CLASSIFICATION OF INTEGRAL MODULAR DATA UP TO RANK 13

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Abstract. This paper classifies all integral modular data up to rank 13 (all can be categorified). Furthermore, it also classifies all integral half-Frobenius fusion rings up to rank 12. We find that each perfect integral modular fusion category up to rank 13, as well as every perfect integral half-Frobenius fusion ring up to rank 12, is trivial. We have also refined the categorifiable non-pointed odd-dimensional modular data at ranks below 25 to three possible items, all of rank 17, FPdim 225, and type [[1,3],[3,8],[5,6]], filling gaps in the literature. For rank 25, we have narrowed down the perfect case to 5 types.

Our initial key insight is that the Egyptian fractions, which are typically employed to list possible types, can be chosen with squared denominators. We then develop several type criteria as initial filters. In particular, we establish that the number of distinct basic FPdims in a non-trivial perfect fusion ring must be at least 4. To obtain the fusion rings, we solve the dimension and associativity equations using new features on Normaliz created specifically for this purpose. The S-matrices (if they exist) are obtained by self-transposing the character table, while the T-matrices are derived by solving the Anderson-Moore-Vafa equations. Finally, we verify the extended axioms of modular data.

From rank 13 onward, the types were further restricted by additional properties unique to the modular case, which involved the universal grading, congruence representations of the modular group and Galois action, leading to critical arithmetic constraints. In particular, we get that, up to rank 21, a prime divisor of the global FPdim does not exceed the rank, and more strongly up to rank 15 in the non-pointed case, does not exceed half the rank.

1. Introduction

In this paper, we assume that all fusion categories are defined over the complex field. The concept of an integral modular fusion category has been extensively studied, as detailed in the references at the beginning of [14]. In [5], they have been classified up to rank 6 (all pointed), with Egyptian fractions playing a crucial role. The approach that enables us to extend this classification up to rank 13 in our work hinges on the observation that it is sufficient to consider Egyptian fractions with squared denominators. This restriction significantly reduces the combinatorial complexity. To illustrate this point, consider that the number of Egyptian fractions (summing to 1) of length $n = 1, 2, \dots, 8$ is $1, 1, 3, 14, 147, 3462, 294314, 159330691$, respectively (as per [44]). In contrast, when limited to squared denominators, the counts are $1, 0, 0, 1, 0, 1, 1, 4$, respectively (refer to [1]).

We begin by recalling the concept of a fusion ring and its fundamental results in §2.1, with reference to [18, Chapter 3]. As defined in [20], a fusion ring $\mathcal{F}$ is termed *s-Frobenius* if for every basic element b, the ratio $\mathrm{FPdim}(\mathcal{F})^s/\mathrm{FPdim}(b)$ is an algebraic integer. According to [18, Proposition 8.14.6], the Grothendieck ring of a modular fusion category is 1/2-Frobenius (denoted *half-Frobenius* in the rest of the paper). Consider $\mathcal{F}$ to be an integral half-Frobenius fusion ring with a basis $\{b_1, \dots, b_r\}$, FPdim D, and type $[d_1, \dots, d_r]$, where $1 = d_1 \leq d_2 \leq \cdots \leq d_r$ and $d_i = \mathrm{FPdim}(b_i)$. Thus d_i^2 is a divisor of D, for all i. There exists a unique square-free integer q such that $D = qs^2$, implying that each d_i is a divisor of s. Let s_i denote the positive integer s/d_i. Given that $D = \sum_{i=1}^r d_i^2$, we arrive at the following representation of q as an Egyptian fraction with squared denominators:

$$q = \sum_{i=1}^{r} \frac{1}{s_i^2}.$$

We have classified all such Egyptian fractions up to $r = 13$ using SageMath, as will be discussed in §4, where a method to constrain to $q \leq r/4$ is also described. Since $s_1 = s$, we have $d_i = s_1/s_i$, and we may assume that s_i is a divisor of s_1, for all i. As detailed in §4, this leads us to consider only 9025 types up to rank 13.

The subsequent phase entails implementing new criteria for identifying a type that emerges from a fusion ring, as delineated in §5. Particularly, we establish:

Theorem 5.1. *The minimum number of distinct basic* FPdims *in a non-trivial perfect fusion ring is four.*

The proof of these criteria predominantly relies on modular arithmetic and serves to rule out approximately 62% of the types up to rank 13.

To address the remaining types, we classify all possible fusion data $(N_{i,j}^k)$, as defined in §2.1, for each type (d_i), utilizing our fusion ring solver described in §6. We begin by reducing the number of variables, leveraging the Unit axiom of fusion data and the Frobenius reciprocity. The main challenge, denoted as "patching", involves integrating the associativity equations $\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t$ (which are non-linear) as efficiently as possible into the (linear) solving process of the dimension equations $d_i d_j = \sum_k N_{i,j}^k d_k$, which are positive linear Diophantine equations. This approach was implemented using Normaliz [10], on which we developed new features dedicated to the classification of fusion rings, as explained in §6.1, and for more details, see Appendix H of the manual [11].

This step culminates in a classification of all the half-Frobenius integral fusion rings up to rank 12, tallying exactly 10628 instances derived from 71 types, and proves the absence of any non-trivial perfect integral half-Frobenius fusion rings up to rank 12 (see §7). From rank 13 onward, the types were further restricted by additional properties coming from more advanced results on modular fusion categories §8. This adjustment was necessary because we encountered computational limits for classifying all half-Frobenius fusion rings. Consequently, the result became less general at the fusion ring level compared to what we get up to rank 12. In the non-perfect case, we applied specific *universal grading* techniques, as discussed in §8.1 and based on [30, Proposition VI.2]. Additionally, we explored congruence representations of the modular group (see §8.2) and Galois actions (see §8.3). In particular, we provide two proofs (the shorter one applies [17, Theorem II (iii)]) of (folklore) Theorem 8.4: any prime factor p of the dimension norm of a modular fusion category with rank r, it holds that $p \leq 2r+1$. This inequality is optimal, and the examples for which the equality holds are classified in [33]. Regarding the integral case, discussions with Eric Rowell and Andrew Schopieray indicated that the rank r can be substituted with the multiplicity m of a certain basic FPdim, leading to the inequality $p \leq 2m+1$, see Theorem 8.5, required to exclude some hard types at rank 13. Here is a stronger version:

Theorem 8.7. *For an integral modular fusion category, let S be the set of odd prime factors of the global* FPdim*. There is a partition (S_i) of S, and multiplicities (m_i) of* some *distinct basic* FPdims *such that*

$$m_i \geq \frac{1}{2} \operatorname{lcm}_{p \in S_i}(p-1).$$

It was crucial to prove Proposition 11.8 for the odd-dimensional case, and also the following theorem from §9.2:

Theorem 1.1. *Let $\mathcal{C}$ be an integral modular fusion category of rank r. For any prime factor p of* $\mathrm{FPdim}(\mathcal{C})$*, the following bounds hold:*

(1) If $r \leq 21$, then $p \leq r$.
(2) If $r \leq 15$ and $\mathcal{C}$ is non-pointed, then $p \leq r/2$.

These inequalities are conjectured to hold without rank restriction in §9.2. These conjectures are proved for $\mathcal{Z}(\mathrm{Rep}(G))$, across all finite groups G, in §9.1, among other results.

The next step is to classify all possible modular data related to the fusion rings found up to rank 13. The definition of modular data we employ (refer to §2.2) is informed by the key attributes of a modular fusion category, specifically a pseudo-unitary one, as our research is centered on the integral case (see [18, Proposition 9.6.5]). We can limit our attention to commutative fusion rings since a modular fusion category, being braided, possesses a commutative Grothendieck ring (although our classification encompasses 213 noncommutative fusion rings as well; see §7).

First, we examine the S-matrices: for a given commutative fusion ring, we take its eigentable (as defined in Definition 2.7) and consider it as a matrix, retaining only those with cyclotomic elements—such fusion rings are termed *cyclotomic*. If suitable renormalization and permutation yield a self-transpose matrix (detailed in §3.1), we call the fusion ring as *self-transposable*; if not, it is dismissed. From this, we infer that there are precisely 69 self-transposable, cyclotomic, half-Frobenius, integral fusion rings up to rank 12, originating from 27 types, which is fewer than 0.7% of the 10628 identified in the initial stage. At rank 13, we reduced to 6 types (more restricted) with such fusion rings.

Moving on to the T-matrices: for the fusion rings that remain, we solve the Anderson-Moore-Vafa equations (see §2.2) in the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. We preserve only those S- and T-matrices that satisfy all the conditions of Definition 2.11. The use of our so-called *magic criterion* was pivotal for several big cases and could lead to an interesting theoretical reformulation, see Question 3.2. Ultimately, we arrive at $19+64$ modular data, derived from $5+18$ fusion rings and $3+13$ types (non-pointed + pointed).

Remark 1.2. *Every pointed modular fusion category corresponds to a metric group (G,q)—a finite Abelian group G equipped with a non-degenerate quadratic form $q : G \to \mathbb{C}^*$, represented by the T-matrix, as described in* [18, §8.4].

The modular data (MD) mentioned in §12 encompass S- and T-matrices, central charge, fusion data, and second Frobenius-Schur indicators for the non-pointed case. For the pointed case, however, it includes only the T-matrices. The following theorem provides a concise overview:

Theorem 1.3. *There are* 19 *MD of non-pointed integral modular fusion categories up to rank* 13*, given by:*

- *Rank* 8, FPdim 36*, type* $[1,1,2,2,2,2,3,3]$*:*
  - 6 *MD with central charge $c=0$ from $\mathcal{Z}(\mathrm{Vec}_{S_3}^{\omega})$, see* [23],
  - 2 *MD with $c=4$ from $(C_3^2+0)^{C_2}$, see* [22, point (b) on page 983].
- *Rank* 10, FPdim 36*, type* $[1,1,1,2,2,2,2,2,2,3]$*:*
  - 3 *MD with $c=4$ from $SU(3)_3$, its complex conjugate and a zesting, see* [16, §6.3.1].

- *Rank* 11*,* FPdim 32*, type* $[1,1,1,1,2,2,2,2,2,2,2]$*:*
  - 8 *MD with* $c = \pm 1$ *from* $SO(8)_2$*, conjugates and zestings, see §3.3.*

There are 64 *MD of pointed modular fusion categories up to rank* 13*; here are their number per group* G*:*

G	C_1	C_2	C_3	C_2^2	C_4	C_5	C_6	C_7	C_2^3	$C_2 \times C_4$	C_8	C_3^2	C_9	C_{10}	C_{11}	$C_2 \times C_6$	C_{12}	C_{13}
$\#MD$	1	2	2	5	4	2	4	2	4	4	4	2	2	4	2	10	8	2

There is no other integral modular data up to rank 13 *(i.e. all categorifiable as above).*

Question 1.4. *Is there a modular data without categorification?*

It should be noted that [30] provides an interesting (non-integral) candidate of rank 11 (in its introduction) relevant to Question 1.4, and it also recovers Theorem 1.3 up to rank 12. This theorem yields the following consequence:

Theorem 1.5. *Every perfect integral modular fusion category up to rank* 13 *is trivial.*

In fact, we obtained the following more general result within the context of fusion rings up to rank 12:

Theorem 1.6. *Every perfect integral half-Frobenius fusion ring up to rank* 12 *is trivial.*

The proof of Theorem 1.6 for ranks up to 9 is straightforward, following the list provided in §7.1 combined with an extended version of the Nichols-Richmond theorem applied to fusion rings, as detailed in the proof of [34, Theorem 11]. This is due to the consistent presence of a non-trivial basic element with FPdim ≤ 2. However, proving the theorem for ranks up to 12 necessitates the employment of type criteria, as discussed in §5, and the use of a fusion ring solver, elaborated in §6.

It should be noted that the Drinfeld center of the representation category of any non-Abelian finite simple group G—and, more broadly, any centerless perfect group—is a perfect (though not simple) integral modular fusion category denoted as $\mathcal{Z}(\mathrm{Rep}(G))$ with FPdim $= |G|^2$. For further information, see [12, §11.1]. Thus, the Grothendieck ring of $\mathcal{Z}(\mathrm{Rep}(A_5))$, of rank 22 and type $[[1,1],[3,2],[4,1],[5,1],[12,10],[15,4],[20,3]]$, constitutes a perfect integral half-Frobenius fusion ring. Consequently, Theorem 1.6 cannot be extended to all ranks; however, it remains an open question whether its simple version can be:

Question 1.7. *Is there a non-pointed simple integral half-Frobenius fusion ring?*

A negative response to Question 1.7 would imply a negative answer to the renowned [19, Question 2] in the simple case, due to a result in [26], which states that every simple integral fusion category is weakly group-theoretical if and only if every simple integral modular fusion category is pointed. With this in mind, we propose the following question:

Question 1.8. *Is there a non-pointed simple integral modular fusion category?*

For further insights into Question 1.8 at the fusion ring level, [35, Corollary 6.16] adds a constraint: the absence of any basic elements with a prime-power FPdim. It is worth noting that Theorem 1.6 cannot be generalized to all ranks, even with this added constraint. This is because the Grothendieck ring of $\mathcal{Z}(\mathrm{Rep}(A_7))$, which is a perfect integral half-Frobenius fusion ring of rank 74 and type

$$[[1,1],[6,1],[10,2],[14,2],[15,1],[21,1],[35,1],[70,9],[105,4],[210,20],[280,9],[360,14],[504,5],[630,4]],$$

satisfies this constraint (but consider Question 7.2). If necessary, Question 1.7 could be refined to include this constraint and the property of commutativity, and even the more advanced constraints mentioned above.

Employing similar techniques, along with [14, Remark 4.3] and [35, Corollary 6.16], we are able to prove the following (see §11):

Theorem 1.9. *There are three MD of non-pointed integral modular fusion categories at rank below* 25*, given by:*

- *Rank* 17*,* FPdim 225*, type* $[1,1,1,3,3,3,3,3,3,3,3,5,5,5,5,5,5]$*:*
  - 3 *MD with central charge* $c = 4$ *from* 2 *fusion rings (see §13 for the details).*

Remark 1.10. *Note that Theorem 1.9 does align with* [3, Theorem 4.2, proof of Case (viii) $\mathrm{FPdim}(\mathcal{C}_{pt}) = p$] *as well as* [14, Theorem 6.3 (b), proof of Case $|\mathcal{G}(\mathcal{C})| = 3$]*. But Remark 11.6 points out gaps in these proofs. Following our paper,* [14] *was corrected on arXiv, and* [21] *introduces modular category models for these new MD.*

Finally, this paper narrows down the possible rank 25 odd-dimensional perfect types to 5 ones, see Proposition 11.8.

Contents

2. Fusion Data and Modular Data

In this section, we review the concepts of fusion data and modular data, along with the essential results. For further details, we refer the reader to [18].

2.1. Fusion Data.

The concept of fusion data expands upon the idea of a finite group.

Definition 2.1. Fusion data *consist of a finite set* $\{1, 2, ..., r\}$ *with an involution* $i \mapsto i^*$*, and nonnegative integers* $N_{i,j}^k$ *satisfying the following conditions for all* i, j, k, t*:*

- *(Associativity)* $\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t$,
- *(Unit)* $N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$,
- *(Dual)* $N_{i^*,j}^1 = N_{j,i^*}^1 = \delta_{i,j}$,
- *(Anti-involution)* $N_{i,j}^k = N_{j^*,i^*}^{k^*}$.

Note that $1^* = 1$*. We may represent the fusion data simply as* $(N_{i,j}^k)$*.*

Proposition 2.2 (Frobenius Reciprocity)**.** *For all* i, j, k*,* $N_{i,j}^k = N_{k,j^*}^i = N_{k^*,i}^{j^*} = N_{j^*,i^*}^{k^*} = N_{j,k^*}^{i^*} = N_{i^*,k}^j$.

Proof. Starting with (Associativity) and setting $t = 1$, we have $\sum_s N_{i,j}^s N_{s,k}^1 = \sum_s N_{j,k}^s N_{i,s}^1$. Applying (Dual), we get $\sum_s N_{i,j}^s \delta_{s,k^*} = \sum_s N_{j,k}^s \delta_{s,i^*}$. Consequently, $N_{i,j}^{k^*} = N_{j,k}^{i^*}$. Substituting k^* with k, we obtain $N_{i,j}^k = N_{j,k^*}^{i^*}$, which equals N_{k,j^*}^i by (Anti-involution). The proposition follows by iterating the equality $N_{i,j}^k = N_{k,j^*}^i$. □

Remark 2.3. *We can construct data that satisfy the first three axioms of Definition 2.1 but not the fourth, proving it is not superfluous. However, (Unit) is redundant when combined with the other axioms, as it is not utilized in the proof of Proposition 2.2. Taken together, (Dual) and (Frobenius Reciprocity) trivially imply (Unit).*

A *fusion ring* $\mathcal{R}$ is a free $\mathbb{Z}$-module equipped with a finite basis $\mathcal{B} = \{b_1, \dots, b_r\}$ and a fusion product defined by

$$b_i b_j = \sum_k N_{i,j}^k b_k,$$

where $(N_{i,j}^k)$ constitutes fusion data, and a $*$-structure given by $b_i^* := b_{i^*}$. The four axioms for fusion data translate to the following for all i, j, k:

- $(b_i b_j) b_k = b_i (b_j b_k)$,
- $b_1 b_i = b_i b_1 = b_i$,
- $\tau(b_i b_j^*) = \delta_{i,j}$,
- $(b_i b_j)^* = b_j^* b_i^*$,

where $\tau(x)$ is the coefficient of b_1 in the decomposition of $x \in \mathcal{R}$. Consequently, $\mathcal{R}_\mathbb{C} := \mathcal{R} \otimes_\mathbb{Z} \mathbb{C}$ becomes a finite-dimensional unital $*$-algebra, with τ extending linearly to a trace (i.e., $\tau(xy) = \tau(yx)$) and an inner product defined by $\langle x, y \rangle := \tau(xy^*)$. Here, $\langle x, b_i \rangle$ is the coefficient of b_i in the decomposition of x.

Theorem 2.4 (Frobenius-Perron Theorem)**.** *Given a fusion ring $\mathcal{R}$ with basis $\mathcal{B}$ and the corresponding finite-dimensional unital $*$-algebra $\mathcal{R}_\mathbb{C}$ as defined above, there exists a unique $*$-homomorphism $d : \mathcal{R}_\mathbb{C} \to \mathbb{C}$ such that $d(\mathcal{B}) \subset \mathbb{R}_{>0}$.*

The value $d(b_i)$ is termed the *Frobenius-Perron dimension* of b_i, denoted as $\mathrm{FPdim}(b_i)$ or simply d_i. The sum $\sum_i d_i^2$ is referred to as the Frobenius-Perron dimension of $\mathcal{R}$, denoted $\mathrm{FPdim}(\mathcal{R})$. The sequence $[d_1, d_2, \dots, d_r]$ is called the *type* of $\mathcal{R}$. A fusion ring $\mathcal{R}$ is described as:

- *Frobenius* (or 1-Frobenius, or of Frobenius type) if $\frac{\mathrm{FPdim}(\mathcal{R})}{\mathrm{FPdim}(b_i)}$ is an algebraic integer for all i,
- *integral* if $\mathrm{FPdim}(b_i)$ is an integer for all i,
- *pointed* if $\mathrm{FPdim}(b_i) = 1$ for all i,
- *commutative* if $b_i b_j = b_j b_i$ for all i, j, meaning $N_{i,j}^k = N_{j,i}^k$.

The *multiplicity* of $\mathcal{R}$ is the maximum value among $N_{i,j}^k$, and its *rank* is r, the size of the basis.

Remark 2.5. *Fusion data enable a representation of its corresponding fusion ring. Consider the matrices $M_i = (N_{i,j}^k)_{k,j}$. By the Associativity axiom in Definition 2.1, we verify that $M_i M_j = \sum_k N_{i,j}^k M_k$. Additionally, M_1 is the identity matrix, and Frobenius Reciprocity ensures that the adjoint matrix M_i^* is M_{i^*}. According to Frobenius-Perron Theorem, the operator norm $\|M_i\|$ equals* $\mathrm{FPdim}(b_i)$.

Remark 2.6. *The concept of fusion data is a combinatorial reformulation of the fusion ring notion, so any property applicable to a fusion ring is also applicable to its fusion data.*

Definition 2.7 (Eigentable)**.** *Given commutative fusion data $(N_{i,j}^k)$, consider the corresponding fusion matrices $M_i = (N_{i,j}^k)_{k,j}$. The commutativity and the property that $M_i^* = M_{i^*}$ render these matrices normal and thus simultaneously diagonalizable. Let (D_i) denote their simultaneous diagonalization, where $D_i = \mathrm{diag}(\lambda_{i,j})$. We can select $\lambda_{i,1} = \|M_i\| = d_i$. The matrix $(\lambda_{i,j})$ is termed the* eigentable *(or character table) of the fusion data, and the values $c_j := \sum_i |\lambda_{i,j}|^2$ are called the* formal codegrees.

Lemma 2.8. *Let $M \in M_n(\mathbb{Z}_{\geq 0})$. The matrix M is a permutation matrix if and only if $\|M\| = 1$.*

Proof. Consider an orthonormal basis $\{e_1, \dots, e_n\}$ for which the entries of M are non-negative integers. If M is not a permutation matrix, then one of the following cases must occur:

(0) there exists i for which $Me_i = 0$,
(1) there exist i, j such that $\langle Me_i, e_j \rangle > 1$,
(2) there exist i, j, k with $j \neq k$, such that $\langle Me_i, e_j \rangle = \langle Me_i, e_k \rangle = 1$,
(3) there exist i, j, k with $i \neq j$, such that $Me_i = Me_j = e_k$.

However, case (0) implies $\|Me_i\|/\|e_i\| = 0$, while case (1) leads to $\|Me_i\|/\|e_i\| > 1$. In case (2), it follows that $\|Me_i\|/\|e_i\| \geq \sqrt{2}$. Likewise, case (3) implies $\|M(e_i + e_j)\|/\|e_i + e_j\| = \sqrt{2}$. Each of these cases indicates that $\|M\| > 1$. Conversely, if M is a permutation matrix, it trivially follows that $\|M\| = 1$. □

Corollary 2.9. *For two basic elements x, y of a fusion ring with* $\mathrm{FPdim}(x) = 1$*, both xy and yx are basic elements, and* $\mathrm{FPdim}(xy) = \mathrm{FPdim}(yx) = \mathrm{FPdim}(y)$.

Proof. This follows directly from Remark 2.5, Lemma 2.8, and the fact that FPdim is a ring homomorphism. □

Corollary 2.10. *A fusion ring is pointed if and only if its basis forms a finite group under the fusion product.*

2.2. **Modular Data.** Broadly speaking, modular data refers to a fusion data together with two matrices, S and $T = (t_{i,j})$, that generate a projective representation of the modular group $\mathrm{SL}(2, \mathbb{Z})$. To provide a more detailed description, we draw upon [29, Theorem 2.1] and [18, §8.13, §8.18]. Let $\mathbf{i}$ be the imaginary unit.

Definition 2.11. *Given a fusion ring $\mathcal{R}$ of rank r, type $[d_1, \dots, d_r]$, and fusion data $(N_{i,j}^k)$, let $\mathsf{d} := \mathrm{FPdim}(\mathcal{R})$ and $\zeta_n := \exp(2\pi\mathbf{i}/n)$. A (pseudounitary)* modular data *for $\mathcal{R}$ consists of two matrices $S, T \in M_r(\mathbb{C})$ satisfying:*

- *S and T are symmetric, T is unitary and diagonal with $T_{1,1} = 1$, $S_{1,i} = d_i$ for all i, and $SS^* = \mathsf{d}\mathbf{1}$.*
- *Verlinde formula: $N_{i,j}^k = \frac{1}{\mathsf{d}} \sum_l \frac{S_{li} S_{lj} \overline{S_{lk}}}{d_l}$.*
- *Twist: let θ_i be $T_{i,i}$, then $\sum_k N_{i,j}^k \theta_k d_k = \theta_i \theta_j S_{i,j}$.*
- *Ribbon structure: $\theta_i = \theta_{i^*}$ (see Remark 2.13).*
- *Central charge: $p_\pm := \sum_{i=1}^r d_i^2 (\theta_i)^{\pm 1}$. The ratio p_+/p_- is a root of unity, and $p_+ = \sqrt{\mathsf{d}}\zeta_8^c$ for some rational number c, referred to as the **central charge**, determined modulo 8.*
- *The matrices S and T afford a projective representation of $\mathrm{SL}(2, \mathbb{Z})$: we have $(ST)^3 = p_+ S^2$, $\frac{S^2}{\mathsf{d}} = C$, $C^2 = \mathbf{1}$, where C is the permutation matrix associated with the involution $i \to i^*$ and satisfies $\mathrm{Tr}(C) > 0$.*
- *Cauchy theorem: the set of distinct prime factors of $\mathrm{ord}(T)$ is identical to the distinct prime factors of $\mathrm{norm}(\mathsf{d})$, where $\mathrm{norm}(x)$ denotes the product of the distinct Galois conjugates of the algebraic number x.*
- *Cyclotomic integers: for all i, j, the elements $S_{i,j}$, $S_{i,j}/d_j$ and $T_{i,i}$ are cyclotomic integers. The conductor of $S_{i,j}$ divides $\mathrm{ord}(T)$, which in turn divides $\mathsf{d}^{5/2}$, and there exists j such that $S_{i,j}/d_j \in \mathbb{R}_{\geq 1}$, for all i.*
- *Frobenius-Schur indicators: for every i and for all $n \geq 1$, the sum $\nu_n(i) := \frac{1}{\mathsf{d}} \sum_{j,k} N_{j,k}^i (d_j \theta_j^n) \overline{(d_k \theta_k^n)}$ is a cyclotomic integer with a conductor that divides both n and $\mathrm{ord}(T)$. Additionally, $\nu_1(i) = \delta_{i,1}$ and $\nu_2(i) = \pm\delta_{i,i^*}$.*
- *Anderson-Moore-Vafa equations: $T_{i,i} = e^{2\pi\mathbf{i}t_i}$, and $\forall i, j, k, l$, the following equation holds in the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$:*
$$\left(\sum_{p=1}^r N_{i,j}^p N_{p,k}^l\right)(t_i + t_j + t_k + t_l) = \sum_{p=1}^r \left(N_{i,j}^p N_{p,k}^l + N_{i,k}^p N_{j,p}^l + N_{j,k}^p N_{i,p}^l\right) t_p.$$
*The **topological spin** of the i-th basic element is the representative $s_i \in (-1/2, 1/2]$ of $t_i \in \mathbb{Q}/\mathbb{Z}$.*

We could question the necessity of each component in Definition 2.11, particularly whether the Anderson-Moore-Vafa equations can be inferred from the other assumptions.

Remark 2.12. *The Verlinde formula, in conjunction with results from* [26, §2]*, implies that the fusion ring $\mathcal{R}$ is commutative. Together with S symmetric and the identity $SS^* = \mathrm{FPdim}(\mathcal{R})\mathbf{1}$, it can be deduced that $\mathcal{R}$ is self-transposable (as discussed in §3.1). Moreover, according to the proof presented in* [18, Proposition 8.14.6]*, $\mathcal{R}$ is also half-Frobenius.*

Remark 2.13. *A modular tensor category $\mathcal{C}$ possesses a ribbon structure, which means that the twist $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$ satisfies the condition $(\theta_X)^* = \theta_{X^*}$ for every object X within $\mathcal{C}$. Let (X_i) represent the set of simple objects (up to isomorphism) within $\mathcal{C}$. Schur's lemma guarantees that $\theta_{X_i} = \theta_i \mathrm{id}_{X_i}$, where the scalar θ_i is consistent with the one described in Definition 2.11. Owing to the ribbon structure, we deduce the following:*
$$\theta_{i^*}\mathrm{id}_{X_{i^*}} = \theta_{X_{i^*}} = (\theta_{X_i})^* = (\theta_i \mathrm{id}_{X_i})^* = \theta_i (\mathrm{id}_{X_i})^* = \theta_i \mathrm{id}_{X_{i^*}}.$$
From this, it follows that $\theta_{i^*} = \theta_i$ for all simple objects X_i.

This paper primarily addresses integral fusion categories, implying that d is an integer and $\mathrm{norm}(\mathsf{d}) = \mathsf{d}$. Such categories are pseudounitary and, consequently, spherical as well as pivotal (see [18]). In contexts that are not pseudounitary, Definition 2.11 would require modifications (as suggested in [29, Theorem 2.1]) because the equality $S_{1,i} = \mathrm{FPdim}(b_i)$ may not be valid.

It should be noted that the definition of modular data provided here is so stringent that, as of now, no instances exist that lack a categorification, leading to Question 1.4.

3. From Fusion Data to Modular Data

This section elucidates the classification of all potential modular data associated with a given set of fusion data. Initially, we may consider the fusion data to be commutative and half-Frobenius (refer to Remark 2.12).

3.1. **S-matrix.** Consider a commutative fusion data $(N_{i,j}^k)$ of rank r, eigentable $(\lambda_{i,j})$, and formal codegrees (c_j) as defined in Definition 2.7. The objective here is to identify all permutations q of the set $\{1,\dots,r\}$ such that:

- $q(1) = 1$,
- $d_{q(i)} = d_i$ for all i,
- The matrix $S = (\sqrt{c_1/c_j}\lambda_{i,q(j)})$ is symmetric (i.e. self-transpose).

Remark 3.1. *The symmetric requirement implies that*

$$\sqrt{c_1/c_j} = \sqrt{c_1/c_j}\lambda_{1,q(j)} = \sqrt{c_1/c_1}\lambda_{j,q(1)} = d_j,$$

hence we can infer that $c_1/c_j = d_j^2$ for all j, as shown in [37, Example 2.9].

If such a permutation q exists (Remark 3.1 can serve as an effective necessary condition), the fusion data are referred to as *self-transposable*. This property is exceedingly rare, rendering this step a potent sieve. Using the Verlinde formula, one can reconstruct the fusion data from S. It is important to note that we need only consider *cyclotomic* fusion data, i.e. whose eigentable entries are all cyclotomic.

3.2. **T-matrix.** For the remaining fusion rings $\mathcal{R}$ with fusion data $(N_{i,j}^k)$, we address the Anderson-Moore-Vafa equations:

$$\left(\sum_{p=1}^{r} N_{i,j}^p N_{p,k}^l\right)(t_i + t_j + t_k + t_l) = \sum_{p=1}^{r}\left(N_{i,j}^p N_{p,k}^l + N_{i,k}^p N_{j,p}^l + N_{j,k}^p N_{i,p}^l\right) t_p$$

within the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. For each valid solution $t = (t_i) \in (\mathbb{Q}/\mathbb{Z})^r$, if any, the corresponding T-matrix is $\mathrm{diag}(e^{2\pi \mathbf{i} t_i})$.

The solutions to the aforementioned equations are determined using the following method: Initially, the matrix reformulation is represented as $At = 0$, where A is an $m \times n$ matrix over $\mathbb{Z}$ with $m = r^4$ and $n = r$. Subsequently, the Smith normal form is employed, denoted as $D = UAV$, in which U and V are invertible matrices over $\mathbb{Z}$ of sizes $m \times m$ and $n \times n$, respectively, and D is a diagonal $m \times n$ matrix $(\alpha_i \delta_{i,j})$, where the integer α_i is divisible by α_{i+1} for all $i < r$, and $\delta_{i,j}$ is the Kronecker delta. The solutions to $Dx = 0$ are precisely represented by the vectors (k_i/α_i), where $0 \le k_i < \alpha_i$. Consequently, we have $U^{-1}DV^{-1}t = 0$, which simplifies to $DV^{-1}t = 0$. Therefore, the solutions can be expressed as $t = Vx$.

A complete listing of potential T-matrices requires considering all vectors (k_i/α_i), where $0 \le k_i < \alpha_i$. As a result, there are $p = \prod_i \alpha_i$ possible combinations. This task remains manageable up to rank 11. However, in certain case of rank above, the value of p becomes too large. But a miraculous circumstance arises (referred to as the **magic criterion**): for all such cases, if one abstractly considers the T-matrix with variables (k_i), then for every determined S-matrix S in §3.1, the abstract product $(ST)^3$ consistently exhibits a zero where it should not, specifically at an entry (i, i^*) for some i. This is because $(ST)^3/p_+ = S^2 = \mathtt{d}C$, where C is the duality matrix (realizing the involution $i \to i^*$, and thus $C_{i,i^*} = 1$, non-zero), as defined in Definition 2.11. The function `MagicCriterion` (also covered by the function `STmatrix`) can verify this.

Question 3.2. *Can the aforementioned* magic criterion *be reformulated at the level of fusion data?*

3.3. **Model by Zesting.** This subsection seeks to model certain modular data indicated in Theorem 1.3, predominantly through the process of zesting as delineated in [16]. The ensuing proposition is attributed to Eric C. Rowell.

Proposition 3.3. *The eight modular data delineated in §12.1.5 are derived from $SO(8)_2$, its conjugates, and zestings.*

Proof sketch. Commencing with $SO(8)_2$, one identifies that it is graded by the group $G = C_2 \times C_2$. This enables to twist the braiding by a bicharacter: the braiding is altered to $B(\deg(X), \deg(Y))c_{X,Y}$, where B represents the bicharacter. Correspondingly, the twists must be adjusted. This action exemplifies a specialized instance of braided (or ribbon) zesting. The resultant effect is the multiplication of specific rows and columns of the S-matrix by a sign. Upon inspecting the S-matrices itemized in §12.1.5, the rationale behind these variations should become apparent. Complex conjugation preserves the S-matrix while altering the T-matrix, thus providing a comprehensive explanation (notably, complex conjugation modifies the underlying fusion category). □

4. Egyptian Fractions with Squared Denominators

A (q, r)-Egyptian fraction with squared denominators is defined as a sum of the form:

$$q = \sum_{i=1}^{r} \frac{1}{s_i^2},$$

where $q, r, s_i \in \mathbb{Z}_{\ge 1}$ and the sequence satisfies $s_1 \ge s_2 \ge \cdots \ge s_r \ge 1$. Additionally, in the context of classifying potential types of Grothendieck rings for modular integral fusion categories (or more broadly, half-Frobenius integral fusion rings), we can assume that each s_i is a divisor of s_1 for all i. By repeatedly subtracting 1 from both q and r as necessary, we can further assume that $s_i \geq 2$ for all i. Subsequently, we can complete the list of (q, r)-Egyptian fractions with squared denominators by including the $(q-k, r-k)$ ones, augmented by k times the number 1 in their sum. Using this technique, we can assume that $q \leq r/4$.

The following steps outline our methodology:

- Employ the function `ModularRep` provided in §4.1 for $1 \leq r \leq 13$ and $1 \leq q \leq r/4$.
- Refine the classification by incorporating additional 1s as described previously.
- Construct all possible types using $d_i = s_1/s_i$. The resulting list is presented in §7.1.

4.1. SageMath Code.

```
def ModularRep(q,r):
    L=all_rep(q, r)
    P=[]
    for l in L:
        if l[0]!=1: # those starting with 1 should be considered with q-1.
            k=0
            for ll in l:
                if l[-1]%ll!=0:
                    k=1
                    break
            if k==0:
                lll=[l[-1]/ll for ll in l]
                lll.sort()
                Di=sum([i^2 for i in lll])
                P.append(lll+[[sqrt(Di)]])
    return P

def res_rep(s, N):
    def succ(t):
        s0, m = t
        if s0==0 or len(m)>=N:
            return []
        p = numerator(s0)
        q = denominator(s0)
        if len(m)==N-1:
            if p==1 and is_square(q):
                r = q.isqrt()
                if r>=m[-1]:
                    return [(0,m+(r,))]
            return []
        L = max(m[-1], ((q-1)//p).isqrt()+1)
        U = floor((N-len(m))/s0).isqrt()
        if len(m)==N-2:
            S = []
            try:
                two_squares(p)
                two_squares(q)
            except:
                return S
            q2 = q^2
            for r in (L..U):
                d = p*r^2-q
                if d>0 and q2%d==0:
                    r2 = (q2//d + q)//p
                    if is_square(r2):
                        S.append( (0,m+(r,r2.isqrt())) )
            return S
```

```
        if len(m)==N-3:
            t = p*q
            a = valuation(t,2)
            if a%2==0 and (t>>a)%8==7:
                return []
        return ( (s0-1/r^2, m+(r,)) for r in (L..U) )
    return RecursivelyEnumeratedSet(seeds=[(s-1/r^2,(r,)) for r in range(1,floor(N/s).isqrt()+1)], \
    successors=succ, structure='forest')

def all_rep(s, N):
    return res_rep(s,N).map_reduce(lambda t: {t[1]} if t[0]==0 and len(t[1])==N else set(), set.union, \
    set() )

def count_rep(s, N):
    return res_rep(s,N).map_reduce(lambda t: int(t[0]==0 and len(t[1])==N))
```

5. Type Criteria

In this section, we delineate criteria that were employed to exclude certain candidates from being the type of a fusion ring. A *type* refers to a list denoted by $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$, where the conditions $1 = d_1 < d_2 < \cdots < d_s$ and $m_i \geq 1$ for all indices i are satisfied. Such a type is characterized as:

- *trivial* if $t = [[1, 1]]$,
- *pointed* if $t = [[1, m]]$ for some m,
- *perfect* if $m_1 = 1$,
- *integral* if each d_i is an integer.

A type $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$ may sometimes be represented simply as

$$[d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_s, \dots, d_s],$$

where each d_i appears m_i times. Thus, we can rephrase the notation for a type of rank r as $[d_1, \dots, d_r]$ with the condition $1 = d_1 \leq d_2 \leq \cdots \leq d_r$.

The criteria described herein are proved using modular arithmetic, and arranged in order of increasing computational complexity. They are group together in the function `TypeCriteria` in the script `TypeCriteria.sage` in [41]. They permit to exclude 5608 types among the 9025 types presented in §7.1, i.e. more than 62%, in just one minute. Here is their counting:

Rank	1	2	3	4	5	6	7	8	9	10	11	12	13
# Types	1	1	1	1	2	3	3	7	11	42	144	812	7997
# Excluded Types	0	0	0	0	0	1	1	3	5	26	85	520	4967

For the remaining types, we will utilize the fusion ring solver, as elaborated in §6.

5.1. Small Perfect Type.

This subsection is dedicated to prove:

Theorem 5.1. *A perfect integral fusion ring of the type $[[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$, with $s \leq 3$, is trivial.*

Proof. If $s = 1$, then the type $t = [[1, 1]]$ is trivial. If $s = 2$, then we have a type $t = [[1, 1], [d, n]]$ with $d > 1$ and $n \geq 1$. However, should a fusion ring of type t exist, let b be a basic element with $\mathrm{FPdim}(b) = d$. When applying FPdim to the decomposition of bb^*, we obtain that $d^2 = 1 + kd$ for some integer $k \geq 1$. Reducing this equation modulo d yields $0 = 1 \mod d$, which is contradictory to $d > 1$. Lastly, if $s = 3$, then the type is $t = [[1, 1], [a, m], [b, n]]$, with $1 < a < b$ and $m, n \geq 1$. Suppose $\mathcal{R}$ is a fusion ring of this type, with basic elements $1, a_1, \dots, a_m, b_1, \dots, b_n$.

Claim 5.2. *The greatest common divisor of a and b, denoted $a \wedge b$, is 1.*

Proof. Let $d = a \wedge b$. Then $\mathrm{FPdim}(a_i a_i^*) = a^2 = 1 + \alpha a + \beta b$, but since d divides both a and b, we have $0 = 1 \mod d$, which implies $d = 1$. □

Claim 5.3. *For every i, there exists j such that $\langle a_i a_i^*, b_j \rangle \neq 0$.*

Proof. If this were not the case, then $a^2 = 1 + \alpha a$, leading to $1 = 0 \mod a$, which contradicts the fact that $a > 1$. □

Claim 5.4. *If $k \neq i^*$, then $\langle a_i a_k, b_j \rangle = 0$.*

Proof. If the claim were false, then $a^2 = \alpha a + \beta b$ with $\beta \neq 0$, which would mean that $\beta b = 0 \mod a$. However, since $a \wedge b = 1$ (indicating that b is invertible modulo a), we get $\beta = b^{-1} \times 0 = 0 \mod a$. Therefore, $\beta = ka$ for some $k \geq 1$. Now, since $a^2 = \alpha a + \beta b \geq \beta b = kab \geq ab$, we would have $a^2 \geq ab$, which contradicts the fact that $b > a$. □

Claim 5.5. $a_{i^*}b_j = ba_{i^*}$.

Proof. By Frobenius reciprocity and Claim 5.4, if $k \neq i^*$ then $\langle a_{i^*}b_j, a_k\rangle = 0$. Claim 5.3 ensures that $\langle a_{i^*}b_j, a_{i^*}\rangle \neq 0$. We know $\mathrm{FPdim}(a_{i^*}b_j) = ab = \alpha a + \beta b$, with $\alpha \geq 1$, leading to the conclusion that $\beta = 0 \mod a$. Hence, $\beta = ka$ for some $k \geq 0$. As a result, $ab = \alpha a + kab$, which simplifies to $(1-k)ab = \alpha a > 0$. This implies $(1-k) > 0$ and thus $k < 1$. Therefore, $k = 0$ and $\beta = 0$. Combining the initial part of this proof with $\beta = 0$ indicates that $a_{i^*}b_j = \alpha a_{i^*}$, where α must equal b (determined by applying FPdim). □

Claim 5.5, together with Frobenius reciprocity, leads us to deduce that $\langle a_i a_{i^*}, b_j\rangle = b$, which means that $a^2 \geq b^2$. This is in contradiction with $a < b$. □

Remark 5.6. *Theorem 5.1 is not extendable to $s = 4$ because the representation category of the alternating group A_5, denoted $\mathrm{Rep}(A_5)$, is of type $[[1,1],[3,2],[4,1],[5,1]]$.*

By applying Theorem 5.1 to the list presented in §4, we can exclude the following four types (up to rank 13): $[[1,1],[2,2],[3,3]]$, $[[1,1],[2,6],[5,3]]$, $[[1,1],[2,2],[3,7]]$, $[[1,1],[3,7],[4,5]]$.

Corollary 5.7. *A non-trivial perfect integral fusion ring has a rank of at least 4.*

Proof. Suppose there is a perfect integral fusion ring with a rank less than 4. Then its type would be $[[d_1,m_1],\dots,[d_s,m_s]]$ with $s \leq r = \sum_i m_i \leq 3$, which contradicts Theorem 5.1. □

Let us also mention the following folklore result:

Proposition 5.8. *There is no non-trivial perfect integral fusion category with* FPdim $p^a q^b$.

Proof. Assume the existence of a non-trivial perfect integral fusion category $\mathcal{C}$ with $\mathrm{FPdim}(\mathcal{C}) = p^a q^b$. Let $\mathcal{S}$ be a non-trivial simple fusion subcategory of $\mathcal{C}$. By Lagrange's theorem [18, Theorem 7.17.6], $\mathrm{FPdim}(\mathcal{S}) = p^c q^d$. Thus, $\mathcal{S}$ is solvable by [19, Theorem 1.6], so it is weakly group-theoretical (and non-pointed), and thus by [19, Proposition 9.11], $\mathrm{FPdim}(\mathcal{S}) = \mathrm{Rep}(G)$ with G being a non-abelian finite simple group. However, this contradicts Burnside's theorem [8] since $|G| = p^c q^d$. □

Theorem 5.1 and Proposition 5.8 serve as categorification criterion checked by the function `SmallPerfect`.

5.2. Gcd Criterion.

Lemma 5.9. *Consider a non-pointed fusion ring of type $[d_1, d_2, \dots, d_r]$. For all i such that $d_i > 1$, let Z_i be the set of indices $j \neq 1$ for which N_{i,i^*}^j is nonzero, and let g_i be $\gcd(d_j \mid j \in Z_i)$. Then it holds that $d_i^2 \equiv 1 \pmod{g_i}$ and $\gcd(d_i, g_i) = 1$.*

Proof. First, note that Z_i is non-empty, which implies that $g_i \neq 0$. According to the Frobenius-Perron theorem, the dimension equation, and the Dual axiom, we have

$$d_i^2 = d_i d_{i^*} = \sum_k d_k N_{i,i^*}^k = 1 + \sum_{j \in Z_i} d_j N_{i,i^*}^j = 1 + K g_i,$$

where K is some integer. Consequently, $d_i^2 \equiv 1 \pmod{g_i}$, and $0 \equiv 1 \pmod{\gcd(d_i, g_i)}$. The lemma follows. □

Proposition 5.10. *Consider a non-trivial perfect fusion ring of type $[d_1, d_2, \dots, d_r]$. Take $i > 1$, let Z'_i be the set of indices $j \neq 1$ for which $d_j < d_i^2$, and let g'_i be $\gcd(d_j \mid j \in Z'_i)$. Then $g'_i = 1$. In particular, $\gcd(d_2, \dots, d_r) = 1$.*

Proof. Note that if N_{i,i^*}^j is nonzero, then $d_i^2 \geq d_j$. Hence, following the notation in Lemma 5.9, Z_i is included in Z'_i, and as a result, g'_i divides g_i. Due to perfectness, we have $d_i > 1$, implying $d_i^2 > d_i$ and therefore i belongs to Z'_i. Consequently, g'_i divides d_i. However, according to Lemma 5.9, $g'_i = 1$. For the final assertion, note that $\gcd(d_2, \dots, d_r)$ is a divisor of $g'_2 = 1$. □

The SageMath code implementing the criterion from Proposition 5.10 can be found in the function `GcdCriterion` within the file `TypeCriteria.sage`, available at [41]. This criterion excludes more than 37% of the perfect types listed in §4, for example, $[1,2,2,6,6,9,9,9]$. Here is the count per rank:

Rank	8	9	10	11	12	13
# Excluded Perfect Types	1	1	7	19	212	2474

5.3. **Type Test.** Let's consider a type $t = [d_1, \dots, d_r]$ with $1 = d_1 \le \dots \le d_r$ and $d_2 > 1$ (signifying that it is perfect). If there is an index i and $g_i > 1$ such that g_i divides every d_j not equal to 1 or d_i, and d_i is coprime with g_i, then assume a fusion ring of this type exists with a basis $\{b_1, \dots, b_r\}$ where $d_k = \mathrm{FPdim}(b_k)$.

Lemma 5.11. *For every j with $d_j \neq 1$ and $d_j \neq d_i$, the following equation holds:*

$$\sum_{k;\ d_k = d_i} N_{j,j^*}^k \equiv -1/d_i \mod g_i.$$

Proof. For each j with $d_j \neq 1$ and $d_j \neq d_i$, we have:

$$b_j b_{j^*} = b_1 + \sum_{k;\ d_k = d_i} N_{j,j^*}^k b_k + \sum_{k;\ d_k \neq 1, d_i} N_{j,j^*}^k b_k.$$

By applying FPdim and reducing modulo g_i, we obtain:

$$0 = 1 + x d_i \mod g_i,$$

where d_i has a multiplicative inverse modulo g_i. Therefore, $x \equiv -1/d_i \mod g_i$. □

Given an integer a_{d_i} such that $0 \le a_{d_i} < g_i$ and $a_{d_i} \equiv -1/d_i \mod g_i$, let S be the set containing all such d_i. From Lemma 5.11, for every $j \neq 1$, the inequality below must hold:

$$d_j^2 \ge 1 + \sum_{d \in S \setminus \{d_j\}} a_d d,$$

thus if the inequality does not hold, t cannot be a type of a fusion ring. Furthermore, if the set $\{k \mid d_k = d_j\}$ is a singleton, we can use a stronger inequality:

$$d_j^2 \ge 1 + b_j d_j + \sum_{d \in S \setminus \{d_j\}} a_d d,$$

with $0 \le b_j < g_j^2$ and $b_j \equiv d_j - \frac{1}{d_j} \mod g_j^2$.

The SageMath code implementing this criterion can be found in the function `TypeTest` within the file `TypeCriteria.sage`, available at [41]. This criterion helped to exclude a certain number of perfect types per rank in the list from §4, as shown in the table below:

Rank	8	9	10	11	12	13
#Excluded Perfect Types	1	1	12	37	249	2380

5.4. **Local Criterion.** Consider a type $t = [[d_1, m_1], [d_2, m_2], \dots, [d_s, m_s]]$. Assume the existence of $g, i_0 > 1$ such that g divides each d_i for all indices i not in the set $\{1, i_0\}$, and d_{i_0} is coprime with g. Let $(d, m) := (d_{i_0}, m_{i_0})$. If it corresponds to a fusion ring with a basis $\{b_{1-m_1}, \dots, b_0, b_1, \dots, b_{r-1}\}$, where b_0 is the unit, $\mathrm{FPdim}(b_i) = 1$ for $i \le 0$, and $\mathrm{FPdim}(b_j) = d$ for $j \in \{1, \dots, m\}$, then the following lemma applies:

Lemma 5.12. *For each $i \in \{1, \dots, m\}$, the equation below is valid:*

$$\sum_{j,k=1}^{m} N_{i,j}^k \equiv md - \frac{m_1}{d} \mod g^2,$$

and for all $j > m$, the integer g divides $\sum_{k=1}^m N_{i,j}^k$.

Proof. For any $i \in \{1, \dots, m\}$ and $j > m$, since $\mathrm{FPdim}(b_i) \neq \mathrm{FPdim}(b_j)$, by Corollary 2.9 and Frobenius reciprocity, we have:

$$b_i b_j = \sum_{k \ge 1} N_{i,j}^k b_k = \sum_{k=1}^{m} N_{i,j}^k b_k + \dots,$$

Applying FPdim and reducing modulo g, we conclude that:

$$d \sum_{k=1}^{m} N_{i,j}^k \equiv 0 \mod g,$$

which implies that g divides $\sum_{k=1}^m N_{i,j}^k$. For each $i \in \{1, \dots, m\}$, the sum over the basis elements yields:

$$b_{i^*} \sum_{k=1}^{m} b_k = \sum_{s \le 0} b_s + \sum_{j=1}^{m} \left(\sum_{k=1}^{m} N_{i,j}^k \right) b_j + \sum_{j > m} \left(\sum_{k=1}^{m} N_{i,j}^k \right) b_j.$$

After applying FPdim, we obtain $md^2 = m_1 + xd + yg^2$, hence $x \equiv md - \frac{m_1}{d} \mod g^2$. □

For a type t, we can analyze the partitions of $md^2 - xd - m_1$ in the form $\sum_{i \notin \{1,i_0\}} a_i d_i$, with $x \equiv md - \frac{1}{d} \mod g^2$ and $a_i \equiv 0 \mod g$. The SageMath code performing this analysis can be found in the function `LocalCriterion` within the file specified earlier, also available at [41]. This criterion, which can rule out types when no suitable partitions are found, is further detailed in the following example.

Example 5.13. *Consider the type* $t = [[1,1],[1295,2],[3990,1],[4218,1],[24605,1],[42180,1],[98420,2],[147630,3]]$. *We can apply Lemma 5.12 to the triples* $(d,m,g) = (1295,2,19),(3990,1,37),(4218,1,5)$. *Subsequently, we obtain* $md - \frac{1}{d} \equiv 126, 1135, 11 \pmod{g^2}$ *for each respective triple. The application of the function* `LocalCriterion` *to the triple* $(d,m,g) = (1295,2,19)$ *enables us to eliminate the type* t *in less than one second.*

```
sage: %time LocalCriterion(T, 1295, 2, 19)
CPU times: user 640 ms, sys: 0 ns, total: 640 ms
Wall time: 982 ms
[]
```

However, we cannot employ the triple $(d,m,g) = (3990,1,37)$, *as it yields 55 solutions.*

```
sage: L = LocalCriterion(T, 3990, 1, 37)
sage: len(L)
55
```

The application of LocalCriterion to the list in §4 led to the exclusion of several types per rank, as summarized in the following table:

Rank	6	7	8	9	10	11	12	13
# Excluded Types	1	1	3	5	21	63	344	2852
# Excluded Perfect Types	1	1	2	2	14	37	238	2173

It is noteworthy that this criterion alone suffices to eliminate all perfect types up to rank 9. Therefore, it can be stated conclusively that no non-trivial perfect integral half-Frobenius fusion rings, and thus no non-trivial perfect modular integral fusion categories, exist up to rank 9. The use of a fusion ring solver as detailed in §6 can extend this conclusion to rank 12, as discussed in §7, and then to rank 13, see §10.

6. Enhanced Fusion Ring Solver Using Normaliz

A fusion ring solver is a computational tool designed to receive a particular type as input and output all corresponding fusion rings of that type. Initially, §6.1 provides a brief introduction to the highly intuitive user interface of Normaliz [10] from version 3.10.2, and §6.2 offers an overview of Normaliz's goals and outlines the adjustments made to support the unique linear and polynomial constraints specific to fusion rings. The last two subsections §6.3 and §6.4 contains the SageMath/Normaliz approach *before* version 3.10.2 (and show the system of equations explicitly). They introduce two versions of a fusion ring solver: the full version, which is discussed in §6.3 and addresses both dimension equations and associativity equations, and the partition (intermediate) version, which is detailed in §6.4 and focuses on a simplified set of dimension equations through the implementation of a partition.

6.1. **Normaliz user interface for fusion rings.** Starting from version 3.10.2, Normaliz [10] offers a streamlined user interface for computing fusion rings. This is illustrated through the input file named `bracket_4.in`, found in the `example` directory of the Normaliz distribution:

```
amb_space auto
fusion_type
[1,1,2,3,3,6,6,8,8,8,12,12]
fusion_duality
[0,1,2,3,4,5,6,7,8,9,11,10]
```

It's important to note that in the duality, indices start from 0. From this input, Normaliz generates the linear and quadratic equations defining the fusion data. The default computational goal for this input is `FusionRings`.

To run this command on Linux or MacOS, use the following command line syntax, assuming `example` is the current directory. For some progress information on the terminal, you can add the `-c` flag. On a modern laptop, the computation typically takes less than 10 seconds and requires about 2.4 GB of RAM.

```
path/to/normaliz bracket_4
```

A brief informal explanation of the algorithm used to solve the system of equations is presented in §6.2.

The results are detailed in the `bracket_4.out` file, beginning with a preamble:

```
148 fusion rings up to isomorphism
0 simple fusion rings up to isomorphism
148 nonsimple fusion rings up to isomorphism
Embedding dimension 231
dehomogenization
```

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ... 0 0 0 0 0 1
```

The 148 fusion rings correspond to the orbits of the set of lattice points with respect to the symmetries of the equation system. These symmetries are observed under permutations of the type vector that adhere to the Frobenius-Perron equations and are compatible with the duality. Put simply, we have identified 148 pairwise nonisomorphic fusion rings, classified according to their type and duality. They are automatically categorized into simple and nonsimple fusion rings. To limit the computation exclusively to simple fusion rings, one can modify the input file by including the `SimpleFusionRings` option.

The term *embedding dimension* refers to the number of coordinates utilized during the computation. The rationale behind the selection of these coordinates is discussed in §6.3. The final component, represented by the number 1 in the dehomogenization process, signifies that the equations' right-hand side corresponds to the last coordinate of the solutions. This component is not included in the fusion data.

The latter part of the output file details the fusion rings represented by lattice points:

```
0 simple fusion rings up to isomorphism:

148 nonsimple fusion rings up to isomorphism:
0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 ... 1 3 3 1 1 1
...
```

To generate fusion data from the lattice points, add the `FusionData` option in the input file. The result includes a list of fusion data for each fusion ring, presented as a series of matrices M_i, where $i = 1, \ldots, r$ and r is the rank of the fusion ring ($r = 12$ in our example). The matrix M_i comprises the elements N_{ij}^k, with row index k and column index j.

The input file for solving the dimension partition version (see §6.4), for instance, `bracket_3_part.in`, is as follows:

```
amb_space auto
fusion_type_for_partition
[1,1,2,3,3,6,6,8,8,8,12,12]
```

Here, the default computation goal is `SingleLatticePoint`, focusing on the solvability of the system.

For additional information and further options, refer to Appendix H of the Normaliz manual (`Normaliz.pdf`), available in the `doc` directory of the Normaliz distribution or online at [11]. This includes details on restricting computations to fusion rings that meet certain criteria for modular categorification.

6.2. **Normaliz and its approach to fusion rings.** Normaliz [10] is an open source software for discrete convex geometry and its algebraic aspects. Readers are referred to Bruns and Gubeladze [7] for detailed terminology and a comprehensive discussion. Normaliz is designed to solve Diophantine systems of linear inequalities, equations, and congruences with integer coefficients. Additionally, it calculates enumerative information such as multiplicities (which correspond to geometric volumes) and Hilbert series. Objects in Normaliz can be defined either by generators, such as the extreme rays of cones, bases of lattices, and vertices of polytopes, or by constraints like inequalities, equations, and congruences. For systems with coefficients in real algebraic number fields, Normaliz can execute fundamental operations like convex hull computation and its dual, vertex enumeration. Moreover, it is capable of computing lattice points within (bounded) polytopes over real algebraic number fields, facilitating applications to non-integral fusion rings. In the context of fusion rings, it is crucial that lattice points within polytopes can be subjected to constraints imposed by polynomial equations and inequalities. Each release of Normaliz includes source code, comprehensive documentation, sample examples, a testing suite, and pre-compiled binaries for Linux, Mac OS, and MS Windows systems.

For lattice points in generic polytopes denoted by P, Normaliz employs the project-and-lift algorithm. It sequentially projects P onto coordinate hyperplanes until reaching zero dimensions and then lifts the lattice points back up. If P' is a projection of P onto a coordinate hyperplane, then the lattice points of P are projected to lattice points in P', and if $x \in P'$ is a lattice point within P', its preimages are the lattice points in a line segment. Polynomial constraints can be introduced as soon as the lifting process reaches the highest coordinate present in the constraint.

In its standard form, the project-and-lift method is suitable for only minor cases of fusion rings. For satisfactory performance, the algorithm has been tailored to the special linear and polynomial constraint structure specific to fusion rings. Each linear equation is inhomogeneous with nonnegative coefficients and a positive right-hand side. We can refer to the set of coordinates that appear in the equation with positive coefficients as a "patch". These patches encompass the entire set of coordinates, and thus the linear equations, when restricted to the nonnegative orthant, delineate a polytope P. Solutions to a linear equation, confined to its patch, are ascertained using the project-and-lift technique previously outlined, and the lattice points in P are derived by combining these local solutions along matching components. In essence, we begin with the solutions of one of the equations and progressively extend them patch by patch. The sequence in which patches are integrated into the extension process is pivotal. Normaliz includes options that allow alteration of the sequence, as detailed later on.

In the partition version (§6.4), the input file is only made of simplified dimension (linear) equations. Particularly for this case, it is critical to recognize a secondary, implicit constraint type: congruences extracted from the linear equations by taking successive residue classes modulo their coefficients. By default, each congruence involves only the coordinates pertaining to the patch of its originating equation. Nonetheless, since congruences only involve a subset of these coordinates, they frequently pertain to other patches or combinations thereof, potentially significantly limiting their number of solutions. Our current classification up to rank 13 would not have been achievable without meticulous utilization of the congruences.

When polynomial equations of degree two or higher are in play, Normaliz endeavors to determine an optimal patch extension order that allows these equations to be applied as early as feasible. Users can influence this order by either insisting on the "linear" input order or by directing Normaliz to employ "weights" that gauge the anticipated solution count for each patch and prioritize those with lower weight. Regardless of whether polynomial equations are present, users can request an order based on the applicability of congruences. This order can also be weight-dependent.

Some computations for simple rank 13 were executed on the high-performance cluster (HPC) at Osnabrück by early splitting of partial solutions into parts, which were then processed separately. Despite the rather basic approach of using a static subdivision without intercommunication between running instances of Normaliz, the HPC proved to be advantageous.

6.3. **Full Version.**

Remark 6.1. *All the processes outlined in this subsection have been fully automated in the recently released Normaliz 3.10.2* [10]*, see §6.1. Appendix H of its manual* [11] *specifically addresses the computation of the fusion rings for a specified type.*

Consider a fusion ring with the basis $\{b_1, \dots, b_r\}$. As described in §2.1, for all indices i, j:

$$b_i b_j = \sum_k N_{i,j}^k b_k,$$

and by applying FPdim, we obtain the type $[d_1, \dots, d_r]$ and the corresponding *dimension equations*:

$$d_i d_j = \sum_k N_{i,j}^k d_k.$$

The objective is to resolve these r^2 linear positive Diophantine equations, where (d_i) are specified and $(N_{i,j}^k)$ represent r^3 variables, using Normaliz. Now, we can decrease the variable count to roughly $(r-1)^3/6$ by invoking the Unit axiom ($N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$) from the Definition 2.1 of fusion data, as well as the Frobenius reciprocity (Proposition 2.2).

A critical factor in accelerating computation is the strategic use of associativity equations (non-linear)

$$\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t,$$

in the most effective manner possible during the solving process of the aforementioned linear Diophantine equations. While the optimal approach is not confirmed, the method we employ is highly efficient (refer to §6.2 for further details).

In practice, for a given type $L = [d_1, d_2, \dots, d_r]$, utilize the function `TypeToNormaliz`, the SageMath code for which can be found at [41]. This function generates input files (.in), one for each potential duality map $i \to i^*$. Place these files in a directory alongside the normaliz.exe and run_normaliz.bat files available at [41], and execute run_normaliz (note the existence of a more recent and faster Linux version used for our latest computations). This process yields output files (.out) containing all potential solutions (if any exist). The remaining task is to convert these solutions into fusion data, considering isomorphism. We demonstrate how this can be done with the following example. Take the type $L = [1, 1, 2]$ of the character ring of S_3. When `TypeToNormaliz` is applied, it generates the file [1,1,2][0,1,2].in with the content as follows:

```
amb_space 4
inhom_equations 4
1 2 0 0 0
0 1 2 0 -2
0 1 2 0 -2
0 0 1 2 -3
LatticePoints
convert_equations
nonnegative
polynomial_equations 2
x[2]^2 - x[1]*x[3] + x[3]^2 - x[2]*x[4] - 1;
-x[2]^2 + x[1]*x[3] - x[3]^2 + x[2]*x[4] + 1;
```

The upper part encodes the linear Diophantine equations, and the lower part lists the associativity equations. Following the execution of run_normaliz, the file [1,1,2][0,1,2].out is produced, containing:

```
1 lattice points in polytope (module generators) satisfying polynomial constraints:
 0 0 1 1 1
```

Here, we encounter a single solution, but there could be multiple in general (as seen in the subsequent example). Next, remove the final '1' from each line of the solution and convert it into a list of lists:

```
sage: LL=[[0,0,1,1]]
```

Collect the lists for the type and the duality map:

```
sage: L=[1,1,2]
sage: d=[0,1,2]
```

Finally, to obtain all the fusion data up to isomorphism, apply the function `ListToFusion`:

```
sage: ListToFusion(LL,L,d)
[[[[1, 0, 0], [0, 1, 0], [0, 0, 1]],
  [[0, 1, 0], [1, 0, 0], [0, 0, 1]],
  [[0, 0, 1], [0, 0, 1], [1, 1, 1]]]]
```

The result is the fusion data of $\mathrm{ch}(S_3)$, which can ultimately be formatted in TeX as follows:

$$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix},\quad\begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix},\quad\begin{matrix}0&0&1\\0&0&1\\1&1&1\end{matrix}$$

Now, applying the same procedure with the type $L=[1,5,5,5,6,7,7]$, we obtain four input files. Only the file corresponding to the trivial duality map yields solutions, with its output file containing:

```
6 lattice points in polytope (module generators) satisfying polynomial constraints:
 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 0 3 1 2 1
 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 2 1 1
 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 1 3 0 1
 1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 0 3 1 2 1
 1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 2 1 1
 1 1 0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 1 3 0 1
```

All files can be accessed at [41]. Ultimately, we acquire the following two sets of fusion data, up to isomorphism:

$$\begin{matrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{matrix},\ \begin{matrix}0&1&0&0&0&0&0\\1&1&0&1&0&1&1\\0&0&1&0&1&1&1\\0&1&0&0&1&1&1\\0&0&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{matrix},\ \begin{matrix}0&0&1&0&0&0&0\\0&0&1&0&1&1&1\\1&1&1&0&0&1&1\\0&0&0&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{matrix},\ \begin{matrix}0&0&0&1&0&0&0\\0&1&0&0&1&1&1\\0&0&0&1&1&1&1\\1&0&1&1&0&1&1\\0&1&1&0&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{matrix},\ \begin{matrix}0&0&0&0&1&0&0\\0&0&1&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&0&1&1&1\\1&1&1&1&1&1&1\\0&1&1&1&1&2&1\\0&1&1&1&1&1&2\end{matrix},\ \begin{matrix}0&0&0&0&0&1&0\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&2&1\\1&1&1&1&2&0&3\\0&1&1&1&1&3&1\end{matrix},\ \begin{matrix}0&0&0&0&0&0&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&2\\0&1&1&1&1&3&1\\1&1&1&1&2&1&2\end{matrix}$$

$$\begin{matrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{matrix},\ \begin{matrix}0&1&0&0&0&0&0\\1&1&0&1&0&1&1\\0&0&1&0&1&1&1\\0&1&0&0&1&1&1\\0&0&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{matrix},\ \begin{matrix}0&0&1&0&0&0&0\\0&0&1&0&1&1&1\\1&1&1&0&0&1&1\\0&0&0&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{matrix},\ \begin{matrix}0&0&0&1&0&0&0\\0&1&0&0&1&1&1\\0&0&0&1&1&1&1\\1&0&1&1&0&1&1\\0&1&1&0&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\end{matrix},\ \begin{matrix}0&0&0&0&1&0&0\\0&0&1&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&0&1&1&1\\1&1&1&1&1&1&1\\0&1&1&1&1&2&1\\0&1&1&1&1&1&2\end{matrix},\ \begin{matrix}0&0&0&0&0&1&0\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&2&1\\1&1&1&1&2&1&2\\0&1&1&1&1&2&2\end{matrix},\ \begin{matrix}0&0&0&0&0&0&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&2\\0&1&1&1&1&2&2\\1&1&1&1&2&2&1\end{matrix}$$

6.4. **Dimension Partition Version.** This method is applicable primarily for types denoted by

$$T=[[1,m_1],[d_2,m_2],\dots,[d_s,m_s]],$$

where s is not exceedingly large. This is because we can streamline the dimension equations by grouping elements that share the same dimension (i.e. dimension partition). However, the conversion of the associativity equations remains an open challenge. This version is intended to serve as an intermediary step to the full version for suitable types. Its utility lies in its ability to circumvent certain computational complexities by breaking symmetries. For the time being, it functions as a criterion; that is, if this version fails to yield a solution, the full version will similarly lack a solution.

We can reframe the type as $[1,d_{1,1},\dots,d_{1,n_1},d_{2,1},\dots,d_{2,n_2},\dots,d_{s,1},d_{s,n_s}]$, where $d_{i,a}=d_i$, $d_1=1=d_{0,1}$, and $n_i=m_i-\delta_{1,i}$. The dimension equations are then expressed as follows:

$$d_{i,a}d_{j,b}=\sum_{k,c}N_{i,a,j,b}^{k,c}d_{k,c}.$$

Let us define $D_i:=\sum_{a=1}^{n_i}d_{i,a}=n_id_i$ and $M_{i,j}^k:=\sum_{a,b,c}N_{i,a,j,b}^{k,c}$, which simplifies the equations to:

$$D_iD_j=\sum_{a,b}\sum_{k,c}N_{i,a,j,b}^{k,c}d_{k,c}=\sum_k(\sum_{a,b,c}N_{i,a,j,b}^{k,c})d_k=\sum_k M_{i,j}^k d_k.$$

Consequently, we are tasked with solving the linear positive Diophantine equations:

$$n_i d_i n_j d_j = \sum_k M_{i,j}^k d_k,$$

where (d_i, n_i) are predetermined, and the variables $(M_{i,j}^k)$ are reduced to roughly $s^3/6$ by employing the dimension partition variant of the Unit axiom and Frobenius reciprocity. After grouping by dimension, the duality map becomes straightforward (that is, $i^* = i$). Note that we have not yet derived a satisfactory dimension partition version of the associativity axiom, but about the other ones:

Lemma 6.2. *The following equalities hold:*

- *(Unit)* $M_{i,0}^j = M_{0,i}^j = \delta_{i,j} m_i$
- *(Dual)* $M_{i,j}^0 = M_{j,i}^0 = \delta_{i,j} m_i$
- *(Frobenius reciprocity)* $M_{i,j}^k = M_{i,k}^j = M_{j,k}^i = M_{j,i}^k = M_{k,i}^j = M_{k,j}^i$.

Proof. The proof is straightforward. ☐

In practice, one should follow the procedure outlined in §6.3 up to the generation of output files but replace the function `TypeToNormaliz` with `TypeToPreNormaliz`. For instance, consider the type $L = [1, 6, 12, 12, 15, 15, 15, 20, 20, 30, 30, 60]$.

Remark 6.3. *While this version utilizes the dimension partition of the type, alternative versions could explore other pertinent partitions.*

7. Half-Frobenius Integral Fusion Rings up to rank 12

This section focuses on classifying all half-Frobenius integral fusion rings up to rank 12. Initially, we considered 1028 types derived from Egyptian fractions with squared denominators, as discussed in §7.1. From these, we identified 10628 fusion rings originating from just 71 types, detailed in §7.2. Among them, we found 213 noncommutative fusion rings. Ultimately, only 69 fusion rings, from 27 types, are commutative, cyclotomic, and self-transposable, as outlined in §7.3.

7.1. **List of possible types.** Based solely on Egyptian fractions with squared denominators, we found 1028 possible types up to rank 12 (in fact, 9025 ones up to rank 13). Below is the counting per rank:

Rank	1	2	3	4	5	6	7	8	9	10	11	12	13
#Types	1	1	1	1	2	3	3	7	11	42	144	812	7997
#Perfect Types	1	0	0	0	0	1	1	2	2	24	88	591	6517

The ratio of perfect types (refer to §5) exhibits an increasing trend, e.g. 18% for rank 9, but 81% for rank 13. This leads us to question whether this ratio tends to 1 as the rank goes to infinity.

The list of all such (non-pointed) types up to rank 9 is provided (those up to rank 13 can be found online at [41]):

- Rank 5: [[1, 1, 1, 1, 2]],
- Rank 6: [[1, 1, 1, 1, 2, 2], [1, 2, 2, 3, 3, 3]],
- Rank 7: [[1, 1, 1, 1, 2, 2, 2], [1, 2, 2, 3, 3, 3, 6]],
- Rank 8: [[1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4], [1, 1, 2, 2, 2, 2, 3, 3], [1, 1, 3, 3, 4, 6, 6, 6], [1, 2, 2, 3, 3, 3, 6, 6], [1, 2, 2, 6, 6, 9, 9, 9]],
- Rank 9: [[1, 1, 1, 1, 1, 1, 1, 1, 2], [1, 1, 1, 1, 1, 2, 3, 3, 3], [1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6], [1, 1, 3, 3, 4, 6, 6, 6, 12], [1, 1, 4, 9, 9, 12, 18, 18, 18], [1, 2, 2, 3, 3, 3, 6, 6, 6], [1, 2, 2, 6, 6, 9, 9, 9, 18]].

7.2. **List of fusion rings.** The set of types given in §7.1 was critically reduced by the type criteria in §5. Subsequently, we classified all the possible fusion rings of the remaining types utilizing the fusion ring solver discussed in §6. We ended with 71 types, 10628 fusion rings, 213 among them being noncommutative. Here is the counting per rank.

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Types	1	1	1	1	2	2	2	4	5	9	15	28
#Fusion Rings	1	1	1	2	3	6	9	23	105	158	1218	9101
#Noncommutative	0	0	0	0	0	1	0	4	5	7	38	158

The number of types per rank is relatively modest when contrasted with the table in §7.1. Below is the list of (non-pointed) types for each rank, arranged in lexicographic order:

- Rank 5: [[1, 1, 1, 1, 2]],
- Rank 6: [[1, 1, 1, 1, 2, 2]],
- Rank 7: [[1, 1, 1, 1, 2, 2, 2]],
- Rank 8: [[1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4], [1, 1, 2, 2, 2, 2, 3, 3]],
- Rank 9: [[1, 1, 1, 1, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6]],
- Rank 10: [[1, 1, 1, 1, 1, 1, 1, 1, 1, 3], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6]],

- Rank 11: [[1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 4], [1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 6], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8], [1, 1, 1, 1, 2, 4, 4, 4, 4, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 12], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12, 12], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18]],
- Rank 12: [[1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2], [1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 4, 4], [1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 6], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 4], [1, 1, 1, 1, 2, 2, 2, 2, 4, 6, 6, 6], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4, 4], [1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8, 8], [1, 1, 1, 1, 2, 2, 2, 8, 8, 12, 12, 12], [1, 1, 1, 1, 2, 4, 4, 4, 4, 6, 6, 12], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 12, 24], [1, 1, 1, 1, 2, 8, 18, 18, 24, 36, 36, 36], [1, 1, 1, 1, 3, 3, 3, 3, 4, 4, 6, 6], [1, 1, 1, 1, 4, 4, 6, 6, 6, 12, 12, 12], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18, 36], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 6, 6], [1, 1, 1, 2, 2, 2, 3, 4, 4, 4, 6, 6], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6, 12], [1, 1, 1, 3, 6, 8, 8, 8, 8, 8, 8, 12], [1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 12], [1, 1, 2, 2, 2, 2, 6, 6, 6, 6, 9, 9], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12, 12], [1, 1, 2, 3, 3, 6, 6, 8, 8, 8, 12, 12], [1, 1, 2, 6, 6, 6, 6, 10, 10, 10, 15, 15]].

As you can see, Theorem 1.6 has already been proven at this point. It is noteworthy that the perfect integral modular fusion category, and therefore half-Frobenius, $\mathcal{Z}(\mathrm{Rep}(A_5))$ has an FPdim of $60^2 = 3600$, a rank of 22, and a type of $[[1,1],[3,2],[4,1],[5,1],[12,10],[15,4],[20,3]]$, calculated using [28, Section 3] and GAP.

Question 7.1. *Is there a perfect integral half-Frobenius fusion ring/category with a rank less than* 22*?*

Note that the perfect integral modular fusion category $\mathcal{Z}(\mathrm{Rep}(A_7))$, with FPdim $(7!/2)^2$, rank 74, and type:

$$[[1,1],[6,1],[10,2],[14,2],[15,1],[21,1],[35,1],[70,9],[105,4],[210,20],[280,9],[360,14],[504,5],[630,4]],$$

notably lacks any basic elements whose FPdim is a prime-power.

Question 7.2. *Is there a perfect integral half-Frobenius fusion ring/category, without any basic elements of prime-power* FPdim*, that has a rank lower than* 74*?*

7.3. **Commutative, cyclotomic and self-transposable.** Among the 10628 half-Frobenius integral fusion rings up to rank 12 discovered above, only 99.3% are commutative cyclotomic and self-transposable (see §3.1), i.e., 69 fusion rings from 27 types, here is their counting per rank:

Rank	1	2	3	4	5	6	7	8	9	10	11	12
#Types	1	1	1	1	1	1	2	2	2	4	5	6
#Fusion Rings	1	1	1	2	1	1	3	7	4	11	13	24

The types mentioned above, restricted to the non-pointed ones, are listed below:

- Rank 7: [1, 1, 1, 1, 2, 2, 2],
- Rank 8: [1, 1, 2, 2, 2, 2, 3, 3],
- Rank 9: [1, 1, 1, 1, 4, 4, 6, 6, 6],
- Rank 10:[1, 1, 1, 1, 2, 2, 2, 4, 4, 4], [1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6],
- Rank 11:[1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 6, 6, 8, 12, 12, 12], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18],
- Rank 12:[1, 1, 1, 1, 2, 8, 18, 18, 24, 36, 36, 36], [1, 1, 1, 3, 6, 8, 8, 8, 8, 8, 8, 12], [1, 1, 2, 2, 2, 2, 6, 6, 6, 6, 9, 9], [1, 1, 2, 3, 3, 6, 6, 8, 8, 8, 12, 12], [1, 1, 2, 6, 6, 6, 6, 10, 10, 10, 15, 15].

The list of fusion rings referenced above can be found in [41]. They were classified utilizing the list from §7.2 in conjunction with the function `preSmatrix` contained within the file `ModularData.sage`, also available at [41].

8. Advanced results on modular fusion categories

From rank 13 onwards, it becomes impractical to classify all half-Frobenius integral fusion rings using our current technology. Thus, the types were further restricted by additional properties coming from more advanced results on modular fusion categories, which involved the universal grading §8.1, congruence representations of the modular group §8.2 and Galois action §8.3.

8.1. **Universal Grading.** Let G de a finite group. A *G-grading* of a fusion ring R is given by a partiton of its basis $B = \sqcup_{g \in G} B_g$ such that:

- For any $x \in B_g$ and any $y \in B_{g'}$, the basic components of xy belong to $B_{gg'}$.
- For any $x \in B_g$, x^* is in $B_{g^{-1}}$.

A G-grading is called *faithful* if B_g is non-empty for all $g \in G$. Consequently, by [18, Theorem 3.5.2], $\mathrm{FPdim}(B_g) := \sum_{x \in B_g} \mathrm{FPdim}(x)^2$ is constant in g. The faithful grading with the largest group is called the *universal grading*. By [18, Lemma 8.22.9], the universal grading group of the Grothendieck ring of modular fusion category is $G = B_{pt}$, the group of basic element with FPdim $= 1$ (see Corollary 2.10). All these modular constraints lead to the following definition:

Definition 8.1. *Let t be a* type*, i.e., a sorted list of integers starting with* 1*. Let r be the length of t. Let p be the number of entries equal to* 1 *in t. Let $D := \sum_{d \in t} d^2$. A* modular partition *of t is a list L of lists such that:*

- *t is the sorted concatenation of the lists in L,*

- *L is lexicographically sorted,*
- *the lists in L are sorted,*
- *L has p elements,*
- *p divides d,*
- *for all l in L, then $\sum_{d \in l} d^2 = d/p$,*

A solution for t may be called a partitioned type.

The function classifying all the modular partitions of a given type is named `ModularPartitions` in `TypeCriteria.sage` in [41]. Here are a few examples with 0, 1, 2, and 3 solutions:

```
sage: %attach TypeCriteria.sage
sage: L0=[1,1,1,1,2]
sage: ModularPartitions(L0)
[]
sage: L1=[1, 1, 1, 3, 12, 12, 30, 40, 40, 40, 40, 40, 40, 60]
sage: ModularPartitions(L1)
[[[1, 1, 1, 3, 12, 12, 30, 60], [40, 40, 40], [40, 40, 40]]]
sage: L2=[1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4]
sage: ModularPartitions(L2)
[[[1, 1, 1, 1, 4], [2, 2, 2, 2, 2], [2, 4], [2, 4]],
 [[1, 1, 1, 1, 2, 2, 2, 2], [2, 4], [2, 4], [2, 4]]]
sage: L3=[1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 6, 6]
sage: ModularPartitions(L3)
[[[1, 1, 2, 2, 2, 2, 3, 3, 6], [3, 3, 3, 3, 6]],
 [[1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3], [6, 6]],
 [[1, 2, 2, 3, 3, 3, 6], [1, 2, 2, 3, 3, 3, 6]]]
```

The classification of fusion rings constrained to such a grading is a coming Normaliz feature, already available for the C_2-grading, see §6.

Theorem 8.2. *Let $\mathcal{C}$ be an integral modular fusion category. Let R be its Grothendieck ring with basis B. Let $G = B_{pt}$ be the universal grading group. Let $t_g := (\mathrm{FPdim}(x))_{x \in B_g}$. Let $\mathcal{C}_e$ be the fusion subcategory corresponding to B_e. Then:*

(0) If $\mathcal{C}_e$ is perfect then it is modular,
(1) If $B_{pt} \subset B_e$ and t_e has an entry with odd multiplicity, then $\forall g \neq e$, every entry of t_g has multiplicity ≥ 2,
(2) If condition (1) is satisfied, $p := |B_{pt}|$ is prime, and an entry d in t_e appears with multiplicity m, then p divides d or m,
(3) If condition (1) is satisfied, $p := |B_{pt}|$ is prime, then there exists a modular fusion category $\mathcal{D}$ with $\mathrm{FPdim} = \mathrm{FPdim}(\mathcal{C})/p^2$ and type t'_e, where t'_e is a reduction of t_e that involves mapping p identical entries $x, \dots, x$ to a single entry x, or alternatively, mapping one entry x which is a multiple of p to p entries $x/p, \dots, x/p$.

Proof. The sentence (0) is contained in [30, Proposition VI.2]. By [30, Proposition VI.2 (b)], and following the notation of this reference, condition (1) forces the invertible objects to be bosons, so $\mathcal{C}_{pt}$ is Tannakian, thus (1) follows from [30, Proposition VI.2 (d)]. Finally, (3) follows from [30, Proposition VI.2 (e)], citing the *modularization* of [6], as a group of prime order must be cyclic; and (3) implies (2) trivially. □

Remark 8.3. *Since the Grothendieck ring of a modular fusion category $\mathcal{C}$ is half-Frobenius, for each entry x of its type, x^2 divides $\mathrm{FPdim}(\mathcal{C})$. However, regarding Theorem 8.2 (3), it is often the case that an entry x in t_e does not satisfy the condition that x^2 divides $\mathrm{FPdim}(\mathcal{C})/p^2$. Therefore, this entry must be split into p entries $x/p, \dots, x/p$ in t'_e. This helps to reduce the number of possible t'_e.*

The function `GradingCriteria` in `TypeCriteria.sage` automates the use of Theorem 8.2, it also iterates over possible types of modularization and rechecks `Theorem4Check` and `TypeCriteria`.

Here are three examples demonstrating the exclusion criteria, each corresponding to a point of Theorem 8.2.

(0) In the partitioned type $[[1,2,2,3,3,3,3,3],[1,2,2,3,6]]$, the neutral component is perfect, but we already know that there is no perfect integral modular fusion category of rank 8,
(1) In $[[1,1,1,1,2,2,2,3,10,10],[15],[15],[15]]$, the pointed part is in the neutral component t_e, and the entry 2 has multiplicity three (odd) in t_e, but 15 appears with multiplicity one in some non-neutral components.
(2) In $[[1,1,2,2,3,3,5,6,6,10,15],[15,15]]$, the pointed part is in the neutral component t_e, and the entry 5 has multiplicity one (odd) in t_e, but 5 is not divisible by the prime $2 = |B_{pt}|$.

8.2. **Congruence Representation.** This subsection reviews some applications of congruence representations of the modular group to modular fusion categories, leading to a proof of the folklore Theorem 8.4. Although a more concise proof is presented later in §8.3, the current exposition is meant to be informative and to serve for future research.

As discussed in [29, Section 3], a modular fusion category $\mathcal{C}$ is associated with modular data (S,T), which gives a projective representation of

$$\mathrm{SL}(2,\mathbb{Z}) = \langle s,t \mid (st)^3 = s^2, s^4 = e\rangle.$$

This representation can be lifted to a usual (linear) representation ρ by utilizing the linear characters (i.e. one-dimensional representations), forming a cyclic group of order 12. This representation is r-dimensional—where r represents the rank of $\mathcal{C}$—and is *congruence*. This means it factors through $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$, for some n whose smallest one is called the *level*. The level is determined as $\mathrm{ord}(\rho(t))$ and satisfies

$$\mathrm{ord}(T) \mid \mathrm{ord}(\rho(t)) \mid 12\mathrm{ord}(T).$$

A finite-dimensional congruence representation ρ of level n is completely reducible, hence it can be broken down into a direct sum of irreducible representations of $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$. It's important to note that this includes only those irreducible representations that do not further factor through $\mathrm{SL}(2,\mathbb{Z}/d\mathbb{Z})$ for any proper divisor d of n. Nevertheless, if $n = \prod_i p_i^{n_i}$ represents the prime factorization of n, then $\rho = \bigotimes_i \rho_i$ with each ρ_i being a congruence representation of level $p_i^{n_i}$.

For deeper applications, note that [31] proves that the finite-dimensional congruence representations are equivalent to *symmetric* ones, which are classified in [32].

The dimensions d of the irreducible finite-dimensional congruence representations at level $n = p^a$ are provided in the table at the end of [36]. Observe that $d \geq (n-1)/2$ [with equality only if $a = 1$], leading to $p \leq p^a = n \leq 2d+1$. Given that the rank r of the modular fusion category is the sum of dimensions d of such irreducible representations, it follows that $d \leq r$ and therefore $p \leq 2r+1$.

According to the Cauchy theorem in [4], the set S of prime factors of $\mathrm{ord}(T)$ coincides with the prime factors of the global dimension norm N of the modular fusion category (of rank r). The prime numbers p lastly mentioned (satisfying $p \leq 2r+1$) constitute the set S' of prime factors of the level n of the congruence representation. Thus, $S \subseteq S' \subseteq S \cup \{2,3\}$, since $\mathrm{ord}(T) \mid n \mid 12\mathrm{ord}(T)$ and $12 = 2^2 \cdot 3$. Hence, for all prime factors $p \neq 2,3$ of N, it follows that $p \leq 2r+1$. The inequality trivially holds for $p = 2,3$. This end our first proof of Theorem 8.4, without using [17, Theorem II (iii)].

8.3. **Galois Action.** Let (s,t) be a *normalized* modular data. A Galois automorphism σ induces a permutation $X \to \sigma(X)$ on the simple objects, and acts as follows on dim, s and t:

(1) $\sigma(\dim(X)^2) = \frac{\sigma(\dim(\mathcal{C}))}{\dim(\mathcal{C})}\dim(\sigma(X))^2$, see [18],
(2) $\sigma(s_{X,Y}^2) = s_{X,\sigma(Y)}^2$, see [18],
(3) $\sigma^2(t_X) = t_{\sigma(X)}$, see [17, Theorem II (iii)].

See for example [42, Section 2] for an explicit normalization of the modular data.

Theorem 8.4. *For any prime factor p of the dimension norm of a modular fusion category with rank r, it holds that $p \leq 2r+1$.*

Proof. If $p = 2,3$ then $p \leq 2r+1$ trivially as $r \geq 1$. Let $p \neq 2,3$ be a prime factor of the global dimension norm. By Cauchy theorem in [4], p divides ord(t). So there must be a simple object X such that p divides the conductor of t_X, thus the orbit of $(\sigma^2(t_X))$ has at least $(p-1)/2$ distinct elements, because the group of units in $\mathbb{Z}/p\mathbb{Z}$ is cyclic of order $p-1$, so it has an element g with $\mathrm{ord}(g^2) = (p-1)/2$. So by (3), $r \geq (p-1)/2$, i.e., $p \leq 2r+1$. □

Here is an example of type of rank $r = 13$ and $\mathrm{FPdim} = 2^4 3^2 5^2 7^2 19^2 37^2$ excluded Theorem 8.4:

$$[1, 777, 1036, 1295, 3990, 4218, 24605, 42180, 98420, 98420, 147630, 147630, 147630],$$

because $p = 37 > 2r+1 = 27$.

Here is a stronger version in the integer case (shared by Eric Rowell and Andrew Schopieray):

Theorem 8.5. *For an integral modular fusion category, for every prime p dividing the global* FPdim*, there is a basic* FPdim *of multiplicity m such that $p \leq 2m+1$.*

Proof. Consider the orbit $(\sigma^2(t_X))$ with at least $(p-1)/2$ distinct elements from above proof of Theorem 8.4. By (3), the orbit $(\sigma(X))$ has also at least $(p-1)/2$ distinct elements. By applying (1) on the (weakly) integral case, we get that $\sigma(\mathrm{FPdim}(X)) = \mathrm{FPdim}(X)$. Thus all simple objects in the orbit $(\sigma(X))$ has the same FPdim, so the multiplicity m of this basic FPdim satisfies $m \geq (p-1)/2$, i.e., $p \leq 2m+1$. □

Here is an example of type of rank 13 and FPdim $= 2^4 3^6 5^2 7^2 17^2$ excluded by Theorem 8.5 (but not Theorem 8.4):

$$[1, 238, 459, 540, 595, 918, 5355, 9180, 21420, 21420, 32130, 32130, 32130],$$

because $p = 17 > 2m + 1 = 7$, where $m = 3$ is the largest multiplicty of a basic FPdim.

Remark 8.6. *The assertion of Theorem 8.5 can be generalized beyond the integral assumption, albeit with the necessity to substitute the usual multiplicity with the Galois-multiplicity, that is, the count of Galois-conjugate basic* dim *of the given basic* dim*.*

Here is an even stronger version of Theorem 8.5:

Theorem 8.7. *For an integral modular fusion category, let S be the set of odd prime factors of the global* FPdim*. There is a partition (S_i) of S, and multiplicities (m_i) of* some *distinct basic* FPdims *such that*

$$m_i \geq \frac{1}{2} \operatorname*{lcm}_{p \in S_i} (p-1).$$

Proof. Let S_X be the set of odd prime divisors of t_X. By Cauchy theorem in [4], the union of the sets S_X over all the simple objects X is exactly S. Let λ be the *Carmichael function*, i.e. the exponent of the multiplicative group of integers modulo n. It is well-known that if $n = \prod_i p_i^{n_i}$ is the prime factorization of n, then $\lambda(n) = \mathrm{lcm}_i\, \lambda(p_i^{n_i})$, whereas for p_i odd then $\lambda(p_i^{n_i}) = \varphi(p_i^{n_i}) = (p_i - 1)p_i^{n_i - 1}$, where φ is the Euler totient function. The orbit $(\sigma(X))$ has at least $\frac{1}{2}\lambda(\prod_{p \in S_X} p)$ distinct elements. Therefore, by above and the proof of Theorem 8.5, the multiplicity of $\mathrm{FPdim}(X)$ is at least $\frac{1}{2} \mathrm{lcm}_{p \in S_X}(p-1)$. The result follows. $\square$

Here is an example of type of rank 25 and FPdim $= 3^4 5^2 7^2 11^2 13^2$ excluded by Theorem 8.7 (but not Theorem 8.5):

$$[[1, 1], [39, 2], [231, 2], [273, 2], [1001, 2], [1287, 2], [3465, 2], [4095, 2], [9009, 2], [15015, 8]],$$

because the largest prime factor, $p = 13$, is insufficient for the largest multiplicity of 8 since $(13 - 1)/2 = 6$, which is less than 8. Therefore, the subset S_i that includes 13 must also include another prime to increase $\mathrm{lcm}_{p_i \in S_i}(p - 1)$. Primes 3, 5, and 7 do not contribute to an increase; although they could be part of S_i, another prime is needed. Thus, 11 must be included in S_i. Now, $\mathrm{lcm}(13 - 1, 11 - 1)/2 = 60$, which is greater than 8, presenting a contradiction.

The checking of Theorems 8.5 and 8.7 are automated by the function `Theorem3Check` and `Theorem4Check` in `TypeCriteria.sage` in [41].

9. Stronger arithmetic constraints

In this section, we will first prove some arithmetic constraints on the rank of the Drinfeld center of $\mathrm{Rep}(G)$, for any finite group G. Then, we will discuss how far they can be generalized to any integral modular fusion categories.

9.1. **Rank of $\mathcal{Z}(\mathrm{Rep}(G))$.** This subsection is inspired from discussions with Geoff Robinson and Dave Benson in [38, 39]. The goal is to simplify the group theoretic way to express the rank of $\mathcal{Z}(\mathrm{Rep}(G))$, for any finite group G, and to provide some bounds involving the prime divisors of $|G|$. Consequently, this class of integral modular fusion categories satisfies Conjectures 9.8 and 9.11. We will adopt the following notations throughout our discussion:

- $Z(G)$ denotes the center of G,
- Γ_G represents a complete set of conjugacy class representatives,
- c_G is the rank of $\mathrm{Rep}(G)$, corresponding to the total number of conjugacy classes, so $|\Gamma_G|$,
- r_G refers to the rank of $\mathcal{Z}(\mathrm{Rep}(G))$.

Lemma 9.1. *Let G be a finite group. Then*

$$r_G = \sum_{a \in \Gamma_G} c_{C_G(a)}. \tag{1}$$

Proof. According to [13] or [28, Section 3], the rank r_G is determined by the number of irreducible characters within the centralizers of class representatives of G, which is precisely the equality (1). $\square$

Let us simplify how to express the rank r_G.

Lemma 9.2. *The rank r_G is the number of conjugacy classes of pairs of commuting elements of G, i.e. the cardinality of the set*

$$A_G := \{c(a_1, a_2) \mid a_1, a_2 \in G \text{ with } a_1 a_2 = a_2 a_1\},$$

where

$$c(a_1, a_2) := \{(g a_1 g^{-1}, g a_2 g^{-1}) \mid g \in G\}.$$

Proof. By the equality in Lemma 9.4, it suffices to establish a bijection between the set A_G and the set

$$B_G := \{(a,\beta) \mid a \in \Gamma_G \text{ and } \beta \text{ is a conjugacy class within } C_G(a)\}.$$

Given $c(a_1,a_2) \in A_G$, we associate the element $(a_1, \{ha_2h^{-1} \mid h \in C_G(a_1)\}) \in B_G$. We merely need to confirm that if $a_1 = ga_1g^{-1}$, then a_2 and ga_2g^{-1} are conjugates in $C_G(a_1)$, which is apparent since $a_1 = ga_1g^{-1}$ means that $g \in C_G(a_1)$. Given $(a,\beta) \in B_G$, we associate the element $c(a,b) \in A_G$, where $b \in \beta$. We only need to verify that if $b' \in \beta$, then $c(a,b') = c(a,b)$. Note that $b' = hbh^{-1}$, where $h \in C_G(a)$. Therefore, $c(a,b) = c(hah^{-1}, hbh^{-1}) = c(a,b')$ because $hah^{-1} = a$, given that $h \in C_G(a)$. □

Recall that a prime p divides $\mathrm{FPdim}(\mathcal{Z}(\mathrm{Rep}(G))) = |G|^2$ if and only if it divides $|G|$.

Proposition 9.3. *Let G be a finite group. For every prime p dividing $|G|$, then $r_G \geq p$.*

Proof. This proof is due to Dave Benson. The pair of commuting elements (g, g^i) for $1 \leq i \leq \mathrm{ord}(g)$ are all in distinct conjugacy classes, so by Lemma 9.2, $r_G \geq \mathrm{ord}(g)$ for all g in G. By Cauchy's theorem, there is an element of order p dividing $|G|$. Thus, $r_G \geq p$. □

The number of conjugacy classes of pairs of commuting elements in the alternating group A_n is $1, 1, 9, 14, 22, 44, 74$ for $n = 1, \dots, 7$, respectively, see [40].

Lemma 9.4. *Let G be a finite group. Then*

$$r_G \geq |Z(G)|c_G + \sum_{g \in \Gamma_G \setminus Z(G)} \mathrm{ord(g)}, \tag{2}$$

$$r_G \geq \sum_{g \in \Gamma_G} \mathrm{ord(g)}. \tag{3}$$

Proof. In general, we have

$$c_{C_G(a)} \geq |Z(C_G(a))| \geq \mathrm{ord}(a) \tag{4}$$

but if $a \in Z(G)$ then $C_G(a) = G$ and so $c_{C_G(a)} = c_G$. The inequalities (2) and (3) follow from (1). □

Theorem 9.5. *Let G be a finite group. For every prime p dividing $|G|$, then $r_G \geq 2p$.*

Proof. By Cauchy's theorem, there is an element g of order p. If $p = 2$ then $r_G \geq 3$ by (3). So if $r_G \leq 3$, then $r_G = 3$ and $|\Gamma_G| = 2$ by (3), thus $G = C_2$, contradiction. So $r_G \geq 4 = 2p$.

Therefore, we can assume that p is odd. If g^2 is not in the conjugacy class of g then, by (1) and (4),

$$r_G \geq c_{C_G(g)} + c_{C_G(g^2)} \geq |Z(C_G(g)| + |Z(C_G(g^2)| \geq 2p,$$

because $x \in Z(C_G(x))$ and $\mathrm{ord}(g^2) = p$, as p is odd. Thus, we can assume the existence of h in G such that $hgh^{-1} = g^2$, but then $h^ngh^{-n} = g^{2^n}$. Fermat's little theorem states that $2^{p-1} \equiv 1 \mod p$, and the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic of order $p-1$. So $h^ngh^{-n} = g$ for $n = p-1$, and $\{h^ngh^{-n} \mid n = 1, \dots, p-1\} = \langle g \rangle \setminus \{e\}$. Thus $p-1$ divides $\mathrm{ord}(h)$. But g and h have different order, so cannot be in the same conjugacy class, so by (3), $r_G \geq \mathrm{ord}(e) + \mathrm{ord}(g) + \mathrm{ord}(h) \geq 1 + p + p - 1 = 2p$. □

The equality $r_G = 2p$ is realized by $(G,p) = (C_2, 2)$. Out of this example, we can get even better:

Theorem 9.6. *Let G be a finite group. For every prime p dividing $|G|$, then $r_G \geq 3p - 1 - \delta_{G,C_2}$.*

Proof. If G is Abelian then $r_G = |G|^2$ by (2), so $r_G < 3p-1$ implies $G = C_2$, and $r_G = 4 = 3p-2$. The non-Abelian case (by far harder) is due to Geoff Robinson in its answer in [39]. □

The equality $r_G = 3p-1$ is realized by $(G,p) = (S_3, 3)$. We can expect even better in the non-solvable case:

Conjecture 9.7. *For every non-solvable finite group G, and for every prime p dividing the order of G, the inequality $r_G \geq 5p - 3$ holds.*

This has been verified for all non-solvable groups of order less than 1920 and for all non-Abelian finite simple groups of order less than 10^8. The equality is achieved for $G = A_5$ and $\mathrm{PSL}(2,7)$. Geoff has shown that $r \geq 5p-1$ for any non-solvable groups without a self-centralizing Sylow p-subgroup of order p, where p the largest prime factor of $|G|$.

9.2. **Integral modular fusion categories.** Referring to the notations used previously, recall that Theorem 8.4 asserts that $p \leq 2r+1$, which is enhanced for the integral case by Theorem 8.5 to $p \leq 2m+1$, and also consult Theorem 8.7 for a more robust version. Now, let's explore the extent to which the arithmetic constraints discussed in §9.1 can be applied to integral modular fusion categories. The subsequent conjecture aims to expand upon Proposition 9.3.

Conjecture 9.8. *For any prime number p that divides the global dimension of a integral modular fusion category with rank r, then $p \leq r$.*

Proposition 9.9. *The statement of Conjecture 9.8 is true up to rank* 21.

Proof. The proof largely relies on computer assistance. The script detailed in §4.1 can be adapted to focus on types that comply with Theorem 8.7 yet contradict Conjecture 9.8. The specialized script is (or will be) available at [2]. Consequently, we identified exactly 187 potential types up to rank 21, as listed in `UpToRank21par.txt` in [41]. Specifically, we found none up to rank 13, and then $1, 4, 22, 2, 28, 0, 8, 122$ types at ranks 14 through 21, respectively. Of these, only 40 types pass `TypeCriteria` from §5. Among these, only 28 types pass `GradingCriteria` from §8.1. They are ultimately excluded by our fusion ring solver's partition version in §6.4, limited to one second per type. The detailed computational data is accessible in `InvestUpToRank21par.txt` in [41]. ☐

Remark 9.10. *The extension of Proposition 9.9 to rank* 22 *is in progress. About the rank* 23*, we can reduce to the following type of* $\mathrm{FPdim} = 2^4 3^8 5^4 29^2$ *which requires deeper examination:*

$$[[1,1],[540,1],[725,1],[1450,1],[2610,1],[3132,1],[8100,1],[26100,1],[58725,14],[78300,1]].$$

If Conjecture 9.8 is true then it is optimal as demonstrated by the pointed examples of prime rank. Thus, we could expect better for the non-pointed case. The following conjecture generalizes Theorem 9.5.

Conjecture 9.11. *For any prime number p that divides the global dimension of a non-pointed integral modular fusion category with rank r, then $p \leq r/2$.*

Proposition 9.12. *The statement of Conjecture 9.11 is true up to rank* 15.

Proof. The proof is mainly computer-assisted, but computationally a bit harder. The specialized script, as for the proof of Proposition 9.9, provides 3094 non-pointed types up to rank 15 satisfying Theorem 8.7 but contradicting Conjecture 9.11, available in `UpToRank15PAHR.txt`. We found none up to rank 8, and then $6, 36, 250, 2266, 45, 491$ types at ranks 9 through 15, respectively. The irregularity between rank 13 and rank 14 comes from the fact that if $p > 14/2 = 7$, then $p \geq 11$ because 9 is not prime. Of these, only 1256 types pass `TypeCriteria` from §5. Among these, only 894 types pass `GradingCriteria` from §8.1. The use of our fusion ring solver's partition version from §6.4, limited to one second per type, reduced the rest to just 14 types, then 12, 5, 4 ones with 10, 100, 1000 seconds per type respectively. Next, we applied our full fusion ring solver's partition version from §6.3, requiring to specified the duality, so our 4 types becames 40 cases, we simplify with the commutativity option. After the first round, limited to 10 seconds per case, there remain 8 cases from 2 types, ultimately excluded in about two hours all. The detailed computational data is accessible in `InvestUpToRank15pahr.txt` in [41]. ☐

With our current knowledge, it would not be reasonable to conjecture a generalization of Theorem 9.6 suggesting that $p \leq (r+1)/3$ in the non-pointed case, or of Conjecture 9.7 suggesting that $p \leq (r+3)/5$ in the non-solvable case. But it makes sense to explore in this direction.

Question 9.13. *Is it true that for any prime number p dividing the global dimension of a non-solvable integral modular fusion category with rank r, the inequality $p \leq (r+3)/5$ holds?*

The following proposition is related to Question 7.1.

Proposition 9.14. *An affirmative response to Question 9.13 would indicate that the minimum rank required for a non-trivial perfect integral modular fusion category is* 22.

Proof. By [19, Proposition 4.5 (iv)], a non-trivial perfect integral fusion category $\mathcal{C}$ is non-solvable. Thus by [19, Theorem 1.6], $\mathrm{FPdim}(\mathcal{C})$ must have at least three distinct prime factors, so there must be a prime factor $p \geq 5$. Assume that $\mathcal{C}$ is modular of rank r, then an affirmative response to Question 9.13 implies that $r \geq 5p-3 \geq 22$. Finally, the bound is realized by $\mathcal{Z}(\mathrm{Rep}(A_5))$ having rank 22. ☐

Here is what we can deduce in the perfect integral case:

Corollary 9.15. *For any prime number p that divides the global dimension of a non-trivial perfect integral modular fusion category with rank r, then $p \leq 2r-5$.*

Proof. By Theorem 5.1, the number of distinct basic FPdims is at least 4. Let p be the biggest prime divisor of the global dimension, then by Theorem 8.5, there is a basic FPdim of multiplicity $m \geq (p-1)/2$. Thus $r \geq m+3 = (p+5)/2$. The result follows. ☐

10. Proof of Theorem 1.3

This section proves Theorem 1.3 by utilizing the advanced results from Sections 8 and 9, and with computer assistance. The process commenced with the 9012 non-pointed types from §7.1. The application of Proposition 9.12 reduces to 1722 types only, all satisfying Theorem 8.7. Of these, only 1413 types pass `GcdCriterion` from §5.2. Following this, 1260 types pass `TypeTest` from §5.3, and then 861 types pass `LocalCriterion` from §5.4, with a time limit of one second per type. This leaves 390 non-perfect types and 471 perfect types. Among these, only 85 non-perfect types pass `GradingCriteria` from §8.1. The use of our fusion ring solver's partition version from §6.4 to these 556 remaining types, limited to one second per type, left 8 types uncompleted plus 67 ones with a solution (i.e. `SingleLatticePoint`, as explained in §6.1). Next, with 10, 100 and 1000 seconds per type, the counting became 4, 3 and 2 types uncompleted respectively, plus 70 ones with a solution. After considering all the possible duality maps (see §2.1), these 72 remaining types produce 1044 cases, reducing to 948 uncompleted cases plus 27 with a solution, by our fusion ring solver's full version from §6.3, with 1 second per case, then $86 + 107$, $33 + 133$, and $19 + 141$, with 10, 100 and 1000 seconds per case, respectively. These 160 remaining cases come from 33 types (each one from fusion rings), distributed as follows: 1, 1, 2, 3, 5, 8, 13 types at ranks 7 to 13, respectively. Observe that it marks the beginning of the Fibonacci sequence, but it should be a meaningless coincidence since we have already found at least 22 types at this step at rank 14 (see 10.1). Here are the types classified by rank:

- Rank 7: [1, 1, 1, 1, 2, 2, 2],
- Rank 8: [1, 1, 2, 2, 2, 2, 3, 3],
- Rank 9: [1, 1, 1, 1, 4, 4, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6],
- Rank 10: [1, 1, 1, 2, 2, 2, 2, 2, 2, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6],
- Rank 11: [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 2, 4, 4, 4, 4, 6, 6], [1, 1, 1, 1, 4, 4, 12, 12, 18, 18, 18], [1, 1, 1, 3, 4, 4, 4, 4, 4, 4, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6],
- Rank 12: [1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2], [1, 1, 1, 1, 3, 3, 3, 3, 4, 4, 6, 6], [1, 1, 1, 3, 6, 8, 8, 8, 8, 8, 8, 12], [1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 6], [1, 1, 2, 2, 2, 2, 6, 6, 6, 6, 9, 9], [1, 1, 2, 3, 3, 6, 6, 8, 8, 8, 12, 12], [1, 1, 2, 6, 6, 6, 6, 10, 10, 10, 15, 15],
- Rank 13: [1, 1, 1, 1, 2, 2, 4, 4, 4, 4, 4, 4, 6], [1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 10, 10, 10], [1, 1, 1, 1, 4, 4, 6, 6, 6, 6, 6, 6, 6], [1, 1, 1, 1, 4, 4, 6, 12, 12, 12, 12, 18, 18], [1, 1, 1, 1, 4, 4, 12, 12, 36, 36, 54, 54, 54], [1, 1, 1, 3, 6, 12, 16, 16, 16, 16, 16, 16, 24], [1, 1, 1, 3, 12, 12, 20, 20, 20, 20, 20, 20, 30], [1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 6], [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 6, 6], [1, 1, 2, 2, 2, 2, 3, 3, 12, 12, 18, 18, 18], [1, 1, 2, 2, 2, 2, 6, 6, 6, 6, 9, 9, 18], [1, 1, 2, 3, 3, 4, 4, 4, 6, 6, 12, 12, 12], [1, 1, 2, 3, 3, 24, 30, 30, 40, 40, 40, 60, 60].

As you can see, Theorem 1.5 has already been proven at this point. Let us exclude some types of rank 13 by hand.

Lemma 10.1. *If* $T_2 = [1, 1, 1, 3, 12, 12, 20, 20, 20, 20, 20, 20, 30]$*, and then so is* $T = [1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 10, 10, 10]$.

Proof. The type T_2 admits a unique possible modular partition as $[[1, 1, 1, 3, 12, 12, 30], [20, 20, 20], [20, 20, 20]]$ and $\text{FPdim} = 2^4 3^2 5^2$. But the modularization of the neutral component must be half-Frobenius of $\text{FPdim} = 2^4 5^2$, so must be of type T (all the multiple of 3 must split), which is what we wanted to prove. ☐

Note that the type T in Lemma 10.1 will be excluded later, so we can consider T_2 as excluded.

Lemma 10.2. *A fusion category of rank* 13 *with any of the following types admits no modular structure.*

- $T_3 = [1, 1, 1, 3, 6, 12, 16, 16, 16, 16, 16, 16, 24]$,
- $T_4 = [1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 6]$,
- $T_5 = [1, 1, 2, 2, 2, 2, 3, 3, 6, 6, 6, 6, 6]$,
- $T_6 = [1, 1, 2, 2, 2, 2, 3, 3, 12, 12, 18, 18, 18]$.
- $T_7 = [1, 1, 2, 2, 2, 2, 6, 6, 6, 6, 9, 9, 18]$.

Proof. The proof is similar to the one of Lemma 10.1. First T_3, with $\text{FPdim} = 2^8 3^2$, admits a unique possible modular partition as $[[1, 1, 1, 3, 6, 12, 24], [16, 16, 16], [16, 16, 16]]$, so the modularization of its neutral component must be of type $[1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 8, 8, 8]$, because the multiple of 3 must split. But this last type of rank 13 is not in above list. Next, T_4 has three possible modular partitions:

$$[[1, 2, 2, 3, 3, 3, 3, 3], [1, 2, 2, 3, 6]], [[1, 1, 2, 2, 2, 2, 6], [3, 3, 3, 3, 3, 3]], [[1, 1, 2, 2, 2, 2, 3, 3, 3, 3], [3, 3, 6]].$$

The first is excluded by Theorem 8.2 (0) and above list. For both the second and third, the modularization of the neutral component must be of $\text{FPdim} = 3^3$ and type $[1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3]$, which admits no modular partition. Next, T_5, with $\mathrm{FPdim} = 2^3 3^3$, admits a unique possible modular partition as $[[1,1,2,2,2,2,3,3,6,6],[6,6,6]]$, so the modularization of its neutral component must be of type $[1,1,1,1,1,1,1,1,1,3,3,3,3,3]$, which admits no modular partition. Next, T_6, with $\mathrm{FPdim} = 2^4 3^4$, admits a unique possible modular partition as $[[1,1,2,2,2,2,3,3,12,12,18],[18,18]]$, so the modularization of its neutral component must be of $\mathrm{FPdim} = 2^2 3^4$ (so all multiple of 4 must split) and type $[[1,1+4a],[2,2-a],[3,1],[6,4],[9,2]]$, with $a = 0,1,2$, but none of them admit a modular partition. Next, T_7, with $\mathrm{FPdim} = 2^3 3^4$, admits a unique possible modular partition as $[[1,1,2,2,2,2,6,6,6,6,9,9],[18]]$, so the modularization of its neutral component must be of $\mathrm{FPdim} = 2^1 3^4$ (so all multiple of 2 must split) and type $[[1,9],[3,8],[9,1]]$, which admits no modular partition. □

Remark 10.3. *The purpose of Lemmas 10.1 and 10.2 is to render the proof of Theorem 1.3 more theoretical, thereby reducing the need for an extensive classification of the fusion rings of the types T_i by Normaliz to exclude them. But, this classification has been previously undertaken and was notably challenging for certain types.*

From the classification of commutative cyclotomic self-transposable half-Frobenius fusion rings in §7.3, we can exclude 8 more types up to rank 12, and at rank 13, among the remaining $13-7$ types, only $[1,1,1,1,4,4,4,4,4,4,10,10,10]$, $[1,1,1,1,4,4,12,12,36,36,54,54,54]$ and $[1,1,2,3,3,24,30,30,40,40,40,60,60]$ admit such a fusion ring. Thus, there remain 14 non-pointed types to consider up to rank 13.

Remark 10.4. *In order to get all the possible fusion rings for these types (instead of just one), it is required to remove the option* `SingleLatticePoint` *on Normaliz. Nevertheless, by §8.1, we can restrict to the fusion rings which are graded by an abelian group of order the size of the pointed part. Currently, Normaliz supports only C_2-grading (though support for the general case is under development).*

Make a text files with all the commutative cyclotomic self-transposable half-Frobenius fusion rings for these 14 types, and put it on GitHub, and add its name here - Sebastien

Next, all the remaining types are excluded through the application of `MagicCriterion` from §3.2 to their fusion rings, with the exception of the following six types:

- $[1,1,1,1,2,2,2]$,
- $[1,1,2,2,2,2,3,3]$,
- $[1,1,1,2,2,2,2,2,2,3]$,
- $[1,1,1,1,2,2,2,2,2,2,2]$,
- $[1,1,2,2,2,2,6,6,6,6,9,9]$,
- $[1,1,2,6,6,6,6,10,10,10,15,15]$.

All the types above can be (and were) handle by computer, but some ones can by excluded by hand also.

Lemma 10.5. *A fusion category of type $T_8 = [1,1,2,2,2,2,6,6,6,6,9,9]$ admits no modular structure.*

Proof. The proof is similar to the one of Lemma 10.1. First T_8, with $\mathrm{FPdim} = 2^2 3^4$, admits two possible modular partitions, $[[1,2,2,6,6,9],[1,2,2,6,6,9]]$ and $[[1,1,2,2,2,2,6,6,6,6],[9,9]]$. The first is excluded by Theorem 8.2 (0) and above list. About the second, the modularization of the neutral component, of $\mathrm{FPdim} = 3^4$, must be of type $[[1,9],[3,8]]$, which is excluded by Theorem 1.9. □

Lemma 10.6. *A fusion category of type $T_9 = [1,1,1,1,2,2,2]$ or $T_{10} = [1,1,2,6,6,6,6,10,10,10,15,15]$ admits no modular structure.*

Proof. The proof is primarily computer-based. Regarding T_9, we discovered 8 commutative fusion rings using Normaliz; two of these are both cyclotomic and self-transposable. One passed the `MagicCriterion`, but `STmatrix2` did not provide a solution. Similarly, for T_{10}, we found 8 commutative C_2-graded fusion rings with Normaliz. Among these, four are cyclotomic and self-transposable, and they passed the `MagicCriterion`, yet `STmatrix2` yielded no solution. The computation in both cases was very quick. □

Remark 10.7. *The type T_{10} with $\mathrm{FPdim} = 2^2 3^2 5^2$, admits a unique modular partition $[[1,1,2,6,6,6,6,10,10,10],[15,15]]$, and the modularization of the neutral component must be of type $[[1,3],[3,8],[5,6]]$ which admits a modular structure by Theorem §1.9. So it cannot be excluded by hand this way.*

Finally, modular data were derived for each of the three remaining non-pointed types. These were classified by applying `STmatrix` or `STmatrix2` (following all the criteria specified in Definition 2.11) to their commutative, cyclotomic, self-transposable, half-Frobenius fusion rings identified in §7.3, which also passed the `MagicCriterion`. The pointed cases were addressed by using `STmatrixCo` on a list of fusion rings for the abelian groups of order up to 13. We concluded with $19+64$ sets of modular data, originating from $5+18$ fusion rings representing $3+13$ types (non-pointed + pointed), up to rank 13. This proves Theorem 1.3. The full details are available in §12 and at [41].

Remark 10.8. *When considering isomorphism classes, it is appropriate to adopt a* normal form *by sorting the basic elements according to their* FPdim and *spin, and limit basis permutations to those preserving both.*

10.1. **About Ranks 14 and 15.** There remain 46 open types for the simple rank 14 case. The set of prime divisors of their FPdim is always $\{2, 3, 5, 7\}$, i.e. their FPdim is always of the form $2^a 3^b 5^c 7^d$ with $abcd$ nonzero. The list is available in `SimpleRank14.txt` in [41].

11. The Odd-Dimensional Case

For an overview of the current state of knowledge on odd-dimensional modular fusion categories, we refer the reader to [14, 15]. A foundational result in this area establishes that an odd-dimensional modular fusion category $\mathcal{C}$ is equivalent to being maximally non self-dual (MNSD), meaning that its only self-dual simple object is the unit object. Let $(d_i)_{i \in I}$ represent the FPdim of the simple objects in $\mathcal{C}$, considered up to isomorphism. Since d_i^2 is a divisor of the odd $\mathrm{FPdim}(\mathcal{C})$, each d_i must be odd. Furthermore, the equation $\sum_{i \in I} d_i^2 = \mathrm{FPdim}(\mathcal{C})$ implies that the rank $r = |I|$ must also be odd. This reduces our investigation to Egyptian fractions of the form $q = \sum_{i=1}^{r} \frac{1}{s_i^2}$, where $q, r, s_i \in \mathbb{Z}_{\geq 1}$, $s_1 \geq \cdots \geq s_r \geq 1$, and both r and s_i are odd. Additionally, s_i divides s_1 for all i, and $s_{2k} = s_{2k+1}$. This yields the expression

$$q = \frac{1}{s_1^2} + \sum_{k=1}^{(r-1)/2} \frac{2}{s_{2k}^2}.$$

Since each s_i is odd, we have $s_i^2 \equiv 1 \pmod 8$, which implies $q \equiv r \pmod 8$ and that q is odd as well. Utilizing a similar technique as in §4, we can assume $s_i > 1$ (hence $s_i \geq 3$), by completing the classification with additional 1s if necessary. Consequently, we can assume $q \leq r/9$. For $r < 27$, this allows us to deduce that $q = 1$, and therefore $r \equiv 1 \pmod 8$, which narrows the possibilities for r to $1, 9, 17, 25$ (up to completing by 1s).

Remark 11.1. *This strategy can be extended. For instance, by adding eighteen 3s to complete the classification, we may assume that $s_i = 5$ for $i + 16 \leq r$, which leads to $q \leq 16/9 + (r - 16)/25$. If $r < 47$ (which becomes 51 because $q \equiv r \pmod 8$), we can assume that $q = 1$. However, this extended strategy will not be applied in this paper.*

Consequently, for all $r < 25$, we have compiled the following list of all possible non-pointed types (as for §4):

- [[1, 9], [3, 8], [9, 2a]],
- [[1, 7], [3, 2], [5, 8], [15, 2a]],
- [[1, 3], [3, 8], [5, 6], [15, 2a]],
- [[1, 1], [3, 2], [7, 2], [9, 4], [21, 8], [63, 2a]],
- [[1, 1], [9, 4], [25, 2], [45, 2], [75, 8], [225, 2a]],

where $a \geq 0$ represents the number of 1s added for completion. It is noteworthy that these ranks are $17 + 2a$, which corroborates a result from [14] stating that any odd-dimensional modular fusion category with rank less than 17 is pointed. Further, [14, Remark 4.3] states that any perfect odd-dimensional modular fusion category is a Deligne product of simple ones. From the preceding analysis, a non-pointed one must have a rank of at least 17, meaning a perfect non-simple one must have a rank of at least 289 $(= 17^2)$. Therefore, a perfect one with a rank less than 289 must be simple and cannot have non-trivial simple objects of prime-power FPdim, as shown in [35, Corollary 6.16]. Consequently, the previously mentioned perfect types are excluded. It follows that:

Theorem 11.2. *Every perfect odd-dimensional modular fusion category of rank less than 25 is trivial, and so everyone of rank less than 625 $(= 25^2)$ is simple.*

Proof of Theorem 1.9

Proof. By Theorem 11.2, there remain to address the non-perfect types above. Their rank is always $17 + 2a < 25$, which implies $a < 4$. As outlined in §8.1, the modular grading results in a partition indexed by the pointed part, with each component having the same FPdim, in particular the FPdim of the pointed part divides the global FPdim.

- First, let's examine the type $[[1, 9], [3, 8], [9, 2a]]$. The FPdim for this type is $81(1 + 2a)$. Consequently, each partition component must have $\mathrm{FPdim} = 9(1 + 2a)$. If $a > 0$, a component with 9 must have $\mathrm{FPdim} \geq 81$. This leads to $81 \leq 9(1 + 2a)$, resulting in $a \geq 4$, contradiction. Thus, $a = 0$. The modular partition must be
$$[[1, 1, 1, 1, 1, 1, 1, 1, 1], [3], [3], [3], [3], [3], [3], [3], [3]],$$
which contradicts Theorem 8.2 (1).
- Regarding the second type $[[1, 7], [3, 2], [5, 8], [15, 2a]]$, the FPdim of the pointed part equaling 7 is not a divisor of the global $\mathrm{FPdim} = 225(1 + 2a)$, for $0 \leq a < 4$, except $a = 3$. Therefore, each partition component must have $\mathrm{FPdim} = 225(1 + 2 \times 3)/7 = 225 = 15^2$, so the modular partition must be
$$[[1, 1, 1, 1, 1, 1, 1, 3, 3, 5, 5, 5, 5, 5, 5, 5, 5], [15], [15], [15], [15], [15], [15]],$$
which contradicts Theorem 8.2 (1).

- Lastly, consider the third type $[[1,3],[3,8],[5,6],[15,2a]]$. If $a=1$ then the modular partition must be
$$[[1,1,1,3,3,3,3,3,3,3,3,5,5,5,5,5,5],[15],[15]],$$
which contradicts Theorem 8.2 (1). If $a=2$, then there is no modular partition. If $a=3$, then the global FPdim $=3^2 5^2 7$, but its powerless prime factor $p=7$ does not divide $\mathrm{FPdim}(\mathcal{C}_{pt})=3$, contradicting [12, Theorem 1.17]. Alternatively, see Remark 11.3. Hence, $a=0$. Following this, applying the fusion ring solver outlined in §6.3 to the type $[[1,3],[3,8],[5,6]]$ yields two fusion rings. Applying `STmatrix2` to these fusion rings provides 3 modular data, detailed in §13. ☐

Remark 11.3. *Here is an alternative proof for the case $a=3$ in the last item above. The neutral component $\mathcal{C}_e$ must be of type $[[1,3],[3,8],[15,2]]$ and $\mathrm{FPdim}=3^1 5^2 7$, so its modularization $\mathcal{M}$ would have $\mathrm{FPdim}=5^2 7$, hence cannot have basic $\mathrm{FPdim}=3$, thus must be of type $[[1,25],[5,6]]$, but each component of the modular partition (for $\mathcal{M}$) would have $\mathrm{FPdim}=7$, and so $5^2 \le 7$, contradiction.*

Definition 11.4. *A modular data is called* anomaly-free *if its Gauss sums are equal ($p_+ = p_-$), see Definition 2.11.*

Lemma 11.5. *A modular data is anomaly-free if and only if $p_+ = \pm\sqrt{\dim}$, if and only if the central charge $c \in \{0,4\}$.*

Proof. Recall from Definition 2.11 that $p_\pm := \sum_{i=1}^r d_i^2(\theta_i)^{\pm 1}$. Thus p_+ and p_- are complex-conjugate. So anomaly-free is equivalent to p_+ real. Now, $p_+ = \sqrt{\dim}\zeta_8^c$, thus it is real if and only if $\zeta_8^c = \pm 1$, if and only if $c=0$ or 4. ☐

Remark 11.6. *As highlighted in Remark 1.10, gaps have been identified in the literature:*

(1) *In* [3, Theorem 4.2, proof of Case (viii) $\mathrm{FPdim}(\mathcal{C}_{pt})=p$], *on page 727, the assertion that the anomaly-freeness (as defined in their reference* [17]*, see Definition 11.4) necessarily leads to $p_+ = pq$ is incorrect, see Lemma 11.5, it may also be $-pq$, as for the MD described in §13, thus also allowing $p|(q+1)$.*

(2) *In* [14, Theorem 6.3 (b), proof of Case $|\mathcal{G}(\mathcal{C})|=3$], *on page 1936, the deduction "Hence $l \le 24$" in the seventh last line is accurate, except when $c_{X_1}=1$, which permits $l=5$, thereby accommodating the type* $[[1,3],[3,8],[5,6]]$.

Following our paper, [14] was corrected on arXiv, and [21] introduces modular categorifications for these new MD.

Remark 11.7. *Regarding the modular categorifications $\mathcal{C}$ of these new MD in §13, a discussion with Sebastian Burciu revealed that the Grothendieck ring of the braided adjoint fusion subcategory $\mathcal{D}=\mathcal{C}_{ad}$, which is of rank* 11, FPdim 75*, type* $[[1,3],[3,8]]$*, and basis $\{b_g\}_{g\in C_3} \cup \{x_i, x_i^*\}_{i\in\{1,2,3,4\}}$, is equal to* $\mathrm{ch}(G)$*, where $G = C_5^2 \rtimes C_3$ is the unique non-Abelian finite group of order* 75*. In fact, an application of §6.3 shows a unique MNSD cyclotomic fusion ring of this type. Furthermore, $\mathcal{D}$ is not symmetric, as indicated by the S-matrices mentioned in §13. In fact, the Müger center $\mathcal{Z}_2(\mathcal{D})$ is $\mathcal{C}_{pt}$, pointed of rank* 3*. Therefore, $\mathcal{D}$ can be* $\mathrm{Rep}(G)$*, albeit with an unusual braiding (see* [9]*), or, more broadly, a Jordan-Larson category* [25] *with an* FPdim *of* 3×5^2*. Finally, according to the S-matrices again, $\mathcal{C}$ is a minimal modular extension of $\mathcal{D}$, see* [27] *and* [24, §1.1].

There are non-pointed and non-perfect odd-dimensional modular fusion categories of rank 25, exemplified by $\mathcal{Z}(\mathrm{Vec}^{\omega}_{C_7\rtimes C_3})$. Furthermore, [15] demonstrates that, up to equivalence, no additional such examples exist. Consequently, our attention must now turn to the examination of the perfect case, which is the simple case by Theorem 11.2. Moreover, $q=1$; otherwise, the type would be a completion (via the process of adding 1s to the Egyptian fraction with squared denominators) of a perfect type with a rank less than 25, however, as previously mentioned, all such types possess some entries that are prime-power, so non-conform to the simple case as outlined in [35, Corollary 6.16]. We have identified precisely 91 possible types (listed in [41]) using the aforementioned method combined with [35, Corollary 6.16]. Subsequent application of the type criteria from §5 reduces this to 29 types, and the fusion ring solver in §6.4 quickly eliminates 8 more types. Among the remaining 21 types, just 5 listed in Proposition 11.8, pass Theorem 8.7.

Proposition 11.8. *A perfect odd-dimensional modular fusion category of rank* 25*, if any, must have one of the following* 5 *types:*

1. $[[1,1],[39,4],[65,2],[189,2],[315,2],[585,2],[1365,2],[2457,2],[4095,8]]$,
2. $[[1,1],[75,2],[91,4],[175,2],[585,2],[975,2],[2275,2],[4095,2],[6825,8]]$,
3. $[[1,1],[75,2],[91,4],[175,2],[975,2],[2275,2],[2925,4],[6825,8]]$,
4. $[[1,1],[99,2],[231,2],[385,2],[675,2],[10395,4],[28875,2],[51975,2],[86625,8]]$,
5. $[[1,1],[135,4],[165,2],[189,2],[315,2],[385,2],[1155,2],[2079,2],[3465,8]]$.

12. Catalogue of Integral Modular Data Up To Rank 13

This section presents a comprehensive catalogue of integral modular data (MD) up to rank 13. Notably, all the MD listed here are categorifiable, meaning each corresponds to the MD of an integral modular fusion category. These corresponding categorical models are identified in Theorem 1.3. We will begin by listing the non-pointed MD in §12.1, followed by the pointed MD in §12.2. Additionally, these data can be accessed in a machine-readable format in [41].

12.1. **Non-Pointed Modular Data Up To Rank 13.** The data are organized in the following sequence:

- Rank,
- Type,
- Fusion coefficients,
- List of topological spins (s_i) giving the T-matrix (see Definition 2.11),
- Cyclotomic conductor of the S-matrix,
- Cyclotomic conductor of the T-matrix (its order),
- Central charge,
- List of 2nd Frobenius-Schur indicators,
- S-matrix.

12.1.1. *Rank* 8*, type* $[1,1,2,2,2,2,3,3]$*, first fusion ring.*

$$\begin{matrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{matrix},\quad \begin{matrix}0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{matrix},\quad \begin{matrix}0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&0&1&0&0\\0&0&0&1&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&1&0&1&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{matrix},\quad \begin{matrix}0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&1&1&0&0&0\\1&1&1&0&0&0&0&0\\0&0&1&0&0&1&0&0\\0&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{matrix},\quad \begin{matrix}0&0&0&0&1&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{matrix},\quad \begin{matrix}0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&1&0&0&0\\0&0&0&0&1&1&0&0\\0&0&1&1&0&0&0&0\\1&1&0&1&0&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{matrix},\quad \begin{matrix}0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\1&0&1&1&1&1&0&0\\0&1&1&1&1&1&0&0\end{matrix},\quad \begin{matrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&1&1&1&1&1&0&0\\1&0&1&1&1&1&0&0\end{matrix}$$

- $[0,0,-2/9,4/9,0,1/9,1/2,0]$, 9, 18, 0, $[1,1,1,1,1,1,1,1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&2\zeta_9^4+2\zeta_9^5&2\zeta_9^2+2\zeta_9^7&-2&2\zeta_9+2\zeta_9^8&0&0\\2&2&2\zeta_9^2+2\zeta_9^7&2\zeta_9+2\zeta_9^8&-2&2\zeta_9^4+2\zeta_9^5&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&2\zeta_9+2\zeta_9^8&2\zeta_9^4+2\zeta_9^5&-2&2\zeta_9^2+2\zeta_9^7&0&0\\3&-3&0&0&0&0&3&-3\\3&-3&0&0&0&0&-3&3\end{bmatrix},$$

- $[0,0,-4/9,-1/9,0,2/9,1/2,0]$, 9, 18, 0, $[1,1,1,1,1,1,1,1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&2\zeta_9+2\zeta_9^8&2\zeta_9^4+2\zeta_9^5&-2&2\zeta_9^2+2\zeta_9^7&0&0\\2&2&2\zeta_9^4+2\zeta_9^5&2\zeta_9^2+2\zeta_9^7&-2&2\zeta_9+2\zeta_9^8&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&2\zeta_9^2+2\zeta_9^7&2\zeta_9+2\zeta_9^8&-2&2\zeta_9^4+2\zeta_9^5&0&0\\3&-3&0&0&0&0&3&-3\\3&-3&0&0&0&0&-3&3\end{bmatrix},$$

- $[0,0,-2/9,4/9,0,1/9,-1/4,1/4]$, 9, 36, 0, $[1,1,1,1,1,1,-1,-1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&2\zeta_9^4+2\zeta_9^5&2\zeta_9^2+2\zeta_9^7&-2&2\zeta_9+2\zeta_9^8&0&0\\2&2&2\zeta_9^2+2\zeta_9^7&2\zeta_9+2\zeta_9^8&-2&2\zeta_9^4+2\zeta_9^5&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&2\zeta_9+2\zeta_9^8&2\zeta_9^4+2\zeta_9^5&-2&2\zeta_9^2+2\zeta_9^7&0&0\\3&-3&0&0&0&0&-3&3\\3&-3&0&0&0&0&3&-3\end{bmatrix},$$

- $[0,0,-4/9,-1/9,0,2/9,-1/4,1/4]$, 9, 36, 0, $[1,1,1,1,1,1,-1,-1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&2\zeta_9+2\zeta_9^8&2\zeta_9^4+2\zeta_9^5&-2&2\zeta_9^2+2\zeta_9^7&0&0\\2&2&2\zeta_9^4+2\zeta_9^5&2\zeta_9^2+2\zeta_9^7&-2&2\zeta_9+2\zeta_9^8&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&2\zeta_9^2+2\zeta_9^7&2\zeta_9+2\zeta_9^8&-2&2\zeta_9^4+2\zeta_9^5&0&0\\3&-3&0&0&0&0&-3&3\\3&-3&0&0&0&0&3&-3\end{bmatrix}$$

12.1.2. *Rank* 8*, type* $[1,1,2,2,2,2,3,3]$*, second fusion ring.*

$$\begin{smallmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{smallmatrix},\ \begin{smallmatrix}0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{smallmatrix},\ \begin{smallmatrix}0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&1&0&0&0&0&0\\0&0&0&0&1&1&0&0\\0&0&0&1&0&1&0&0\\0&0&0&1&1&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{smallmatrix},\ \begin{smallmatrix}0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&1&0&0\\1&1&0&1&0&0&0&0\\0&0&1&0&0&1&0&0\\0&0&1&0&1&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&1&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\\0&0&0&1&1&0&0&0\\0&0&1&0&1&0&0&0\\0&0&1&1&0&0&0&0\\1&1&0&0&0&1&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\1&0&1&1&1&1&0&0\\0&1&1&1&1&1&0&0\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&1&1&1&1&1&0&0\\1&0&1&1&1&1&0&0\end{smallmatrix}$$

- $[0,0,-1/3,1/3,0,0,1/2,0]$, 1, 6, 0, $[1,1,1,1,1,1,1,1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&-2&4&-2&-2&0&0\\2&2&4&-2&-2&-2&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&-2&-2&-2&4&0&0\\3&-3&0&0&0&0&3&-3\\3&-3&0&0&0&0&-3&3\end{bmatrix}$$

- $[0,0,-1/3,-1/3,1/3,1/3,1/2,0]$, 1, 6, 4, $[1,1,1,1,1,1,-1,-1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&-2&4&-2&-2&0&0\\2&2&4&-2&-2&-2&0&0\\2&2&-2&-2&-2&4&0&0\\2&2&-2&-2&4&-2&0&0\\3&-3&0&0&0&0&-3&3\\3&-3&0&0&0&0&3&-3\end{bmatrix}$$

- $[0,0,-1/3,1/3,0,0,-1/4,1/4]$, 1, 12, 0, $[1,1,1,1,1,1,-1,-1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&-2&4&-2&-2&0&0\\2&2&4&-2&-2&-2&0&0\\2&2&-2&-2&4&-2&0&0\\2&2&-2&-2&-2&4&0&0\\3&-3&0&0&0&0&-3&3\\3&-3&0&0&0&0&3&-3\end{bmatrix}$$

- $[0,0,-1/3,-1/3,1/3,1/3,-1/4,1/4]$, 1, 12, 4, $[1,1,1,1,1,1,1,1]$,

$$\begin{bmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&-2&4&-2&-2&0&0\\2&2&4&-2&-2&-2&0&0\\2&2&-2&-2&-2&4&0&0\\2&2&-2&-2&4&-2&0&0\\3&-3&0&0&0&0&3&-3\\3&-3&0&0&0&0&-3&3\end{bmatrix}$$

12.1.3. *Rank* 10*, type* $[1,1,1,2,2,2,2,2,2,3]$*, first fusion ring.*

$$\begin{smallmatrix}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\end{smallmatrix},\ \begin{smallmatrix}0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\end{smallmatrix},\ \begin{smallmatrix}0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\end{smallmatrix},\ \begin{smallmatrix}0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0&1&0\\0&0&0&0&1&1&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&1&0&0\\0&0&0&1&0&0&1&0&0&0\\0&1&0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\end{smallmatrix},$$

$$\begin{smallmatrix}0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&0&0&1&1&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&1&0&0\\0&0&0&0&1&1&0&0&1&0\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&1&1&0&0&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&1&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&1&0\\0&0&0&0&1&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1&0&0\\0&0&0&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&1&1&0&0&1&0\end{smallmatrix},\ \begin{smallmatrix}0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\\0&0&0&0&1&1&0&0&1&0\\0&0&0&1&0&0&1&1&0&0\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\\1&1&1&0&0&0&0&0&0&2\end{smallmatrix}$$

- $[0, 0, 0, 1/9, 4/9, -2/9, -2/9, 4/9, 1/9, 1/2]$, 9, 18, 4, $[1, 0, 0, 0, 0, 0, 0, 0, 0, 1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\
1 & 1 & 1 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_3^2 & 2\zeta_3 & 3 \\
1 & 1 & 1 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_3 & 2\zeta_3^2 & 3 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2\zeta_9^7 & -2\zeta_9^5 & 2\zeta_9^2+2\zeta_9^5 & 2\zeta_9^4+2\zeta_9^7 & -2\zeta_9^4 & -2\zeta_9^2 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2\zeta_9^5 & 2\zeta_9^4+2\zeta_9^7 & -2\zeta_9^7 & -2\zeta_9^2 & 2\zeta_9^2+2\zeta_9^5 & -2\zeta_9^4 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_9^2+2\zeta_9^5 & -2\zeta_9^7 & -2\zeta_9^4 & -2\zeta_9^5 & -2\zeta_9^2 & 2\zeta_9^4+2\zeta_9^7 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_9^4+2\zeta_9^7 & -2\zeta_9^2 & -2\zeta_9^5 & -2\zeta_9^4 & -2\zeta_9^7 & 2\zeta_9^2+2\zeta_9^5 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2\zeta_9^4 & 2\zeta_9^2+2\zeta_9^5 & -2\zeta_9^2 & -2\zeta_9^7 & 2\zeta_9^4+2\zeta_9^7 & -2\zeta_9^5 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2\zeta_9^2 & -2\zeta_9^4 & 2\zeta_9^4+2\zeta_9^7 & 2\zeta_9^2+2\zeta_9^5 & -2\zeta_9^5 & -2\zeta_9^7 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{bmatrix}$$

- $[0, 0, 0, 2/9, -1/9, -4/9, -4/9, -1/9, 2/9, 1/2]$, 9, 18, 4, $[1, 0, 0, 0, 0, 0, 0, 0, 0, 1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\
1 & 1 & 1 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_3 & 2\zeta_3^2 & 3 \\
1 & 1 & 1 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_3^2 & 2\zeta_3 & 3 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2\zeta_9^5 & 2\zeta_9^4+2\zeta_9^7 & -2\zeta_9^7 & -2\zeta_9^2 & 2\zeta_9^2+2\zeta_9^5 & -2\zeta_9^4 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_9^4+2\zeta_9^7 & -2\zeta_9^2 & -2\zeta_9^5 & -2\zeta_9^4 & -2\zeta_9^7 & 2\zeta_9^2+2\zeta_9^5 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2\zeta_9^7 & -2\zeta_9^5 & 2\zeta_9^2+2\zeta_9^5 & 2\zeta_9^4+2\zeta_9^7 & -2\zeta_9^4 & -2\zeta_9^2 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2\zeta_9^2 & -2\zeta_9^4 & 2\zeta_9^4+2\zeta_9^7 & 2\zeta_9^2+2\zeta_9^5 & -2\zeta_9^5 & -2\zeta_9^7 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_9^2+2\zeta_9^5 & -2\zeta_9^7 & -2\zeta_9^4 & -2\zeta_9^5 & -2\zeta_9^2 & 2\zeta_9^4+2\zeta_9^7 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2\zeta_9^4 & 2\zeta_9^2+2\zeta_9^5 & -2\zeta_9^2 & -2\zeta_9^7 & 2\zeta_9^4+2\zeta_9^7 & -2\zeta_9^5 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{bmatrix}$$

12.1.4. *Rank* 10*, type* $[1, 1, 1, 2, 2, 2, 2, 2, 2, 3]$*, second fusion ring.*

$$\begin{matrix}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\end{matrix},\quad
\begin{matrix}0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\end{matrix},\quad
\begin{matrix}0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\end{matrix},\quad
\begin{matrix}0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0&1&0\\0&0&0&0&1&1&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix},\quad
\begin{matrix}0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&1&0&0\\0&0&0&1&0&0&1&0&0&0\\0&1&0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix},$$

$$\begin{matrix}0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&0&0&1&1&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&1&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix},\quad
\begin{matrix}0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&1&1&0&0&0&0\\0&0&0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix},\quad
\begin{matrix}0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&1&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&1&0\\0&0&0&0&1&0&0&0&1&0\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\end{matrix},\quad
\begin{matrix}0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&1&0&0\\0&0&0&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&0&0&0\\0&0&0&0&1&1&0&0&1&0\end{matrix},\quad
\begin{matrix}0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\\0&0&0&0&1&1&0&0&1&0\\0&0&0&1&0&0&1&1&0&0\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&1&0&0&1&0\\1&1&1&0&0&0&0&0&0&2\end{matrix}$$

- $[0, 0, 0, 0, 1/3, -1/3, -1/3, 1/3, 0, 1/2]$, 3, 6, 4, $[1, 0, 0, 0, 0, 0, 0, 0, 0, 1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\
1 & 1 & 1 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_3^2 & 2\zeta_3 & 3 \\
1 & 1 & 1 & 2\zeta_3 & 2\zeta_3^2 & 2\zeta_3^2 & 2\zeta_3 & 2\zeta_3 & 2\zeta_3^2 & 3 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2 & -2\zeta_3 & -2\zeta_3^2 & -2\zeta_3 & -2\zeta_3^2 & -2 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2\zeta_3 & -2\zeta_3 & -2 & -2 & -2\zeta_3^2 & -2\zeta_3^2 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2\zeta_3^2 & -2 & -2\zeta_3^2 & -2\zeta_3 & -2 & -2\zeta_3 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2\zeta_3 & -2 & -2\zeta_3 & -2\zeta_3^2 & -2 & -2\zeta_3^2 & 0 \\
2 & 2\zeta_3^2 & 2\zeta_3 & -2\zeta_3^2 & -2\zeta_3^2 & -2 & -2 & -2\zeta_3 & -2\zeta_3 & 0 \\
2 & 2\zeta_3 & 2\zeta_3^2 & -2 & -2\zeta_3^2 & -2\zeta_3 & -2\zeta_3^2 & -2\zeta_3 & -2 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{bmatrix}$$

12.1.5. *Rank* 11*,* $[1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2]$.

$$\begin{matrix}1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\end{matrix},\quad
\begin{matrix}0&1&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\end{matrix},\quad
\begin{matrix}0&0&1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&1&0&0&0&0&0&0\end{matrix},\quad
\begin{matrix}0&0&0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1\end{matrix},\quad
\begin{matrix}0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0&0&0&0\\1&0&0&1&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&1&0&0\\0&0&0&0&0&1&0&0&0&1&0\\0&0&0&0&1&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&0&1&0\\0&0&0&0&0&0&1&0&1&0&0\\0&1&1&0&0&0&0&1&0&0&0\end{matrix},$$

$$\begin{matrix} 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 1&0&1&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&1&0&1&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0 \end{matrix},\quad \begin{matrix} 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 1&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 0&0&1&1&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1&0 \end{matrix},\quad \begin{matrix} 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 1&1&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&1 \end{matrix},$$

$$\begin{matrix} 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&1&1&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 1&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1&0 \end{matrix},\quad \begin{matrix} 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 0&1&0&1&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0 \end{matrix},\quad \begin{matrix} 0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1\\ 0&1&1&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&1&0&0\\ 1&0&0&1&0&0&0&1&0&0&0 \end{matrix}$$

- $[0,0,0,0,-7/16,-3/16,-3/16,1/4,5/16,5/16,1/16]$, 8, 16, 1, $[1,1,1,1,1,-1,-1,1,-1,-1,1]$,

$$\begin{bmatrix} 1&1&1&1&2&2&2&2&2&2&2\\ 1&1&1&1&-2&-2&2&2&2&-2&-2\\ 1&1&1&1&-2&2&-2&2&-2&2&-2\\ 1&1&1&1&2&-2&-2&2&-2&-2&2\\ 2&-2&-2&2&2\zeta_8-2\zeta_8^3&0&0&0&0&0&-2\zeta_8+2\zeta_8^3\\ 2&-2&2&-2&0&-2\zeta_8+2\zeta_8^3&0&0&0&2\zeta_8-2\zeta_8^3&0\\ 2&2&-2&-2&0&0&-2\zeta_8+2\zeta_8^3&0&2\zeta_8-2\zeta_8^3&0&0\\ 2&2&2&2&0&0&0&-4&0&0&0\\ 2&2&-2&-2&0&0&2\zeta_8-2\zeta_8^3&0&-2\zeta_8+2\zeta_8^3&0&0\\ 2&-2&2&-2&0&2\zeta_8-2\zeta_8^3&0&0&0&-2\zeta_8+2\zeta_8^3&0\\ 2&-2&-2&2&-2\zeta_8+2\zeta_8^3&0&0&0&0&0&2\zeta_8-2\zeta_8^3 \end{bmatrix}$$

- $[0,0,0,0,-7/16,-3/16,1/16,1/4,-7/16,5/16,1/16]$, 8, 16, 1, $[1,1,1,1,-1,1,-1,1,-1,1,-1]$,

$$\begin{bmatrix} 1&1&1&1&2&2&2&2&2&2&2\\ 1&1&1&1&-2&-2&2&2&2&-2&-2\\ 1&1&1&1&-2&2&-2&2&-2&2&-2\\ 1&1&1&1&2&-2&-2&2&-2&-2&2\\ 2&-2&-2&2&2\zeta_8-2\zeta_8^3&0&0&0&0&0&-2\zeta_8+2\zeta_8^3\\ 2&-2&2&-2&0&-2\zeta_8+2\zeta_8^3&0&0&0&2\zeta_8-2\zeta_8^3&0\\ 2&2&-2&-2&0&0&2\zeta_8-2\zeta_8^3&0&-2\zeta_8+2\zeta_8^3&0&0\\ 2&2&2&2&0&0&0&-4&0&0&0\\ 2&2&-2&-2&0&0&-2\zeta_8+2\zeta_8^3&0&2\zeta_8-2\zeta_8^3&0&0\\ 2&-2&2&-2&0&2\zeta_8-2\zeta_8^3&0&0&0&-2\zeta_8+2\zeta_8^3&0\\ 2&-2&-2&2&-2\zeta_8+2\zeta_8^3&0&0&0&0&0&2\zeta_8-2\zeta_8^3 \end{bmatrix}$$

- $[0,0,0,0,-7/16,1/16,1/16,1/4,-7/16,-7/16,1/16]$, 8, 16, 1, $[1,1,1,1,1,1,1,1,1,1,1]$,

$$\begin{bmatrix} 1&1&1&1&2&2&2&2&2&2&2\\ 1&1&1&1&-2&-2&2&2&2&-2&-2\\ 1&1&1&1&-2&2&-2&2&-2&2&-2\\ 1&1&1&1&2&-2&-2&2&-2&-2&2\\ 2&-2&-2&2&2\zeta_8-2\zeta_8^3&0&0&0&0&0&-2\zeta_8+2\zeta_8^3\\ 2&-2&2&-2&0&2\zeta_8-2\zeta_8^3&0&0&0&-2\zeta_8+2\zeta_8^3&0\\ 2&2&-2&-2&0&0&2\zeta_8-2\zeta_8^3&0&-2\zeta_8+2\zeta_8^3&0&0\\ 2&2&2&2&0&0&0&-4&0&0&0\\ 2&2&-2&-2&0&0&-2\zeta_8+2\zeta_8^3&0&2\zeta_8-2\zeta_8^3&0&0\\ 2&-2&2&-2&0&-2\zeta_8+2\zeta_8^3&0&0&0&2\zeta_8-2\zeta_8^3&0\\ 2&-2&-2&2&-2\zeta_8+2\zeta_8^3&0&0&0&0&0&2\zeta_8-2\zeta_8^3 \end{bmatrix}$$

- $[0,0,0,0,-3/16,5/16,5/16,1/4,-3/16,-3/16,5/16]$, 8, 16, 1, $[1,1,1,1,1,1,1,1,1,1,1]$,

$$\begin{bmatrix} 1&1&1&1&2&2&2&2&2&2&2\\ 1&1&1&1&-2&-2&2&2&2&-2&-2\\ 1&1&1&1&-2&2&-2&2&-2&2&-2\\ 1&1&1&1&2&-2&-2&2&-2&-2&2\\ 2&-2&-2&2&-2\zeta_8+2\zeta_8^3&0&0&0&0&0&2\zeta_8-2\zeta_8^3\\ 2&-2&2&-2&0&-2\zeta_8+2\zeta_8^3&0&0&0&2\zeta_8-2\zeta_8^3&0\\ 2&2&-2&-2&0&0&-2\zeta_8+2\zeta_8^3&0&2\zeta_8-2\zeta_8^3&0&0\\ 2&2&2&2&0&0&0&-4&0&0&0\\ 2&2&-2&-2&0&0&2\zeta_8-2\zeta_8^3&0&-2\zeta_8+2\zeta_8^3&0&0\\ 2&-2&2&-2&0&2\zeta_8-2\zeta_8^3&0&0&0&-2\zeta_8+2\zeta_8^3&0\\ 2&-2&-2&2&2\zeta_8-2\zeta_8^3&0&0&0&0&0&-2\zeta_8+2\zeta_8^3 \end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, 7/16, 7/16, -1/4, -1/16, -1/16, 3/16]$, 8, 16, -1, $[1, 1, 1, 1, 1, -1, -1, 1, -1, -1, 1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\
2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 \\
2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\
2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\
2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3
\end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, 7/16, -5/16, -1/4, 3/16, -1/16, 3/16]$, 8, 16, -1, $[1, 1, 1, 1, -1, 1, -1, 1, -1, 1, -1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\
2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 \\
2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\
2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\
2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3
\end{bmatrix}$$

- $[0, 0, 0, 0, -5/16, -5/16, -5/16, -1/4, 3/16, 3/16, 3/16]$, 8, 16, -1, $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\
2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 \\
2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\
2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\
2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3
\end{bmatrix}$$

- $[0, 0, 0, 0, -1/16, -1/16, -1/16, -1/4, 7/16, 7/16, 7/16]$, 8, 16, -1, $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$,

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & -2 & -2 & 2 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 2 & -2 & -2 & 2 & -2 & -2 & 2 \\
2 & -2 & -2 & 2 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 \\
2 & -2 & 2 & -2 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
2 & 2 & -2 & -2 & 0 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 2\zeta_8-2\zeta_8^3 & 0 & 0 \\
2 & -2 & 2 & -2 & 0 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3 & 0 \\
2 & -2 & -2 & 2 & -2\zeta_8+2\zeta_8^3 & 0 & 0 & 0 & 0 & 0 & 2\zeta_8-2\zeta_8^3
\end{bmatrix}$$

12.2. **Pointed Modular Data Up To Rank 13.** In light of Remark 1.2, it suffices to specify only the Abelian groups and their corresponding T-matrices, which encapsulate the topological spins in this context. Let us denote by C_n the cyclic group of order n. The complete data set, akin to the format presented in §12.1, is available at [41].

- C_1: [0],
- C_2: [0, 1/4], [0, -1/4],
- C_3: [0, 1/3, 1/3], [0, -1/3, -1/3],
- C_2^2: [0, 1/2, 0, 0], [0, -1/4, 1/4, 0], [0, 1/2, 1/4, 1/4], [0, 1/2, 1/2, 1/2], [0, -1/4, -1/4, 1/2],

- C_4: [0, 1/2, 1/8, 1/8], [0, 1/2, 3/8, 3/8], [0, 1/2, -3/8, -3/8], [0, 1/2, -1/8, -1/8],
- C_5: [0, 1/5, -1/5, -1/5, 1/5], [0, 2/5, -2/5, -2/5, 2/5],
- C_6: [0, -1/4, 1/12, 1/3, 1/3, 1/12], [0, -1/4, 5/12, -1/3, -1/3, 5/12], [0, 1/4, -5/12, 1/3, 1/3, -5/12], [0, 1/4, -1/12, -1/3, -1/3, -1/12],
- C_7: [0, 1/7, -3/7, 2/7, 2/7, -3/7, 1/7], [0, 3/7, -2/7, -1/7, -1/7, -2/7, 3/7],
- C_2^3: [0, -1/4, 1/4, 1/4, 1/4, 1/2, 0, 0], [0, -1/4, -1/4, -1/4, 1/4, 1/2, 0, 0], [0, -1/4, 1/2, 1/2, 1/4, 1/2, 1/4, 1/4], [0, -1/4, 1/2, -1/4, 1/2, -1/4, 1/2, 1/4],
- $C_2 \times C_4$: [0, 1/2, -1/4, 1/4, 1/8, 3/8, 3/8, 1/8], [0, 1/2, 1/4, -1/4, 1/8, -1/8, -1/8, 1/8], [0, 1/2, -1/4, 1/4, 3/8, -3/8, -3/8, 3/8], [0, 1/2, -1/4, 1/4, -3/8, -1/8, -1/8, -3/8],
- C_8: [0, 0, 1/4, -7/16, 1/16, 1/16, -7/16, 1/4], [0, 0, 1/4, -3/16, 5/16, 5/16, -3/16, 1/4], [0, 0, -1/4, -5/16, 3/16, 3/16, -5/16, -1/4], [0, 0, -1/4, -1/16, 7/16, 7/16, -1/16, -1/4],
- C_3^2: [0, 0, 0, 1/3, -1/3, -1/3, 1/3, 0, 0], [0, 1/3, 1/3, -1/3, -1/3, -1/3, -1/3, 1/3, 1/3],
- C_9: [0, 0, 1/9, -2/9, 4/9, 4/9, -2/9, 1/9, 0], [0, 0, 2/9, -4/9, -1/9, -1/9, -4/9, 2/9, 0],
- C_{10}: [0, 1/4, 1/20, -1/5, 1/5, 9/20, 9/20, 1/5, -1/5, 1/20], [0, -1/4, 3/20, 2/5, -2/5, 7/20, 7/20, -2/5, 2/5, 3/20], [0, -1/4, -9/20, -1/5, 1/5, -1/20, -1/20, 1/5, -1/5, -9/20], [0, 1/4, -7/20, 2/5, -2/5, -3/20, -3/20, -2/5, 2/5, -7/20],
- C_{11}: [0, 1/11, 4/11, -2/11, 5/11, 3/11, 3/11, 5/11, -2/11, 4/11, 1/11], [0, 2/11, -3/11, -4/11, -1/11, -5/11, -5/11, -1/11, -4/11, -3/11, 2/11],
- $C_2 \times C_6$: [0, -1/4, 0, 1/4, -1/3, 5/12, -1/3, -1/12, -1/12, -1/3, 5/12, -1/3],[0, 1/2, -1/4, -1/4, -1/3, 1/6, 5/12, 5/12, 5/12, 5/12, 1/6, -1/3], [0, 1/2, 1/2, 1/2, -1/3, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, -1/3], [0, 1/2, 1/4, 1/4, -1/3, 1/6, -1/12, -1/12, -1/12, -1/12, 1/6, -1/3], [0, 1/2, 0, 0, -1/3, 1/6, -1/3, -1/3, -1/3, -1/3, 1/6, -1/3], [0, 1/2, 1/2, 1/2, 1/3, -1/6, -1/6, -1/6, -1/6, -1/6, -1/6, 1/3], [0, 1/4, 1/2, 1/4, 1/3, -5/12, -1/6, -5/12, -5/12, -1/6, -5/12, 1/3],[0, 0, 1/2, 0, 1/3, 1/3, -1/6, 1/3, 1/3, -1/6, 1/3, 1/3],[0, -1/4, 0, 1/4, 1/3, 1/12, 1/3, -5/12, -5/12, 1/3, 1/12, 1/3], [0, -1/4, 1/2, -1/4, 1/3, 1/12, -1/6, 1/12, 1/12, -1/6, 1/12, 1/3],
- C_{12}: [0, 1/2, -1/8, -1/3, 1/6, -11/24, -11/24, -11/24, -11/24, 1/6, -1/3, -1/8], [0, 1/2, -1/8, 1/3, -1/6, 5/24, 5/24, 5/24, 5/24, -1/6, 1/3, -1/8], [0, 1/2, -3/8, -1/3, 1/6, 7/24, 7/24, 7/24, 7/24, 1/6, -1/3, -3/8], [0, 1/2, -3/8, 1/3, -1/6, -1/24, -1/24, -1/24, -1/24, -1/6, 1/3, -3/8], [0, 1/2, 3/8, -1/3, 1/6, 1/24, 1/24, 1/24, 1/24, 1/6, -1/3, 3/8], [0, 1/2, 3/8, 1/3, -1/6, -7/24, -7/24, -7/24, -7/24, -1/6, 1/3, 3/8], [0, 1/2, 1/8, -1/3, 1/6, -5/24, -5/24, -5/24, -5/24, 1/6, -1/3, 1/8], [0, 1/2, 1/8, 1/3, -1/6, 11/24, 11/24, 11/24, 11/24, -1/6, 1/3, 1/8],
- C_{13}: [0, 1/13, 4/13, -4/13, 3/13, -1/13, -3/13, -3/13, -1/13, 3/13, -4/13, 4/13, 1/13], [0, 2/13, -5/13, 5/13, 6/13, -2/13, -6/13, -6/13, -2/13, 6/13, 5/13, -5/13, 2/13].

13. Non-Pointed Odd-Dimensional Modular Data of Rank 17

The data are organized as in §12.1. Additionally, it is accessible in a format compatible with computers within the file `Rank17MNSD.txt` in [41]. About their categorification, see Remark 1.10, 11.6 and 11.7.

13.1. **Type [[1,3],[3,8],[5,6]], First Fusion Ring.**

The whole fusion data would be too big to be entirely displayed here. So here is a compressed version. Consider its basis

$$B = \{b_g\}_{g \in C_3} \cup \{x_i, x_i^*\}_{i \in \{1,2,3,4\}} \cup \{y_g, y_g^*\}_{g \in C_3},$$

where $\mathrm{FPdim}(b_g) = 1$, $\mathrm{FPdim}(x_i) = 3$ and $\mathrm{FPdim}(y_g) = 5$. Here are the fusion rules:

- $b_{g_1} b_{g_2} = b_{g_1 g_2}$, $b_g^* = b_{g^{-1}}$,
- $b_g x_i = x_i$,
- $b_{g_1} y_{g_2} = y_{g_1 g_2}$,
- $x_i x_j$ and $x_i x_j^* - \delta_{i,j} \sum_{g \in C_3} b_g$ are sum of x_k or x_k^*, with multiplicites written below (see also Remark 11.7):

$$\begin{matrix} 0&2&0&0&0&0&0&1 \\ 0&0&1&1&0&0&0&0 \\ 1&0&0&0&1&0&1&0 \\ 1&0&0&0&0&1&1&0 \\ 0&0&1&0&1&0&0&1 \\ 0&0&0&1&0&1&0&1 \\ 0&1&0&0&1&1&0&0 \\ 0&0&1&1&0&0&1&0 \end{matrix}, \begin{matrix} 0&0&1&1&0&0&0&0 \\ 2&0&0&0&0&0&1&0 \\ 0&1&0&0&1&0&0&1 \\ 0&1&0&0&0&1&0&1 \\ 0&0&1&0&1&0&1&0 \\ 0&0&0&1&0&1&1&0 \\ 0&0&1&1&0&0&0&1 \\ 1&0&0&0&1&1&0&0 \end{matrix}, \begin{matrix} 1&0&0&0&1&0&1&0 \\ 0&1&0&0&1&0&0&1 \\ 0&0&0&2&1&0&0&0 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&1&1&1 \\ 1&1&0&1&0&0&0&0 \\ 1&0&1&0&0&1&0&0 \\ 0&1&1&0&0&1&0&0 \end{matrix}, \begin{matrix} 1&0&0&0&0&1&1&0 \\ 0&1&0&0&0&1&0&1 \\ 0&0&0&0&0&0&1&1 \\ 0&0&2&0&0&1&0&0 \\ 1&1&1&0&0&0&0&0 \\ 0&0&0&0&1&0&1&1 \\ 1&0&0&1&1&0&0&0 \\ 0&1&0&1&1&0&0&0 \end{matrix}, \begin{matrix} 0&0&1&0&1&0&0&1 \\ 0&0&1&0&1&0&1&0 \\ 0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0 \\ 0&0&0&1&0&2&0&0 \\ 1&1&0&0&0&0&0&0 \\ 0&1&0&1&0&0&1&0 \\ 1&0&0&1&0&0&0&1 \end{matrix}, \begin{matrix} 0&0&0&1&0&1&0&1 \\ 0&0&0&1&0&1&1&0 \\ 1&1&0&1&0&0&0&0 \\ 0&0&0&0&1&0&1&1 \\ 1&1&0&0&0&0&0&0 \\ 0&0&1&0&2&0&0&0 \\ 0&1&1&0&0&0&1&0 \\ 1&0&1&0&0&0&0&1 \end{matrix}, \begin{matrix} 0&1&0&0&1&1&0&0 \\ 0&0&1&1&0&0&0&1 \\ 1&0&1&0&0&1&0&0 \\ 1&0&0&1&1&0&0&0 \\ 0&1&0&1&0&0&1&0 \\ 0&1&1&0&0&0&1&0 \\ 1&0&0&0&0&0&0&2 \\ 0&0&0&0&1&1&0&0 \end{matrix}, \begin{matrix} 0&0&1&1&0&0&1&0 \\ 1&0&0&0&1&1&0&0 \\ 0&1&1&0&0&1&0&0 \\ 0&1&0&1&1&0&0&0 \\ 1&0&0&1&0&0&0&1 \\ 1&0&1&0&0&0&0&1 \\ 0&0&0&0&1&1&0&0 \\ 0&1&0&0&0&0&2&0 \end{matrix},$$

- $x_i y_g = x_i^* y_g = \sum_{g \in C_3} y_g$,
- $y_{g_1} y_{g_2} = y^*_{(g_1 g_2)^{-1}} + 2 \sum_{g_1 g_2 g \neq e} y_g^*$,
- $y_{g_1} y_{g_2}^* = b_{g_1 g_2^{-1}} + \sum_i (x_i + x_i^*)$,

the other rules follows by commutativity and duality. Now we write the rest of the modular data as in §12.1:

- $[0, 0, 0, -2/5, -2/5, -1/5, -1/5, 1/5, 1/5, 2/5, 2/5, -1/3, -1/3, 0, 0, 1/3, 1/3]$, 15, 15, 4, $[1, 0, \ldots, 0]$,

$$\begin{bmatrix}
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 5 & 5 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 \\
3 & 3 & 3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_3^2 & -5\zeta_3 & -5\zeta_3 & -5\zeta_3^2 & -5 & -5 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_3 & -5\zeta_3^2 & -5\zeta_3^2 & -5\zeta_3 & -5 & -5 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_3 & -5\zeta_3^2 & -5 & -5 & -5\zeta_3^2 & -5\zeta_3 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_3^2 & -5\zeta_3 & -5 & -5 & -5\zeta_3 & -5\zeta_3^2 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & -5 & -5\zeta_3^2 & -5\zeta_3 & -5\zeta_3 & -5\zeta_3^2 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & -5 & -5\zeta_3 & -5\zeta_3^2 & -5\zeta_3^2 & -5\zeta_3
\end{bmatrix}$$

13.2. **Type [[1,3],[3,8],[5,6]], Second Fusion Ring.** The basis and fusion rules are the same as in §13.1, except that there is $h \in C_3 \setminus \{e\}$ such that

- $y_{g_1} y_{g_2} = b_h (y^*_{(g_1 g_2)^{-1}} + 2 \sum_{g_1 g_2 g \neq e} y^*_g)$,

which sounds like a zesting. The element b_h can equivalently be the second or the third basic element (here we chose the second one).

- $[0,0,0,-2/5,-2/5,-1/5,-1/5,1/5,1/5,2/5,2/5,-2/9,-2/9,1/9,1/9,4/9,4/9]$, 45, 45, 4, $[1,0,\dots,0]$,

$$\begin{bmatrix}
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 5 & 5 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 \\
3 & 3 & 3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^4 & -5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & -5\zeta_9^7 & -5\zeta_9^2 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^5 & -5\zeta_9^4 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & -5\zeta_9^2 & -5\zeta_9^7 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & -5\zeta_9^7 & -5\zeta_9^2 & -5\zeta_9^4 & -5\zeta_9^5 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & -5\zeta_9^2 & -5\zeta_9^7 & -5\zeta_9^5 & -5\zeta_9^4 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^7 & -5\zeta_9^2 & -5\zeta_9^4 & -5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^2 & -5\zeta_9^7 & -5\zeta_9^5 & -5\zeta_9^4 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7
\end{bmatrix}$$

- $[0,0,0,-2/5,-2/5,-1/5,-1/5,1/5,1/5,2/5,2/5,-4/9,-4/9,-1/9,-1/9,2/9,2/9]$, 45, 45, 4, $[1,0,\dots,0]$,

$$\begin{bmatrix}
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 5 & 5 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 & 5\zeta_3 & 5\zeta_3^2 \\
3 & 3 & 3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 3\zeta_5^2+6\zeta_5^4 & 6\zeta_5+3\zeta_5^3 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 3\zeta_5+6\zeta_5^2 & 6\zeta_5^3+3\zeta_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 6\zeta_5+3\zeta_5^3 & 3\zeta_5^2+6\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5-3\zeta_5^4 & -3\zeta_5^2-3\zeta_5^3 & -3\zeta_5^2-3\zeta_5^3 & 6\zeta_5^3+3\zeta_5^4 & 3\zeta_5+6\zeta_5^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & -5\zeta_9^5 & -5\zeta_9^4 & -5\zeta_9^2 & -5\zeta_9^7 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & -5\zeta_9^4 & -5\zeta_9^5 & -5\zeta_9^7 & -5\zeta_9^2 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^5 & -5\zeta_9^4 & -5\zeta_9^2 & -5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^4 & -5\zeta_9^5 & -5\zeta_9^7 & -5\zeta_9^2 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 \\
5 & 5\zeta_3^2 & 5\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^2 & -5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & 5\zeta_9^4+5\zeta_9^7 & -5\zeta_9^5 & -5\zeta_9^4 \\
5 & 5\zeta_3 & 5\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\zeta_9^7 & -5\zeta_9^2 & 5\zeta_9^4+5\zeta_9^7 & 5\zeta_9^2+5\zeta_9^5 & -5\zeta_9^4 & -5\zeta_9^5
\end{bmatrix}$$

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