

# APPENDICES OF CLASSIFYING INTEGRAL GROTHENDIECK RINGS UP TO RANK 5 AND BEYOND

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## Appendix A. INTEGRAL DRINFELD RINGS

**A.1. Up to rank 5.** This section presents the comprehensive list of integral Drinfeld rings up to rank 5—including their global FPdim, type, duality, formal codegrees, and fusion data. Copy-pastable data can be found in the file `GeneralUpToRank5DataOnly.txt`, located in the `Data/General` directory of [7]. For each case, either an explicit categorification is provided, or a reference is given to a theoretical result ruling out its existence.

A.1.1. *Rank 1.* Trivial case

A.1.2. *Rank 2.*

(1) FPdim 2, type  $[1, 1]$ , duality  $[0, 1]$ , fusion data:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[2, 2]$ ,
- Property: simple,
- Categorification:  $\text{Rep}(C_2)$ .

A.1.3. *Rank 3.*

(1) FPdim 3, type  $[1, 1, 1]$ , duality  $[0, 2, 1]$ , fusion data:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[3, 3, 3]$ ,
- Property: simple,
- Categorification:  $\text{Rep}(C_3)$ .

(2) FPdim 6, type  $[1, 1, 2]$ , duality  $[0, 1, 2]$ , fusion data:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Formal codegrees:  $[2, 3, 6]$ ,
- Categorification:  $\text{Rep}(S_3)$ .

A.1.4. *Rank 4.*

(1) FPdim 4, type  $[1, 1, 1, 1]$ , duality  $[0, 1, 2, 3]$ , fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Formal codegrees:  $[4, 4, 4, 4]$ ,
- Categorification:  $\text{Rep}(C_2^2)$ .

(2) FPdim 4, type  $[1, 1, 1, 1]$ , duality  $[0, 1, 2, 3]$ , fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Formal codegrees:  $[4, 4, 4, 4]$ ,
- Categorification:  $\text{Rep}(C_4)$ .

(3) FPdim 10, type  $[1, 1, 2, 2]$ , duality  $[0, 1, 2, 3]$ , fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- Formal codegrees:  $[2, 5, 5, 10]$ ,
- Categorification:  $\text{Rep}(D_5)$ .



































- Formal codegrees:  $[3, 3, 6, 8, 42, 57, 3192]$ ,
- Property: simple, non-1-Frobenius, non-3-positive
- Categorification: open, non-braided, non-unitary

(5) FPdim 4284, type  $[1, 17, 17, 20, 28, 35, 36]$ , duality  $[0, 2, 1, 3, 4, 5, 6]$ , fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 3 & 2 & 3 & 4 & 5 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 & 6 & 4 & 4 \\ 0 & 3 & 3 & 3 & 4 & 6 & 6 \\ 0 & 2 & 3 & 3 & 4 & 5 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 & 3 & 3 \\ 0 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 2 & 2 & 2 & 6 & 4 & 4 \\ 1 & 3 & 3 & 6 & 1 & 7 & 8 \\ 0 & 4 & 4 & 4 & 7 & 8 & 8 \\ 0 & 4 & 4 & 4 & 8 & 8 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 2 & 3 & 4 & 5 & 5 \\ 0 & 3 & 3 & 4 & 6 & 6 & 6 \\ 0 & 4 & 4 & 4 & 7 & 8 & 8 \\ 1 & 5 & 5 & 6 & 8 & 10 & 10 \\ 0 & 5 & 5 & 6 & 8 & 10 & 11 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 3 & 3 & 4 & 5 & 5 \\ 0 & 3 & 3 & 4 & 4 & 6 & 6 \\ 0 & 4 & 4 & 4 & 8 & 8 & 8 \\ 0 & 5 & 5 & 6 & 8 & 10 & 11 \\ 1 & 5 & 5 & 6 & 8 & 10 & 11 \end{bmatrix}$$

- Formal codegrees:  $[3, 4, 7, 7, 9, 51, 4284]$ ,
- Property: simple, non-1-Frobenius, non-3-positive
- Categorification: open, non-braided, non-unitary

These fusion rings also invite questions in the spirit of Question 1.13.

**A.4. Ranks 8 and 9.** There are 792 integral 1-Frobenius Drinfeld rings of rank 8 with  $\text{FPdim} \leq 25000$ , and 1292 such rings of rank 9 with  $\text{FPdim} \leq 2000$ . Copy-pastable data can be found in the files `1FrobR8d25000.txt` and `1FrobR9d2000.txt`, located in the `Data/General` directory of [7].

**A.5. Addressing divisibility.** The classification at rank  $r$  begins with the list of length  $r$  Egyptian fractions representing the possible

$$\sum_V \sum_{i=1}^{n_V^2} \frac{1}{n_V f_V} = 1,$$

where the  $(f_V)$  are the formal codegrees of the Drinfeld ring; see §2.2. Under the Drinfeld assumption, each formal codegree  $f_V$  divides  $f_1$ . Moreover, the Drinfeld ring is commutative if and only if  $n_V = 1$  for all  $V$ . In this case, we can restrict to Egyptian fractions of length  $r$  satisfying the divisibility condition  $f_V \mid f_1$  for all  $V$ .

In the noncommutative setting, it may happen that  $n_V f_V$  does not divide  $f_1$  for some  $V$ . However, as verified in Appendix C, no such exceptions occur up to rank 8 (but they do appear at rank 9; see §2.2). Therefore, for ranks  $r \leq 8$ , it is safe to restrict to Egyptian fractions of length  $r$  that satisfy the divisibility condition.

At rank 9, Lemma 9.3 shows that the complexified noncommutative Drinfeld ring must be isomorphic to either  $\mathbb{C} \oplus M_2(\mathbb{C})^2$  or  $\mathbb{C}^5 \oplus M_2(\mathbb{C})$ . The exception corresponds to Egyptian fractions of length 5 or 7 with one or two terms having  $n_V = 2$ , and at least one violating the divisibility condition  $n_V f_V \mid f_1$ .

We verified that for  $\text{FPdim} \leq 32000$ , the values arising from these exceptional cases are already covered by those for the Egyptian fractions of length 9 satisfying the divisibility condition. Details of this computation can be found in the file `InvestNCRank9Except.txt` of the `Data/EgyptianFractionsDiv/Except` folder of [7].

## Appendix B. MNSD DRINFELD RINGS

As established in §8, the Grothendieck ring of any odd-dimensional integral fusion category over  $\mathbb{C}$  is an MNSD integral Drinfeld ring (Definition 8.6). This section provides a complete classification of such rings up to rank 9.

**B.1. Up to rank 5.** There are four MNSD integral Drinfeld rings up to rank 5, contained in §A.1, namely the Grothendieck rings of  $\text{Rep}(G)$ , with  $G = C_1, C_3, C_5, C_7 \rtimes C_3$ .

**B.2. Rank 7.** Here is the complete list of 4 MNSD integral Drinfeld ring of rank 7. Copy-pastable data can be found in the file `MNSDRank7DataOnly.txt.txt`, located in the `Data/Odd` directory of [7].

(1) FPdim 7, type  $[1, 1, 1, 1, 1, 1, 1]$ , duality  $[0, 6, 5, 4, 3, 2, 1]$ , fusion data: see §A.3.1.

(2) FPdim 39, type  $[1, 1, 1, 3, 3, 3, 3]$ , duality  $[0, 2, 1, 6, 5, 4, 3]$ , fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Formal codegrees:  $[3, 3, 13, 13, 13, 13, 39]$ ,
- Categorification:  $\text{Rep}(C_{13} \rtimes C_3)$ .

(3) FPdim 55, type  $[1, 1, 1, 1, 1, 5, 5]$ , duality  $[0, 4, 3, 2, 1, 6, 5]$ , fusion data:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Formal codegrees:  $[5, 5, 5, 5, 11, 11, 55]$ ,
- Categorification:  $\text{Rep}(C_{11} \rtimes C_5)$ .









(3) FPdim 20, type  $[1, 1, 1, 1, 2, 2, 2, 2]$ , duality  $[0, 1, 2, 3, 4, 5, 7, 6]$ , fusion data:

[illegible]

- Formal codegrees:  $[4, 4, 5_2, 20, 20]$ ,
- Property: noncommutative,
- Categorification:  $\mathcal{C}(F_5, 1, D_5, 1)$ ,  $\mathcal{C}(F_5, 1, C_2, 1)$ .

(4) FPdim 20, type  $[1, 1, 1, 1, 2, 2, 2, 2]$ , duality  $[0, 1, 3, 2, 4, 5, 7, 6]$ , fusion data:

[illegible]

- Formal codegrees:  $[4, 4, 5_2, 20, 20]$ ,
- Property: noncommutative,
- Categorification:

(5) FPdim 24, type  $[1, 1, 1, 1, 1, 1, 3, 3]$ , duality  $[0, 1, 2, 3, 5, 4, 6, 7]$ , fusion data:

[illegible]

- Formal codegrees:  $[3_2, 8, 12, 12, 24]$ ,
- Property: noncommutative,
- Categorification: excluded by Lemma 5.3.

(6) FPdim 24, type  $[1, 1, 1, 1, 1, 1, 3, 3]$ , duality  $[0, 1, 2, 3, 5, 4, 6, 7]$ , fusion data:

[illegible]

- Formal codegrees:  $[3_2, 8, 8, 24, 24]$ ,
- Property: noncommutative,
- Categorification:  $\mathcal{C}(S_4, 1, A_4, 1), \mathcal{C}(S_4, 1, S_3, 1)$ .

(7) FPdim 24, type  $[1, 1, 1, 1, 1, 1, 3, 3]$ , duality  $[0, 1, 2, 3, 5, 4, 7, 6]$ , fusion data:

[illegible]

- Formal codegrees:  $[3_2, 8, 12, 12, 24]$ ,
- Property: noncommutative,
- Categorification: excluded by Lemma 5.2.

(8) FPdim 78, type  $[1, 1, 1, 1, 1, 1, 6, 6]$ , duality  $[0, 1, 2, 3, 5, 4, 6, 7]$ , fusion data:

[illegible]

- Formal codegrees:  $[3_2, 6, 13, 13, 78]$ ,
- Property: noncommutative,
- Categorification: excluded by Lemma 5.2.

(9) FPdim 96, type  $[1, 1, 1, 3, 4, 4, 4, 6]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

[illegible]

(10) FPdim 168, type  $[1, 1, 1, 2, 2, 2, 3, 12]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

[illegible]

(11) FPdim 168, type  $[1, 1, 1, 3, 3, 7, 7, 7]$ , duality  $[0, 2, 1, 4, 3, 5, 6, 7]$ , fusion data:

[illegible]

(12) FPdim 240, type  $[1, 1, 1, 1, 1, 1, 3, 15]$ , duality  $[0, 1, 2, 3, 5, 4, 6, 7]$ , fusion data:

[illegible]

(13) FPdim 240, type  $[1, 1, 1, 3, 6, 8, 8, 8]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

[illegible]

(14) FPdim 600, type  $[1, 1, 1, 2, 2, 2, 3, 24]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

[illegible]

(15) FPdim 816, type  $[1, 1, 1, 3, 6, 16, 16, 16]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

[illegible]

- Formal codegrees:  $[3_2, 4, 16, 51, 816]$ ,
- Property: noncommutative,
- Categorification:



(16) FPdim 960, type  $[1, 1, 1, 3, 4, 4, 30]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 1 0
0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 1 0
0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 1 0 0 0 0 0	1 1 1 2 0 0 0 0 0	0 0 0 0 1 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 0 0 3 0
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 1 1 0	1 0 0 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 1 0 1 1 1 1 1 0	0 0 0 0 0 0 0 4 0
0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 1 1 0	1 0 0 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 1 0 1 1 1 1 1 0	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 1 1 0	0 0 1 1 1 1 1 1 0	0 1 0 1 1 1 1 1 0	0 1 0 1 1 1 1 1 0	1 0 0 1 1 1 1 1 0	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 1 0	0 0 0 0 0 0 1 1 0	0 0 1 1 1 1 1 1 0	0 0 1 1 1 1 1 1 0	0 1 0 1 1 1 1 1 0	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	1 1 1 3 4 4 4 4 28

- Formal codegrees:  $[3_2, 4, 15, 64, 960]$ ,
- Property: noncommutative,
- Categorification:

(17) FPdim 1806, type  $[1, 1, 1, 1, 1, 1, 6, 42]$ , duality  $[0, 1, 2, 3, 5, 4, 6, 7]$ , fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0	0 0 0 0 0 1 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	1 1 1 1 1 1 5 0 0	0 0 0 0 0 0 0 0 6
0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 6	1 1 1 1 1 1 6 4 1

- Formal codegrees:  $[3_2, 6, 7, 43, 1806]$ ,
- Property: noncommutative,
- Categorification:

(18) FPdim 2184, type  $[1, 1, 1, 3, 12, 26, 26, 26]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 1 0 0 0 0	1 1 1 2 0 0 0 0 0	0 0 0 0 3 0 0 0 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 3 0 0 0 0	1 1 1 3 1 1 0 0 0	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4
0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	1 0 0 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8	0 1 0 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 1 0 1 4 8 8 8 8	1 0 0 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 0 1 1 4 8 8 8 8	0 1 0 1 4 8 8 8 8	1 0 0 1 4 8 8 8 8	0 0 1 1 4 8 8 8 8

- Formal codegrees:  $[3_2, 4, 13, 168, 2184]$ ,
- Property: noncommutative,
- Categorification:

(19) FPdim 3660, type  $[1, 1, 1, 3, 4, 4, 4, 60]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 1 0 0 0 0	1 1 1 2 0 0 0 0 0	0 0 0 0 3 0 0 0 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 0 0 3 0
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 3 0 0 0 0	1 1 1 3 1 1 0 0 0	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4
0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	1 0 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 1 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 0 1 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 0 1 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	1 1 1 3 4 4 4 4 59

- Formal codegrees:  $[3_2, 4, 15, 61, 3660]$ ,
- Property: noncommutative,
- Categorification:

(20) FPdim 8268, type  $[1, 1, 1, 3, 12, 52, 52, 52]$ , duality  $[0, 2, 1, 3, 4, 5, 6, 7]$ , fusion data:

1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0	0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0	0 0 0 0 1 0 0 0 0	0 0 0 0 1 0 0 0 0	1 1 1 2 0 0 0 0 0	0 0 0 0 3 0 0 0 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0	0 0 0 0 0 1 1 1 0
0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 0 1 0 0 0	0 0 0 0 3 0 0 0 0	1 1 1 3 1 1 0 0 0	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4	0 0 0 0 0 4 4 4 4
0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	1 0 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 1 0 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 0 1 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	0 0 0 0 0 0 0 4 0
0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1	0 0 0 0 0 1 1 1 1	0 0 0 0 0 4 4 4 4	0 0 1 1 4 7 17 17 17	0 0 1 1 4 7 17 17 17	0 1 0 1 4 7 17 17 17	1 0 0 1 4 7 17 17 17

- Formal codegrees:  $[3_2, 4, 13, 159, 8268]$ ,
- Property: noncommutative,
- Categorification:

**C.4. Rank 9.** There are 83 integral 1-Frobenius, noncommutative Drinfeld rings of rank 9 with FPdim  $\leq 10000$ . Copy-pastable data can be found in the file `1FrobR9NCd10000DataOnly.txt`, located in the `Data/Noncommutative` directory of [7].

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