 0 0 0 0 0 0 4
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 1 1 0	0 0 1 1 1 1 1 1 0	0 1 0 1 1 1 1 1 0	0 1 0 1 1 1 1 1 0	1 0 0 1 1 1 1 1 0	0 0 0 0 0 0 0 0 4
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	1 1 1 3 4 4 4 4 28

- Formal codegrees: $[3_2, 4, 15, 64, 960]$,
- Property: noncommutative,
- Categorification:

(17) FPdim 1806, type $[1, 1, 1, 1, 1, 1, 6, 42]$, duality $[0, 1, 2, 3, 5, 4, 6, 7]$, fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 1 0 0	1 0 0 0 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 0 0 1 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 1 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	1 1 1 1 1 1 5 0	0 0 0 0 0 0 0 0 6
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	1 1 1 1 1 1 6 41

- Formal codegrees: $[3_2, 6, 7, 43, 1806]$,
- Property: noncommutative,
- Categorification:

(18) FPdim 2184, type $[1, 1, 1, 3, 12, 26, 26, 26]$, duality $[0, 2, 1, 3, 4, 5, 6, 7]$, fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	1 1 1 2 0 0 0 0 0	0 0 0 0 3 0 0 0 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 0 0 0 3
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 3 0 0 0	1 1 1 3 11 0 0 0 0	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4
0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	1 0 0 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8	0 1 0 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 1 0 1 4 8 8 8 8	1 0 0 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 1 1	0 0 0 0 0 4 4 4 4	0 0 1 1 4 8 8 8 8	0 1 0 1 4 8 8 8 8	1 0 0 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8

- Formal codegrees: $[3_2, 4, 13, 168, 2184]$,
- Property: noncommutative,
- Categorification:

(19) FPdim 3660, type $[1, 1, 1, 3, 4, 4, 4, 60]$, duality $[0, 2, 1, 3, 4, 5, 6, 7]$, fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	1 1 1 2 0 0 0 0 0	0 0 0 0 3 0 0 0 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 0 0 0 3
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 3 0 0 0	1 1 1 3 11 0 0 0 0	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4
0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	1 0 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 1 0 1 4 7 17 17 17	1 0 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 1 1	0 0 0 0 0 4 4 4 4	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	1 0 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17

- Formal codegrees: $[3_2, 4, 15, 61, 3660]$,
- Property: noncommutative,
- Categorification:

(20) FPdim 8268, type $[1, 1, 1, 3, 12, 52, 52, 52]$, duality $[0, 2, 1, 3, 4, 5, 6, 7]$, fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	1 1 1 2 0 0 0 0 0	0 0 0 0 3 0 0 0 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 0 0 0 3
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 3 0 0 0	1 1 1 3 11 0 0 0 0	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4
0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	1 0 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 1 0 1 4 7 17 17 17	1 0 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 1 1	0 0 0 0 0 4 4 4 4	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	1 0 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17

- Formal codegrees: $[3_2, 4, 13, 159, 8268]$,
- Property: noncommutative,
- Categorification:

C.4. Rank 9. There are 83 integral 1-Frobenius, noncommutative Drinfeld rings of rank 9 with FPdim ≤ 10000 . Copy-pastable data can be found in the file `1FrobR9NCd10000DataOnly.txt`, located in the `Data/Noncommutative` directory of [7].

REFERENCES

- [1] P. BRUILLARD, C.M. ORTIZ-MARRERO, *Classification of rank 5 premodular categories*. J. Math. Phys. 59 (2018), no. 1, 011702, 8 pp.
- [2] S. BURCIU, S. PALCOUX, *Structure constants, Isaacs property and extended Haagerup fusion categories*. Comm. Algebra 53 (2025), no. 4, 1438–1452.
- [3] M. IZUMI, *A Cuntz algebra approach to the classification of near-group categories*. Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday, 222–343, Proc. Centre Math. Appl. Austral. Nat. Univ., 46, Austral. Nat. Univ., Canberra, 2017.
- [4] M. IZUMI, H. KOSAKI, *Kac algebras arising from composition of subfactors: general theory and classification*. Mem. Amer. Math. Soc. 158 (2002), no. 750, 198 pp.
- [5] Z. LIU, S. PALCOUX, Y. REN, *Triangular prism equations and categorification*, arXiv:2203.06522 (2022).

- [6] Z. LIU, S. PALCOUX, Y. REN, *Classification of Grothendieck rings of complex fusion categories of multiplicity one up to rank six*. Lett. Math. Phys. 112 (2022), no. 3, Paper No. 54, 37 pp.
- [7] S. PALCOUX, *Fusion Categories Repository*, GitHub, <https://github.com/sebastienpalcoux/Fusion-Categories/tree/main/IntegralPaper>

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